Research Article

Mathematical Methods in the Applied Sciences

Received 1 March 2016 Published online 3 August 2016 in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.4112
MOS subject classification: 39A70; 47A40; 47A68; 47B36; 47B39

Factorization of the transition matrix for the general Jacobi system

Tuncay Aktosun[a*†] and Abdon E. Choque-Rivero[b]

Communicated by P. Sacks

The Jacobi system on a full-line lattice is considered when it contains additional weight factors. A factorization formula is derived expressing the scattering from such a generalized Jacobi system in terms of the scattering from its fragments. This is performed by writing the transition matrix for the generalized Jacobi system as an ordered matrix product of the transition matrices corresponding to its fragments. The resulting factorization formula resembles the factorization formula for the Schrödinger equation on the full line. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: Jacobi system; discrete systems; scattering; scattering from fragments; transition matrix; factorization formula

1. Introduction

Consider the Schrödinger equation on the full-line $\mathbf{R} := (-\infty, +\infty)$ given by

$$-\psi''(k,x) + V(x)\,\psi(k,x) = k^2\,\psi(k,x), \qquad x \in \mathbf{R}, \tag{1.1}$$

where the prime denotes the x-derivative and the potential V is real valued and satisfies

$$\int_{-\infty}^{\infty} dx\,(1+|x|)\,|V(x)| < +\infty. \tag{1.2}$$

The restriction given in (1.2) allows us [1–6] to develop a mathematical theory of scattering for (1.1), assures the existence of scattering solutions, and also guarantees that there are at most a finite number of bound states.

There are two particular solutions to (1.1). The first is the Jost solution from the left, denoted by $f_l(k,x)$, satisfying the spatial asymptotics

$$f_l(k,x) = e^{ikx}\,[1+o(1)], \qquad x \to +\infty, \tag{1.3}$$

and the other is the Jost solution from the right, denoted by $f_r(k,x)$, satisfying the spatial asymptotics

$$f_r(k,x) = e^{-ikx}\,[1+o(1)], \qquad x \to -\infty. \tag{1.4}$$

The scattering coefficients corresponding to V are obtained through the spatial asymptotics

$$f_l(k,x) = \frac{1}{T(k)}\,e^{ikx} + \frac{L(k)}{T(k)}\,e^{-ikx} + o(1), \qquad x \to -\infty, \tag{1.5}$$

$$f_r(k,x) = \frac{1}{T(k)}\,e^{-ikx} + \frac{R(k)}{T(k)}\,e^{ikx} + o(1), \qquad x \to +\infty, \tag{1.6}$$

[a] Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019-0408, U.S.A.
[b] Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Ciudad Universitaria, C.P. 58048 Morelia, Michoacán, Mexico
* Correspondence to: Tuncay Aktosun, Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019-0408, U.S.A.
† E-mail: aktosun@uta.edu

where T is the transmission coefficient, R is the reflection coefficient from the right, and L is the reflection coefficient from the left. Corresponding to the potential V, let us use $\Lambda(k)$ to denote the 2×2 transition matrix defined for $k \in \mathbf{R}$ as

$$\Lambda(k) := \begin{bmatrix} \frac{1}{T(k)} & -\frac{R(k)}{T(k)} \\ \frac{L(k)}{T(k)} & \frac{1}{T(-k)} \end{bmatrix}. \tag{1.7}$$

Let us fragment the real axis $\mathbf{R}$ into N pieces as

$$\mathbf{R} = (x_0, x_1] \cup (x_1, x_2] \cup \cdots \cup (x_{N-1}, x_N],$$

where $N \geq 2$, and we have

$$x_0 := -\infty, \quad x_1 < x_2 < \cdots < x_{N-1}, \quad x_N := +\infty.$$

We then obtain the fragmentation of the potential V into N pieces as

$$V(x) = \begin{cases} V_1(x), & x \in (x_0, x_1], \\ V_2(x), & x \in (x_1, x_2], \\ \vdots & \vdots \\ V_N(x), & x \in (x_{N-1}, x_N], \end{cases} \tag{1.8}$$

where for $j = 1, 2, \ldots, N$ we have defined

$$V_j(x) := \begin{cases} V(x), & x \in (x_{j-1}, x_j], \\ 0, & x \notin (x_{j-1}, x_j]. \end{cases}$$

Let us use $f_{lj}(k, x)$ and $f_{rj}(k, x)$ to denote the Jost solution from the left and from the right, respectively, corresponding to the potential V_j. Let us also use T_j, R_j, and L_j for the respective scattering coefficients corresponding to V_j. Thus, $f_{lj}(k, x)$ and $f_{rj}(k, x)$ satisfy (1.1) when V is replaced with V_j, and they satisfy (1.3) and (1.4), respectively. They also satisfy (1.5) and (1.6), respectively, with the scattering coefficients T, R, and L replaced with T_j, R_j, and L_j, respectively. Let $\Lambda_j(k)$ be the corresponding transition matrix for V_j defined in a similar way as in (1.7) as

$$\Lambda_j(k) := \begin{bmatrix} \frac{1}{T_j(k)} & -\frac{R_j(k)}{T_j(k)} \\ \frac{L_j(k)}{T_j(k)} & \frac{1}{T_j(-k)} \end{bmatrix}.$$

Corresponding to the fragmentation (1.8), we have the factorization formula [7–13]:

$$\Lambda(k) = \Lambda_1(k)\, \Lambda_2(k) \cdots \Lambda_N(k), \qquad k \in \mathbf{R}, \tag{1.9}$$

where the right-hand side consists of the product of the 2×2 transition matrices in the order indicated.

The factorization formula (1.9) and its various generalizations [7, 8, 14] allow us to understand how the scattering from a system develops in terms of the scattering from the components of that system. It has important applications in various areas, such as determining the phase [15, 16] of a complex-valued reflection coefficient based on amplitude measurements, determining material properties of thin films via neutron reflectometry [17–20], quantum wires [9, 21], quantum computing [21], and quantum scattering from coupled systems [22].

Our goal in this paper is to derive the analog of the factorization formula (1.9) for some difference equations related to (1.1) and its generalizations. In particular, we will do so for the general Jacobi system on the full-line lattice given by [23]

$$a(n+1)\, \psi(\lambda, n+1) + b(n)\, \psi(\lambda, n) + a(n)\, \psi(\lambda, n-1) = \lambda\, w(n)\, \psi(\lambda, n), \qquad n \in \mathbf{Z}, \tag{1.10}$$

where $\mathbf{Z}$ denotes the set of integers, λ is the spectral parameter, and $a(n)$, $b(n)$, and $w(n)$ are some real coefficients that may depend on the location in the lattice. Due to the presence of $w(n)$ in (1.10), we call it the general Jacobi system, whereas the Jacobi system [23–27] corresponds to the case $w(n) \equiv 1$.

The system given in (1.10) is used as a model for various physical systems. It describes the behavior of particles in a one-dimensional lattice where each particle may experience a local force as well as a force from the nearest neighbors. By including the weight factors $w(n)$, we can use (1.10) to describe wave propagation in a lattice where the propagation speed may depend on the location.

The coefficients $a(n)$, $b(n)$, and $w(n)$ appearing in (1.10) are assumed to belong to class $\mathcal{A}$ specified below. The resulting restrictions on the coefficients allow us to develop a mathematical scattering theory for (1.10) and to assure the finiteness of the number of bound states associated with (1.10).

Definition 1.1
The coefficients $a(n)$, $b(n)$, and $w(n)$ belong to class $\mathcal{A}$ if they satisfy the following properties:

(1) They are real valued, and $a(n) \neq 0$ and $w(n) > 0$ for $n \in \mathbf{Z}$.
(2) They have the finite limits a_∞, b_∞, and w_∞, respectively, as $n \to \pm\infty$, with $a_\infty \neq 0$ and $w_\infty > 0$.
(3) They satisfy the restriction

$$\sum_{n=-\infty}^{+\infty} |n| \left(\left| \frac{a(n+1)}{\sqrt{w(n)\, w(n+1)}} - \frac{a_\infty}{w_\infty} \right| + \left| \frac{b(n)}{w(n)} - \frac{b_\infty}{w_\infty} \right| \right) < +\infty. \tag{1.11}$$

We remark that the restriction (1.11) is the analog of (1.2). Such a restriction can be obtained from the mathematical theory [28, 29] available for the case $w(n) \equiv 1$.

Our paper is organized as follows. In Section 2, we introduce the alternate parameter z related to the spectral parameter λ via (2.1). We introduce the Jost solution from the left and from the right, respectively, and also introduce two other solutions related to the Jost solutions. With the help of such solutions, the scattering coefficients are obtained as functions of z, and their basic properties are provided. In Section 3, we introduce the transition matrix related to the scattering coefficients as in (3.1). Next, we fragment the full-line lattice into N pieces. We then provide our fundamental result, namely, the factorization formula (3.5) expressing the relationship between the transition matrix for the full-line lattice and the transition matrices for the fragments. The proof of the factorization formula is first given when $N = 2$ and then extended to an arbitrary number of fragments using mathematical induction.

2. The general Jacobi system and scattering coefficients

In this section, in preparation for the derivation of the analog of the factorization formula (1.9), we introduce the scattering coefficients for the general Jacobi system (1.10) and present their relevant properties that are needed later on.

Instead of using the spectral parameter λ in (1.10), we can equivalently use the parameter z, which is related to λ as

$$\lambda = \frac{a_\infty\,(z+z^{-1})+b_\infty}{w_\infty}, \tag{2.1}$$

where a_∞, b_∞, and w_∞ are the real constants appearing in Definition 1.1. Let us define

$$\lambda_{\min} := \frac{-2\,|a_\infty|+b_\infty}{w_\infty}, \quad \lambda_{\max} := \frac{2\,|a_\infty|+b_\infty}{w_\infty}.$$

When $a_\infty < 0$, the transformation $\lambda \mapsto z$ in (2.1) maps the real λ-axis to the boundary of the upper half of the unit disk in such a way that the interval $\lambda \in (-\infty, \lambda_{\min})$ is mapped to the real interval $z \in (0,1)$, the real interval $\lambda \in (\lambda_{\min}, \lambda_{\max})$ is mapped to $z = e^{i\theta}$ with $\theta \in (0,\pi)$, and the real interval $\lambda \in (\lambda_{\max}, +\infty)$ is mapped to the real interval $z \in (-1,0)$, while $\lambda = \lambda_{\min}$ is mapped to $z = 1$, and $\lambda = \lambda_{\max}$ is mapped to $z = -1$. On the other hand, when $a_\infty > 0$, the transformation $\lambda \mapsto z$ maps the real λ-axis to the boundary of the lower half of the unit disk in such a way that the interval $\lambda \in (-\infty, \lambda_{\min})$ is mapped to the real interval $z \in (-1,0)$, the real interval $\lambda \in (\lambda_{\min}, \lambda_{\max})$ is mapped to $z = e^{i\theta}$ with $\theta \in (-\pi, 0)$, and the real interval $\lambda \in (\lambda_{\max}, +\infty)$ is mapped to the real interval $z \in (0,1)$, while $\lambda = \lambda_{\min}$ is mapped to $z = -1$, and $\lambda = \lambda_{\max}$ is mapped to $z = 1$.

Using (2.1) in (1.10), we can write (1.10) in terms of the parameter z as

$$a(n+1)\,\phi(z,n+1) + b(n)\,\phi(z,n) + a(n)\,\phi(z,n-1) = \frac{w(n)}{w_\infty}\,\left[a_\infty(z+z^{-1})+b_\infty\right]\phi(z,n), \tag{2.2}$$

where $n \in \mathbf{Z}$ and the z-values now occur on the unit circle $\mathbf{T}$ in the complex z-plane given by $|z| = 1$. The unperturbed equation corresponding to (2.2) is obtained by replacing $a(n)$, $b(n)$, and $w(n)$ with their limiting values a_∞, b_∞, and w_∞, respectively, and we have

$$\mathring{\phi}(z,n+1) + \mathring{\phi}(z,n-1) = (z+z^{-1})\,\mathring{\phi}(z,n), \qquad n \in \mathbf{Z}. \tag{2.3}$$

The difference equation (2.2) with the coefficients belonging to class $\mathcal{A}$ specified in Definition 1.1 has [23] two linearly independent solutions, namely, the Jost solution from the left $f_{\mathrm{l}}(z,n)$ satisfying

$$f_{\mathrm{l}}(z,n) = z^n\,[1+o(1)], \qquad n \to +\infty, \tag{2.4}$$

and the Jost solution from the right $f_{\mathrm{r}}(z,n)$ satisfying

$$f_{\mathrm{r}}(z,n) = z^{-n}\,[1+o(1)], \qquad n \to -\infty. \tag{2.5}$$

The scattering coefficients T, R, and L are now functions of the variable z, and they are obtained as in (1.5) and (1.6) from the spatial asymptotics of the Jost solutions as

$$f_{\mathrm{l}}(z,n) = \frac{1}{T(z)}\,z^n + \frac{L(z)}{T(z)}\,z^{-n} + o(1), \qquad n \to -\infty, \tag{2.6}$$

$$f_{\mathrm{r}}(z,n) = \frac{1}{T(z)}\,z^{-n} + \frac{R(z)}{T(z)}\,z^{n} + o(1), \qquad n \to +\infty. \tag{2.7}$$

There are two other solutions to (2.2) related to the Jost solutions. Let us use $g_{\mathrm{l}}(z,n)$ and $g_{\mathrm{r}}(z,n)$ to denote them and introduce them as

$$g_{\mathrm{l}}(z,n) := f_{\mathrm{l}}(z^{-1},n), \quad g_{\mathrm{r}}(z,n) := f_{\mathrm{r}}(z^{-1},n). \tag{2.8}$$

Because z and z^{-1} appear symmetrically in (2.2), it follows that $g_{\mathrm{l}}(z,n)$ and $g_{\mathrm{r}}(z,n)$ satisfy (2.2). Using (2.8) in (2.4) and (2.5), it follows that $g_{\mathrm{l}}(z,n)$ and $g_{\mathrm{r}}(z,n)$ satisfy the respective asymptotics

$$g_{\mathrm{l}}(z,n) = z^{-n}\,[1+o(1)], \qquad n \to +\infty, \tag{2.9}$$

$$g_r(z,n) = z^n \left[1 + o(1)\right], \qquad n \to -\infty. \tag{2.10}$$

The Wronskian $[\phi(z,n);\zeta(z,n)]$ of any two solutions $\phi(z,n)$ and $\zeta(z,n)$ to (2.2) is defined [23] as

$$[\phi(z,n);\zeta(z,n)] := a(n+1)\left(\phi(z,n)\,\zeta(z,n+1) - \phi(z,n+1)\,\zeta(z,n)\right), \tag{2.11}$$

and it is known [23] that the value of the Wronskian is independent of n and hence can be evaluated at any n-value or as $n \to \pm\infty$. We have introduced the scattering coefficients through the spatial asymptotics of the Jost solutions via (2.6) and (2.7). Alternatively, the scattering coefficients can be obtained by using some Wronskians involving $f_l(z,n)$, $f_r(z,n)$, $g_l(z,n)$, and $g_r(z,n)$. It is possible to determine the basic properties of the scattering coefficients by evaluating such Wronskians as $n \to +\infty$ and also as $n \to -\infty$ and by equating the resulting expressions.

Theorem 2.1
Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Let $f_l(z,n)$ and $f_r(z,n)$ be the Jost solution from the left and from the right, respectively, to (2.2), and let $T(z)$, $R(z)$, and $L(z)$ be the corresponding scattering coefficients appearing in (2.6) and (2.7). Then:

(1) The z-domain of the Jost solutions $f_l(z,n)$ and $f_r(z,n)$ can be extended to $z \in \mathbf{T}$ with the help of

$$f_l(z^{-1},n) = f_l(z^*,n) = f_l(z,n)^*, \quad f_r(z^{-1},n) = f_r(z^*,n) = f_r(z,n)^*, \qquad z \in \mathbf{T}, \tag{2.12}$$

where the asterisk denotes complex conjugation.

(2) The z-domain of the scattering coefficients is $z \in \mathbf{T}$, and we have for $z \in \mathbf{T}$

$$T(z^{-1}) = T(z^*) = T(z)^*, \quad R(z^{-1}) = R(z^*) = R(z)^*, \quad L(z^{-1}) = L(z^*) = L(z)^*. \tag{2.13}$$

Furthermore, for $z \in \mathbf{T}$, the scattering coefficients satisfy

$$\frac{1}{T(z)\,T(z^{-1})} - \frac{L(z)\,L(z^{-1})}{T(z)\,T(z^{-1})} = 1, \tag{2.14}$$

$$\frac{1}{T(z)\,T(z^{-1})} - \frac{R(z)\,R(z^{-1})}{T(z)\,T(z^{-1})} = 1, \tag{2.15}$$

$$\frac{R(z^{-1})}{T(z^{-1})} = -\frac{L(z)}{T(z)}, \quad \frac{L(z^{-1})}{T(z^{-1})} = -\frac{R(z)}{T(z)}, \tag{2.16}$$

$$T(z)^2 - R(z)\,L(z) = \frac{T(z)}{T(z^{-1})}. \tag{2.17}$$

Proof
As stated in Definition 1.1, the coefficients $a(n)$, $b(n)$, and $w(n)$ and their limiting values a_∞, b_∞, and w_∞ are all real valued. For $z \in \mathbf{T}$, we have $z^{-1} = z^*$. When $z \in \mathbf{T}$, replacing z by z^* in (2.2) and then taking the complex conjugate of both sides of the resulting equation, we see that $f_l(z^*,n)^*$ remains a solution to (2.2). Furthermore, $f_l(z^*,n)^*$ satisfies the asymptotics given in (2.4), and hence, $f_l(z^*,n)^* = f_l(z,n)$ when $z \in \mathbf{T}$. By a similar argument, we get $f_r(z^*,n)^* = f_r(z,n)$ when $z \in \mathbf{T}$. Hence, (2.12) is proved. Using (2.12) in (2.6) and (2.7) and the fact that $z^{-1} = z^*$ for $z \in \mathbf{T}$, we obtain (2.13). We get (2.14) by evaluating the Wronskian $[f_l(z,n);g_l(z,n)]$ as $n \to +\infty$ and also as $n \to -\infty$ and by equating the resulting expressions, where $g_l(z,n)$ is the quantity appearing in (2.8) and (2.9). For this purpose, using (2.4) and (2.9) in (2.11), we get the value of $[f_l(z,n);g_l(z,n)]$ as $n \to +\infty$ as

$$[f_l(z,n);g_l(z,n)] = a_\infty\,\left(z^{-1} - z\right). \tag{2.18}$$

The same Wronskian, as $n \to -\infty$, is evaluated with the help of (2.6) and the first equality in (2.8) as

$$[f_l(z,n);g_l(z,n)] = a_\infty\,\left(z^{-1} - z\right)\left(\frac{1}{T(z)\,T(z^{-1})} - \frac{L(z)\,L(z^{-1})}{T(z)\,T(z^{-1})}\right). \tag{2.19}$$

Comparing (2.18) and (2.19), we establish (2.14). Similarly, we obtain (2.15) by evaluating the Wronskian $[f_r(z,n);g_r(z,n)]$ as $n \to \pm\infty$ and equating the resulting expressions. We get the first equality in (2.16) by evaluating the Wronskian $[f_l(z,n);g_r(z,n)]$ as $n \to \pm\infty$ and equating the resulting expressions, where $g_r(z,n)$ is the quantity appearing in (2.8) and (2.10). The second equality in (2.16) is obtained by evaluating the Wronskian $[f_r(z,n);g_l(z,n)]$ as $n \to \pm\infty$ and equating the resulting expressions. Finally, we get (2.17) by using the second equality of (2.16) in (2.14). □

3. The factorization formula for the general Jacobi system

Our goal in this section is to establish the analog of (1.9) for the general Jacobi system (2.2). This will be accomplished by fragmenting the full-line lattice $\mathbf{Z}$ and relating the transition matrix for the entire lattice to the transition matrices associated with the fragments. The resulting factorization formula with two fragments is given in (3.6), and the factorization formula for an arbitrary number of fragments is given in (3.5).

For (2.2) with coefficients $\{a(n), b(n), w(n)\}_{n\in\mathbf{Z}}$, we define the transition matrix $\Lambda(z)$ for $z \in \mathbf{T}$ as

$$\Lambda(z) := \begin{bmatrix} \frac{1}{T(z)} & -\frac{R(z)}{T(z)} \\ \frac{L(z)}{T(z)} & \frac{1}{T(z^{-1})} \end{bmatrix}, \tag{3.1}$$

which resembles the transition matrix given in (1.7) for the Schrödinger equation.

Let us partition the set of integers $\mathbf{Z}$ into N ordered subsets as

$$\mathbf{Z} = \{n_0, \dots, n_1\} \cup \{n_1+1, \dots, n_2\} \cup \cdots \cup \{n_{N-1}+1, \dots, n_N\},$$

where $N \geq 2$ and we have

$$n_0 := -\infty, \quad n_1 < n_2 < \cdots < n_{N-1}, \quad n_N := +\infty.$$

We then obtain the fragmentation of the coefficients in (1.10) into N disjoint sets as

$$(a(n), b(n), w(n)) = \begin{cases} (a_1(n), b_1(n), w_1(n)), & n_0 < n \leq n_1, \\ (a_2(n), b_2(n), w_2(n)), & n_1 < n \leq n_2, \\ \vdots & \vdots \\ (a_{N-1}(n), b_{N-1}(n), w_{N-1}(n)), & n_{N-2} < n \leq n_{N-1}, \\ (a_N(n), b_N(n), w_N(n)), & n_{N-1} < n \leq n_N, \end{cases} \tag{3.2}$$

where for $j = 1, 2, \dots, N$ we have defined

$$\begin{cases} a_j(n) := a(n), \quad b_j(n) := b(n), \quad w_j(n) := w(n), & n_{j-1} < n \leq n_j, \\ a_j(n) := a_\infty, \quad b_j(n) := b_\infty, \quad w_j(n) := w_\infty, & n \leq n_{j-1} \text{ or } n > n_j. \end{cases} \tag{3.3}$$

For each fixed j, let us use $f_{lj}(z,n)$ and $f_{rj}(z,n)$ to denote the Jost solutions from the left and from the right, respectively, corresponding to the coefficients $\{a_j(n), b_j(n), w_j(n)\}_{n\in\mathbf{Z}}$. Let us also use T_j, R_j, and L_j for the respective scattering coefficients corresponding to $\{a_j(n), b_j(n), w_j(n)\}_{n\in\mathbf{Z}}$. Let $\Lambda_j(z)$ be the corresponding transition matrix for $\{a_j(n), b_j(n), w_j(n)\}_{n\in\mathbf{Z}}$ defined in a similar way as in (3.1) as

$$\Lambda_j(z) := \begin{bmatrix} \frac{1}{T_j(z)} & -\frac{R_j(z)}{T_j(z)} \\ \frac{L_j(z)}{T_j(z)} & \frac{1}{T_j(z^{-1})} \end{bmatrix}. \tag{3.4}$$

Corresponding to the fragmentation (3.2), we are interested in proving the factorization formula

$$\Lambda(z) = \Lambda_1(z)\,\Lambda_2(z) \cdots \Lambda_N(z), \qquad z \in \mathbf{T}, \tag{3.5}$$

where the right-hand side consists of the product of the 2×2 transition matrices in the ordered indicated. It is enough to prove (3.5) when the number of fragments is two, that is, $N = 2$ in (3.5), because the case $N > 2$ can be obtained via mathematical induction. Thus, we would like to prove that

$$\begin{bmatrix} \frac{1}{T(z)} & -\frac{R(z)}{T(z)} \\ \frac{L(z)}{T(z)} & \frac{1}{T(z^{-1})} \end{bmatrix} = \begin{bmatrix} \frac{1}{T_1(z)} & -\frac{R_1(z)}{T_1(z)} \\ \frac{L_1(z)}{T_1(z)} & \frac{1}{T_1(z^{-1})} \end{bmatrix} \begin{bmatrix} \frac{1}{T_2(z)} & -\frac{R_2(z)}{T_2(z)} \\ \frac{L_2(z)}{T_2(z)} & \frac{1}{T_2(z^{-1})} \end{bmatrix}, \qquad z \in \mathbf{T}. \tag{3.6}$$

In order to prove the factorization formula with two fragments, that is, to prove the formula given in (3.6), we need a series of auxiliary results presented in the next several propositions.

In our first proposition, when $N = 2$ in the partitioning in (3.2), we express the Jost solutions $f_l(z,n)$ and $f_r(z,n)$ in terms of the Jost solution $f_{l2}(z,n)$ and its relative $g_{l2}(z,n)$ for $n \geq n_1$. The result is needed later on in the proof of the factorization formula (3.6).

Proposition 3.1

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that $N = 2$ in the partitioning given in (3.2). Let $f_l(z,n)$ and $f_r(z,n)$ be the Jost solution from the left and from the right, respectively, to (2.2), and let f_{l2} be the Jost solution from the left for (2.2) with the coefficients $\{a_2(n), b_2(n), w_2(n)\}_{n\in\mathbf{Z}}$ specified in (3.3). Let $g_{l2}(z,n)$ be the quantity related to $f_{l2}(z,n)$ as in the first equality in (2.8), namely,

$$g_{l2}(z,n) := f_{l2}(z^{-1}, n). \tag{3.7}$$

Then, for $n \geq n_1$, we have

$$\begin{bmatrix} f_l(z,n) & f_r(z,n) \\ f_l(z,n+1) & f_r(z,n+1) \end{bmatrix} = \begin{bmatrix} f_{l2}(z,n) & \alpha(z)\, f_{l2}(z,n) + \beta(z)\, g_{l2}(z,n) \\ f_{l2}(z,n+1) & \alpha(z)\, f_{l2}(z,n+1) + \beta(z)\, g_{l2}(z,n+1) \end{bmatrix}, \tag{3.8}$$

where

$$\alpha(z) = \frac{R(z)}{T(z)}, \quad \beta(z) = \frac{1}{T(z)}. \tag{3.9}$$

Proof

Because of the second line of (3.2), it follows that, for $n \geq n_1 + 1$, both $f_l(z,n)$ and $f_{l2}(z,n)$ satisfy the same equation, namely, (2.2), and the same asymptotic condition, namely, (2.4). Thus, we obtain

$$f_l(z,n) = f_{l2}(z,n), \qquad n \geq n_1 + 1.$$

Because z and z^{-1} appear symmetrically in (2.2), it follows from (3.7) that $f_{l2}(z,n)$ and $g_{l2}(z,n)$ both satisfy (2.2) with $n \geq n_1 + 1$, and hence $f_r(z,n)$ can be expressed as a linear combination of $f_{l2}(z,n)$ and $g_{l2}(z,n)$ for $n \geq n_1 + 1$. Thus, (3.8) holds for $n \geq n_1$ provided we can prove that

$$f_l(z,n_1) = f_{l2}(z,n_1), \quad f_r(z,n_1) = \alpha(z)\, f_{l2}(z,n_1) + \beta(z)\, g_{l2}(z,n_1). \tag{3.10}$$

Using (2.2) at $n = n_1+1$ with $f_l(z,n)$ and $\{a(n), b(n), w(n)\}_{n\in\mathbf{Z}}$ and also using (2.2) at $n = n_1+1$ with $f_{l2}(z,n)$ and $\{a_2(n), b_2(n), w_2(n)\}_{n\in\mathbf{Z}}$, we obtain

$$a(n_1+1)\, f_l(z,n_1) = a_2(n_1+1)\, f_{l2}(z,n_1),$$

and hence, from the facts that $a(n_1+1) = a_2(n_1+1)$ and $a(n_1+1) > 0$, we get the first equality in (3.10). Proceeding similarly, we get the second equality in (3.10). By letting $n \to +\infty$ on both sides of (3.8) and using (2.4), (2.7), and (2.9), we get (3.9). □

In the next proposition, when $N = 2$ in the partitioning in (3.2), we express the Jost solutions $f_l(z,n)$ and $f_r(z,n)$ in terms of the Jost solution $f_{r1}(z,n)$ and its relative $g_{r1}(z,n)$ for $n \leq n_1$. The result is needed to prove the factorization formula given in (3.6).

Proposition 3.2

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that $N = 2$ in the partitioning given in (3.2). Let $f_l(z,n)$ and $f_r(z,n)$ be the Jost solution from the left and from the right, respectively, to (2.2), and let f_{r1} be the Jost solution from the right for (2.2) with the coefficients $\{a_1(n), b_1(n), w_1(n)\}_{n\in\mathbf{Z}}$ given in (3.3). Let $g_{r1}(z,n)$ be the quantity related to $f_{r1}(z,n)$ as in the second equality in (2.8), namely,

$$g_{r1}(z,n) := f_{r1}(z^{-1},n). \tag{3.11}$$

Then, for $n \leq n_1 - 1$, we have

$$\begin{bmatrix} f_l(z,n) & f_r(z,n) \\ f_l(z,n+1) & f_r(z,n+1) \end{bmatrix} = \begin{bmatrix} \gamma(z)\, g_{r1}(z,n) + \epsilon(z)\, f_{r1}(z,n) & f_{r1}(z,n) \\ \gamma(z)\, g_{r1}(z,n+1) + \epsilon(z)\, f_{r1}(z,n+1) & f_{r1}(z,n+1) \end{bmatrix}, \tag{3.12}$$

and we also have

$$f_r(z,n_1+1) = \frac{a_\infty}{a(n_1+1)}\, f_{r1}(z,n_1+1), \tag{3.13}$$

$$f_l(z,n_1+1) = \frac{a_\infty}{a(n_1+1)} \left[\gamma(z)\, g_{r1}(z,n_1+1) + \epsilon(z)\, f_{r1}(z,n_1+1)\right], \tag{3.14}$$

where

$$\gamma(z) = \frac{L(z)}{T(z)}, \quad \epsilon(z) = \frac{1}{T(z)}. \tag{3.15}$$

Proof

Because of the first line of (3.2), it follows that, for $n \leq n_1 - 1$, both $f_r(z,n)$ and $f_{r1}(z,n)$ satisfy the same equation, namely, (2.2), and the same asymptotic condition, namely, (2.5). Thus, we obtain

$$f_r(z,n) = f_{r1}(z,n), \qquad n \leq n_1 - 1. \tag{3.16}$$

From (2.2) with $n = n_1 - 1$, it then follows that (3.16) actually holds for $n = n_1$ as well and we obtain

$$f_r(z,n) = f_{r1}(z,n), \qquad n \leq n_1.$$

From the second equation in (2.8), it follows that $f_{r1}(z,n)$ and $g_{r1}(z,n)$ both satisfy (2.2) with $n \leq n_1 - 1$, and hence $f_l(z,n)$ can be expressed as a linear combination of $f_{r1}(z,n)$ and $g_{r1}(z,n)$ for $n \leq n_1$, which is proved by proceeding in a similar way as in the proof of Proposition 3.1. In a similar way, using (2.2) with $n = n_1$ and the facts that we have (3.16), $a(n) > 0$, and $a_1(n_1+1) = a_\infty$, we obtain (3.13). Similarly, using (2.2) with $n = n_1$ and making use of the equality of the $(1,1)$-entries in (3.12) for $n = n_1$, we obtain (3.14). Finally, letting $n \to -\infty$ in (3.12) and using (2.5), (2.6), and (2.10), we obtain (3.15). □

The result in the next proposition is needed in the proof of Theorem 3.6.

Proposition 3.3

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that $N = 2$ in the partitioning given in (3.2). Let $f_{l2}(z,n)$ be the Jost solution from the left for (2.2) with the coefficients $\{a_2(n), b_2(n), w_2(n)\}_{n\in\mathbf{Z}}$ given in (3.3). Let $T_2(z)$, $R_2(z)$, and $L_2(z)$ be the scattering coefficients associated with the coefficients $\{a_2(n), b_2(n), w_2(n)\}_{n\in\mathbf{Z}}$. Then, we have

$$f_{l2}(z,n) = \frac{1}{T_2(z)}\, z^n + \frac{L_2(z)}{T_2(z)}\, z^{-n}, \qquad n \leq n_1, \tag{3.17}$$

$$f_{l2}(z,n_1+1)=\frac{a_\infty}{a(n_1+1)}\left[\frac{1}{T_2(z)}\,z^{n_1+1}+\frac{L_2(z)}{T_2(z)}\,z^{-n_1-1}\right]. \tag{3.18}$$

Proof

When $n\le n_1-1$ in (2.2), the Jost solution $f_{l2}(z,n)$ satisfies (2.3) with $n\le n_1-1$. Furthermore, from the analog of (2.6) for $f_{l2}(z,n)$, we already have (3.17) for $n\le n_1-1$. Then, (3.2) with $n=n_1-1$ implies that (3.17) actually holds for $n=n_1$ as well. Thus, (3.17) is proved. Then, using (2.2) with $n=n_1$ and utilizing (3.17) for $n=n_1$ and $n=n_1-1$, after some simplifications, we obtain (3.18). □

We need the following analog of Proposition 3.3, which is needed in the proof of the factorization formula given in (3.6).

Proposition 3.4

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that $N=2$ in the partitioning given in (3.2). Let f_{r1} be the Jost solution from the right for (2.2) with the coefficients $\{a_1(n),b_1(n),w_1(n)\}_{n\in\mathbf{Z}}$ appearing in (3.3). Let $T_1(z)$, $R_1(z)$, and $L_1(z)$ be the scattering coefficients associated with the coefficients $\{a_1(n),b_1(n),w_1(n)\}_{n\in\mathbf{Z}}$. Then, we have

$$f_{r1}(z,n)=\frac{1}{T_1(z)}\,z^{-n}+\frac{R_1(z)}{T_1(z)}\,z^{n},\qquad n\ge n_1. \tag{3.19}$$

Proof

When $n\ge n_1+1$ in (2.2), the Jost solution $f_{r1}(z,n)$ satisfies (2.3) with $n\ge n_1+1$. Furthermore, from the analog of (2.7) for $f_{r1}(z,n)$, we already have (3.19) for $n\ge n_1+1$. Then, (2.3) with $n=n_1+1$ implies that (3.19) actually holds for $n=n_1$ as well. □

The next proposition is needed in the proof of Theorem 3.6.

Proposition 3.5

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that $N=2$ in the partitioning given in (3.2). Let f_{r1} be the Jost solution from the right for (2.2) with the coefficients $\{a_1(n),b_1(n),w_1(n)\}_{n\in\mathbf{Z}}$ appearing in (3.3). Let $T_1(z)$, $R_1(z)$, and $L_1(z)$ be the scattering coefficients associated with the coefficients $\{a_1(n),b_1(n),w_1(n)\}_{n\in\mathbf{Z}}$. Similarly, let f_{l2} be the Jost solution from the left for (2.2) with the coefficients $\{a_2(n),b_2(n),w_2(n)\}_{n\in\mathbf{Z}}$ appearing in (3.3). Let $T_2(z)$, $R_2(z)$, and $L_2(z)$ be the scattering coefficients associated with the coefficients $\{a_2(n),b_2(n),w_2(n)\}_{n\in\mathbf{Z}}$. Furthermore, let g_{r1} and g_{l2} be the quantities appearing in (3.11) and (3.7), respectively. Then, we have

$$\begin{bmatrix} f_{l2}(z,n_1) & g_{l2}(z,n_1)\\ f_{l2}(z,n_1+1) & g_{l2}(z,n_1+1)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \frac{a_\infty}{a(n_1+1)}\end{bmatrix}\begin{bmatrix} z^{n_1} & z^{-n_1}\\ z^{n_1+1} & z^{-n_1-1}\end{bmatrix}\begin{bmatrix}\frac{1}{T_2(z)} & \frac{L_2(z^{-1})}{T_2(z^{-1})}\\ \frac{L_2(z)}{T_2(z)} & \frac{1}{T_2(z^{-1})}\end{bmatrix}, \tag{3.20}$$

$$\begin{bmatrix} g_{r1}(z,n_1) & f_{r1}(z,n_1)\\ g_{r1}(z,n_1+1) & f_{r1}(z,n_1+1)\end{bmatrix}=\begin{bmatrix} z^{n_1} & z^{-n_1}\\ z^{n_1+1} & z^{-n_1-1}\end{bmatrix}\begin{bmatrix}\frac{1}{T_1(z^{-1})} & \frac{R_1(z)}{T_1(z)}\\ \frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z)}\end{bmatrix}. \tag{3.21}$$

Proof

We obtain (3.20) by using (3.7), (3.17), and (3.18). Similarly, we get (3.21) by using (3.11) and (3.19). □

In the next theorem, we prove the factorization formula when there are two fragments. The factorization formula (3.5) with N fragments can then be proved via mathematical induction.

Theorem 3.6

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that $N=2$ in the partitioning given in (3.2). Let f_{r1} be the Jost solution from the right for (2.2) with the coefficients $\{a_1(n),b_1(n),w_1(n)\}_{n\in\mathbf{Z}}$ appearing in (3.3). Let $T_1(z)$, $R_1(z)$, and $L_1(z)$ be the scattering coefficients associated with the coefficients $\{a_1(n),b_1(n),w_1(n)\}_{n\in\mathbf{Z}}$. Similarly, let f_{l2} be the Jost solution from the left for (2.2) with the coefficients $\{a_2(n),b_2(n),w_2(n)\}_{n\in\mathbf{Z}}$ appearing in (3.3). Let $T_2(z)$, $R_2(z)$, and $L_2(z)$ be the scattering coefficients associated with the coefficients $\{a_2(n),b_2(n),w_2(n)\}_{n\in\mathbf{Z}}$. Furthermore, let g_{r1} and g_{l2} be the quantities appearing in (3.11) and (3.7), respectively. Then, the factorization formula given in (3.6) holds.

Proof

We will derive (3.6) by evaluating the left-hand side of (3.8) when $n=n_1$ in two different ways and by equating the resulting expressions. The first expression will be obtained by using (3.8), and the second will be obtained by using (3.12). From (3.8) and (3.9), we get

$$\begin{bmatrix} f_{l}(z,n_1) & f_{r}(z,n_1)\\ f_{l}(z,n_1+1) & f_{r}(z,n_1+1)\end{bmatrix}=\begin{bmatrix} f_{l2}(z,n_1) & g_{l2}(z,n_1)\\ f_{l2}(z,n_1+1) & g_{l2}(z,n_1+1)\end{bmatrix}\begin{bmatrix}1 & \frac{R(z)}{T(z)}\\ 0 & \frac{1}{T(z)}\end{bmatrix}. \tag{3.22}$$

On the other hand, from (3.12)–(3.15), we get

$$\begin{bmatrix} f_{l}(z,n_1) & f_{r}(z,n_1)\\ f_{l}(z,n_1+1) & f_{r}(z,n_1+1)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \frac{a_\infty}{a(n_1+1)}\end{bmatrix}\begin{bmatrix} g_{r1}(z,n_1) & f_{r1}(z,n_1)\\ g_{r1}(z,n_1+1) & f_{r1}(z,n_1+1)\end{bmatrix}\begin{bmatrix}\frac{1}{T(z)} & 0\\ \frac{L(z)}{T(z)} & 1\end{bmatrix}. \tag{3.23}$$

Thus, the right-hand side of (3.22) must be equal to the right-hand side of (3.23). Using (3.20) on the right-hand side of (3.22) and using (3.21) on the right-hand side of (3.23) and equating the resulting expressions, after some simplification, we obtain

$$\begin{bmatrix} \frac{1}{T_2(z)} & \frac{L_2(z^{-1})}{T_2(z^{-1})} \\ \frac{L_2(z)}{T_2(z)} & \frac{1}{T_2(z^{-1})} \end{bmatrix} \begin{bmatrix} 1 & \frac{R(z)}{T(z)} \\ 0 & \frac{1}{T(z)} \end{bmatrix} = \begin{bmatrix} \frac{1}{T_1(z^{-1})} & \frac{R_1(z)}{T_1(z)} \\ \frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z)} \end{bmatrix} \begin{bmatrix} \frac{1}{T(z)} & 0 \\ \frac{L(z)}{T(z)} & 1 \end{bmatrix},$$

or equivalently, we have

$$\begin{bmatrix} \frac{1}{T_1(z^{-1})} & \frac{R_1(z)}{T_1(z)} \\ \frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z)} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{T_2(z)} & \frac{L_2(z^{-1})}{T_2(z^{-1})} \\ \frac{L_2(z)}{T_2(z)} & \frac{1}{T_2(z^{-1})} \end{bmatrix} = \begin{bmatrix} \frac{1}{T(z)} & 0 \\ \frac{L(z)}{T(z)} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{R(z)}{T(z)} \\ 0 & \frac{1}{T(z)} \end{bmatrix}^{-1}. \tag{3.24}$$

In a straightforward way, we evaluate the inverse matrices appearing in (3.24) as

$$\begin{bmatrix} 1 & \frac{R(z)}{T(z)} \\ 0 & \frac{1}{T(z)} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -R(z) \\ 0 & T(z) \end{bmatrix}, \tag{3.25}$$

$$\begin{bmatrix} \frac{1}{T_1(z^{-1})} & \frac{R_1(z)}{T_1(z)} \\ \frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z)} \end{bmatrix}^{-1} = \frac{1}{\frac{1}{T_1(z)}\,\frac{1}{T_1(z^{-1})} - \frac{R_1(z)}{T_1(z)}\,\frac{R_1(z^{-1})}{T_1(z^{-1})}} \begin{bmatrix} \frac{1}{T_1(z)} & -\frac{R_1(z)}{T_1(z)} \\ -\frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z^{-1})} \end{bmatrix}. \tag{3.26}$$

From (2.15), we see that the determinantal quantity appearing as a coefficient on the right-hand side of (3.26) is equal to 1. Thus, (3.26) simplifies to

$$\begin{bmatrix} \frac{1}{T_1(z^{-1})} & \frac{R_1(z)}{T_1(z)} \\ \frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z)} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{T_1(z)} & -\frac{R_1(z)}{T_1(z)} \\ -\frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z^{-1})} \end{bmatrix}. \tag{3.27}$$

Using (3.25) and (3.27) in (3.24), we obtain

$$\begin{bmatrix} \frac{1}{T_1(z)} & -\frac{R_1(z)}{T_1(z)} \\ -\frac{R_1(z^{-1})}{T_1(z^{-1})} & \frac{1}{T_1(z^{-1})} \end{bmatrix} \begin{bmatrix} \frac{1}{T_2(z)} & \frac{L_2(z^{-1})}{T_2(z^{-1})} \\ \frac{L_2(z)}{T_2(z)} & \frac{1}{T_2(z^{-1})} \end{bmatrix} = \begin{bmatrix} \frac{1}{T(z)} & 0 \\ \frac{L(z)}{T(z)} & 1 \end{bmatrix} \begin{bmatrix} 1 & -R(z) \\ 0 & T(z) \end{bmatrix}. \tag{3.28}$$

Using (2.16) on the left-hand side of (3.28), we see that the left-hand side of (3.28) is equal to the matrix product $\Lambda_1(z)\,\Lambda_2(z)$, where $\Lambda_1(z)$ and $\Lambda_2(z)$ are the transition matrices defined in (3.4). On the other hand, with the help of (2.7) we see that the right-hand side in (3.28) is equal to the transition matrix $\Lambda(z)$ defined in (3.1). Thus, (3.6) is established. □

Via mathematical induction, the factorization formula (3.5) with *N* fragments holds, as stated in the next corollary.

Corollary 3.7

Assume that the coefficients in (2.2) belong to class $\mathcal{A}$. Further, assume that we have the partitioning specified in (3.2) with the coefficients $\{a_j(n), b_j(n), w_j(n)\}_{n\in\mathbf{Z}}$ appearing in (3.3) for $1 \le j \le N$ for any positive integer $N \ge 2$. Let $T_j(z)$, $R_j(z)$, and $L_j(z)$ be the scattering coefficients associated with the coefficients $\{a_j(n), b_j(n), w_j(n)\}_{n\in\mathbf{Z}}$. Then, the factorization formula given in (3.5) holds, where $\Lambda(z)$ and $\Lambda_j(z)$ are the transition matrices defined in (3.1) and (3.4), respectively.

Acknowledgements

The first author expresses his gratitude to the Institute of Physics and Mathematics of the Universidad Michoacana de San Nicolás de Hidalgo, México, for its hospitality. This work was completed with the support of the Visiting Distinguished Professor 2014 grant of the Mexican Academy of Sciences. The second author was partially supported by CIC-UMNSH and SNI-Conacyt.

References

1. Chadan K, Sabatier PC. *Inverse Problems in Quantum Scattering Theory* 2nd ed. Springer: New York, 1989.
2. Deift P, Trubowitz E. Inverse scattering on the line. *Communications on Pure and Applied Mathematics* 1979; **32**:121–251.
3. Faddeev LD. Properties of the *S*-matrix of the one-dimensional Schrödinger equation. *American Mathematical Society Translations* 1967; **65**(ser. 2):139–166.
4. Gel'fand IM, Levitan BM. On the determination of a differential equation from its spectral function. *American Mathematical Society Translations* 1955; **1**(ser. 2):253–304.
5. Levitan BM. *Inverse Sturm Liouville Problems*. VNU Science Press: Utrecht, 1987.
6. Marchenko VA. *Sturm-Liouville Operators and Applications*. Birkhäuser: Basel, 1986.
7. Aktosun T. A factorization of the scattering matrix for the Schrödinger equation and for the wave equation in one dimension. *Journal of Mathematical Physics* 1992; **33**:3865–3869.
8. Aktosun T, Klaus M, van der Mee C. Factorization of scattering matrices due to partitioning of potentials in one-dimensional Schrödinger-type equations. *Journal of Mathematical Physics* 1996; **37**:5897–5915.
9. Kostrykin V, Schrader R. Kirchhoff's rule for quantum wires. *Journal of Physics A* 1996; **37**:595–630.
10. Kostrykin V, Schrader R. The generalized star product and the factorization of scattering matrices on graphs. *Journal of Mathematical Physics* 2001; **42**:1563–1598.

11. Redheffer R. Difference equations and functional equations in transmission line theory. In *Modern Mathematics for the Engineer*. 2nd series, Beckenbach EF (ed). McGraw-Hill: New York, 1961; Ch 12.
12. Redheffer R. On the relation of transmission-line theory to scattering and transfer. *Journal of Mathematics and Physics* 1962; **41**:1–41.
13. Sassoli de Bianchi M, Di Ventra M. Differential equations and factorization property for the one-dimensional Schrödinger equation with position-dependent mass. *European Journal of Physics* 1995; **16**:260–265.
14. Aktosun T. Factorization and small-energy asymptotics for the radial Schrödinger equation. *Journal of Mathematical Physics* 2000; **41**:4262–4270.
15. Aktosun T, Sacks PE. Phase recovery with nondecaying potentials. *Inverse Problems* 2000; **16**:821–838.
16. Aktosun T, Sacks PE. Potential splitting and numerical solution of the inverse scattering problem on the line. *Mathematical Methods in the Applied Sciences* 2002; **25**:347–355.
17. Aktosun T, Sacks PE. Inversion of reflectivity data for nondecaying potentials. *SIAM Journal on Applied Mathematics* 2000; **60**:1340–1356.
18. Majkrzak CF, Berk NF. Exact determination of the phase in neutron reflectometry. *Physical Review B* 1995; **52**:10827–10830.
19. Majkrzak CF, Berk NF. Exact determination of the phase in neutron reflectometry by variation of the surrounding medium. *Physical Review B* 1998; **58**:15416–15418.
20. Majkrzak CF, Berk NF, Dura J, Satija SK, Karim A, Pedulla J, Deslattes RD. Direct inversion of specular reflectometry. *Physica B* 1998; **241–243**:1101–1103.
21. Kostrykin V, Schrader R. Kirchhoff's rule for quantum wires. II: The inverse problem with possible applications to quantum computers. *Fortschritte der Physik* 2000; **48**:703–716.
22. Kiers KA, van Dijk W. Scattering in one dimension: the coupled Schrödinger equation, threshold behaviour and Levinson's theorem. *Journal of Mathematical Physics* 1999; **32**:6033–6058.
23. Case KM, Kac M. A discrete version of the inverse scattering problem. *Journal of Mathematical Physics* 1973; **14**:594–603.
24. Akhiezer N. *The Classical Moment Problem*. Oliver and Boyd: London, 1965.
25. Atkinson F. *Discrete and Continuous Boundary Problems*. Academic Press: New York, 1964.
26. Simon B. The classical moment problem as a self-adjoint finite difference operator. *Advances in Mathematics* 1998; **137**:82–103.
27. Teschl G. *Jacobi Operators and Completely Integrable Nonlinear Lattices*. American Mathematical Society: Providence, RI, 2000.
28. Guseinov GS. The determination of an infinite Jacobi matrix from the scattering data. *Soviet mathematics - Doklady* 1976; **17**:596–600.
29. Guseinov GS. The inverse problem of scattering theory for a second-order difference equation on the whole axis. *Soviet mathematics - Doklady* 1976; **17**:1684–1688.