

SPECIAL ISSUE ARTICLE

Wave Dynamics, Integrability and Beyond

The Marchenko Method for Soliton Solutions to the Sawada–Kotera Equation

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Associated with the third-order linear differential operator, we present the Marchenko integral equation using as input the bound-state poles of a transmission coefficient and the time-evolved bound-state dependency constants. We derive the N -soliton solution to the Sawada–Kotera equation, for an arbitrary positive integer N , by recovering that soliton solution from the solution to our Marchenko integral equation. Our method explains the origin of the $2N$ real parameters appearing in the N -soliton solution formula obtained by the ad-hoc method of Hirota. We show that N of those parameters are related to the N bound-state poles of the left transmission coefficient and the remaining N parameters are related to the bound-state dependency constants. Our Marchenko integral equation corresponds to the “GLM (Gel’fand–Levitan–Marchenko) integral equation” Kaup relentlessly but unsuccessfully tried to obtain.

AMS SUBJECT CLASSIFICATION (2020): 34A55, 34M50, 35C08, 37K10**1 | Introduction**

In this paper, we present the derivation of the N -soliton solution, where N is an arbitrary positive integer, to the integrable evolution equation known as the Sawada–Kotera equation [1], that is,

$$Q_t + Q_{xxxxx} + 5Q_x Q_{xx} + 5Q Q_{xxx} + 5Q^2 Q_x = 0, \quad x, t \in \mathbb{R}, \quad (1.1)$$

with x and t being the spacial and temporal independent variables, respectively, taking values on the real line $\mathbb{R}$. The subscripts in (1.1) denote the corresponding partial derivatives. The mathematical significance of our paper comes from the fact that we obtain the N -soliton solution to (1.1) from the solution to a linear integral equation associated with the

Dedicated to the memory of David J. Kaup, who started the study of the inverse scattering problem for the third-order equation.

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inverse scattering problem for the third-order linear differential operator on the full line. We refer to our linear integral equation as the Marchenko integral equation.

A Marchenko integral equation [2–8] is used to solve the inverse scattering problems related to various linear differential operators such as those associated with the KdV (Korteweg–de Vries) equation [9] and the NLS (nonlinear Schrödinger) equation [10]. The Marchenko integral equation for the KdV equation arises in the analysis of the inverse scattering problem for the second-order linear differential operator, and the Marchenko system of linear integral equations arises in the analysis of a linear differential operator associated with a system of first-order equations. Our primary goal in this paper is to establish the Marchenko integral equation corresponding to the N -soliton solution to (1.1) in such a way that our method also applies to other integrable evolution equations associated with the third-order linear differential operator, and those evolution equations include the Kaup–Kupershmidt equation [11–13], the good Boussinesq equation [14], the bad Boussinesq equation [15, 16], and a modified version of the bad Boussinesq equation not containing the second x -derivative term [17, 18].

In order to obtain solutions to (1.1), Kaup initiated [11] the study of the inverse scattering problem for the third-order linear operator. He derived [11] the one-soliton solution to the Sawada–Kotera equation as well as the one-soliton solution to the Kaup–Kupershmidt equation. He tried to formulate the scattering theory for the third-order operator, but as he indicated [11] he was unsuccessful even in defining the scattering coefficients properly. The “scattering matrix” a_{mn} introduced in (2.9)–(2.11) of [11] is not the analog of the scattering matrix $S(k)$ used in the second-order case, but it is the analog of the transition matrix $\Lambda(k)$ used in the second-order case. For the distinction between the scattering matrix $S(k)$ and the transition matrix $\Lambda(k)$ in the second-order case, we refer the reader to (1.2) of [19] for $S(k)$ and to (2.1) of [19] for $\Lambda(k)$. On p. 190 in [11], Kaup stated that “Due to its complexity, at the present time we leave many questions unanswered. We shall point out what these are in a summary at the end.” Kaup tried to deal with various other difficulties. On p. 203 in [11], Kaup stated that “With all of this, we may now obtain integral ‘dispersion relations’ for all of our eigenfunctions in terms of the others and the scattering data. To do this, we shall assume compact support, so that all functions are entire.” For other difficulties Kaup encountered, we refer the reader to Sections 3 and 5 of his 1980 paper [11]. On p. 2706 of [20], Hirota stated that “Kaup [11] tried to solve the inverse scattering problem of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, but was unsuccessful.”

Kaup relentlessly but unsuccessfully tried [12] to find the analog of the Marchenko integral equation corresponding to (1.1). Kaup’s strong desire and unsuccessful efforts in this direction were described in the last section entitled “Unsolved Problems” in his 2002 paper [12]. He stated that “Now we will discuss what the hard problems are. They are hard because they haven’t been solved. This is not a complete list, but is the start of a list of problems in need of solutions. Although we have a solution for the IST of the cubic eigenvalue problem, there still is no equivalent of the GLM equations for this system. What we do have is the solution of the Riemann–Hilbert problem for the eigenfunctions. What is next needed is a representation of the eigenfunctions in terms of some transformation kernels (like the $K(x, y; t)$ for the KdV). Once these are known, then the equivalent GLM equations will follow.” The GLM (Gel’fand–Levitan–Marchenko) equation Kaup wanted to obtain is exactly the Marchenko equation we present in our paper for the Sawada–Kotera equation (1.1) with its N -soliton solutions. The IST of the cubic eigenvalue problem mentioned by Kaup refers to the inverse scattering transform for the third-order linear differential operator.

A Marchenko integral equation is at times referred to as the GLM equation, as Kaup mentioned it in [12], but this is a misnomer [21]. The Gel’fand–Levitan integral equation and the Marchenko integral equation are two separate integral equations [3, 5, 7, 8] used to solve various inverse spectral and scattering problems, respectively. The kernel and the nonhomogeneous term in the Gel’fand–Levitan integral equation [22] are related to the spectral measure [2, 3, 5, 7, 8, 22]. The Gel’fand–Levitan integral equation is normally used to solve an inverse problem on a half line or on a finite interval, and the integral in the Gel’fand–Levitan integral equation is over a finite interval such as $[0, x]$. On the other hand, the kernel and the nonhomogeneous term in the Marchenko integral equation [4, 6, 23] are related to the Fourier transform of the scattering matrix or of a reflection coefficient and the bound-state information [2, 3, 5, 7, 8, 22]. The Marchenko integral equation is used to solve an inverse scattering problem either on a half line or on the full line, and the integral in the Marchenko equation is over a semi-infinite interval containing either $+\infty$ or $-\infty$ at one of its end points.

The N -soliton solution to the Sawada–Kotera equation was first provided by Hirota [20], in which the dressing method [24] of Shabat and Zakharov was used. In [20], Hirota formulated an integral equation whose solution yields soliton solutions to the BKP (B-type Kadomtsev–Petviashvili) equation. In that N -soliton solution [20], by letting $y = 0$ and using a partial integration in the x -variable, one obtains the N -soliton solution to the Sawada–Kotera equation (1.1). However, Hirota’s integral equation presented in [20] is not the analog of a Marchenko integral equation for which Kaup was looking. There are two main issues with Hirota’s integral equation. The first issue is that its integrand involves Hirota’s bilinear form [25]. The second issue is that the kernel and the nonhomogeneous term in that integral equation are specified in an

ad-hoc manner with no connection to the third-order linear differential operator or to any physical quantities. In other words, the input to Hirota's integral equation does not use any scattering data set related to the third-order operator. Unsatisfied with the complications due to the use of Hirota's bilinear form, Hereman and Nuseir [26] provided a simplified version of Hirota's method without needing a bilinear representation. Although the N -soliton solution is formulated by the simplified version of Hirota's bilinear method, it still remains an ad-hoc method, and the parameters appearing in the N -soliton solution have not been shown to be related to any physical parameters. The unsatisfactory aspects of Hirota's integral equation in [20] motivated Parker [27] to try to use a modification of the dressing method so that Hirota's integral equation could be transformed into a linear integral equation resembling a Marchenko integral equation. Although Parker was not successful, his paper [27] is still informative and relevant.

We refer the reader to the papers [11, 17, 28–30] for the general study where the scattering and inverse scattering problems associated with the third-order linear operator are analyzed. We recently presented a method [28, 31] for the derivation of the N -soliton solution formula for the Sawada–Kotera equation by analyzing the inverse scattering problem for the third-order linear differential operator. In this method, by formulating a Riemann–Hilbert problem in the complex plane and by solving that Riemann–Hilbert problem in a closed form, we obtain an explicit expression for the N -soliton solution, for any positive integer N , to the Sawada–Kotera equation. The input used to solve the aforementioned Riemann–Hilbert problem consists of the bound-state information for the third-order operator in the reflectionless case, namely, the bound-state values of the spectral parameter and the bound-state dependency constants. A summary of the derivation of that N -soliton solution formula via our formulation of the Riemann–Hilbert problem is presented in Section 4 in our present paper. We remark that, in solving inverse scattering problems, it is customary [3, 5, 7] to formulate a Riemann–Hilbert problem by recovering a sectionally analytic function from the jump determined by the scattering coefficients and the bound-state information. In our own formulation of the Riemann–Hilbert problem for the third-order linear operator, we recover a sectionally meromorphic function from the jump solely determined by the reflection coefficients. This approach, that is, the detachment of the bound-state information from the jump, offers a significant advantage in solving the inverse scattering problem for the third-order linear operator in the reflectionless case.

In the present paper, by using the appropriate Fourier transformation on our aforementioned Riemann–Hilbert problem formulated in [28, 31], we derive our Marchenko integral equation associated with the N -soliton solution to the Sawada–Kotera equation. The input to our Marchenko integral equation is the same input used for our Riemann–Hilbert problem. In fact, we show that our relevant input data set can be expressed in terms of $2N$ positive parameters, as in the case where a typical N -soliton solution formula to an integrable evolution equation is expressed in a closed form by using $2N$ parameters associated with N simple bound-state energy values and N time-evolved bound-state normalization constants. We refer to this procedure as the Marchenko method as it is similar to the procedure used in the Marchenko method to solve other integrable evolution equations [32–36] via the inverse scattering transform method [37]. In this case, our Marchenko integral equation has a separable kernel, and hence, it can be solved explicitly in a closed form by using the methods of linear algebra. We then show how the N -soliton solution formula is obtained from the solution to our Marchenko integral equation. By comparing the N -soliton solution formula obtained from the solution to our Riemann–Hilbert problem or the solution to our Marchenko integral equation with the corresponding expression obtained from Hirota's method, we reveal how the parameters in Hirota's method are related to the bound-state information for the third-order linear operator in the reflectionless case.

Our paper is organized as follows. In Section 2, we provide a summary of the connection between the Sawada–Kotera equation and the third-order linear differential operator, by making the distinction between the two cases SK1 and SK2, where the two potentials in the linear equation are related to each other in two specific manners. In Section 3, we briefly describe the scattering problem for the third-order linear operator in the reflectionless case, and we indicate how the two potentials are obtained from the large spectral asymptotics of the so-called left and right Jost solutions to the third-order linear equation. In Section 4, we establish our Riemann–Hilbert problem associated with the inverse scattering problem for the third-order linear operator in the reflectionless case. We use as input the input data set consisting of the bound-state poles and the bound-state dependency constants, and we show how the solution to the Riemann–Hilbert problem yields the two potentials in the third-order linear operator. In Section 5, we show how the restrictions on the time-evolved dependency constants yield the N -soliton solution to the Sawada–Kotera equation. In Section 6, we derive our Marchenko integral equation associated with the third-order linear operator in the reflectionless case. This is done by using the appropriate Fourier transformation on our Riemann–Hilbert problem. By using the same input used in the Riemann–Hilbert problem, we show how the solution to the Marchenko integral equation yields the potentials in the third-order operator in the reflectionless case. We then show that the appropriate restrictions on the time-evolved dependency constants yield the N -soliton solution to the Sawada–Kotera equation. Finally, in Section 7, we provide the conclusion by summarizing the significance of our results and the relevant planned work in the near future.

To help the reader quickly concentrate on the main issue in our paper, that is, the establishment of the Marchenko integral equation and the construction of the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation, we provide the following guidance. We associate the Sawada–Kotera equation (1.1) with two cases of the third-order linear equation (2.4). We refer to the first case as the SK1 case and to the second as the SK2 case. In both cases, the potential Q in (2.4) corresponds to the unknown Q in the Sawada–Kotera equation. In the SK1 case the potential P in (2.4) is given by $P \equiv 0$, and in the SK2 case the potential P in (2.4) is related to Q as $P = Q_x$. In Section 6, we establish the Marchenko integral equation (6.27) associated with the third-order linear equation (2.4). The kernel $\Omega(\zeta, y)$ in that Marchenko integral equation is given in (6.28) and that kernel is constructed by using the bound-state poles of the left transmission coefficient $T_1(k)$ and the time-evolved bound-state dependency constants D_j and D_j^* in (4.5) and (4.6), respectively, for $1 \leq j \leq \mathbf{N}$. The nonhomogeneous term in the Marchenko integral equation is given by $\Omega(0, y)$, and hence that nonhomogeneous term is also constructed by using the same input data set used to construct the integral kernel. The time-evolved dependency constants D_j and D_j^* can equivalently be expressed in terms of the time-independent dependency constants E_j and E_j^* as in (4.7) or they can be expressed in terms of the modified-dependency constants γ_j and γ_j^* appearing in (4.19) and (5.1). We use an asterisk to denote complex conjugation. As in (5.1), we write γ_j in terms of its real and imaginary parts r_j and s_j , respectively. The modified dependency constants γ_j and γ_j^* are uniquely determined in each of the SK1 and SK2 cases by using the ratios s_j/r_j in (5.34) and (5.35), respectively, in terms of the real parameters r_j for $1 \leq j \leq \mathbf{N}$. In fact, instead of using the real parameters r_j , we can use the positive parameters c_j given in (5.42) and (5.43) in the SK1 and SK2 cases, respectively. Thus, the input to the Marchenko integral equation is equivalent to $\{k_j, c_j\}_{j=1}^{\mathbf{N}}$. Consequently, in each of the cases of SK1 and SK2, we have the equivalence of the input data sets indicated in the diagram

$$\{k_j, D_j\}_{j=1}^{\mathbf{N}} \longleftrightarrow \{k_j, E_j\}_{j=1}^{\mathbf{N}} \longleftrightarrow \{k_j, \gamma_j\}_{j=1}^{\mathbf{N}} \longleftrightarrow \{k_j, r_j\}_{j=1}^{\mathbf{N}} \longleftrightarrow \{k_j, c_j\}_{j=1}^{\mathbf{N}}. \quad (1.2)$$

Using our input data set, we construct the kernel $\Omega(\zeta, y)$ and the nonhomogeneous term $\Omega(0, y)$ of the Marchenko integral equation (6.27). In this case, the Marchenko integral equation has a separable kernel, and hence we obtain the solution to (6.27) explicitly in a closed form, by using the methods of linear algebra, as in (6.41). We then recover the potential Q as in (6.45), where $\det[\mathbf{m}(x)]$ is the quantity expressed as in (6.47) and $\Delta(x)$ is the quantity in (5.44), explicitly expressed in terms of the input data set $\{k_j, c_j\}_{j=1}^{\mathbf{N}}$. Thus, the potential Q , constructed as in (6.45) or equivalently as in (5.46), yields the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation (1.1).

2 | The Sawada–Kotera Equation

In this section, we relate the Sawada–Kotera equation (1.1) to the third-order ordinary differential operator associated with (2.4). This is because our goal is to obtain soliton solutions to (1.1) by analyzing the scattering and inverse scattering for the corresponding third-order linear operator in the reflectionless case. For the analysis of the scattering problem for the third-order operator when the reflection coefficients are not zero, we refer the reader to [28, 38].

Being an integrable nonlinear evolution equation, the Sawada–Kotera equation (1.1) can be derived [28] by using either of the two Lax pairs (L_1, A_1) and (L_2, A_2) given by

$$\begin{cases} L_1 := D^3 + QD, \\ A_1 := 9D^5 + 15QD^3 + 15Q_x D^2 + (10Q_{xx} + 5Q^2)D, \end{cases} \quad (2.1)$$

$$\begin{cases} L_2 := D^3 + QD + Q_x, \\ A_2 := 9D^5 + 15QD^3 + 30Q_x D^2 + (25Q_{xx} + 5Q^2)D + (10Q_{xxx} + 10Q Q_x), \end{cases} \quad (2.2)$$

where we use D to denote the derivative operator by letting $D := d/dx$ and $D^n := d^n/dx^n$ for $n \geq 2$. In other words, the two Lax operator equations [39]

$$\dot{L}_1 + L_1 A_1 - A_1 L_1 = 0, \quad \dot{L}_2 + L_2 A_2 - A_2 L_2 = 0, \quad (2.3)$$

where the overdot denotes the t -derivative, are satisfied provided that the Sawada–Kotera equation (1.1) is satisfied. Letting $L\psi = k^3\psi$, we see that the Sawada–Kotera equation (1.1) is associated with the third-order ordinary linear differential equation

$$\psi''' + Q\psi' + P\psi = k^3\psi, \quad x \in \mathbb{R}, \quad (2.4)$$

where ψ is the wavefunction, k^3 is the spectral parameter, x is the independent variable, the prime denotes the x -derivative, t now appears as a parameter, and the coefficients Q and P are real-valued functions of the independent variable x and the parameter t . We refer to the coefficients Q and P as the potentials. The dependence of Q and P on the parameter t is governed by either of the linear operators A_1 and A_2 in (2.1) and (2.2), respectively. For simplicity, we assume that Q and P in (2.4) belong to the Schwartz class in $x \in \mathbb{R}$ for each fixed t even though our results hold under weaker restrictions on Q and P .

The direct scattering problem for (2.4) consists of the determination of the scattering coefficients and the bound-state information when the potentials Q and P are known. On the other hand, the inverse scattering problem for (2.4) involves the recovery of Q and P from an appropriate input data set including the scattering coefficients and the bound-state information.

From (2.1), (2.2), and (2.4), we see that the linear operator L_1 corresponds to the case $P \equiv 0$ and the linear operator L_2 corresponds to the case $P = Q_x$ and that the same Q appears in both L_1 and L_2 . That same Q also appears in the Sawada–Kotera equation (1.1), whereas P does not appear in (1.1) at all. Consequently, if (1.1) is analyzed without connecting it to the inverse scattering problem for (2.4), we do not need to make a distinction between the case $P \equiv 0$ and the case $P = Q_x$. On the other hand, if the solution to (1.1) involves a method related to the solution of the inverse scattering problem for (2.4), we need to make a distinction between those two cases. We refer the reader to [28, 31] for more information on this issue. As already mentioned, we use SK1 to refer to the case $P \equiv 0$ and we use SK2 to refer to the case $P = Q_x$.

For the analysis of soliton solutions to the Sawada–Kotera equation (1.1), the aforementioned distinction is as follows. When Q is real valued, it is known [28, 31] that the third-order equation (2.4) in the cases of $P \equiv 0$ and $P = Q_x$, respectively, comprises two linear third-order adjoint equations, for which the corresponding transmission coefficients coincide and hence also the bound-state spectral values for the two equations coincide. On the other hand, the bound-state dependency constant at each bound state is not the same for the two equations. Hence, both of the cases $P \equiv 0$ and $P = Q_x$ in (2.4) can be analyzed by using the zero reflection coefficients, the same set of left and right transmission coefficients, but two different sets of bound-state dependency constants. In our paper, we derive the N -soliton solution to (1.1) by solving the inverse scattering problem for (2.4) in the reflectionless case. We use the input data set consisting of a particular left transmission coefficient, the bound-state poles of that left transmission coefficient, and the dependency constants at those bound-state poles by explicitly mentioning whether we use the bound-state dependency constants for SK1 or for SK2.

3 | The Third-Order Equation in the Reflectionless Case

In this section, we present a summary of the basic results for the direct scattering problem for (2.4) in the reflectionless case. We refer the reader to [28, 38] for the general case when the reflection coefficients are not zero. We recall that we assume that the potentials Q and P in (2.4) belong to the Schwartz class in $x \in \mathbb{R}$ for each fixed t .

We divide the complex k -plane into four open sectors $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ as indicated on the left plot of Figure 1 by using the directed half lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$, which are parametrized as

$$\mathcal{L}_1 := \{k \in \mathbb{C} : k = zs \text{ for } s \in [0, +\infty)\}, \quad (3.1)$$

$$\mathcal{L}_2 := \{k \in \mathbb{C} : k = z^2s \text{ for } s \in [0, +\infty)\},$$

$$\mathcal{L}_3 := \{k \in \mathbb{C} : k = -zs \text{ for } s \in [0, +\infty)\}, \quad (3.2)$$

$$\mathcal{L}_4 := \{k \in \mathbb{C} : k = -z^2s \text{ for } s \in [0, +\infty)\},$$

where z is used to denote the special complex number $e^{2\pi i/3}$, which is also expressed as

$$z := -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \quad (3.3)$$

The open sectors $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ are described by using the parametrizations given by

$$\Omega_1 := \left\{ k \in \mathbb{C} : \frac{2\pi}{3} < \arg[k] < \frac{4\pi}{3} \right\},$$

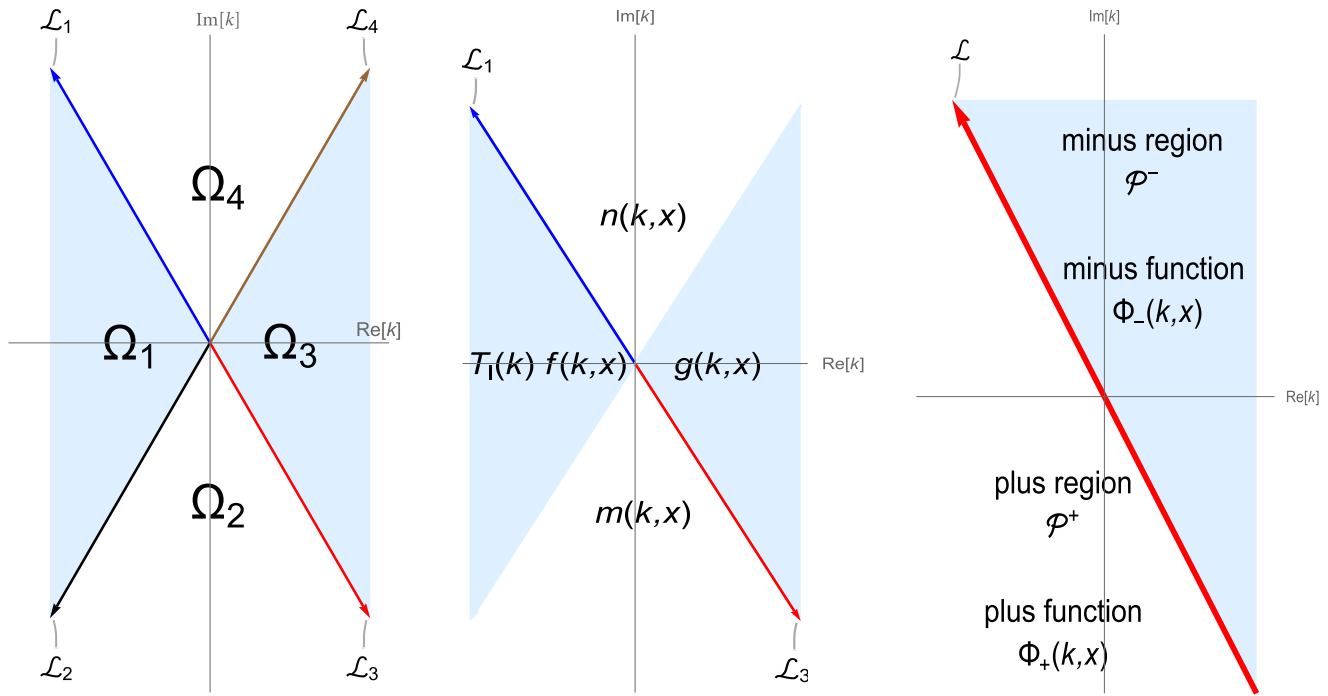


FIGURE 1 | The complex k -plane is divided into the four sectors Ω_1 , Ω_2 , Ω_3 , and Ω_4 as shown on the left plot, with the directed half lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ acting as the boundaries. The k -domains of $T_1(k)f(k, x)$, $m(k, x)$, $g(k, x)$, and $n(k, x)$, respectively, are shown on the middle plot. The right plot shows the plus and minus regions in the complex k -plane separated by the directed full line $\mathcal{L}$, as well as the plus and minus functions in their respective k -domains.

$$\begin{aligned}\Omega_2 &:= \left\{ k \in \mathbb{C} : -\frac{2\pi}{3} < \arg[k] < -\frac{\pi}{3} \right\}, \\ \Omega_3 &:= \left\{ k \in \mathbb{C} : -\frac{\pi}{3} < \arg[k] < \frac{\pi}{3} \right\}, \\ \Omega_4 &:= \left\{ k \in \mathbb{C} : \frac{\pi}{3} < \arg[k] < \frac{2\pi}{3} \right\},\end{aligned}$$

with $\arg[k]$ denoting the argument function taking values in the interval $(-2\pi/3, 4\pi/3)$. We use $\overline{\Omega_1}$, $\overline{\Omega_2}$, $\overline{\Omega_3}$, $\overline{\Omega_4}$ to denote the closures of the open sectors Ω_1 , Ω_2 , Ω_3 , Ω_4 , respectively, where we recall that the closures are obtained by adding the boundaries to the corresponding open sectors.

There are two relevant particular solutions to (2.4), that is, the left Jost solution $f(k, x)$ and the right Jost solution $g(k, x)$. The left Jost solution $f(k, x)$ is the solution to (2.4) satisfying the spacial asymptotics as $x \rightarrow +\infty$ given by

$$\begin{cases} f(k, x) = e^{kx} [1 + o(1)], \\ f'(k, x) = k e^{kx} [1 + o(1)], \\ f''(k, x) = k^2 e^{kx} [1 + o(1)], \end{cases} \quad (3.4)$$

and its k -domain is given by $\overline{\Omega_1}$. The left transmission coefficient $T_1(k)$ appears in the spacial asymptotics of $f(k, x)$ when $x \rightarrow -\infty$, where we have

$$f(k, x) = e^{kx} T_1(k)^{-1} [1 + o(1)], \quad k \in \overline{\Omega_1}. \quad (3.5)$$

The right Jost solution $g(k, x)$ is the solution to (2.4) satisfying the spacial asymptotics as $x \rightarrow -\infty$ given by

$$\begin{cases} g(k, x) = e^{kx} [1 + o(1)], \\ g'(k, x) = k e^{kx} [1 + o(1)], \\ g''(k, x) = k^2 e^{kx} [1 + o(1)], \end{cases} \quad (3.6)$$

and its k -domain is given by $\overline{\Omega_3}$. The right scattering coefficient $T_r(k)$ appears in the spacial asymptotics of $g(k, x)$ when $x \rightarrow +\infty$ as

$$g(k, x) = e^{kx} T_r(k)^{-1} [1 + o(1)], \quad k \in \overline{\Omega_3}. \quad (3.7)$$

In the reflectionless case, the transmission coefficients $T_l(k)$ and $T_r(k)$ have meromorphic extensions from their respective k -domains $\overline{\Omega_1}$ and $\overline{\Omega_3}$ to the entire complex k -plane, and those extensions satisfy [28, 31]

$$T_r(k) = \frac{1}{T_l(k)}, \quad k \in \mathbb{C}.$$

It is known [28] that (2.4) has the particular solutions $m(k, x)$ and $n(k, x)$ with the respective k -domains $\overline{\Omega_2}$ and $\overline{\Omega_4}$, and in the reflectionless case they satisfy the spacial asymptotics given by

$$m(k, x) = e^{kx} [1 + o(1)], \quad x \rightarrow -\infty, \quad k \in \overline{\Omega_2}, \quad (3.8)$$

$$m(k, x) = e^{kx} T_l(z^2 k)^{-1} T_l(zk) [1 + o(1)], \quad x \rightarrow +\infty, \quad k \in \overline{\Omega_2}, \quad (3.9)$$

$$n(k, x) = e^{kx} [1 + o(1)], \quad x \rightarrow -\infty, \quad k \in \overline{\Omega_4}, \quad (3.10)$$

$$n(k, x) = e^{kx} T_l(zk)^{-1} T_l(z^2 k) [1 + o(1)], \quad x \rightarrow +\infty, \quad k \in \overline{\Omega_4}, \quad (3.11)$$

where we recall that z is the cube root of unity appearing in (3.3).

For fixed real values of x and t , the large k -asymptotics of the basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, and $n(k, x)$ are, respectively, given by [28, 31]

$$f(k, x) = e^{kx} \left[1 + \frac{u_1(x)}{k} + \frac{u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_1}, \quad (3.12)$$

$$g(k, x) = e^{kx} \left[1 + \frac{v_1(x)}{k} + \frac{v_2(x)}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_3}, \quad (3.13)$$

$$m(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_2}, \quad (3.14)$$

$$n(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_4}, \quad (3.15)$$

where we have defined

$$u_1(x) := \frac{1}{3} \int_x^\infty dy Q(y), \quad x \in \mathbb{R}, \quad (3.16)$$

$$u_2(x) := -\frac{1}{3} \int_x^\infty dy [Q'(y) - P(y)] + \frac{1}{18} \left[\int_x^\infty dy Q(y) \right]^2, \quad x \in \mathbb{R}, \quad (3.17)$$

$$v_1(x) := -\frac{1}{3} \int_{-\infty}^x dy Q(y), \quad x \in \mathbb{R}, \quad (3.18)$$

$$v_2(x) := \frac{1}{3} \int_{-\infty}^x dy [Q'(y) - P(y)] + \frac{1}{18} \left[\int_{-\infty}^x dy Q(y) \right]^2, \quad x \in \mathbb{R}. \quad (3.19)$$

Using (3.16)–(3.19), we express the potentials Q and P in terms of $u_1(x)$ and $u_2(x)$ as

$$Q(x) = -3 \frac{du_1(x)}{dx}, \quad x \in \mathbb{R}, \quad (3.20)$$

$$P(x) = 3 \left[u_1(x) \frac{du_1(x)}{dx} - \frac{d^2 u_1(x)}{dx^2} - \frac{du_2(x)}{dx} \right], \quad x \in \mathbb{R}, \quad (3.21)$$

or in terms of $v_1(x)$ and $v_2(x)$ as

$$Q(x) = -3 \frac{dv_1(x)}{dx}, \quad x \in \mathbb{R}, \quad (3.22)$$

$$P(x) = 3 \left[v_1(x) \frac{dv_1(x)}{dx} - \frac{d^2 v_1(x)}{dx^2} - \frac{dv_2(x)}{dx} \right], \quad x \in \mathbb{R}.$$

Since (2.4) is a linear homogeneous equation, any constant multiple of a solution is also a solution. In terms of the solutions $f(k, x)$ in (3.4) and (3.5), the solution $g(k, x)$ in (3.6) and (3.7), the solution $m(k, x)$ in (3.8) and (3.9), and the solution $n(k, x)$ in (3.10) and (3.11), we introduce the solution $\Phi_+(k, x)$ to (2.4) with the k -domain $\overline{\Omega_1} \cup \overline{\Omega_2}$ and the solution $\Phi_-(k, x)$ to (2.4) with the k -domain $\overline{\Omega_3} \cup \overline{\Omega_4}$, where we have defined

$$\Phi_+(k, x) := \begin{cases} T_1(k) f(k, x), & k \in \overline{\Omega_1}, \\ m(k, x), & k \in \overline{\Omega_2}, \end{cases} \quad (3.23)$$

$$\Phi_-(k, x) := \begin{cases} g(k, x), & k \in \overline{\Omega_3}, \\ n(k, x), & k \in \overline{\Omega_4}. \end{cases} \quad (3.24)$$

Using the directed half lines $\mathcal{L}_1$ and $\mathcal{L}_3$ defined in (3.1) and (3.2), respectively, we obtain the directed full line $\mathcal{L}$ by letting $\mathcal{L} := \mathcal{L}_1 \cup (-\mathcal{L}_3)$, where we recall that $-\mathcal{L}_3$ is obtained from $\mathcal{L}_3$ by changing its direction. The parametrization of $\mathcal{L}$ is given by

$$\mathcal{L} := \{k \in \mathbb{C} : k = zs \text{ for } s \in (-\infty, +\infty)\}. \quad (3.25)$$

The directed line $\mathcal{L}$ divides the complex k -plane into two half planes $\mathcal{P}^+$ and $\mathcal{P}^-$ as shown on the right plot of Figure 1. The open left-half complex plane $\mathcal{P}^+$ and the open right-half complex plane $\mathcal{P}^-$ can be parametrized as

$$\mathcal{P}^+ := \{k \in \mathbb{C} : k = zs \text{ for } s \in \mathbb{C}^+\},$$

$$\mathcal{P}^- := \{k \in \mathbb{C} : k = zs \text{ for } s \in \mathbb{C}^-\},$$

where we use $\mathbb{C}^+$ and $\mathbb{C}^-$ to denote the upper-half and lower-half complex planes. We refer to $\mathcal{P}^+$ as the plus region and refer to $\mathcal{P}^-$ as the minus region, as indicated on the right plot of Figure 1. We use $\overline{\mathcal{P}^+}$ and $\overline{\mathcal{P}^-}$ to denote their closures. Hence, we have $\overline{\mathcal{P}^+} := \mathcal{P}^+ \cup \mathcal{L}$ and $\overline{\mathcal{P}^-} := \mathcal{P}^- \cup \mathcal{L}$.

In the reflectionless case, it is known [28, 31] that the two quantities $T_1(k)f(k, x)$ and $m(k, x)$ become equal to each other when $k \in \mathcal{L}_2$ and that the two quantities $g(k, x)$ and $n(k, x)$ become equal to each other when $k \in \mathcal{L}_4$. Furthermore, in the reflectionless case, $f(k, x)$, $m(k, x)$, $g(k, x)$, and $n(k, x)$ are each analytic in k in their respective domains Ω_1 , Ω_2 , Ω_3 , and Ω_4 , respectively, and they each are continuous in the closures of those respective sectors. Consequently, in the reflectionless case, we have the following properties for $\Phi_+(k, x)$ defined in (3.23) and $\Phi_-(k, x)$ defined in (3.24) for each fixed $x \in \mathbb{R}$. The solution $\Phi_-(k, x)$ is analytic in $k \in \mathcal{P}^-$ and it is continuous in $k \in \overline{\mathcal{P}^-}$. In the reflectionless case, when the left transmission coefficient $T_1(k)$ has its poles confined to Ω_1 , the solution $\Phi_+(k, x)$ is meromorphic in $k \in \mathcal{P}^+$ with the poles occurring at the poles of $T_1(k)$ in Ω_1 , and it is continuous in $k \in \overline{\mathcal{P}^+}$ except at those poles.

4 | The Riemann–Hilbert Problem Related to the N-Soliton Solution

Since our main goal in this paper is to obtain the N-soliton solution to the Sawada–Kotera equation (1.1) with real-valued Q and P , the corresponding transmission coefficients for (2.4) must be chosen in a particular way. We choose the left transmission coefficient $T_1(k)$ as

$$T_1(k) = \frac{\Gamma(k)}{\Gamma(-k)}, \quad k \in \mathbb{C}, \quad (4.1)$$

where $\Gamma(k)$ is given by

$$\Gamma(k) := \prod_{j=1}^N (k + k_j)(k + k_j^*), \quad k \in \mathbb{C}, \quad (4.2)$$

by recalling that the asterisk denotes complex conjugation. From (4.1) and (4.2), we see that the poles of $T_1(k)$ are simple and they occur at $k = k_j$ and $k = k_j^*$ for $1 \leq j \leq \mathbf{N}$. As indicated in Section 3, we would like to choose the poles of $T_1(k)$ in the sector Ω_1 and we would like the corresponding potentials Q and P to be real valued. These requirements impose the necessary but not sufficient restrictions on the location of each k_j so that we must have $\arg[k_j] = 5\pi/6$ or $\arg[k_j] = 7\pi/6$. Since the poles occur at $k = k_j$ and $k = k_j^*$ for $1 \leq j \leq \mathbf{N}$, there is no loss of generality in using $\arg[k_j] = 7\pi/6$. Then, we automatically have $\arg[k_j^*] = 5\pi/6$. Because we only consider simple bound states, the restriction can be presented as

$$k_j = iz\eta_j, \quad 1 \leq j \leq \mathbf{N}, \quad (4.3)$$

where each η_j is positive and we recall that z is the special complex constant in (3.3). Without loss of generality, we can assume that $0 < \eta_1 < \dots < \eta_{\mathbf{N}}$.

From (4.1) and (4.2), we see that the left transmission coefficient $T_1(k)$ is defined on the whole complex k -plane and that it has $2\mathbf{N}$ poles located at $k = k_j$ and $k = k_j^*$, respectively, for $1 \leq j \leq \mathbf{N}$. The large k -asymptotics of $T_1(k)$ is given by

$$T_1(k) = 1 + \frac{2\Sigma_{\mathbf{N}}}{k} + \frac{2\Sigma_{\mathbf{N}}^2}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ in } \mathbb{C},$$

where we have defined

$$\Sigma_{\mathbf{N}} := \sum_{j=1}^{\mathbf{N}} (k_j + k_j^*). \quad (4.4)$$

From (4.4) we see that $\Sigma_{\mathbf{N}}$ is a real-valued constant.

With the left transmission coefficient given in (4.1), it is known [28, 31] that (2.4) has bound states at $k = k_j$ and $k = k_j^*$ for $1 \leq j \leq \mathbf{N}$. In other words, (2.4) has a square integrable solution at each of those $2\mathbf{N}$ k -values. In fact, $f(k_j, x)$ and $f(k_j^*, x)$ each correspond to a bound-state solution to (2.4). Since (2.4) is linear and homogeneous, any constant multiple of $f(k_j, x)$ is also a bound-state solution at $k = k_j$ and any constant multiple of $f(k_j^*, x)$ is also a bound-state solution at $k = k_j^*$. It is known [28, 31] that $g(zk_j, x)$ is a bound-state solution to (2.4) at $k = k_j$ and that $g(z^2k_j^*, x)$ is a bound-state solution to (2.4) at $k = k_j^*$. Thus, there exists a constant D_j , which we call the bound-state dependency constant at $k = k_j$, such that

$$f(k_j, x) = D_j g(zk_j, x), \quad x \in \mathbb{R}. \quad (4.5)$$

We recall that we suppress the t -dependency in our notation for the potentials Q and P and the Jost solutions $f(k, x)$ and $g(k, x)$. It is known [28, 31] that the bound-state dependency constant at $k = k_j^*$ in this case is given by D_j^* , that is, we have

$$f(k_j^*, x) = D_j^* g(z^2k_j^*, x), \quad x \in \mathbb{R}. \quad (4.6)$$

When the potentials Q and P are evolved in time in a way compatible with the linear operator A_1 or A_2 defined in the second lines of (2.1) and (2.2), respectively, the time evolution of the dependency constants D_j and D_j^* are given by [28, 31]

$$D_j = E_j e^{-9 \left[k_j^5 + \left(k_j^* \right)^5 \right] t}, \quad D_j^* = E_j^* e^{-9 \left[k_j^5 + \left(k_j^* \right)^5 \right] t}, \quad (4.7)$$

where E_j and E_j^* denote the values of the dependency constants D_j and D_j^* , respectively, at $t = 0$.

In the reflectionless case, it is known [28, 31] that the solution $\Phi_+(k, x)$ and the solution $\Phi_-(k, x)$ coincide on the intersection of their respective domains $\overline{\mathcal{P}^+}$ and $\overline{\mathcal{P}^-}$. We recall from the right plot in Figure 1 that the intersection of those domains is given by the directed full line $\mathcal{L}$ defined in (3.25). Thus, we have

$$\Phi_+(k, x) = \Phi_-(k, x), \quad k \in \mathcal{L}. \quad (4.8)$$

We remark that (4.8) constitutes a Riemann–Hilbert problem as follows. The left transmission coefficient $T_1(k)$ described in (4.1) and (4.2) is used as input to our Riemann–Hilbert problem. For each fixed $x \in \mathbb{R}$, we are interested in finding a sectionally meromorphic function $\Phi(k, x)$ in such a way that $\Phi_+(k, x)$ is the meromorphic section in the region $\mathcal{P}^+$

with the poles coinciding with the poles of $T_1(k)$ there and that $\Phi_-(k, x)$ is the analytic section in the region $\mathcal{P}^-$. As seen from (3.12)–(3.15), (3.23), and (3.24), the sectionally meromorphic function $\Phi(k, x)$ has the behavior $1 + O(1/k)$ as $k \rightarrow \infty$ in $\mathbb{C}$. As indicated on the right plot of Figure 1, it is appropriate to refer to $\mathcal{P}^+$ and $\mathcal{P}^-$ as the plus and minus regions, respectively, and refer to $\Phi_+(k, x)$ and $\Phi_-(k, x)$ as the plus and minus functions, respectively. We refer the reader to [28, 38] for the generalization of that Riemann–Hilbert problem when the reflection coefficients are nonzero.

For the solution of the Riemann–Hilbert problem given in (4.8), we refer the reader to [28, 31]. In order to have a unique solution, it is appropriate to use the input $\{k_j, E_j\}_{j=1}^N$. The solution steps are summarized as follows:

(a) We multiplying both sides of (4.8) by $e^{-kx} \Gamma(-k)$, and we get

$$e^{-kx} \Gamma(-k) \Phi_+(k, x) = e^{-kx} \Gamma(-k) \Phi_-(k, x), \quad k \in \mathcal{L}, \quad (4.9)$$

where we recall that $\Gamma(k)$ is the quantity defined in (4.2). For each fixed $x \in \mathbb{R}$, from the properties of $\Phi_+(k, x)$ and $\Phi_-(k, x)$, it follows that the left-hand side of (4.9) is analytic in $k \in \mathcal{P}^+$ and the right-hand side is analytic in $k \in \mathcal{P}^-$. As a consequence of the equality in (4.9), those two sides are analytic continuations of each other and that each side is entire with their respective analytic continuations in $k \in \mathbb{C}$. Furthermore, using the generalized Liouville theorem [40], we conclude that each side of (4.9) is equal to a monic polynomial in k of degree $2N$, where the coefficients may depend on x and t . We suppress the t -dependence of those coefficients and write the solution to our Riemann–Hilbert problem as

$$\Phi_+(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\mathcal{P}^+}, \quad x \in \mathbb{R}, \quad (4.10)$$

$$\Phi_-(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\mathcal{P}^-}, \quad x \in \mathbb{R}, \quad (4.11)$$

where $V(k)$ is the row vector with the $2N$ components defined as

$$V(k) := [k^{2N-1} \quad k^{2N-2} \quad \dots \quad k \quad 1], \quad k \in \mathbb{C}, \quad (4.12)$$

and $\mathbf{A}(x)$ is a column vector with the $2N$ entries that are functions of x and t . We write $\mathbf{A}(x)$ in terms of its components as

$$\mathbf{A}(x) = \begin{bmatrix} A_{2N-1}(x) \\ A_{2N-2}(x) \\ \vdots \\ A_1(x) \\ A_0(x) \end{bmatrix}, \quad x \in \mathbb{R}. \quad (4.13)$$

(b) Using (3.23) and (4.10), with the help of (4.1) we obtain

$$f(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(k)}, \quad k \in \overline{\Omega_1}, \quad x \in \mathbb{R}. \quad (4.14)$$

Similarly, from (3.24) and (4.11) we get

$$g(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\Omega_3}, \quad x \in \mathbb{R}. \quad (4.15)$$

From (4.14) and (4.15), we observe that $f(k, x)$ and $g(k, x)$ have meromorphic extensions from their respective original k -domains of analyticity to the entire complex k -plane.

- (c) Using (4.14) and (4.15) in (4.5) and (4.6), we obtain a linear algebraic system of $2\mathbf{N}$ equations in the $2\mathbf{N}$ unknowns $A_l(x)$ for $0 \leq l \leq 2\mathbf{N} - 1$. With the help of (4.7), we write that linear system as

$$\begin{cases} e^{k_j x} \frac{k_j^{2\mathbf{N}} + V(k_j) \mathbf{A}(x)}{\Gamma(k_j)} = E_j e^{-9 \left[k_j^5 + \left(k_j^* \right)^5 \right] t} e^{z k_j x} \frac{(z k_j)^{2\mathbf{N}} + V(z k_j) \mathbf{A}(x)}{\Gamma(-z k_j)}, \\ e^{k_j^* x} \frac{(k_j^*)^{2\mathbf{N}} + V(k_j^*) \mathbf{A}(x)}{\Gamma(k_j^*)} = E_j^* e^{-9 \left[k_j^5 + \left(k_j^* \right)^5 \right] t} e^{z^2 k_j^* x} \frac{(z^2 k_j^*)^{2\mathbf{N}} + V(z^2 k_j^*) \mathbf{A}(x)}{\Gamma(-z^2 k_j^*)}, \end{cases} \quad (4.16)$$

where we have $1 \leq j \leq \mathbf{N}$. We remark that the second line in (4.16) is obtained by taking the complex conjugate of the first line due to the fact that $\mathbf{A}(x)$ is real valued and we have

$$\Gamma(k^*) = \Gamma(k)^*, \quad V(k^*) = V(k)^*, \quad k \in \mathbb{C}, \quad (4.17)$$

which directly follow from (4.2) and (4.12), respectively.

- (d) We introduce the quantities χ_j and γ_j as

$$\chi_j := e^{-\left(k_j + k_j^* \right) x - 9 \left[k_j^5 + \left(k_j^* \right)^5 \right] t}, \quad 1 \leq j \leq \mathbf{N}, \quad (4.18)$$

$$\gamma_j := -E_j \frac{\Gamma(k_j)}{\Gamma(-z k_j)}, \quad 1 \leq j \leq \mathbf{N}. \quad (4.19)$$

From (4.18), it follows that χ_j is a real-valued function of x and t and that we have $\chi_j \rightarrow 0$ as $x \rightarrow -\infty$ for each fixed t . From (4.19), we observe that γ_j is a complex-valued constant directly proportional to the dependency constant E_j at $t = 0$. We refer to γ_j as the modified dependency constant at $t = 0$ for the bound state at $k = k_j$. From (4.7) and the first equality of (4.17), it follows that γ_j^* corresponds to the modified dependency constant at $t = 0$ for the bound state at $k = k_j^*$. Using (4.18) and (4.19) in (4.16), we write the linear algebraic system (4.16) as

$$\begin{cases} k_j^{2\mathbf{N}} + V(k_j) \mathbf{A}(x) = -\gamma_j \chi_j [(z k_j)^{2\mathbf{N}} + V(z k_j) \mathbf{A}(x)], & 1 \leq j \leq \mathbf{N}, \\ (k_j^*)^{2\mathbf{N}} + V(k_j^*) \mathbf{A}(x) = -\gamma_j^* \chi_j [(z^2 k_j^*)^{2\mathbf{N}} + V(z^2 k_j^*) \mathbf{A}(x)], & 1 \leq j \leq \mathbf{N}. \end{cases} \quad (4.20)$$

- (e) The linear algebraic system (4.20) can be written as

$$\mathbf{M}(x) \mathbf{A}(x) = -\mathbf{B}(x), \quad (4.21)$$

where we have defined the $2\mathbf{N} \times 2\mathbf{N}$ matrix $\mathbf{M}(x)$ and the column vector $\mathbf{B}(x)$ with $2\mathbf{N}$ entries as

$$\mathbf{M}(x) := \begin{bmatrix} m_{2\mathbf{N}-1}(k_1) & m_{2\mathbf{N}-2}(k_1) & \cdots & m_1(k_1) & m_0(k_1) \\ m_{2\mathbf{N}-1}(k_1^*) & m_{2\mathbf{N}-2}(k_1^*) & \cdots & m_1(k_1^*) & m_0(k_1^*) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{2\mathbf{N}-1}(k_{\mathbf{N}}) & m_{2\mathbf{N}-2}(k_{\mathbf{N}}) & \cdots & m_1(k_{\mathbf{N}}) & m_0(k_{\mathbf{N}}) \\ m_{2\mathbf{N}-1}(k_{\mathbf{N}}^*) & m_{2\mathbf{N}-2}(k_{\mathbf{N}}^*) & \cdots & m_1(k_{\mathbf{N}}^*) & m_0(k_{\mathbf{N}}^*) \end{bmatrix}, \quad (4.22)$$

$$\mathbf{B}(x) := \begin{bmatrix} m_{2\mathbf{N}}(k_1) \\ m_{2\mathbf{N}}(k_1^*) \\ \vdots \\ m_{2\mathbf{N}}(k_{\mathbf{N}}) \\ m_{2\mathbf{N}}(k_{\mathbf{N}}^*) \end{bmatrix}, \quad (4.23)$$

with the entries in (4.22) and (4.23) defined as

$$\begin{cases} m_l(k_j) := k_j^l + (zk_j)^l \gamma_j \chi_j, & 1 \leq j \leq \mathbf{N}, \quad 0 \leq l \leq 2\mathbf{N}, \\ m_l(k_j^*) := (k_j^*)^l + (z^2 k_j^*)^l \gamma_j^* \chi_j, & 1 \leq j \leq \mathbf{N}, \quad 0 \leq l \leq 2\mathbf{N}. \end{cases} \quad (4.24)$$

From (4.24) we observe that

$$m_l(k_j^*) = m_l(k_j)^*, \quad 1 \leq j \leq \mathbf{N}, \quad 0 \leq l \leq 2\mathbf{N}. \quad (4.25)$$

We remark that we have suppressed the t -dependence in our notation for $\mathbf{M}(x)$, $\mathbf{B}(x)$, and $m_l(k_j)$. In fact, the only dependence on x and t in those three quantities is through the scalar function χ_j for $1 \leq j \leq \mathbf{N}$.

- (f) We recall that the input to our Riemann–Hilbert problem (4.8) is $\{k_j, E_j\}_{j=1}^{\mathbf{N}}$. From (4.2) and (4.19), it follows that our input data set is equivalent to $\{k_j, \gamma_j\}_{j=1}^{\mathbf{N}}$. From (4.22)–(4.25), we see that $\mathbf{M}(x)$ and $\mathbf{B}(x)$ are each uniquely determined by our input data set. We solve (4.21) and obtain the column vector $\mathbf{A}(x)$ in (4.13) as

$$\mathbf{A}(x) = -\mathbf{M}(x)^{-1} \mathbf{B}(x). \quad (4.26)$$

We refer the reader to [28, 31] for further details and other equivalent expressions for $\mathbf{A}(x)$ given in terms of the input data set $\{k_j, \gamma_j\}_{j=1}^{\mathbf{N}}$. From (3.12), (3.20), (3.21), and (4.14), it follows that the potential Q is determined by $A_{2\mathbf{N}-1}(x)$ alone and the potential P is determined by $A_{2\mathbf{N}-1}(x)$ and $A_{2\mathbf{N}-2}(x)$ only. The remaining entries $A_j(x)$ when we have $0 \leq j \leq 2\mathbf{N} - 3$ are used only to determine the solutions to the third-order equation (2.4). In the special case when $\mathbf{N} = 1$, we only have $A_1(x)$ and $A_0(x)$.

- (g) Having determined $A_{2\mathbf{N}-1}(x)$ and $A_{2\mathbf{N}-2}(x)$ via (4.26) by using the input data set $\{k_j, \gamma_j\}_{j=1}^{\mathbf{N}}$, we next determine the potentials Q and P in (2.4). Using (4.2), (4.12), and (4.14), we obtain the asymptotics of $f(k, x)$ when $k \rightarrow \infty$ in $\mathbb{C}$ as

$$f(k, x) = e^{kx} \left[1 + \frac{A_{2\mathbf{N}-1}(x) - \Sigma_{\mathbf{N}}}{k} + \frac{A_{2\mathbf{N}-2}(x) - \Sigma_{\mathbf{N}} A_{2\mathbf{N}-1}(x) + \Pi_{\mathbf{N}}}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad (4.27)$$

where $\Sigma_{\mathbf{N}}$ is the constant in (4.4) and $\Pi_{\mathbf{N}}$ is the constant defined as

$$\begin{aligned} \Pi_{\mathbf{N}} := & k_1(k_1 + k_1^* + \cdots + k_{\mathbf{N}} + k_{\mathbf{N}}^*) + k_1^*(k_1^* + k_2 + \cdots + k_{\mathbf{N}} + k_{\mathbf{N}}^*) \\ & + k_2(k_2 + k_2^* + \cdots + k_{\mathbf{N}} + k_{\mathbf{N}}^*) + \cdots + k_{\mathbf{N}}(k_{\mathbf{N}} + k_{\mathbf{N}}^*) + k_{\mathbf{N}}^*(k_{\mathbf{N}}^*). \end{aligned} \quad (4.28)$$

Comparing (3.12) with (4.27), we get

$$u_1(x) = A_{2\mathbf{N}-1}(x) - \Sigma_{\mathbf{N}}, \quad (4.29)$$

$$u_2(x) = A_{2\mathbf{N}-2}(x) - \Sigma_{\mathbf{N}} A_{2\mathbf{N}-1}(x) + \Pi_{\mathbf{N}}. \quad (4.30)$$

With the help of (3.20), (3.21), (4.29), and (4.30), we recover Q and P appearing in (2.4) as

$$Q(x) = -3 \frac{dA_{2\mathbf{N}-1}(x)}{dx}, \quad (4.31)$$

$$P(x) = 3 \left(A_{2\mathbf{N}-1}(x) \frac{dA_{2\mathbf{N}-1}(x)}{dx} - \frac{d^2 A_{2\mathbf{N}-1}(x)}{dx^2} - \frac{dA_{2\mathbf{N}-2}(x)}{dx} \right). \quad (4.32)$$

Since the potentials Q and P vanish as $x \rightarrow +\infty$, it follows from (3.16) and (3.17) that $u_1(x)$ and $u_2(x)$ each vanish as $x \rightarrow +\infty$. Consequently, from (4.29) and (4.30), we get

$$\lim_{x \rightarrow +\infty} A_{2\mathbf{N}-1}(x) = \Sigma_{\mathbf{N}}, \quad \lim_{x \rightarrow +\infty} A'_{2\mathbf{N}-1}(x) = 0, \quad \lim_{x \rightarrow +\infty} A_{2\mathbf{N}-2}(x) = \Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}. \quad (4.33)$$

5 | The N-Soliton Solution to the Sawada–Kotera Equation

Our goal in this section is to provide an explicit expression for the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation (1.1). We achieve our goal by solving the inverse scattering problem for the linear equation (2.4) in the reflectionless case with the input data set consisting of the bound-state poles of the transmission coefficient $T_1(k)$ in (4.1) and the corresponding bound-state dependency constants. The reason for the choice of the k_j -values given in (4.3) is due to the fact that the soliton solution Q must be real valued. There is a further restriction that we must impose on our solution to the inverse scattering problem, namely, we must have $P \equiv 0$ in the SK1 case and have $P = Q_x$ in the SK2 case.

In this section, we show that the appropriate restrictions in the SK1 and SK2 cases can be imposed by using certain constraints on the complex-valued modified bound-state dependency constants γ_j appearing in (4.19). A lengthy initial part of our section is devoted to derive the appropriate restrictions on the modified dependency constants γ_j . Those restrictions amount to expressing the ratio of the imaginary part to the real part of each γ_j in terms of the bound-state parameters $k_1, k_2, \dots, k_N$ appearing in (4.3). Consequently, we express the $\mathbf{N}$ -soliton solution Q to (1.1) by using the parameters k_j and the real parts of γ_j for $1 \leq j \leq \mathbf{N}$. Later, we replace each real part of γ_j by the parameter c_j by using (5.42) and (5.43) in the SK1 and SK2 cases, respectively, for $1 \leq j \leq \mathbf{N}$. We then express the $\mathbf{N}$ -soliton solution to the SK equation in terms of the $2\mathbf{N}$ real parameters in the set $\{\eta_j, c_j\}_{j=1}^{\mathbf{N}}$, where the parameters η_j are distinct and positive. The advantage of using c_j instead of the real part of γ_j in our input data set is the following. The function Q does not contain any singularity when x and t take all values in $\mathbb{R}$ if each parameter c_j takes any positive value, whereas to avoid any singularity in Q , the real part of γ_j may need to be restricted to positive or negative values or to a constrained range.

In Section 4, we have solved the inverse scattering problem for (2.4) in the reflectionless case by using the input data set $\{k_j, E_j\}_{j=1}^{\mathbf{N}}$ or the equivalent data set $\{k_j, \gamma_j\}_{j=1}^{\mathbf{N}}$. We have recovered the Jost solutions $f(k, x)$ and $g(k, x)$ and the potentials Q and P as in (4.14), (4.15), (4.31), and (4.32), respectively, where $\mathbf{A}(x)$ is expressed in terms of our input data set via (4.26). In the determination of each of $A_{2\mathbf{N}-1}(x)$ and $A_{2\mathbf{N}-2}(x)$, we have used the modified dependency constants γ_j and γ_j^* for $1 \leq j \leq \mathbf{N}$. In order for Q to satisfy the Sawada–Kotera equation (1.1), as indicated in Section 2, the potential P must satisfy the restriction $P \equiv 0$ in the SK1 case and the restriction $P = Q_x$ in the SK2 case.

Let us use r_j and s_j to denote the real and imaginary parts of the complex-valued γ_j so that we have

$$\gamma_j = r_j \left(1 + i \frac{s_j}{r_j} \right), \quad \gamma_j^* = r_j \left(1 - i \frac{s_j}{r_j} \right), \quad 1 \leq j \leq \mathbf{N}. \quad (5.1)$$

Our motivation behind separating the complex-valued dependency constant γ_j into its real and imaginary parts r_j and s_j , respectively, can be explained as follows. The $\mathbf{N}$ -soliton solution Q to the Sawada–Kotera equation (1.1) is a real-valued function of x and t containing the parameters related to the bound-state poles and the bound-state dependency constants. Hence, it is natural that we use real-valued parameters when we express the real-valued solution Q to (1.1). From (4.3), we see that we can use the positive parameter η_j instead of the complex-valued parameter k_j for $1 \leq j \leq \mathbf{N}$. Similarly, instead of using the complex-valued parameter γ_j , we can use the real parameters r_j and s_j for $1 \leq j \leq \mathbf{N}$. It turns out that the two aforementioned restrictions in the SK1 and SK2 cases comprise the specification of the ratios s_j/r_j for $1 \leq j \leq \mathbf{N}$ in each case. Consequently, the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation (1.1) is determined by the input data set $\{k_j, r_j\}_{j=1}^{\mathbf{N}}$ when the ratios s_j/r_j for $1 \leq j \leq \mathbf{N}$ are specified. Since the $\mathbf{N}$ constants k_j appear symmetrically in the input data set, without loss of generality, it is sufficient to determine the ratio s_1/r_1 . Then, using the symmetry, we can get the remaining ratios s_j/r_j for $2 \leq j \leq \mathbf{N}$. We have the explicit formula for the ratio s_1/r_1 presented in (6.14) of [31] covering both the SK1 and SK2 cases. In this section, we use a slightly different method, but still a general method, to determine the ratios s_1/r_1 in each of the SK1 and SK2 cases.

We know from (4.31) and (4.32) that either of the two aforementioned restrictions in the cases of SK1 and SK2, respectively, involves the corresponding restrictions on $A_{2\mathbf{N}-1}(x)$ and $A_{2\mathbf{N}-2}(x)$ only. By using Cramer's rule, from (4.21) we get

$$A_{2\mathbf{N}-1}(x) = -\frac{\det[\mathbf{M}_1(x)]}{\det[\mathbf{M}(x)]}, \quad (5.2)$$

$$A_{2\mathbf{N}-2}(x) = -\frac{\det[\mathbf{M}_2(x)]}{\det[\mathbf{M}(x)]}, \quad (5.3)$$

where $\mathbf{M}_1(x)$ is the $2\mathbf{N} \times 2\mathbf{N}$ matrix obtained by replacing the first column of the matrix $\mathbf{M}(x)$ with the column vector $\mathbf{B}(x)$ and that $\mathbf{M}_2(x)$ denotes the $2\mathbf{N} \times 2\mathbf{N}$ matrix obtained by replacing the second column of $\mathbf{M}(x)$ with $\mathbf{B}(x)$. It is known [28, 31] that the determinant of $\mathbf{M}_1(x)$ can be expressed in terms of the determinant of $\mathbf{M}(x)$ and the x -derivative of $\det[\mathbf{M}(x)]$

as

$$\det[\mathbf{M}_1(x)] = \Sigma_{\mathbf{N}} \det[\mathbf{M}(x)] + \frac{d \det[\mathbf{M}(x)]}{dx}, \quad (5.4)$$

where we recall that $\Sigma_{\mathbf{N}}$ is the constant defined in (4.4). Using (5.4) in (5.2), we get

$$A_{2\mathbf{N}-1}(x) = -\Sigma_{\mathbf{N}} - \frac{1}{\det[\mathbf{M}(x)]} \frac{d \det[\mathbf{M}(x)]}{dx}, \quad (5.5)$$

and then, using (5.5) in (4.31), we express the potential Q in terms of the determinant of $\mathbf{M}(x)$ as

$$Q(x) = 3 \frac{d}{dx} \left(\frac{1}{\det[\mathbf{M}(x)]} \frac{d \det[\mathbf{M}(x)]}{dx} \right). \quad (5.6)$$

From (4.22) and (4.24), we see that the quantity $\det[\mathbf{M}(x)]$ is a polynomial of degree $2\mathbf{N}$ in the $2\mathbf{N}$ variables $\gamma_1 \chi_1, \gamma_1^* \chi_1, \dots, \gamma_{\mathbf{N}} \chi_{\mathbf{N}}, \gamma_{\mathbf{N}}^* \chi_{\mathbf{N}}$, where each of those $2\mathbf{N}$ variables appears at most to the first power. The appearance of each χ_j in $\det[\mathbf{M}(x)]$ is symmetrical. By letting $\chi_j = 0$ for $2 \leq j \leq \mathbf{N}$, from (4.22) we obtain

$$\det[\mathbf{M}(x)] = \det \begin{bmatrix} k_1^{2\mathbf{N}-1} + (zk_1)^{2\mathbf{N}-1} \gamma_1 \chi_1 & \cdots & k_1 + zk_1 \gamma_1 \chi_1 & 1 + \gamma_1 \chi_1 \\ (k_1^*)^{2\mathbf{N}-1} + (z^2 k_1^*)^{2\mathbf{N}-1} \gamma_1^* \chi_1 & \cdots & k_1^* + z^2 k_1^* \gamma_1^* \chi_1 & 1 + \gamma_1^* \chi_1 \\ \vdots & \ddots & \vdots & \vdots \\ k_{\mathbf{N}}^{2\mathbf{N}-1} & \cdots & k_{\mathbf{N}} & 1 \\ (k_{\mathbf{N}}^*)^{2\mathbf{N}-1} & \cdots & k_{\mathbf{N}}^* & 1 \end{bmatrix}. \quad (5.7)$$

Let us write (5.7) as

$$\det[\mathbf{M}(x)] = \nu + [\nu_{10} \gamma_1 + \nu_{01} \gamma_1^*] \chi_1 + \nu_{11} \gamma_1 \gamma_1^* \chi_1^2, \quad (5.8)$$

where the coefficients ν , ν_{10} , ν_{01} , and ν_{11} are explicitly expressed in terms of the parameters in the set $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$. The coefficient ν is obtained by letting $\chi_1 = 0$ in (5.7), from which we have

$$\nu := \det \begin{bmatrix} k_1^{2\mathbf{N}-1} & k_1^{2\mathbf{N}-2} & \cdots & k_1 & 1 \\ (k_1^*)^{2\mathbf{N}-1} & (k_1^*)^{2\mathbf{N}-2} & \cdots & k_1^* & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{\mathbf{N}}^{2\mathbf{N}-1} & k_{\mathbf{N}}^{2\mathbf{N}-2} & \cdots & k_{\mathbf{N}} & 1 \\ (k_{\mathbf{N}}^*)^{2\mathbf{N}-1} & (k_{\mathbf{N}}^*)^{2\mathbf{N}-2} & \cdots & k_{\mathbf{N}}^* & 1 \end{bmatrix}. \quad (5.9)$$

The right-hand side of (5.9) can be expressed as a Vandermonde coefficient as an ordered product, as indicated in (5.49) and (5.50) of [31]. We have

$$\nu = [(k_1 - k_1^*)(k_1 - k_2) \cdots (k_1 - k_{\mathbf{N}}^*)][(k_1^* - k_2) \cdots (k_1^* - k_{\mathbf{N}}^*)] \cdots [(k_{\mathbf{N}} - k_{\mathbf{N}}^*)]. \quad (5.10)$$

The coefficient ν_{10} is obtained from ν by replacing k_1 with zk_1 , the coefficient ν_{01} is obtained from ν by replacing k_1^* with $z^2 k_1^*$, and the coefficient ν_{11} is obtained from ν by replacing k_1 with zk_1 and k_1^* with $z^2 k_1^*$. In other words, we have

$$\nu_{10} := \nu \Big|_{k_1 \mapsto zk_1}, \quad \nu_{01} := \nu \Big|_{k_1^* \mapsto z^2 k_1^*}, \quad \nu_{11} := \nu \Big|_{\substack{k_1 \mapsto zk_1 \\ k_1^* \mapsto z^2 k_1^*}}. \quad (5.11)$$

We remark that the variable x appears on the right-hand side of (5.8) only through χ_1 . Thus, when we let $\chi_j = 0$ for $2 \leq j \leq \mathbf{N}$, from (4.18) and (5.8) we obtain

$$\frac{d \det[\mathbf{M}(x)]}{dx} = -(k_1 + k_1^*) [\nu_{10} \gamma_1 + \nu_{01} \gamma_1^*] \chi_1 - 2(k_1 + k_1^*) \nu_{11} \gamma_1 \gamma_1^* \chi_1^2, \quad (5.12)$$

$$\frac{d^2 \det[\mathbf{M}(x)]}{dx^2} = (k_1 + k_1^*)^2 [\nu_{10} \gamma_1 + \nu_{01} \gamma_1^*] \chi_1 + 4(k_1 + k_1^*)^2 \nu_{11} \gamma_1 \gamma_1^* \chi_1^2. \quad (5.13)$$

We recall that the matrix $\mathbf{M}_2(x)$ appearing in (5.3) is obtained from the matrix $\mathbf{M}(x)$ in (4.22) by replacing the second column by the vector $\mathbf{B}(x)$ in (4.23). Consequently, $\det[\mathbf{M}_2(x)]$ is a polynomial of degree $2\mathbf{N}$ in the $2\mathbf{N}$ variables $\gamma_1 \chi_1, \gamma_1^* \chi_1, \dots, \gamma_{\mathbf{N}} \chi_{\mathbf{N}}, \gamma_{\mathbf{N}}^* \chi_{\mathbf{N}}$, where each of those $2\mathbf{N}$ variables appears at most to the first power. The appearance of each χ_j in $\det[\mathbf{M}_2(x)]$ is symmetrical. By letting $\chi_j = 0$ for $2 \leq j \leq \mathbf{N}$, from (4.22) we obtain the reduced form of the matrix with the determinant given by

$$\det[\mathbf{M}_2(x)] = \det \begin{bmatrix} k_1^{2\mathbf{N}-1} + (zk_1)^{2\mathbf{N}-1} \gamma_1 \chi_1 & k_1^{2\mathbf{N}} + (zk_1)^{2\mathbf{N}} \gamma_1 \chi_1 & \cdots & 1 + \gamma_1 \chi_1 \\ (k_1^*)^{2\mathbf{N}-1} + (z^2 k_1^*)^{2\mathbf{N}-1} \gamma_1^* \chi_1 & (k_1^*)^{2\mathbf{N}} + (z^2 k_1^*)^{2\mathbf{N}} \gamma_1^* \chi_1 & \cdots & 1 + \gamma_1^* \chi_1 \\ \vdots & \vdots & \ddots & \vdots \\ k_{\mathbf{N}}^{2\mathbf{N}-1} & k_{\mathbf{N}}^{2\mathbf{N}} & \cdots & 1 \\ (k_{\mathbf{N}}^*)^{2\mathbf{N}-1} & (k_{\mathbf{N}}^*)^{2\mathbf{N}} & \cdots & 1 \end{bmatrix}. \quad (5.14)$$

In analogy with (5.8), we write (5.14) as

$$\det[\mathbf{M}_2(x)] = \mu + [\mu_{10} \gamma_1 + \mu_{01} \gamma_1^*] \chi_1 + \mu_{11} \gamma_1 \gamma_1^* \chi_1^2, \quad (5.15)$$

where the coefficients μ, μ_{10}, μ_{01} , and μ_{11} are explicitly expressed in terms of the parameters in the set $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$. The coefficient μ is obtained by letting $\chi_1 = 0$ in (5.7), from which we have

$$\mu := \det \begin{bmatrix} k_1^{2\mathbf{N}-1} & k_1^{2\mathbf{N}} & \cdots & k_1 & 1 \\ (k_1^*)^{2\mathbf{N}-1} & (k_1^*)^{2\mathbf{N}} & \cdots & k_1^* & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{\mathbf{N}}^{2\mathbf{N}-1} & k_{\mathbf{N}}^{2\mathbf{N}} & \cdots & k_{\mathbf{N}} & 1 \\ (k_{\mathbf{N}}^*)^{2\mathbf{N}-1} & (k_{\mathbf{N}}^*)^{2\mathbf{N}} & \cdots & k_{\mathbf{N}}^* & 1 \end{bmatrix}. \quad (5.16)$$

Using (4.4), (4.28), and (5.16), we write μ in terms of the quantity ν in (5.10) as

$$\mu = (\Pi_{\mathbf{N}} - \Sigma_{\mathbf{N}}^2) \nu. \quad (5.17)$$

The coefficient μ_{10} is obtained from μ in (5.17) by replacing k_1 with zk_1 , the coefficient μ_{01} is obtained from μ by replacing k_1^* with $z^2 k_1^*$, and the coefficient μ_{11} is obtained from μ by replacing k_1 with zk_1 and k_1^* with $z^2 k_1^*$. In other words, we have

$$\mu_{10} := \mu \Big|_{k_1 \mapsto zk_1}, \quad \mu_{01} := \mu \Big|_{k_1^* \mapsto z^2 k_1^*}, \quad \mu_{11} := \mu \Big|_{\substack{k_1 \mapsto zk_1 \\ k_1^* \mapsto z^2 k_1^*}}. \quad (5.18)$$

In order to determine the ratio s_1/r_1 in the case of SK1, we proceed as follows. In this case, since $P \equiv 0$, any antiderivative of the right-hand side of (4.32) must be equal to a constant and that constant must be equal to the value of the chosen antiderivative at $x = +\infty$. Hence, with the help of (4.33), we integrate (4.32) and obtain

$$\frac{1}{2} [A_{2\mathbf{N}-1}(x)]^2 - A'_{2\mathbf{N}-1}(x) - A_{2\mathbf{N}-2}(x) = \Pi_{\mathbf{N}} - \frac{1}{2} \Sigma_{\mathbf{N}}^2, \quad (5.19)$$

where we recall that the prime denotes the x -derivative. Using (5.3) and (5.5) in (5.19), after some simplification, we obtain the equivalent expression given by

$$\begin{aligned} (\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}) \det[\mathbf{M}(x)] + \Sigma_{\mathbf{N}} \frac{d \det[\mathbf{M}(x)]}{dx} + \frac{d^2 \det[\mathbf{M}(x)]}{dx^2} + \det[\mathbf{M}_2(x)] \\ = \frac{1}{2 \det[\mathbf{M}(x)]} \left(\frac{d \det[\mathbf{M}(x)]}{dx} \right)^2. \end{aligned} \quad (5.20)$$

Multiplying both sides of (5.20) by $2 \det[\mathbf{M}(x)]$, we obtain the equivalent equality given by

$$\begin{aligned} 2(\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}})(\det[\mathbf{M}(x)])^2 + 2 \Sigma_{\mathbf{N}} \det[\mathbf{M}(x)] \frac{d \det[\mathbf{M}(x)]}{dx} + 2 \det[\mathbf{M}(x)] \frac{d^2 \det[\mathbf{M}(x)]}{dx^2} \\ + 2 \det[\mathbf{M}(x)] \det[\mathbf{M}_2(x)] - \left(\frac{d \det[\mathbf{M}(x)]}{dx} \right)^2 = 0. \end{aligned} \quad (5.21)$$

If we let $\chi_j = 0$ for $2 \leq j \leq \mathbf{N}$ in (5.21), we see that the left-hand side of (5.21) becomes a polynomial of degree four in χ_1 . Using (5.8), (5.12), (5.13), and (5.15) in (5.21), we observe that the equality in (5.21) is satisfied if and only if the coefficient of χ_1 on the left-hand side is chosen as zero. Using (5.1), we see that the aforementioned coefficient vanishes if and only if we have

$$\frac{s_1}{r_1} = i \frac{\alpha_1 (\nu_{10} + \nu_{01}) + (\mu_{10} + \mu_{01})}{\alpha_1 (\nu_{10} - \nu_{01}) + (\mu_{10} - \mu_{01})}, \quad (5.22)$$

where we have defined

$$\alpha_1 := (\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}) - (k_1 + k_1^*) \Sigma_{\mathbf{N}} + (k_1 + k_1^*)^2. \quad (5.23)$$

Thus, we have shown that the ratio s_1/r_1 in the case of SK1 is uniquely determined by the parameters in the set $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$.

To determine the ratio s_1/r_1 in terms of the parameters in the set $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$ in the case of SK2, we proceed as follows. This time, we have $P = Q_x$ or equivalently we have $P - Q_x \equiv 0$. Using (4.31) and (4.32), we see that the latter equality is equivalent to

$$A_{2\mathbf{N}-1}(x) A'_{2\mathbf{N}-1}(x) - A'_{2\mathbf{N}-2}(x) = 0. \quad (5.24)$$

Integrating (5.24) and using (4.33), we get

$$\frac{1}{2} [A_{2\mathbf{N}-1}(x)]^2 - A_{2\mathbf{N}-2}(x) = \Pi_{\mathbf{N}} - \frac{1}{2} \Sigma_{\mathbf{N}}^2. \quad (5.25)$$

Using (5.3) and (5.5) in (5.25), after some simplification we obtain the equivalent expression

$$(\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}) \det[\mathbf{M}(x)] + \Sigma_{\mathbf{N}} \frac{d \det[\mathbf{M}(x)]}{dx} + \det[\mathbf{M}_2(x)] = -\frac{1}{2 \det[\mathbf{M}(x)]} \left(\frac{d \det[\mathbf{M}(x)]}{dx} \right)^2. \quad (5.26)$$

We remark that (5.26) is the counterpart in the SK2 case for (5.20) given in the case of SK1. Multiplying both sides of (5.26) by $2 \det[\mathbf{M}(x)]$, we obtain the equivalent equality given by

$$\begin{aligned} 2(\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}})(\det[\mathbf{M}(x)])^2 + 2 \Sigma_{\mathbf{N}} \det[\mathbf{M}(x)] \frac{d \det[\mathbf{M}(x)]}{dx} \\ + 2 \det[\mathbf{M}(x)] \det[\mathbf{M}_2(x)] + \left(\frac{d \det[\mathbf{M}(x)]}{dx} \right)^2 = 0. \end{aligned} \quad (5.27)$$

If we let $\chi_j = 0$ for $2 \leq j \leq \mathbf{N}$ in (5.27), we observe that the left-hand side of (5.27) becomes a polynomial of degree four in χ_1 . Using (5.8), (5.12), (5.13), and (5.15) in (5.27), we see that the equality in (5.27) is satisfied if and only if the coefficient of χ_1 on the left-hand side is chosen as zero. Using (5.1), we see that the aforementioned coefficient vanishes if and only if we have

$$\frac{s_1}{r_1} = i \frac{\beta_1 (\nu_{10} + \nu_{01}) + (\mu_{10} + \mu_{01})}{\beta_1 (\nu_{10} - \nu_{01}) + (\mu_{10} - \mu_{01})}, \quad (5.28)$$

where we have defined

$$\beta_1 := (\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}) - (k_1 + k_1^*) \Sigma_{\mathbf{N}}. \quad (5.29)$$

Comparing (5.23) and (5.29), we see that α_1 and β_1 differ from each other by only one term. Thus, we have shown that the ratio s_1/r_1 in the case of SK2 is uniquely determined by the parameters in the set $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$.

As we have already indicated, because of the symmetry in the input values k_j for $1 \leq j \leq \mathbf{N}$, the knowledge of the ratio s_1/r_1 in both SK1 and SK2 cases allows us to uniquely determine all the ratios s_j/r_j for $1 \leq j \leq \mathbf{N}$. By using those ratios in (5.1), we express each γ_j and γ_j^* in terms of the real parameter r_j and the elements in the set $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$. In analogy with (5.11) we define the coefficients ν_{j0} , ν_{0j} , and ν_{jj} as

$$\nu_{j0} := \nu \Big|_{k_j \mapsto zk_j}, \quad \nu_{0j} := \nu \Big|_{k_j^* \mapsto z^2 k_j^*}, \quad \nu_{jj} := \nu \Big|_{\substack{k_j \mapsto zk_j \\ k_j^* \mapsto z^2 k_j^*}}, \quad (5.30)$$

and in analogy with (5.18) we define the coefficients μ_{j0} , μ_{0j} , and μ_{jj} as

$$\mu_{j0} := \mu \Big|_{k_j \mapsto zk_j}, \quad \mu_{0j} := \mu \Big|_{k_j^* \mapsto z^2 k_j^*}, \quad \mu_{jj} := \mu \Big|_{\substack{k_j \mapsto zk_j \\ k_j^* \mapsto z^2 k_j^*}}. \quad (5.31)$$

Similarly, in analogy with (5.23) and (5.29), we define the quantities α_j and β_j as

$$\alpha_j := (\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}) - (k_j + k_j^*) \Sigma_{\mathbf{N}} + (k_j + k_j^*)^2, \quad (5.32)$$

$$\beta_j := (\Sigma_{\mathbf{N}}^2 - \Pi_{\mathbf{N}}) - (k_j + k_j^*) \Sigma_{\mathbf{N}}. \quad (5.33)$$

Then, using (5.30)–(5.33), in analogy with (5.22) and (5.28), respectively, we obtain the ratios s_j/r_j in the SK1 case as

$$\frac{s_j}{r_j} = i \frac{\alpha_j (\nu_{j0} + \nu_{0j}) + (\mu_{j0} + \mu_{0j})}{\alpha_j (\nu_{j0} - \nu_{0j}) + (\mu_{j0} - \mu_{0j})}, \quad 1 \leq j \leq \mathbf{N}, \quad (5.34)$$

and we obtain the ratios s_j/r_j in the SK2 case as

$$\frac{s_j}{r_j} = i \frac{\beta_j (\nu_{j0} + \nu_{0j}) + (\mu_{j0} + \mu_{0j})}{\beta_j (\nu_{j0} - \nu_{0j}) + (\mu_{j0} - \mu_{0j})}, \quad 1 \leq j \leq \mathbf{N}. \quad (5.35)$$

We then recover Q by using (5.6), where $\det[\mathbf{M}(x)]$ is uniquely determined by the input data set $\{k_j, r_j\}_{j=1}^{\mathbf{N}}$. The constructed Q becomes the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation (1.1). We remark that although the parameters r_j when we have $1 \leq j \leq \mathbf{N}$ are each real, the range values for each r_j may need to be restricted to ensure that Q does not have any singularities. We deal with those restrictions by using the following simple procedure and by introducing the positive parameter c_j instead of using r_j for each value of j for $1 \leq j \leq \mathbf{N}$.

We remark that the use of the ratios s_j/r_j in both cases of SK1 and SK2 not only ensures that the resulting $\det[\mathbf{M}(x)]$ via (5.6) yields the potential Q satisfying (1.1), but it also results in a significant simplification in $\det[\mathbf{M}(x)]$. The simplification occurs as follows. The ratio $\det[\mathbf{M}(x)]/\nu$, where ν is the constant appearing in (5.9) and (5.10), becomes the square of a polynomial of degree $\mathbf{N}$ in $\chi_1, \dots, \chi_{\mathbf{N}}$, where each χ_j for $1 \leq j \leq \mathbf{N}$ appears at most to the first power. Furthermore, that polynomial reduces to 1 when we let $\chi_j = 0$ for $1 \leq j \leq \mathbf{N}$. We use $\Delta(x)$ to denote that polynomial. We refer the reader to [28, 31] for further details on the introduction of $\Delta(x)$.

We now briefly describe the properties of $\Delta(x)$. We first remark that the justification for the existence of $\Delta(x)$ can be found in (5.20) in the SK1 case and in (5.26) in the SK2 case. Each term on the left-hand side of (5.20) is a polynomial of degree $2\mathbf{N}$ in $\chi_1, \dots, \chi_{\mathbf{N}}$, whereas the right-hand side is the ratio of a polynomial of degree $4\mathbf{N}$ to a polynomial degree $2\mathbf{N}$. Hence, that right-hand side must be a polynomial of degree $2\mathbf{N}$. This is assured by choosing $\det[\mathbf{M}(x)]$ as

$$\det[\mathbf{M}(x)] = \nu \Delta(x)^2. \quad (5.36)$$

The same argument also applies for the equality in (5.26). In other words, (5.36) must hold also in the case of SK2. We already know each coefficient in $\det[\mathbf{M}(x)]$ explicitly in terms of the parameters in the data set $\{k_j, k_j^*, r_j\}_{j=1}^{\mathbf{N}}$. Furthermore, we know the explicit expression for ν in terms of the parameters $\{k_j, k_j^*\}_{j=1}^{\mathbf{N}}$. Thus, using (5.36), we can determine each coefficient of χ_j in the polynomial $\Delta(x)$. For this, we proceed as follows. We would like to write $\Delta(x)$ as

$$\Delta(x) = 1 + \sum_{j=1}^{\mathbf{N}} c_j \chi_j + \dots, \quad (5.37)$$

where each parameter c_j takes positive values and the ellipsis contains the terms with the second and higher powers in the polynomial in $\chi_1, \dots, \chi_N$. With the help of (5.8), we see that $\det[\mathbf{M}(x)]$ can be written as

$$\det[\mathbf{M}(x)] = \nu + \sum_{j=1}^N \left[\nu_{j0} \gamma_j + \nu_{0j} \gamma_j^* \right] \chi_j + \dots, \quad (5.38)$$

where the ellipsis contains the terms with the second and higher powers in the polynomial in $\chi_1, \dots, \chi_N$ and it is understood that γ_j and γ_j^* on the right-hand side are evaluated as in (5.1) by using the ratios s_j/r_j given by (5.34) and (5.35), respectively, in the SK1 case and in the SK2 case. After some simplification, we evaluate the coefficient of χ_j in (5.38) in the SK1 case as

$$\nu_{j0} \gamma_j + \nu_{0j} \gamma_j^* = \frac{2r_j(\nu_{0j} \mu_{j0} - \nu_{j0} \mu_{0j})}{\alpha_j(\nu_{j0} - \nu_{0j}) + (\mu_{j0} - \mu_{0j})}, \quad 1 \leq j \leq N, \quad (5.39)$$

and in the SK2 case as

$$\nu_{j0} \gamma_j + \nu_{0j} \gamma_j^* = \frac{2r_j(\nu_{0j} \mu_{j0} - \nu_{j0} \mu_{0j})}{\beta_j(\nu_{j0} - \nu_{0j}) + (\mu_{j0} - \mu_{0j})}, \quad 1 \leq j \leq N. \quad (5.40)$$

With the help of (5.36)–(5.38), we see that the positive parameters c_j are given by

$$c_j = \frac{\nu_{j0} \gamma_j + \nu_{0j} \gamma_j^*}{2\nu}, \quad 1 \leq j \leq N, \quad (5.41)$$

with the understanding that γ_j and γ_j^* on the right-hand side are evaluated as in (5.1) by using the ratios s_j/r_j given in (5.34) and (5.35), respectively, in the SK1 and SK2 cases. Using (5.39)–(5.41), in the SK1 case, we obtain

$$c_j = \frac{r_j}{\nu} \left[\frac{(\nu_{0j} \mu_{j0} - \nu_{j0} \mu_{0j})}{\alpha_j(\nu_{j0} - \nu_{0j}) + (\mu_{j0} - \mu_{0j})} \right], \quad 1 \leq j \leq N, \quad (5.42)$$

and in the SK2 case, we get

$$c_j = \frac{r_j}{\nu} \left[\frac{(\nu_{0j} \mu_{j0} - \nu_{j0} \mu_{0j})}{\beta_j(\nu_{j0} - \nu_{0j}) + (\mu_{j0} - \mu_{0j})} \right], \quad 1 \leq j \leq N. \quad (5.43)$$

From (5.42) and (5.43), we see that each parameter c_j in (5.37) is directly proportional to r_j . Hence, in order to express the N -soliton solution to the Sawada–Kotera equation (1.1), we can use the input $\{k_j, c_j\}_{j=1}^N$ instead of using the input $\{k_j, r_j\}_{j=1}^N$. As indicated already, the advantage of using the input $\{k_j, c_j\}_{j=1}^N$ is that the quantity $\Delta(x)$ becomes nonsingular for $x \in \mathbb{R}$ for each $t \in \mathbb{R}$ when we choose each c_j as an arbitrary positive parameter.

Using the value of c_j in (5.37), we obtain the explicit expression for $\Delta(x)$ as

$$\begin{aligned} \Delta(x) = & 1 + \sum_{j=1}^N c_j \chi_j + \sum_{1 \leq j_1 < j_2 \leq N} A_{j_1 j_2} c_{j_1} c_{j_2} \chi_{j_1} \chi_{j_2} \\ & + \sum_{1 \leq j_1 < j_2 < j_3 \leq N} A_{j_1 j_2} A_{j_1 j_3} A_{j_2 j_3} c_{j_1} c_{j_2} c_{j_3} \chi_{j_1} \chi_{j_2} \chi_{j_3} \\ & + \dots + [A_{12} A_{13} \dots A_{(N-1)N} c_1 c_2 \dots c_N] \chi_1 \chi_2 \dots \chi_N, \end{aligned} \quad (5.44)$$

where the right-hand side is a polynomial of degree N in $\chi_1, \dots, \chi_N$ with the last term containing the product $\chi_1 \chi_2 \dots \chi_N$. The double-indexed quantity A_{jl} is determined by the elements in the set $\{k_j\}_{j=1}^N$ explicitly and is given by

$$A_{jl} := \frac{(k_j - k_l)^3 (k_j^3 + k_l^3)}{(k_j + k_l)^3 (k_j^3 - k_l^3)}, \quad 1 \leq j < l \leq N. \quad (5.45)$$

We recall that the k_j -values are chosen as distinct and as in (4.3), where each η_j is a positive parameter. Hence, the quantity A_{jl} in (5.45) is strictly positive. Thus, all the coefficients in (5.44) are positive, and consequently $\Delta(x)$ itself is positive for all $x \in \mathbb{R}$ and each fixed $t \in \mathbb{R}$. We refer the reader to Section 6 of [31] for further details.

Using (5.36) in (5.6), we see that the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation (1.1) is obtained as

$$Q(x) = 6 \frac{d}{dx} \left(\frac{\Delta'(x)}{\Delta(x)} \right). \quad (5.46)$$

We remark that we have used the input $\{k_j, c_j\}_{j=1}^{\mathbf{N}}$ to construct $\Delta(x)$. We refer to the positive parameter c_j appearing in $\Delta(x)$ as the bound-state constant at the bound state at $k = k_j$. Instead of using the complex-valued parameters k_j for $1 \leq j \leq \mathbf{N}$, we can use the positive parameters η_j appearing in (4.3). From (4.3), we see that each k_j is a constant multiple of η_j . Consequently, we can write A_{jl} in (5.45) as

$$A_{jl} := \frac{(\eta_j - \eta_l)^3(\eta_j^3 + \eta_l^3)}{(\eta_j + \eta_l)^3(\eta_j^3 - \eta_l^3)}, \quad 1 \leq j < l \leq \mathbf{N}.$$

Thus, the quantity $\Delta(x)$ can also be uniquely constructed by using the input data set $\{\eta_j, c_j\}_{j=1}^{\mathbf{N}}$, where each of the parameters η_j and c_j is positive for $1 \leq j \leq \mathbf{N}$.

6 | The Marchenko Integral Equation

In this section, we present the Marchenko integral equation related to the $\mathbf{N}$ -soliton solution of the Sawada–Kotera equation (1.1). We recall that we use the input data set $\{k_j, E_j\}_{j=1}^{\mathbf{N}}$ or one of its equivalents in (1.2), where our input data set is related to the scattering data of the third-order equation (2.4) in the reflectionless case and the left transmission coefficient $T_1(k)$ is given in (4.1). We also recall that k_j and k_j^* correspond to the bound-state locations in the complex k -plane and that E_j and E_j^* denote the respective bound-state dependency constants for $1 \leq j \leq \mathbf{N}$. After establishing our Marchenko integral equation, we show how the $\mathbf{N}$ -soliton solution to (1.1) is obtained from the solution to our Marchenko integral equation. As indicated in Section 1, the establishment of the Marchenko integral equation is a significant mathematical result.

For the derivation of our Marchenko integral equation, we proceed in the standard way so that the same procedure can be used to derive the relevant Marchenko integral equations associated with the soliton solutions to the Kaup–Kupershmidt equation, the bad Boussinesq equation, the good Boussinesq equation, and the modified bad Boussinesq equation. We start with the Riemann–Hilbert problem formulated in (4.8). By multiplying both sides of (4.8) with e^{-kx} and subtracting 1 from each side, we obtain

$$e^{-kx} \Phi_+(k, x) - 1 = e^{-kx} \Phi_-(k, x) - 1, \quad k \in \mathcal{L}. \quad (6.1)$$

By using the transformation $k \mapsto s$, where $k = zs$ with z being the special constant defined in (3.3), we map the complex k -plane onto the complex s -plane in a one-to-one and onto manner. We remark that the transformation can also be interpreted as a clockwise rotation in the complex plane around the origin by $2\pi/3$ radians. We note that the plus and minus regions $\mathcal{P}^+$ and $\mathcal{P}^-$ shown on the left plot in Figure 2 are mapped to $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, in the complex s -plane. Here, we use $\mathbb{C}^+$ and $\mathbb{C}^-$ for the upper-half and lower-half complex planes and we let $\overline{\mathbb{C}^+} := \mathbb{C}^+ \cup \mathbb{R}$ and $\overline{\mathbb{C}^-} := \mathbb{C}^- \cup \mathbb{R}$. The directed full line $\mathcal{L}$ is mapped to the directed real axis $\mathbb{R}$ in the complex s -plane.

By using $k = zs$ and letting

$$\begin{aligned} F(x, s) &:= e^{-zsx} \Phi_+(zs, x) - 1, & s \in \mathbb{R}, \\ G(x, s) &:= e^{-zsx} \Phi_-(zs, x) - 1, & s \in \mathbb{R}, \end{aligned} \quad (6.2)$$

as shown in Figure 2, we transform (6.1) to the corresponding Riemann–Hilbert problem in the complex s -plane, and we obtain

$$F(x, s) = G(x, s), \quad s \in \mathbb{R}. \quad (6.3)$$

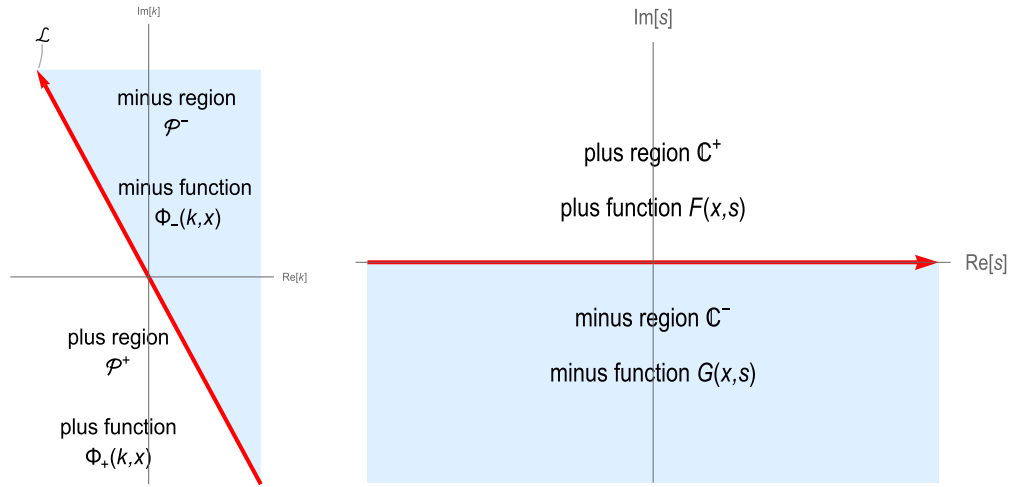


FIGURE 2 | The mapping from the complex k -plane to the complex s -plane transforms the plus and minus regions $\mathcal{P}^+$ and $\mathcal{P}^-$ to $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. The plus and minus functions $\Phi_+(k, x)$ and $\Phi_-(k, x)$ are the components of a sectionally meromorphic function, respectively, in the complex k -plane. The plus and minus functions $F(x, s)$ and $G(x, s)$ are the components of a sectionally meromorphic function, respectively, in the complex s -plane.

From the properties of $\Phi_+(k, x)$ and $\Phi_-(k, x)$ described in Section 4, it follows that, for each fixed $x \in \mathbb{R}$, the quantity $F(x, s)$ is meromorphic with simple poles at $s = z^2 k_j$ and $s = z^2 k_j^*$ for $1 \leq j \leq \mathbf{N}$ in $s \in \mathbb{C}^+$ and continuous in $s \in \overline{\mathbb{C}^+}$, and we have $F(x, s) = O(1/s)$ as $s \rightarrow \infty$ in $s \in \overline{\mathbb{C}^+}$. Similarly, for each fixed $x \in \mathbb{R}$, it follows that the quantity $G(x, s)$ is analytic in $s \in \mathbb{C}^-$ and continuous in $s \in \overline{\mathbb{C}^-}$, and we have $G(x, s) = O(1/s)$ as $s \rightarrow \infty$ in $s \in \overline{\mathbb{C}^-}$. Consequently, the Fourier transforms in $L^2(\mathbb{R})$ with $s \in \mathbb{R}$ for $F(x, s)$ and $G(x, s)$ exist. We define the Fourier transforms $\hat{F}(x, y)$ and $\hat{G}(x, y)$, respectively, by letting

$$\hat{F}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-isy} F(x, s), \quad y \in \mathbb{R}, \quad (6.4)$$

$$\hat{G}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-isy} G(x, s), \quad y \in \mathbb{R}. \quad (6.5)$$

From (6.4) and (6.5), respectively, we get the inverse Fourier transforms given by

$$\begin{aligned} F(x, s) &= \int_{-\infty}^{\infty} dy e^{isy} \hat{F}(x, y), \quad s \in \mathbb{R}, \\ G(x, s) &= \int_{-\infty}^{\infty} dy e^{isy} \hat{G}(x, y), \quad s \in \mathbb{R}. \end{aligned} \quad (6.6)$$

From (6.3)–(6.5), it follows that we have

$$\hat{F}(x, y) = \hat{G}(x, y), \quad y \in \mathbb{R}. \quad (6.7)$$

Using the aforementioned properties of $G(x, s)$ when $s \in \overline{\mathbb{C}^-}$, from (6.6) we conclude that

$$\hat{G}(x, y) = 0, \quad y > 0. \quad (6.8)$$

Then, (6.7) and (6.8) yield

$$\hat{F}(x, y) = 0, \quad y > 0. \quad (6.9)$$

Thus, using (6.8) in (6.6) we obtain

$$G(x, s) = \int_{-\infty}^0 dy e^{isy} \hat{G}(x, y), \quad s \in \mathbb{R}. \quad (6.10)$$

Next, we would like to evaluate $\hat{G}(x, y)$ when $y < 0$. In order to accomplish this, it is sufficient to determine $\hat{F}(x, y)$ for $y < 0$ and then use (6.7). For the determination of $\hat{F}(x, y)$ for $y < 0$, we use (6.4) when $y < 0$ by evaluating the integral there with the help of the residues of the integrand at $s = z^2 k_j$ and $s = z^2 k_j^*$ for $1 \leq j \leq \mathbf{N}$. In terms of the quantity $\Gamma(k)$ in (4.2), we define the related quantity $\Gamma_j(k)$, which differ from $\Gamma(k)$ by the product $(k + k_j)(k + k_j^*)$. Thus, we let

$$\Gamma_j(k) := \frac{\Gamma(k)}{(k + k_j)(k + k_j^*)}, \quad 1 \leq j \leq \mathbf{N}. \quad (6.11)$$

We remark that, in analogy with the first equality of (4.17), the quantity $\Gamma_j(k)$ also satisfies

$$\Gamma_j(k^*) = \Gamma_j(k)^*, \quad k \in \mathbb{C}. \quad (6.12)$$

From (4.1) and (6.11), we see that the left transmission coefficient can be written in terms of $\Gamma_j(k)$ as

$$T_1(k) = \frac{\Gamma(k)}{(k - k_j)(k - k_j^*) \Gamma_j(-k)}, \quad k \in \mathbb{C}. \quad (6.13)$$

Using (6.13), we express the value of the left transmission coefficient in the complex s -plane as

$$T_1(zs) = \frac{z \Gamma(zs)}{(s - z^2 k_j)(s - z^2 k_j^*) \Gamma_j(-zs)}, \quad s \in \mathbb{C}, \quad (6.14)$$

where we have used the fact that $z^3 = 1$. The expression in (6.14) allows us to determine the residues of $T_1(zs)$ at the poles $s = z^2 k_j$ and $s = z^2 k_j^*$, respectively, as

$$\text{Res}[T_1(zs), s = z^2 k_j] = \frac{z^2 \Gamma(k_j)}{(k_j - k_j^*) \Gamma_j(-k_j)}, \quad (6.15)$$

$$\text{Res}[T_1(zs), s = z^2 k_j^*] = -\frac{z^2 \Gamma(k_j^*)}{(k_j - k_j^*) \Gamma_j(-k_j^*)}. \quad (6.16)$$

With the help of (6.15) and (6.16), we evaluate (6.4) for $y < 0$, and we get

$$\begin{aligned} \hat{F}(x, y) = & i \sum_{j=1}^{\mathbf{N}} \text{Res}[T_1(zs), s = z^2 k_j] e^{-k_j x} f(k_j, x) e^{-iz^2 k_j y} \\ & + i \sum_{j=1}^{\mathbf{N}} \text{Res}[T_1(zs), s = z^2 k_j^*] e^{-k_j^* x} f(k_j^*, x) e^{-iz^2 k_j^* y}. \end{aligned} \quad (6.17)$$

Using (6.15) and (6.16) in (6.17), we express $\hat{F}(x, y)$ for $y < 0$ as

$$\hat{F}(x, y) = iz^2 \sum_{j=1}^{\mathbf{N}} \left(\frac{\Gamma(k_j) e^{-k_j x}}{(k_j - k_j^*) \Gamma_j(-k_j)} f(k_j, x) e^{-iz^2 k_j y} - \frac{\Gamma(k_j^*) e^{-k_j^* x}}{(k_j - k_j^*) \Gamma_j(-k_j^*)} f(k_j^*, x) e^{-iz^2 k_j^* y} \right). \quad (6.18)$$

Next, we use (6.7) on the left-hand side and use (4.5) and (4.6) on the right-hand side of (6.18). This yields the expression for $\hat{G}(x, y)$ for $y < 0$ as

$$\hat{G}(x, y) = iz^2 \sum_{j=1}^{\mathbf{N}} \left(\frac{\Gamma(k_j) D_j e^{-k_j x}}{(k_j - k_j^*) \Gamma_j(-k_j)} g(zk_j, x) e^{-iz^2 k_j y} - \frac{\Gamma(k_j^*) D_j^* e^{-k_j^* x}}{(k_j - k_j^*) \Gamma_j(-k_j^*)} g(z^2 k_j^*, x) e^{-iz^2 k_j^* y} \right), \quad (6.19)$$

where we recall that D_j and D_j^* are the time-dependent bound-state dependency constants at $k = k_j$ and $k = k_j^*$. From (4.3), we see that we have

$$k_j^* = -zk_j, \quad 1 \leq j \leq \mathbf{N}. \quad (6.20)$$

Using (4.7), (4.18), (4.19), and (6.20), we express the right-hand side of (6.19) in terms of the modified bound-state dependency constants γ_j and γ_j^* and the quantities χ_j . Hence, for $y < 0$, we get $\hat{G}(x, y)$ as

$$\begin{aligned} \hat{G}(x, y) = & -iz^2 \sum_{j=1}^{\mathbf{N}} \frac{\Gamma(-zk_j) \gamma_j \chi_j}{(k_j - k_j^*) \Gamma_j(-k_j)} e^{-zk_j x} g(zk_j, x) e^{-iz^2 k_j y} \\ & + iz^2 \sum_{j=1}^{\mathbf{N}} \frac{\Gamma(-z^2 k_j^*) \gamma_j^* \chi_j}{(k_j - k_j^*) \Gamma_j(-k_j^*)} e^{-z^2 k_j^* x} g(z^2 k_j^*, x) e^{-iz^2 k_j^* y}. \end{aligned} \quad (6.21)$$

We recall that, in order to obtain the quantity Q satisfying the Sawada–Kotera equation (1.1), we must use the modified bound-state dependency constants γ_j and γ_j^* as in (5.1) by using the ratios s_j/r_j given by (5.34) and (5.35), in the SK1 and SK2 cases, respectively.

Let us introduce the quantity b_j as

$$b_j := \frac{iz^2 \Gamma(-zk_j) \gamma_j \chi_j}{(k_j - k_j^*) \Gamma_j(-k_j)}, \quad 1 \leq j \leq \mathbf{N}. \quad (6.22)$$

Using (4.2), (4.3), (4.18), (4.19), and (6.12), we see that the complex conjugate b_j^* is given by

$$b_j^* = \frac{iz \Gamma(-z^2 k_j^*) \gamma_j^* \chi_j}{(k_j - k_j^*) \Gamma_j(-k_j^*)}, \quad 1 \leq j \leq \mathbf{N}. \quad (6.23)$$

The use of (6.22) and (6.23) in (6.21) yields

$$\hat{G}(x, y) = \sum_{j=1}^{\mathbf{N}} \left(-b_j e^{-zk_j x} g(zk_j, x) e^{-iz^2 k_j y} + z b_j^* e^{-z^2 k_j^* x} g(z^2 k_j^*, x) e^{-iz^2 k_j^* y} \right). \quad (6.24)$$

With the help of (3.24), (6.2), and (6.20), we get

$$e^{-zk_j x} g(zk_j, x) = 1 + \int_{-\infty}^0 d\zeta e^{ik_j \zeta} \hat{G}(x, \zeta), \quad (6.25)$$

$$e^{-z^2 k_j^* x} g(z^2 k_j^*, x) = 1 + \int_{-\infty}^0 d\zeta e^{izk_j^* \zeta} \hat{G}(x, \zeta). \quad (6.26)$$

Using (6.25) and (6.26) in (6.24), we obtain our Marchenko integral equation as

$$\hat{G}(x, y) + \Omega(0, y) + \int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \Omega(\zeta, y) = 0, \quad y < 0, \quad (6.27)$$

where the integral kernel $\Omega(\zeta, y)$ is defined as

$$\Omega(\zeta, y) := \sum_{j=1}^{\mathbf{N}} \left(b_j e^{ik_j \zeta - iz^2 k_j y} - z b_j^* e^{izk_j^* \zeta - iz^2 k_j^* y} \right). \quad (6.28)$$

We remark that we suppress the dependence on x and t in the integral kernel $\Omega(\zeta, y)$. As seen from (6.22) and (6.23), the dependence on x and t in the integral kernel $\Omega(\zeta, y)$ and in the nonhomogeneous term $\Omega(0, y)$ occurs through the quantities χ_j defined in (4.18). We recall that we already suppress the t -dependence in the potentials Q and P , the Jost solutions $f(k, x)$ and $g(k, x)$, and other quantities related to (2.4).

The Marchenko integral equation (6.27) corresponds to the “GLM equation” Kaup always wanted [12] to obtain. We emphasize its simplicity and its resemblance to the Marchenko integral equation for other linear operators. The resemblance can be observed in the case of the Schrödinger equation on the full line by comparing (6.27) with (44) of [2] and also with the penultimate displayed equality on p. 123 of [4] and by comparing the nonhomogeneous term $\Omega(0, y)$ with (45) of [2].

Next, we show how to recover the potential Q in (2.4) from the solution $\hat{G}(x, y)$ to the Marchenko integral equation (6.27). In fact, the potential Q is recovered as

$$Q(x) = 3iz \frac{d\hat{G}(x, 0^-)}{dx}, \quad x \in \mathbb{R}. \quad (6.29)$$

The expression in (6.29) is obtained from (3.13), (3.22), and (6.10) with the help of the first line of (3.24). We refer the reader to page 25 of [38] for the details of the derivation of (6.29). We are interested in obtaining the solution $\hat{G}(x, y)$ to (6.27) when $y < 0$ by using the same input data set used to solve the Riemann–Hilbert problem (4.8). We recall that the relevant input data set is given by $\{k_j, E_j\}_{j=1}^N$ or the equivalent data set $\{k_j, \gamma_j\}_{j=1}^N$, where the modified bound-state dependency constants γ_j are chosen as in (5.1) with the ratios s_j/r_j as in (5.34) in the SK1 case and as in (5.35) in the SK2 case. To recover $\hat{G}(x, y)$ from our input data set, we proceed as follows. From (6.28), we see that the nonhomogeneous term and the integral term in (6.27) are spanned by the basis set $\{e^{-iz^2 k_j y}, e^{-iz^2 k_j^* y}\}_{j=1}^N$. Hence, the first term $\hat{G}(x, y)$ itself in the Marchenko equation (6.27) must be expressed as a linear combination of the elements in the same basis set. Consequently, there must exist the coefficients g_j and h_j for $1 \leq j \leq N$ satisfying

$$\hat{G}(x, y) = \sum_{j=1}^N \left(g_j e^{-iz^2 k_j y} + h_j e^{-iz^2 k_j^* y} \right), \quad y < 0. \quad (6.30)$$

We remark that g_j and h_j are functions of x and t , but we suppress those dependencies in our notation. We use (6.28) and (6.30) in (6.27), and for $y < 0$ we obtain

$$\begin{aligned} & \sum_{j=1}^N \left(g_j e^{-iz^2 k_j y} + h_j e^{-iz^2 k_j^* y} \right) + \sum_{j=1}^N \left(b_j e^{-iz^2 k_j y} - z b_j^* e^{-iz^2 k_j^* y} \right) \\ & + \int_{-\infty}^0 d\zeta \sum_{l=1}^N \left(g_l e^{-iz^2 k_l \zeta} + h_l e^{-iz^2 k_l^* \zeta} \right) \sum_{j=1}^N \left(b_j e^{ik_j \zeta - iz^2 k_j y} - z b_j^* e^{izk_j^* \zeta - iz^2 k_j^* y} \right) = 0. \end{aligned}$$

Since the functions in the basis set $\{e^{-iz^2 k_j y}, e^{-iz^2 k_j^* y}\}_{j=1}^N$ are linearly independent for $y < 0$, each coefficient of the basis elements in (6.27) must vanish. This yields the linear algebraic system of $2N$ equations in the $2N$ unknowns $g_1, h_1, \dots, g_N, h_N$, and we get

$$\begin{cases} g_j + b_j + b_j \sum_{l=1}^N \int_{-\infty}^0 d\zeta \left(g_l e^{ik_j \zeta - iz^2 k_l \zeta} + h_l e^{ik_j \zeta - iz^2 k_l^* \zeta} \right) = 0, & 1 \leq j \leq N, \\ h_j - z b_j^* - z b_j^* \sum_{l=1}^N \int_{-\infty}^0 d\zeta \left(g_l e^{izk_j^* \zeta - iz^2 k_l \zeta} + h_l e^{izk_j^* \zeta - iz^2 k_l^* \zeta} \right) = 0, & 1 \leq j \leq N. \end{cases} \quad (6.31)$$

The integrals in (6.31) can be explicitly evaluated, and (6.31) becomes equivalent to

$$\begin{cases} g_j + \sum_{l=1}^N \left(\frac{b_j}{ik_j - iz^2 k_l} g_l + \frac{b_j}{ik_j - iz^2 k_l^*} h_l \right) = -b_j, & 1 \leq j \leq N, \\ h_j + \sum_{l=1}^N \left(\frac{-z b_j^*}{izk_j^* - iz^2 k_l} g_l + \frac{-z b_j^*}{izk_j^* - iz^2 k_l^*} h_l \right) = z b_j^*, & 1 \leq j \leq N. \end{cases} \quad (6.32)$$

With the help of (4.3), we get

$$\begin{cases} ik_j - iz^2 k_l = -iz^2(k_j^* + k_l), \\ ik_j - iz^2 k_l^* = i(k_j + k_l), \\ izk_j^* - iz^2 k_l = -iz^2(k_j + k_l), \\ izk_j^* - iz^2 k_l^* = -iz^2(k_j + k_l^*). \end{cases} \quad (6.33)$$

Using (6.33) in (6.32), we write (6.32) in the equivalent form as

$$\begin{cases} g_j + \sum_{l=1}^N \left(\frac{izb_j}{k_j^* + k_l} g_l + \frac{-ib_j}{k_j + k_l} h_l \right) = -b_j, & 1 \leq j \leq N, \\ h_j + \sum_{l=1}^N \left(\frac{-iz^2 b_j^*}{k_j + k_l} g_l + \frac{-iz^2 b_j^*}{k_j + k_l^*} h_l \right) = z b_j^*, & 1 \leq j \leq N. \end{cases} \quad (6.34)$$

We can write the linear algebraic system (6.32) in the matrix notation as

$$\mathbf{m}(x) \mathbf{a}(x) = -\mathbf{b}(x), \quad (6.35)$$

where $\mathbf{m}(x)$ is the $2N \times 2N$ coefficient matrix given by

$$\mathbf{m}(x) := \begin{bmatrix} 1 + \frac{izb_1}{k_1^* + k_1} & \frac{-ib_1}{k_1 + k_1} & \frac{izb_1}{k_1^* + k_2} & \frac{-ib_1}{k_1 + k_2} & \cdots & \frac{izb_1}{k_1^* + k_N} & \frac{-ib_1}{k_1 + k_N} \\ \frac{-iz^2 b_1^*}{k_1 + k_1} & 1 + \frac{-iz^2 b_1^*}{k_1 + k_1^*} & \frac{-iz^2 b_1^*}{k_1 + k_2} & \frac{-iz^2 b_1^*}{k_1 + k_2^*} & \cdots & \frac{-iz^2 b_1^*}{k_1 + k_N} & \frac{-iz^2 b_1^*}{k_1 + k_N^*} \\ \frac{izb_2}{k_2^* + k_1} & \frac{-ib_2}{k_2 + k_1} & 1 + \frac{izb_2}{k_2^* + k_2} & \frac{-ib_2}{k_2 + k_2} & \cdots & \frac{izb_2}{k_2^* + k_N} & \frac{-ib_2}{k_2 + k_N} \\ \frac{-iz^2 b_2^*}{k_2 + k_1} & \frac{-iz^2 b_2^*}{k_2 + k_1^*} & \frac{-iz^2 b_2^*}{k_2 + k_2} & 1 + \frac{-iz^2 b_2^*}{k_2 + k_2^*} & \cdots & \frac{-iz^2 b_2^*}{k_2 + k_N} & \frac{-iz^2 b_2^*}{k_2 + k_N^*} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{izb_N}{k_N^* + k_1} & \frac{-ib_N}{k_N + k_1} & \frac{izb_N}{k_N^* + k_2} & \frac{-ib_N}{k_N + k_2} & \cdots & 1 + \frac{izb_N}{k_N^* + k_N} & \frac{-ib_N}{k_N + k_N} \\ \frac{-iz^2 b_N^*}{k_N + k_1} & \frac{-iz^2 b_N^*}{k_N + k_1^*} & \frac{-iz^2 b_N^*}{k_N + k_2} & \frac{-iz^2 b_N^*}{k_N + k_2^*} & \cdots & \frac{-iz^2 b_N^*}{k_N + k_N} & 1 + \frac{-iz^2 b_N^*}{k_N + k_N^*} \end{bmatrix}, \quad (6.36)$$

and $\mathbf{a}(x)$ and $\mathbf{b}(x)$ are the column vectors with $2N$ entries, and they are given by

$$\mathbf{a}(x) := \begin{bmatrix} g_1 \\ h_1 \\ g_2 \\ h_2 \\ \vdots \\ g_N \\ h_N \end{bmatrix}, \quad \mathbf{b}(x) := \begin{bmatrix} b_1 \\ -zb_1^* \\ b_2 \\ -zb_2^* \\ \vdots \\ b_N \\ -zb_N^* \end{bmatrix}. \quad (6.37)$$

From (6.35), we recover the unknown column vector $\mathbf{a}(x)$ as

$$\mathbf{a}(x) = -\mathbf{m}(x)^{-1} \mathbf{b}(x). \quad (6.38)$$

Having obtained $\mathbf{a}(x)$, we use in (6.30) its components g_j and h_j for $1 \leq j \leq \mathbf{N}$, and we get $\hat{G}(x, y)$. For this, we proceed as follows. We write (6.30) in the matrix form as

$$\hat{G}(x, y) = \begin{bmatrix} e^{-iz^2 k_1 y} & e^{-iz^2 k_1^* y} & \dots & e^{-iz^2 k_{\mathbf{N}} y} & e^{-iz^2 k_{\mathbf{N}}^* y} \end{bmatrix} \begin{bmatrix} g_1 \\ h_1 \\ g_2 \\ h_2 \\ \vdots \\ g_{\mathbf{N}} \\ h_{\mathbf{N}} \end{bmatrix}. \quad (6.39)$$

Using (6.37) and (6.38) in (6.39), we obtain

$$\hat{G}(x, y) = - \begin{bmatrix} e^{-iz^2 k_1 y} & e^{-iz^2 k_1^* y} & \dots & e^{-iz^2 k_{\mathbf{N}} y} & e^{-iz^2 k_{\mathbf{N}}^* y} \end{bmatrix} \mathbf{m}(x)^{-1} \mathbf{b}(x). \quad (6.40)$$

Using (15) on p. 12 of [41], we can write the right-hand side of (6.40) as the ratio of two determinants, and we get

$$\hat{G}(x, y) = \frac{\det \left[\begin{array}{c|cccc} 0 & e^{-iz^2 k_1 y} & e^{-iz^2 k_1^* y} & \dots & e^{-iz^2 k_{\mathbf{N}} y} & e^{-iz^2 k_{\mathbf{N}}^* y} \\ \hline \mathbf{b}(x) & & & & \mathbf{m}(x) & \end{array} \right]}{\det[\mathbf{m}(x)]}. \quad (6.41)$$

From (6.41), we have

$$\hat{G}(x, 0^-) = \frac{\det \left[\begin{array}{c|cccc} 0 & 1 & 1 & \dots & 1 & 1 \\ \hline \mathbf{b}(x) & & & & \mathbf{m}(x) & \end{array} \right]}{\det[\mathbf{m}(x)]}. \quad (6.42)$$

In fact, it can be shown that the determinant of the matrix appearing in the numerator on the right-hand side in (6.42) is related to the x -derivative of the determinant of $\mathbf{m}(x)$. We have

$$\det \left[\begin{array}{c|cccc} 0 & 1 & 1 & \dots & 1 & 1 \\ \hline \mathbf{b}(x) & & & & \mathbf{m}(x) & \end{array} \right] = -iz^2 \frac{d \det[\mathbf{m}(x)]}{dx}. \quad (6.43)$$

The proof of (6.43) can be given as follows. Using (6.36) and the second equality of (6.37), we see that the determinant on the left-hand side of (6.43) is a polynomial of degree $2\mathbf{N}$ in the variables $b_1, b_1^*, \dots, b_{\mathbf{N}}, b_{\mathbf{N}}^*$, where each of those $2\mathbf{N}$ variables appears in the polynomial at most to the first power. From (6.36), we know that $\det[\mathbf{m}(x)]$ is also a polynomial of degree $2\mathbf{N}$ in the variables $b_1, b_1^*, \dots, b_{\mathbf{N}}, b_{\mathbf{N}}^*$, where each of those $2\mathbf{N}$ variables appears in the polynomial at most to the first power. Furthermore, from (6.22) and (6.23) we observe that each of b_j and b_j^* is a function of x only through the x -dependence in the term χ_j defined in (4.18) and that each of b_j and b_j^* is a constant multiple of χ_j . Hence, the x -derivative of b_j is a constant multiple of b_j and the x -derivative of b_j^* is a constant multiple of b_j^* . Thus, the right-hand side in (6.43) is also a polynomial of degree $2\mathbf{N}$ in the variables $b_1, b_1^*, \dots, b_{\mathbf{N}}, b_{\mathbf{N}}^*$, where each of those $2\mathbf{N}$ variables appears in the polynomial at most to the first power. In the polynomial equality in (6.43), we observe the equivalence of the corresponding coefficients. Hence, we have shown that (6.43) holds.

Using (6.43) in (6.42) we obtain

$$\hat{G}(x, 0^-) = \frac{-iz^2}{\det[\mathbf{m}(x)]} \frac{d \det[\mathbf{m}(x)]}{dx}. \quad (6.44)$$

Finally, using (6.44) in (6.29), we recover the potential Q in (2.4) as

$$Q(x) = 3 \frac{d}{dx} \left[\frac{1}{\det[\mathbf{m}(x)]} \frac{d \det[\mathbf{m}(x)]}{dx} \right]. \quad (6.45)$$

It can be verified that the determinant of the matrix $\mathbf{m}(x)$ appearing in (6.36) is related to the determinant of the matrix $\mathbf{M}(x)$ appearing in (4.22), and we have

$$\det[\mathbf{M}(x)] = \nu \det[\mathbf{m}(x)], \quad (6.46)$$

where ν is the Vandermonde coefficient appearing in (5.9) and (5.10). The proof of (6.46) is similar to the proof of (6.43), and it can be given as follows. From (5.14) we see that the left-hand side of (6.46) is a polynomial of degree $2N$ in the variables $\gamma_1\chi_1, \gamma_1^*\chi_1, \dots, \gamma_N\chi_N, \gamma_N^*\chi_N$ and that each of those $2N$ variables appears in the polynomial at most to the first power. Moreover, from (6.22) and (6.23), we see that $\gamma_j\chi_j$ is a constant multiple of b_j and that $\gamma_j^*\chi_j$ is a constant multiple of b_j^* . Hence, the left-hand side of (6.46) is a polynomial of degree $2N$ in the variables $b_1, b_1^*, \dots, b_N, b_N^*$, where each of those $2N$ variables appears at most to the first power. From (6.36) we see that $\det[\mathbf{m}(x)]$ is also a polynomial of degree $2N$ in the variables $b_1, b_1^*, \dots, b_N, b_N^*$, where each of those $2N$ variables appears in the polynomial at most to the first power. In the polynomial equality in (6.46), we observe the equivalence of the corresponding coefficients. Thus, we have shown that (6.46) holds.

With the help of (6.46), we see that (6.45) is equivalent to (5.6). We remark that (5.6) and (6.45) hold without restricting the ratios s_j/r_j as in (5.34) or (5.35). Without using those restrictions, the expression Q is not yet related to P as in the SK1 or SK2 cases. Only after we use those restrictions on the ratios s_j/r_j , the resulting expression for Q in (6.45) satisfies the Sawada–Kotera equation (1.1) with $P \equiv 0$ in the SK1 case and with $P = Q_x$ in the SK2 case. Thus, the equality (5.36) holds only after we impose the ratios s_j/r_j in (5.34) for the SK1 case or the ratios s_j/r_j in (5.35) for the SK2 case. Hence, after those ratios are used we get

$$\det[\mathbf{m}(x)] = \Delta(x)^2, \quad (6.47)$$

where $\Delta(x)$ is the quantity in (5.36) and (5.44).

Let us finally remark that in [38] we have presented the Marchenko integral equation for (2.4) in the absence of bound states. The recovery of the potential Q from the solution to the Marchenko integral equation is given in (6.29) and that formula holds both in the presence or absence of bound states. As indicated in the last displayed equation in Section 5 of [38], the recovery of the potential P from the solution to the Marchenko equation is obtained via

$$P(x) = -3z^2 \hat{G}(x, 0^-) \frac{d\hat{G}(x, 0^-)}{dx} + 3iz \frac{d^2\hat{G}(x, 0^-)}{dx^2} - 3z^2 \frac{d\hat{G}_y(x, 0^-)}{dx}, \quad x \in \mathbb{R},$$

where $\hat{G}_y(x, 0^-)$ denotes the value of the partial y -derivative of $\hat{G}(x, y)$ evaluated in the limit $y \rightarrow 0^-$.

7 | Conclusion

In this paper, we have presented the Marchenko integral equation (6.27) for the third-order linear equation (2.4) when the reflection coefficients are all zero. In the reflectionless case the input data set to solve the corresponding inverse scattering problem consists of the bound-state information, which comprises the bound-state poles of a transmission coefficient and the corresponding bound-state dependency constants. We have demonstrated how the relevant input data set yields the solution of the corresponding Riemann–Hilbert problem (4.8). We have also shown how the relevant Marchenko integral equation is obtained by using the appropriate Fourier transformation on the Riemann–Hilbert problem and how the solution to the Marchenko integral equation yields the potentials in (2.4). Furthermore, we have illustrated that, in the reflectionless case, the kernel of the Marchenko integral equation becomes separable, which results in a closed-form solution to the Marchenko equation by using the methods from linear algebra.

The importance of the Marchenko integral equation (6.27) is that, besides offering a solution to the relevant inverse scattering problem for the third-order linear equation (2.4), it yields soliton solutions to the integrable Sawada–Kotera equation (1.1). The method presented here explains the physical origins of the $2N$ parameters appearing in the N -soliton solution to the Sawada–Kotera equation by showing that those parameters come from the bound-state information for the related third-order linear problem (2.4). In the consideration of the Sawada–Kotera equation by using the inverse scattering transform method associated with the third-order equation, there are two distinct cases with $P \equiv 0$ and $P = Q_x$. In each of those two cases, we have indicated how the bound-state dependency constants must be chosen so that the constructed potential Q in the third-order linear problem yields a solution to the Sawada–Kotera equation.

We have derived our Marchenko integral equation yielding the N -soliton solution to the Sawada–Kotera equation (1.1) by employing a procedure used to derive the already known Marchenko integral equations for the second-order linear equation, that is, for the Schrödinger equation, and for various first-order linear systems such as the AKNS system [42]. This standard procedure involves the formulation of a corresponding Riemann–Hilbert problem and then the derivation of the Marchenko integral equation by using a Fourier transformation. Traditionally, in the formulation of a Riemann–Hilbert problem, the jump for the sectionally analytic function is determined by both the reflection coefficients and

the bound-state information. Our own formulation of the Riemann–Hilbert problem comprises the determination of a sectionally meromorphic function with the jump condition that is solely determined by the reflection coefficients. This latter formulation of a Riemann–Hilbert problem is better suited to solve the inverse scattering problem for the third-order linear operator in the reflectionless case. This fact also plays a key role in the formulation of our Marchenko integral equation.

The technique we have developed in our paper to obtain the $\mathbf{N}$ -soliton solution to the Sawada–Kotera equation can readily be applied to other integrable evolution equations associated with the third-order linear problem (2.4). In the near future, we plan to present the relevant formulation of the Riemann–Hilbert problem and the Marchenko integral equation for the $\mathbf{N}$ -soliton solution, for any arbitrary positive integer $\mathbf{N}$, for the Kaup–Kupershmidt equation

$$Q_t + Q_{xxxxx} + \frac{25}{2} Q_x Q_{xx} + 5 Q Q_{xxx} + 5 Q^2 Q_x = 0, \quad x, t \in \mathbb{R},$$

the bad Boussinesq equation

$$Q_{tt} - Q_{xx} + (Q^2)_{xx} - Q_{xxxx} = 0, \quad x, t \in \mathbb{R},$$

the good Boussinesq equation

$$Q_{tt} - Q_{xx} + (Q^2)_{xx} + Q_{xxxx} = 0, \quad x, t \in \mathbb{R},$$

and the modified bad Boussinesq equation

$$Q_{tt} + (Q^2)_{xx} - Q_{xxxx} = 0, \quad x, t \in \mathbb{R},$$

where the $\mathbf{N}$ -soliton solution Q in each of these four cases vanishes as $x \rightarrow \pm\infty$ for each fixed $t \in \mathbb{R}$ and that the $2\mathbf{N}$ parameters in each soliton solution is related to the bound-state poles of a transmission coefficient and the bound-state dependency constants associated with (2.4) with the appropriate choice of the accompanying potential P .

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Conflicts of Interest

The authors have no conflicts of interest to declare.

Data Availability Statement

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References

1. K. Sawada and T. Kotera, “A Method for Finding $\mathbf{N}$ -Soliton Solutions of the KdV Equation and KdV-Like Equation,” *Progress of Theoretical Physics* 51 (1974): 1355–1367.
2. T. Aktosun and M. Klaus, “Chapter 2.2.4: Inverse Theory: Problem on the Line,” in *Scattering*, ed. E. R. Pike and P. C. Sabatier (Academic Press, 2001), 770–785.
3. K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer, 1989).
4. P. Deift and E. Trubowitz, “Inverse Scattering on the Line,” *Communications on Pure and Applied Mathematics* 32 (1979): 121–251.
5. B. M. Levitan, *Inverse Sturm–Liouville Problems* (VNU Science Press, 1987).
6. V. A. Marchenko, “Reconstruction of the Potential Energy From the Phases of the Scattered Waves,” *Doklady Akademii Nauk SSSR* 104 (1955): 695–698.
7. V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Rev. ed. (American Mathematical Society, Chelsea Publishing, 2011).
8. R. G. Newton, “The Marchenko and Gel’fand–Levitan Methods in the Inverse Scattering Problem in One and Three Dimensions,” in *Conference on Inverse Scattering: Theory and Application*, ed. J. B. Bednar, R. Redner, E. Robinson, and A. Weglein (SIAM, 1983), 1–74.

9. D. J. Korteweg and G. de Vries, "On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves," *Philosophical Magazine* 39 (1895): 422–443.
10. V. E. Zakharov and A. B. Shabat, "Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media," *Soviet Physics JETP* 34 (1972): 62–69.
11. D. J. Kaup, "On the Inverse Scattering Problem for Cubic Eigenvalue Problems of the Class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$," *Studies in Applied Mathematics* 62 (1980): 189–216.
12. D. J. Kaup, "The Legacy of the IST," in *The Legacy of the Inverse Scattering Transform in Applied Mathematics*, ed. J. Bona, R. Choudhury, and D. J. Kaup (American Mathematical Society, 2002), 1–14.
13. B. A. Kupershmidt, "A Super Korteweg–de Vries Equation: An Integrable System," *Physics Letters A* 102 (1984): 213–215.
14. C. Charlier and J. Lenells, "The 'Good' Boussinesq Equation: A Riemann–Hilbert Approach," *Indiana University Mathematics Journal* 71 (2022): 1505–1562.
15. J. Boussinesq, "Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond," *Journal de Mathématiques Pures et Appliquées* 17 (1872): 55–108.
16. C. Charlier and J. Lenells, "The Soliton Resolution Conjecture for the Boussinesq Equation," *Journal de Mathématiques Pures et Appliquées* 191 (2024): 103621.
17. P. Deift, C. Tomei, and E. Trubowitz, "Inverse Scattering and the Boussinesq Equation," *Communications on Pure and Applied Mathematics* 35 (1982): 567–628.
18. H. P. McKean, "Boussinesq's Equation as a Hamiltonian System," in *Topics in Functional Analysis*, ed. I. Gohberg and M. Kac (Academic Press, 1978), 217–226.
19. T. Aktosun, "A Factorization of the Scattering Matrix for the Schrödinger Equation and for the Wave Equation in One Dimension," *Journal of Mathematical Physics* 33 (1992): 3865–3869.
20. R. Hirota, "Soliton Solutions to the BKP Equations. II. The Integral Equation," *Journal of the Physical Society of Japan* 58 (1989): 2705–2712.
21. R. G. Newton, "Inverse Scattering. I. One Dimension," *Journal of Mathematical Physics* 21 (1980): 493–505.
22. I. M. Gel'fand and B. M. Levitan, "On the Determination of a Differential Equation From Its Spectral Function," *American Mathematical Society Translations (Ser 2)* 1 (1955): 253–304.
23. L. D. Faddeev, "Properties of the S-Matrix of the One-Dimensional Schrödinger Equation," *American Mathematical Society Translations (Ser 2)* 65 (1967): 139–166.
24. V. E. Zakharov and A. B. Shabat, "A Scheme for Integrating the Nonlinear Equations of Mathematical Physics by the Method of the Inverse Scattering Problem. I," *Functional Analysis and its Applications* 8 (1974): 226–235.
25. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, 2004).
26. W. Hereman and A. Nuseir, "Symbolic Methods to Construct Exact Solutions of Nonlinear Partial Differential Equations," *Mathematics and Computers in Simulation* 43 (1997): 13–27.
27. A. Parker, "A Reformulation of the Dressing Method for the Sawada–Kotera Equation," *Inverse Problem* 17 (2001): 885–895.
28. T. Aktosun, A. E. Choque-Rivero, I. Toledo, and M. Unlu, "Soliton Solutions Associated With a Class of Third-Order Ordinary Linear Differential Operators," *Studies in Applied Mathematics* 154 (2025): e70057.
29. P. J. Caudrey, "The Inverse Problem for the Third Order Equation $u_{xxx} + q(x)u_x + r(x)u = -i\zeta^3u$," *Physics Letters A* 79 (1980): 264–268.
30. I. Toledo, "The Direct and Inverse Scattering Problems for the Third-Order Operator," (PhD thesis, The University of Texas at Arlington, 2024).
31. T. Aktosun, A. E. Choque-Rivero, I. Toledo, and M. Unlu, "Soliton Solutions to the Sawada–Kotera Equation," in *Analysis and Mathematical Physics*, ed. T. Aktosun, S. Avdonin, and R. Weder (De Gruyter, 2026), 1–34.
32. M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, 1991).
33. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, 1981).
34. T. Aktosun, "Inverse Scattering Transform and the Theory of Solitons," in *Encyclopedia of Complexity and Systems Science*, ed. R. A. Meyers (Springer, 2009), 4960–4971.
35. G. L. Lamb Jr, *Elements of Soliton Theory* (Wiley, 1980).
36. S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons* (Consultants Bureau, 1984).
37. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Method for Solving the Korteweg–de Vries Equation," *Physical Review Letters* 19 (1967): 1095–1097.
38. T. Aktosun, I. Toledo, and M. Unlu, "Inverse Scattering Problem for the Third-Order Equation on the Line," *Inverse Problem* 41 (2025): 125008.

39. P. D. Lax, "Integrals of Nonlinear Equations of Evolution and Solitary Waves," *Communications on Pure and Applied Mathematics* 21 (1968): 467–490.
40. W. Rudin, *Real and Complex Analysis*, 3rd ed. (McGraw-Hill, 1987).
41. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1 (Wiley, 1989).
42. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, "The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems," *Studies in Applied Mathematics* 53 (1974): 249–315.