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ABSTRACT

A system of linear integral equations is presented, which is the analog of the system of Marchenko integral equations, to solve the inverse scattering problem for the linear system associated with the DNLS (derivative nonlinear Schrödinger) equations. The corresponding direct and inverse scattering problems are analyzed, and the recovery of the potentials and the Jost solutions from the solution to the Marchenko system is described. When the reflection coefficients are zero, some explicit solution formulas are provided for the potentials and the Jost solutions in terms of a pair of constant matrix triplets representing the bound-state information for any number of bound states and any multiplicities. In the reduced case, when the two potentials in the linear system are related to each other through complex conjugation, the corresponding reduced Marchenko integral equation is obtained. The solution to the DNLS equation is obtained from the solution to the reduced Marchenko integral equation. The theory presented is illustrated with some explicit examples.

I. INTRODUCTION

Our main goal in this paper is to present solutions to the general system of DNLS (derivative nonlinear Schrödinger) equations[1,3,4,27,35–39]

$$\begin{cases} i\,\tilde q_t+\tilde q_{xx}+i(4\delta-\epsilon)\tilde q\tilde q_x\tilde r+4i\delta\tilde q^{\,2}\tilde r_x+\delta(4\delta+\epsilon)\,\tilde q^{\,3}\tilde r^{\,2}=0,\\ i\,\tilde r_t-\tilde r_{xx}+i(4\delta-\epsilon)\tilde q\tilde r\tilde r_x+4i\delta\tilde q_x\tilde r^{\,2}-\delta(4\delta+\epsilon)\,\tilde q^{\,2}\tilde r^{\,3}=0, \end{cases} \tag{1.1}$$

where the subscripts denote the respective partial derivatives, the dependent variables $\tilde q$ and $\tilde r$ are complex-valued functions, the independent variables x and t take values on the real axis $\mathbb{R}$, and the parameters δ and ϵ are complex valued. The nonlinear system (1.1) can be used as a model to describe propagation electromagnetic waves in plasmas, where the quantities $\tilde q$ and $\tilde r$ correspond to two components of the electric field. For clarity and simplicity, we assume that for each fixed $t\in\mathbb{R}$, the scalar quantities $\tilde q(x,t)$ and $\tilde r(x,t)$ belong to the Schwartz class even though our results hold under weaker conditions. We recall that the Schwartz class consists of functions of x decaying to zero faster than any inverse power of $|x|$ as $x\to\pm\infty$, while the derivatives of all orders are continuous everywhere.

The integrability of the nonlinear system (1.1) by the inverse scattering transform method[1,4,24,28,34] is already known because of the existence of a corresponding Lax pair. What is new and significant in our paper is the development and use of the Marchenko method for (1.1), and hence the implementation of recovery of $\tilde q(x,t)$ and $\tilde r(x,t)$ from the corresponding time-evolved scattering data by the Marchenko method. Even though the Marchenko method is available for various other integrable systems such as the NLS system[1,2,4,17,34,40]

$$\begin{cases} iu_t + u_{xx} - 2u^2 v = 0, \\ iv_t - v_{xx} + 2uv^2 = 0, \end{cases}$$

it has not been available for (1.1) so far.

One of the most effective methods used to solve an inverse scattering problem is the Marchenko method.[5,13,18,22,23,29,31–33] This method was originally developed by Marchenko[30] to solve the inverse scattering problem for the half-line Schrödinger equation, where the potential is recovered from the solution to a linear integral equation. Later, Faddeev[23] extended the method of Marchenko to solve the inverse scattering problem for the full-line Schrödinger equation, where again the potential is recovered from the solution to a linear integral equation. The method has also been extended to solve inverse scattering problems associated with some other linear differential equations and linear difference equations as well as systems of such linear equations. In the Marchenko method, the potential is recovered from the solution to a linear integral equation, usually called the Marchenko equation, where the kernel and the nonhomogeneous term are constructed from the scattering dataset with the help of a Fourier transform.

The goal[1,4,24,34] in the inverse scattering transform method consists of the determination of the solutions $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ to (1.1) when the initial values $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ are known. The execution of the integrability of (1.1) in the sense of the inverse scattering transform is equivalent to the use of the following three steps. In the first step, the initial values $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ are associated with a corresponding initial scattering dataset $\tilde{\mathbf{S}}(\zeta,0)$, where ζ is an appropriate spectral parameter. In the second step, the time evolution $\tilde{\mathbf{S}}(\zeta,0) \mapsto \tilde{\mathbf{S}}(\zeta,t)$ of the scattering dataset is described. In the third step, the time-evolved quantities $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ are recovered from the time-evolved scattering dataset $\tilde{\mathbf{S}}(\zeta,t)$. The significance of our paper is that we provide the execution of these three steps by describing the corresponding scattering dataset, by showing the time evolution of the scattering dataset, and by presenting the recovery of $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ from the corresponding solution to our Marchenko system of linear integral equations.

A particular strength of our paper is that we do not assume the simplicity of bound states in the relevant scattering dataset, which is usually artificially assumed in the analysis of (1.1). On the contrary, we deal with any number of bound states having any multiplicities in an elegant way with the help of a pair of matrix triplets. Another strength of our paper is that we provide explicit solution formulas for (1.1) in a closed form corresponding to reflectionless scattering data with any number of bound states and any multiplicities. The use of matrix triplets to describe the bound-state information in the input to the Marchenko method is the most appropriate and elegant way to handle bound states with multiplicities, and this is true in the Marchenko method[6–10,14–16] for all other integrable systems as well. The use of matrix triplets in reflectionless scattering datasets causes the integral kernels in the corresponding Marchenko systems to be separable, and hence it allows for the construction of explicit solution formulas in the reflectionless case corresponding to any number of bound states with any multiplicities. Such formulas are expressed in a compact form in terms of matrix exponentials, and those formulas are valid for any number of bound states and any multiplicities.

The general DNLS system (1.1), when $(\delta,\epsilon) = (-1/4, 1)$, yields the Kaup–Newell system[26] (also called DNLS I system) given by

$$\begin{cases} iq_t + q_{xx} - i(q^2 r)_x = 0, \\ ir_t - r_{xx} - i(qr^2)_x = 0. \end{cases} \tag{1.2}$$

It reduces, when $(\delta,\epsilon) = (0,1)$, to the Chen–Lee–Liu system[19] (also called DNLS II system)

$$\begin{cases} i\tilde{q}_t + \tilde{q}_{xx} - i\tilde{q}\tilde{q}_x\tilde{r} = 0, \\ i\tilde{r}_t - \tilde{r}_{xx} - i\tilde{q}\tilde{r}\tilde{r}_x = 0, \end{cases} \tag{1.3}$$

and it gives us, when $(\delta,\epsilon) = (1/4, 1)$, the Gerdjikov–Ivanov system[25] (also called DNLS III system)

$$\begin{cases} i\tilde{q}_t + \tilde{q}_{xx} + i\tilde{q}^2\tilde{r}_x + \dfrac{1}{2}\tilde{q}^3\tilde{r}^2 = 0, \\ i\tilde{r}_t - \tilde{r}_{xx} + i\tilde{q}_x\tilde{r}^2 - \dfrac{1}{2}\tilde{q}^2\tilde{r}^3 = 0. \end{cases} \tag{1.4}$$

We are not interested in analyzing particular cases of the general DNLS system (1.1) separately because that would be tedious and is not necessary. Instead, we present our method to solve (1.1) with the presence of the two free parameters δ and ϵ, and from that solution, we are able to extract the solution to any specific case by assigning particular values to the parameters.

Even though it is possible to apply the Marchenko method directly on (1.1) in the presence of the two free parameters δ and ϵ, for clarity and simplicity, we instead proceed as follows. We view the Kaup–Newell system (1.2) as the unperturbed system and view (1.1) with the two parameters as the perturbed system. We use a tilde to denote the quantities related to the perturbed system (1.1), and the quantities without a tilde are related to the unperturbed system. This explains why we have written (1.2) without the use of a tilde even though we have used a tilde in (1.3) and (1.4).

Because our method originates in analyzing a pair of linear systems corresponding to the integrable system (1.1), it turns out that it is more appropriate for us to express (1.1) by using three complex-valued parameters a, b, and κ instead of the two complex parameters δ and ϵ in (1.1). Letting

$$\delta = \frac{\kappa(a-b-1)}{4}, \quad \epsilon = \kappa, \tag{1.5}$$

from (1.1), we obtain the equivalent system

$$\begin{cases} i\tilde{q}_t + \tilde{q}_{xx} + i\kappa(a-b-2)\tilde{q}\tilde{q}_x\tilde{r} + i\kappa(a-b-1)\tilde{q}^2\tilde{r}_x + \dfrac{\kappa^2(a-b)(a-b-1)}{4}\tilde{q}^3\tilde{r}^2 = 0, \\ i\tilde{r}_t - \tilde{r}_{xx} + i\kappa(a-b-2)\tilde{q}\tilde{r}\tilde{r}_x + i\kappa(a-b-1)\tilde{q}_x\tilde{r}^2 - \dfrac{\kappa^2(a-b)(a-b-1)}{4}\tilde{q}^2\tilde{r}^3 = 0, \end{cases} \tag{1.6}$$

with still containing only two arbitrary parameters because the new parameters a and b appear in (1.6) in the combined form $a - b$. The advantage of using three relevant parameters in the corresponding linear domain, even though there are only two relevant parameters in the nonlinear domain, will soon be apparent.

If we use the parameters δ and ϵ, from (1.1), it is difficult to see why we single out (1.2) as the unperturbed system with the choice $(\delta, \epsilon) = (1/4, 1)$. On the other hand, the simplicity of (1.2) is easily seen from the equivalent formulation of (1.1) as (1.6) with the new parameters a, b, and κ. Although we could use any particular case of (1.6) as the unperturbed problem instead of (1.2), it is advantageous and the simplest to use (1.2) as the unperturbed nonlinear problem. This is because (1.2) is obtained from (1.6) by using the simplest choice $(a, b, \kappa) = (0, 0, 1)$. We note that the Chen–Lee–Liu system (1.3) corresponds to using $(a, b, \kappa) = (1, 0, 1)$ in (1.6) and the Gerdjikov–Ivanov system (1.4) is obtained by using $(a, b, \kappa) = (1, -1, 1)$. In Example 9.8, we elaborate on the issue that any particular case of the nonlinear system (1.6) could be used as the unperturbed problem instead of the particular nonlinear system (1.2). As already mentioned, in the analysis of the linear system associated with the nonlinear system (1.6), it is advantageous to choose (1.2) as the unperturbed system.

Let $(\mathcal{X}, \mathcal{T})$ be the AKNS pair[1,2,4,34] associated with the unperturbed nonlinear system (1.2) so that the matrix equality

$$\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} = 0, \tag{1.7}$$

yields (1.2). Thus, corresponding to (1.2), we have the pair of unperturbed linear systems given by

$$\Psi_x = \mathcal{X}\Psi, \quad \Psi_t = \mathcal{T}\Psi. \tag{1.8}$$

It can be verified directly that we can choose the AKNS pair (X, T) in (1.7) as

$$\mathcal{X} = \begin{bmatrix} -i\zeta^2 & \zeta q \\ \zeta r & i\zeta^2 \end{bmatrix}, \quad \mathcal{T} = \begin{bmatrix} -2i\zeta^4 - iqr\zeta^2 & 2q\zeta^3 + (iq_x + q^2 r)\zeta \\ 2r\zeta^3 + (-ir_x + qr^2)\zeta & 2i\zeta^4 + iqr\zeta^2 \end{bmatrix}, \tag{1.9}$$

where we use ζ to denote the spectral parameter.

Similarly, let $(\tilde{\mathcal{X}}, \tilde{\mathcal{T}})$ be the AKNS pair associated with the perturbed nonlinear system (1.6) so that the matrix equality

$$\tilde{\mathcal{X}}_t - \tilde{\mathcal{T}}_x + \tilde{\mathcal{X}}\tilde{\mathcal{T}} - \tilde{\mathcal{T}}\tilde{\mathcal{X}} = 0, \tag{1.10}$$

yields (1.6). Hence, corresponding to (1.6), we have the pair of perturbed linear systems given by

$$\tilde{\psi}_x = \tilde{\mathcal{X}}\tilde{\psi}, \quad \tilde{\psi}_t = \tilde{\mathcal{T}}\tilde{\psi}. \tag{1.11}$$

It can again be verified directly that we can choose $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ in (1.10) as

$$\tilde{\mathcal{X}} = \begin{bmatrix} -i\zeta^2 + \dfrac{ib}{2}\tilde{q}\tilde{r} & \kappa\zeta\tilde{q} \\ \dfrac{1}{\kappa}\zeta\tilde{r} & i\zeta^2 + \dfrac{ia}{2}\tilde{q}\tilde{r} \end{bmatrix}, \quad \tilde{\mathcal{T}} = \begin{bmatrix} \tilde{\mathcal{T}}_{11} & \tilde{\mathcal{T}}_{12} \\ \tilde{\mathcal{T}}_{21} & \tilde{\mathcal{T}}_{22} \end{bmatrix}, \tag{1.12}$$

where we have defined

$$\tilde{\mathcal{T}}_{11} := -2i\zeta^4 - i\tilde{q}\tilde{r}\zeta^2 + \frac{b}{2}(\tilde{q}\tilde{r}_x - \tilde{q}_x\tilde{r}) + \frac{ib}{2}\left(b - a + \frac{3}{2}\right)\tilde{q}^2\tilde{r}^2, \tag{1.13}$$

$$\tilde{\mathcal{T}}_{12} := 2\kappa\zeta^3\tilde{q} + \kappa\zeta\left[i\tilde{q}_x + \frac{1}{2}(b - a + 2)\tilde{q}^2\tilde{r}\right], \tag{1.14}$$

$$\tilde{\mathcal{T}}_{21} := \frac{2}{\kappa}\zeta^3\tilde{r} + \frac{1}{\kappa}\zeta\left[-i\tilde{r}_x + \frac{1}{2}(b-a+2)\tilde{q}\tilde{r}^2\right], \tag{1.15}$$

$$\tilde{\mathcal{T}}_{22} := 2i\zeta^4 + i\tilde{q}\tilde{r}\zeta^2 + \frac{a}{2}(\tilde{q}\tilde{r}_x - \tilde{q}_x\tilde{r}) + \frac{ia}{2}\left(b-a+\frac{3}{2}\right)\tilde{q}^2\tilde{r}^2. \tag{1.16}$$

Let us remark that the simplicity of (1.2), and hence its choice as the unperturbed problem, is also seen by comparing the matrices $\mathcal{X}$ and $\tilde{\mathcal{X}}$ appearing in (1.9) and (1.12), respectively. Another simple aspect of $\mathcal{X}$ is that the matrix $\mathcal{X}$ has zero trace, which implies that the left and right transmission coefficients in the corresponding scattering dataset are equal, whereas the trace of the matrix $\tilde{\mathcal{X}}$ is nonzero unless $a + b = 0$.

In our paper, we apply our Marchenko method to the unperturbed linear system given in the first equality of (1.8), and we obtain the solution to the corresponding inverse scattering problem. We then get the solution to the inverse scattering problem for the linear system in the first equality of (1.11) by relating the perturbed linear system to the unperturbed linear system through the transformation expressed as

$$\tilde{\Psi} = \mathcal{G}\Psi. \tag{1.17}$$

In (1.17), the coefficient matrix $\mathcal{G}$ is given by

$$\mathcal{G} := \begin{bmatrix} E(x,t)^b & 0 \\ 0 & E(x,t)^a \end{bmatrix}, \tag{1.18}$$

with the complex-valued scalar quantity $E(x,t)$ defined as

$$E(x,t) := \exp\left(\frac{i}{2}\int_{-\infty}^{x} dz\, q(z,t)\, r(z,t)\right). \tag{1.19}$$

The perturbed potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ appearing in (1.12)–(1.16) are related to the unperturbed potentials $q(x,t)$ and $r(x,t)$ appearing in (1.2) as

$$\tilde{q}(x,t) := \frac{1}{\kappa}\, q(x,t)\, E(x,t)^{b-a}, \quad \tilde{r}(x,t) := \kappa\, r(x,t)\, E(x,t)^{a-b}. \tag{1.20}$$

It might at first seem contradictory to use only two free parameters in each of the nonlinear systems (1.1) and (1.6) and use three free parameters in the corresponding linear system given in the first equality of (1.11). However, there is no contradiction here, and the use of only two free parameters in the nonlinear system (1.6) and the use of three free parameters in the linear system can be explained as follows. As seen from (1.17) and (1.18), for the perturbed linear system in the first equality of (1.11), the use of the three free parameters a, b, and κ is essential because the wavefunction $\tilde{\Psi}$ for the corresponding linear system contains the parameters a and b independently. On the other hand, from (1.20), we see that the parameters a and b appear not separately but together as $a - b$ in the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$. Since the perturbed nonlinear system (1.6) involves only the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ and does not involve the wavefunction $\tilde{\Psi}$, only the two free parameters κ and $a - b$ are used to describe the nonlinear system (1.6).

When the arguments of a function are clearly understood, we may omit those arguments. Hence, we may use E instead of $E(x,t)$, and similarly, we may use q, r, $\tilde{q}$, and $\tilde{r}$ instead of $q(x,t)$, $r(x,t)$, $\tilde{q}(x,t)$, and $\tilde{r}(x,t)$, respectively. Let us remark that, using (1.8), (1.11), and (1.17), we can express $(\mathcal{X}, \mathcal{T})$ and $(\tilde{\mathcal{X}}, \tilde{\mathcal{T}})$ in terms of each other as

$$\tilde{\mathcal{X}} = \mathcal{G}_x\mathcal{G}^{-1} + \mathcal{G}\mathcal{X}\mathcal{G}^{-1}, \quad \tilde{\mathcal{T}} = \mathcal{G}_t\mathcal{G}^{-1} + \mathcal{G}\mathcal{T}\mathcal{G}^{-1},$$

$$\mathcal{X} = -\mathcal{G}^{-1}\mathcal{G}_x + \mathcal{G}^{-1}\tilde{\mathcal{X}}\mathcal{G}, \quad \mathcal{T} = -\mathcal{G}^{-1}\mathcal{G}_t + \mathcal{G}^{-1}\tilde{\mathcal{T}}\mathcal{G}.$$

Our paper is organized as follows. In Sec. II, we provide the relevant results related to the direct scattering problem for the unperturbed linear system (2.1). The relevant quantities include the Jost solutions, the scattering coefficients, and the bound-state information. We use a pair of matrix triplets to describe the bound-state information with any number of bound states and any multiplicities. In Sec. III, we present our Marchenko system of integral equations relevant to the inverse scattering problem for (2.1). We relate the scattering dataset to the kernel of the Marchenko system. We also describe the recovery of potentials and Jost solutions from the solution to the Marchenko system. In Sec. IV, we consider the Marchenko system for (2.1) when the reflection coefficients are zero. In that case, the Marchenko system has a separable kernel, and hence it can be solved explicitly by using the methods from the linear algebra. We present some explicit formulas expressing the corresponding potentials and Jost solutions in terms of the two matrix triplets used as input to the Marchenko system. In Sec. V, we relate the quantities relevant for the perturbed linear system (5.1) to the corresponding relevant quantities for the unperturbed linear system (2.1). In Sec. VI, we present our Marchenko method to obtain the solution to the perturbed nonlinear system (1.6). In Sec. VII, in the reflectionless case, we present explicit formulas for the quantities relevant to the perturbed linear system. In Sec. VIII, we consider the unperturbed linear and nonlinear systems in the special case where the potentials $q(x,t)$ and $r(x,t)$ are related to each other through complex conjugation. Using such a reduction, we obtain the corresponding linear and nonlinear equations and also a scalar Marchenko equation, and we present the recovery of the potential $q(x,t)$ from the solution to that reduced Marchenko equation. Finally, in Sec. IX, we provide some explicit examples to illustrate the theory presented in Secs. I–VIII.

As in the formulation of any inverse scattering problem, the analysis of the inverse scattering problem associated with (1.1) contains four aspects: the existence, uniqueness, reconstruction, and characterization. The existence aspect deals with the question whether there exists at least one pair of potentials $q(x,t)$ and $r(x,t)$ in some class corresponding to a given set of scattering data in a particular class. The uniqueness aspect deals with the question whether there is only one pair of potentials for that corresponding scattering dataset or there are two or more such pairs. The reconstruction aspect is concerned with the recovery of the two potentials from the scattering dataset. The characterization aspect deals with the specification of the class of potentials and the class of scattering datasets so that there is a one-to-one correspondence between the elements of the former and latter classes. In our paper, we only deal with the reconstruction aspect of the inverse problem associated with (1.1). The remaining three aspects are challenging and need to be investigated separately, and we do not deal with those aspects in our paper. The analysis of the inverse problem associated with (1.1) is further complicated due to the fact that the linear differential operator associated with (1.1) is not self-adjoint. In our paper, we do not deal with the problem of the determination of the class of initial values of the potentials $q(x,t)$ and $r(x,t)$ at $t=0$ so that the time-evolved potentials will remain in the same class. Our emphasis is the development and the use of the Marchenko method associated with the DNLS system. We hope that the Marchenko method presented in this paper will motivate other researchers in the field to analyze other aspects of the relevant inverse scattering problem.

II. THE DIRECT SCATTERING PROBLEM FOR THE UNPERTURBED SYSTEM

In this section, we present the basic results related to the direct scattering problem for the unperturbed linear system given in the first equality of (1.8). For convenience, we write it as

$$\frac{d}{dx}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-i\zeta^2 & \zeta\, q(x,t)\\ \zeta\, r(x,t) & i\zeta^2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix},\qquad x\in\mathbb{R}, \tag{2.1}$$

where the quantities α and β are the components of the wavefunction depending on the spatial variable x, the time variable t, and the spectral parameter ζ, and where we recall that the potentials $q(x,t)$ and $r(x,t)$ are assumed to belong to the Schwartz class for each fixed $t\in\mathbb{R}$. The solution to the direct scattering problem for (2.1) consists of the specification of the scattering dataset $\mathbf{S}(\zeta,t)$ corresponding to the potentials $q(x,t)$ and $r(x,t)$ appearing in (2.1). The direct scattering problem is solved as follows. Using $q(x,t)$ and $r(x,t)$ as input to (2.1), we obtain the four particular solutions to (2.1), which are known as the Jost solutions. From the large spatial asymptotics of those four Jost solutions, we get the scattering coefficients. Finally, we obtain $\mathbf{S}(\zeta,t)$ by supplementing the set of scattering coefficients with the bound-state information for (2.1).

We use $\psi(\zeta,x,t)$, $\bar\psi(\zeta,x,t)$, $\phi(\zeta,x,t)$, and $\bar\phi(\zeta,x,t)$ to denote the four Jost solutions to (2.1), where they satisfy the respective spatial asymptotics

$$\psi(\zeta,x,t)=\begin{bmatrix}o(1)\\ e^{i\zeta^2x}[1+o(1)]\end{bmatrix},\qquad x\to+\infty, \tag{2.2}$$

$$\bar\psi(\zeta,x,t)=\begin{bmatrix}e^{-i\zeta^2x}[1+o(1)]\\ o(1)\end{bmatrix},\qquad x\to+\infty, \tag{2.3}$$

$$\phi(\zeta,x,t)=\begin{bmatrix}e^{-i\zeta^2x}[1+o(1)]\\ o(1)\end{bmatrix},\qquad x\to-\infty, \tag{2.4}$$

$$\bar\phi(\zeta,x,t)=\begin{bmatrix}o(1)\\ e^{i\zeta^2x}[1+o(1)]\end{bmatrix},\qquad x\to-\infty. \tag{2.5}$$

We remark that the overbar does not denote complex conjugation.

We have six scattering coefficients associated with (2.1), i.e., the transmission coefficients $T(\zeta,t)$ and $\bar T(\zeta,t)$, the right reflection coefficients $R(\zeta,t)$ and $\bar R(\zeta,t)$, and the left reflection coefficients $L(\zeta,t)$ and $\bar L(\zeta,t)$. Since the trace of the coefficient matrix in (2.1) is zero, the transmission coefficients from the left and from the right are equal to each other, and hence we do not need to use separate notations for the left and right transmission coefficients. The six scattering coefficients are obtained from the spatial asymptotics of the Jost solutions given by

$$\psi(\zeta,x,t)=\begin{bmatrix}\dfrac{L(\zeta,t)}{T(\zeta,t)}\,e^{-i\zeta^2x}[1+o(1)]\\[2mm] \dfrac{1}{T(\zeta,t)}\,e^{i\zeta^2x}[1+o(1)]\end{bmatrix},\qquad x\to-\infty, \tag{2.6}$$

$$\bar{\psi}(\zeta,x,t)=\begin{bmatrix}\dfrac{1}{\bar{T}(\zeta,t)}\,e^{-i\zeta^2x}[1+o(1)]\\ \dfrac{\bar{L}(\zeta,t)}{\bar{T}(\zeta,t)}\,e^{i\zeta^2x}[1+o(1)]\end{bmatrix},\qquad x\to-\infty, \tag{2.7}$$

$$\phi(\zeta,x,t)=\begin{bmatrix}\dfrac{1}{T(\zeta,t)}\,e^{-i\zeta^2x}[1+o(1)]\\ \dfrac{R(\zeta,t)}{T(\zeta,t)}\,e^{i\zeta^2x}[1+o(1)]\end{bmatrix},\qquad x\to+\infty, \tag{2.8}$$

$$\bar{\phi}(\zeta,x,t)=\begin{bmatrix}\dfrac{\bar{R}(\zeta,t)}{\bar{T}(\zeta,t)}\,e^{-i\zeta^2x}[1+o(1)]\\ \dfrac{1}{\bar{T}(\zeta,t)}\,e^{i\zeta^2x}[1+o(1)]\end{bmatrix},\qquad x\to+\infty. \tag{2.9}$$

Let us use the subscripts 1 and 2 to denote the first and second components, respectively, of the Jost solutions. Hence, we introduce the notation

$$\begin{bmatrix}\psi_1(\zeta,x,t)\\ \psi_2(\zeta,x,t)\end{bmatrix}:=\psi(\zeta,x,t),\qquad \begin{bmatrix}\bar{\psi}_1(\zeta,x,t)\\ \bar{\psi}_2(\zeta,x,t)\end{bmatrix}:=\bar{\psi}(\zeta,x,t), \tag{2.10}$$

$$\begin{bmatrix}\phi_1(\zeta,x,t)\\ \phi_2(\zeta,x,t)\end{bmatrix}:=\phi(\zeta,x,t),\qquad \begin{bmatrix}\bar{\phi}_1(\zeta,x,t)\\ \bar{\phi}_2(\zeta,x,t)\end{bmatrix}:=\bar{\phi}(\zeta,x,t). \tag{2.11}$$

We also introduce the auxiliary spectral parameter λ in terms of the spectral parameter ζ as

$$\lambda=\zeta^2,\qquad \zeta=\sqrt{\lambda}, \tag{2.12}$$

with the square root denoting the principal branch of the complex-valued square-root function. We remark that when λ takes values on the real axis, ζ takes values on the real and imaginary axes. We use $\mathbb{C}^+$ and $\mathbb{C}^-$ to denote the upper-half and lower-half, respectively, of the complex plane $\mathbb{C}$, and we let $\overline{\mathbb{C}^+}:=\mathbb{C}^+\cup\mathbb{R}$ and $\overline{\mathbb{C}^-}:=\mathbb{C}^-\cup\mathbb{R}$.

The quantity $E(x,t)$ defined in (1.19) is, in general, complex valued, and hence it does not necessarily have the unit amplitude. From (1.19), it follows that

$$\lim_{x\to-\infty}E(x,t)=1,\qquad \lim_{x\to+\infty}E(x,t)=e^{i\mu/2}, \tag{2.13}$$

where we have defined the complex constant μ as

$$\mu:=\int_{-\infty}^{\infty}dz\,q(z,t)\,r(z,t). \tag{2.14}$$

With the help of (1.2), one can show that μ is a complex constant and its value is independent of t. Hence, from (2.14), we obtain

$$\mu=\int_{-\infty}^{\infty}dz\,q(z,0)\,r(z,0). \tag{2.15}$$

In the following theorem, we summarize the relevant properties of the Jost solutions and the scattering coefficients for (2.1).

Theorem 2.1. *Assume that the potentials $q(x,t)$ and $r(x,t)$ appearing in the first-order system (2.1) belong to the Schwartz class for each fixed $t\in\mathbb{R}$. Let E denote the quantity defined in (1.19), and let μ be the complex constant defined in (2.14). Furthermore, assume that the spectral parameters ζ and λ are related to each other as in (2.12). Then, we have the following:*

(a) *For each fixed pair of x and t in $\mathbb{R}$, the Jost solutions $\psi(\zeta,x,t)$ and $\phi(\zeta,x,t)$ to (2.1) are analytic in the first and third quadrants in the complex ζ-plane and are continuous in the closures of those regions. Similarly, the Jost solutions $\bar{\psi}(\zeta,x,t)$ and $\bar{\phi}(\zeta,x,t)$ are analytic in the second and fourth quadrants in the complex ζ-plane and are continuous in the closures of those regions.*

(b) *The components of the Jost solutions appearing in (2.10) and (2.11) satisfy the following properties. The components $\psi_1(\zeta,x,t)$, $\bar{\psi}_2(\zeta,x,t)$, $\phi_2(\zeta,x,t)$, and $\bar{\phi}_1(\zeta,x,t)$ are odd in ζ; and the components $\psi_2(\zeta,x,t)$, $\bar{\psi}_1(\zeta,x,t)$, $\phi_1(\zeta,x,t)$, and $\bar{\phi}_2(\zeta,x,t)$ are even in ζ. Moreover, for each fixed pair of x and t in $\mathbb{R}$, the four scalar quantities $\psi_1(\zeta,x,t)/\zeta$, $\psi_2(\zeta,x,t)$, $\phi_1(\zeta,x,t)$, and $\phi_2(\zeta,x,t)/\zeta$ are even in ζ; and they are analytic in $\lambda\in\mathbb{C}^+$ and continuous in $\lambda\in\overline{\mathbb{C}^+}$. Similarly, for each fixed pair of x and t in $\mathbb{R}$, the four scalar quantities $\bar{\psi}_1(\zeta,x,t)$, $\bar{\psi}_2(\zeta,x,t)/\zeta$, $\bar{\phi}_1(\zeta,x,t)/\zeta$, and $\bar{\phi}_2(\zeta,x,t)$ are even in ζ; and they are analytic in $\lambda\in\mathbb{C}^-$ and continuous in $\lambda\in\overline{\mathbb{C}^-}$.*

(c) *The transmission coefficients $T(\zeta,t)$ and $\bar T(\zeta,t)$ are independent of t, and hence we have*

$$T(\zeta,t)=T(\zeta,0),\quad \bar T(\zeta,t)=\bar T(\zeta,0),\qquad t\in\mathbb{R}. \tag{2.16}$$

The quantity $T(\zeta,t)$ is continuous in $\zeta\in\mathbb{R}$ and has a meromorphic extension from $\zeta\in\mathbb{R}$ to the first and third quadrants in the complex ζ-plane. Moreover, $T(\zeta,t)$ is an even function of ζ, and hence it is a function of λ in $\overline{\mathbb{C}^+}$. The quantity $1/T(\zeta,t)$ is analytic in $\lambda\in\mathbb{C}^+$ and continuous in λ in $\overline{\mathbb{C}^+}$. Furthermore, $T(\zeta,t)$ is meromorphic in $\lambda\in\mathbb{C}^+$ with a finite number of poles there, where the poles are not necessarily simple but have finite multiplicities. The large ζ-asymptotics of $T(\zeta,t)$ expressed in λ is given by

$$T(\zeta,t)=e^{-i\mu/2}\left[1+O\left(\frac{1}{\lambda}\right)\right],\qquad \lambda\to\infty \text{ in } \overline{\mathbb{C}^+}. \tag{2.17}$$

Similarly, the quantity $\bar T(\zeta,t)$ is continuous in $\zeta\in\mathbb{R}$ and has a meromorphic extension from $\zeta\in\mathbb{R}$ to the second and fourth quadrants in the complex ζ-plane. Furthermore, $\bar T(\zeta,t)$ is an even function of ζ, and hence it is a function of λ in $\overline{\mathbb{C}^-}$. The quantity $1/\bar T(\zeta,t)$ is analytic in $\lambda\in\mathbb{C}^-$ and continuous in λ in $\overline{\mathbb{C}^-}$. Moreover, $\bar T(\zeta,t)$ is meromorphic in $\lambda\in\mathbb{C}^-$ with a finite number of poles, where the poles are not necessarily simple but have finite multiplicities. The large ζ-asymptotics of $\bar T(\zeta,t)$ expressed in λ is given by

$$\bar T(\zeta,t)=e^{i\mu/2}\left[1+O\left(\frac{1}{\lambda}\right)\right],\qquad \lambda\to\infty \text{ in } \overline{\mathbb{C}^-}. \tag{2.18}$$

(d) *For each fixed $t\in\mathbb{R}$, the small ζ-asymptotics of the six scattering coefficients $T(\zeta,t)$, $\bar T(\zeta,t)$, $R(\zeta,t)$, $\bar R(\zeta,t)$, $L(\zeta,t)$, and $\bar L(\zeta,t)$ are expressed in λ as*

$$T(\zeta,t)=1+O(\lambda),\qquad \lambda\to 0 \text{ in } \overline{\mathbb{C}^+}, \tag{2.19}$$

$$\bar T(\zeta,t)=1+O(\lambda),\qquad \lambda\to 0 \text{ in } \overline{\mathbb{C}^-}, \tag{2.20}$$

$$R(\zeta,t)=-\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, r(z,t)+O(\lambda)\right],\qquad \lambda\to 0 \text{ in } \mathbb{R}, \tag{2.21}$$

$$\bar R(\zeta,t)=\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, q(z,t)+O(\lambda)\right],\qquad \lambda\to 0 \text{ in } \mathbb{R}, \tag{2.22}$$

$$L(\zeta,t)=-\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, q(z,t)+O(\lambda)\right],\qquad \lambda\to 0 \text{ in } \mathbb{R},$$

$$\bar L(\zeta,t)=\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, r(z,t)+O(\lambda)\right],\qquad \lambda\to 0 \text{ in } \mathbb{R}.$$

(e) *The time evolutions of the reflection coefficients are given by*

$$R(\zeta,t)=R(\zeta,0)\,e^{4i\lambda^2 t},\quad \bar R(\zeta,t)=\bar R(\zeta,0)\,e^{-4i\lambda^2 t},\qquad \lambda\in\mathbb{R},\quad t\in\mathbb{R}, \tag{2.23}$$

$$L(\zeta,t)=L(\zeta,0)\,e^{-4i\lambda^2 t},\quad \bar L(\zeta,t)=\bar L(\zeta,0)\,e^{4i\lambda^2 t},\qquad \lambda\in\mathbb{R},\quad t\in\mathbb{R}, \tag{2.24}$$

with the understanding that the ζ-domain and the λ-domain are related to each other by the relationship expressed in (2.12). For each fixed $t\in\mathbb{R}$, each of the four reflection coefficients $R(\zeta,t)$, $\bar R(\zeta,t)$, $L(\zeta,t)$, and $\bar L(\zeta,t)$ is continuous when $\lambda\in\mathbb{R}$, is an odd function of ζ, and has the behavior $O(1/\zeta^{5/2})$ as $\lambda\to\pm\infty$. Furthermore, the four functions $R(\zeta,t)/\zeta$, $\bar R(\zeta,t)/\zeta$, $L(\zeta,t)/\zeta$, and $\bar L(\zeta,t)/\zeta$ are even in ζ; are continuous functions of $\lambda\in\mathbb{R}$; and behave as $O(1/\lambda^3)$ as $\lambda\to\pm\infty$.

(f) *For each fixed $t\in\mathbb{R}$, the scattering coefficients satisfy*

$$T(\zeta,t)\,\bar T(\zeta,t)+R(\zeta,t)\,\bar R(\zeta,t)=1,\quad T(\zeta,t)\,\bar T(\zeta,t)+L(\zeta,t)\,\bar L(\zeta,t)=1,\qquad \lambda\in\mathbb{R}. \tag{2.25}$$

(g) *For each fixed $t\in\mathbb{R}$, the left reflection coefficients are determined when the right reflection coefficients and the transmission coefficients are known, and we have*

$$L(\zeta,t)=-\frac{\bar R(\zeta,t)\,T(\zeta,t)}{\bar T(\zeta,t)},\quad \bar L(\zeta,t)=-\frac{R(\zeta,t)\,\bar T(\zeta,t)}{T(\zeta,t)},\qquad \lambda\in\mathbb{R}. \tag{2.26}$$

Conversely, as (2.26) indicates, the right reflection coefficients are determined when the left reflection coefficients and the transmission coefficients are known.

(h) *The bound states for (2.1) correspond to solutions that are square integrable in x. Such solutions cannot occur when λ is real. In particular, there is no bound state at $\zeta = 0$ or equivalently at $\lambda = 0$. A bound state can only occur at a complex value of ζ at which the transmission coefficient $T(\zeta, t)$ has a pole in the interiors of the first or third quadrants in the complex ζ-plane or at which the transmission coefficient $\bar{T}(\zeta, t)$ has a pole in the interiors of the second or the fourth quadrants. Since the parameter ζ appears as ζ^2 in the transmission coefficients $T(\zeta, t)$ and $\bar{T}(\zeta, t)$, the ζ-values corresponding to the bound states must be symmetrically located with respect to the origin in the complex ζ-plane.*

(i) *The number of poles of $T(\zeta, t)$ in the upper-half complex λ-plane is finite, and we use $\{\lambda_j\}_{j=1}^{N}$ to denote the set of those distinct poles, where we use N to denote their number without counting the multiplicities. Similarly, the number of poles of $\bar{T}(\zeta, t)$ in the lower-half complex λ-plane is finite, and we use $\{\bar{\lambda}_j\}_{j=1}^{\bar{N}}$ to denote the set of those distinct poles, where we use $\bar{N}$ to denote their number without counting the multiplicities. The multiplicity of each of those poles is finite, and we use m_j to denote the multiplicity of the pole at $\lambda = \lambda_j$ and use $\bar{m}_j$ to denote the multiplicity of the pole at $\lambda = \bar{\lambda}_j$. As a consequence of (2.16), it follows that each of the quantities λ_j, m_j, $\bar{\lambda}_j$, and $\bar{m}_j$ is independent of t, and hence their values at any time t coincide with the corresponding values at $t = 0$.*

(j) *For each bound state and multiplicity, there corresponds a norming constant. We use the double-indexed constants c_{jk} for $0 \le k \le m_j - 1$ to denote the norming constants at the bound state $\lambda = \lambda_j$ at time $t = 0$, and we use the double-indexed constants $\bar{c}_{jk}$ for $0 \le k \le \bar{m}_j - 1$ to denote the norming constants at the bound state $\lambda = \bar{\lambda}_j$ at time $t = 0$. Thus, the bound-state information at $t = 0$ for (2.1) consists of the two sets given by*

$$\left\{\lambda_j, m_j, \{c_{jk}\}_{k=0}^{m_j-1}\right\}_{j=1}^{N}, \quad \left\{\bar{\lambda}_j, \bar{m}_j, \{\bar{c}_{jk}\}_{k=0}^{\bar{m}_j-1}\right\}_{j=1}^{\bar{N}}. \tag{2.27}$$

(k) *The bound-state information at $t = 0$ specified in (2.27) can be organized by using a pair of matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ by letting*

$$A := \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & A_N \end{bmatrix}, \quad \bar{A} := \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{A}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{A}_{\bar{N}-1} & 0 \\ 0 & 0 & \cdots & 0 & \bar{A}_{\bar{N}} \end{bmatrix}, \tag{2.28}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \vdots \\ \bar{B}_{\bar{N}} \end{bmatrix}, \tag{2.29}$$

$$C := \begin{bmatrix} C_1 & C_2 & \cdots & C_N \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \cdots & \bar{C}_{\bar{N}} \end{bmatrix}, \tag{2.30}$$

where we have defined

$$A_j := \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_j \end{bmatrix}, \qquad 1 \le j \le N, \tag{2.31}$$

$$B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j := \begin{bmatrix} c_{j(m_j-1)} & c_{j(m_j-2)} & \cdots & c_{j1} & c_{j0} \end{bmatrix}, \qquad 1 \le j \le N, \tag{2.32}$$

with A_j being the $m_j \times m_j$ square matrix in the Jordan canonical form with λ_j appearing in the diagonal entries, B_j being the column vector with m_j components that are all zero except for the last entry that is 1, and C_j being the row vector with m_j components containing all the norming constants in the indicated order, and

$$\bar{A}_j := \begin{bmatrix} \bar{\lambda}_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{\lambda}_j & 1 & \cdots & 0 & 0 \\ 0 & 0 & \bar{\lambda}_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{\lambda}_j & 1 \\ 0 & 0 & 0 & \dots & 0 & \bar{\lambda}_j \end{bmatrix}, \qquad 1 \le j \le \bar{N}, \tag{2.33}$$

$$\bar{B}_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \bar{C}_j := \begin{bmatrix} \bar{c}_{j(\bar{m}_j-1)} & \bar{c}_{j(\bar{m}_j-2)} & \cdots & \bar{c}_{j1} & \bar{c}_{j0} \end{bmatrix}, \qquad 1 \le j \le \bar{N}, \tag{2.34}$$

with $\bar{A}_j$ being the $\bar{m}_j \times \bar{m}_j$ square matrix in the Jordan canonical form with $\bar{\lambda}_j$ appearing in the diagonal entries, $\bar{B}_j$ being the column vector with $\bar{m}_j$ components that are all zero except for the last entry that is 1, and $\bar{C}_j$ being the row vector with $\bar{m}_j$ components containing all the norming constants in the indicated order.

(l) *The four matrices A, $\bar{A}$, B, and $\bar{B}$ and each of the four matrices A_j, $\bar{A}_j$, B_j, and $\bar{B}_j$ are all independent of t, and hence their values at any time t coincide with the corresponding values at $t = 0$. On the other hand, the norming constants evolve in time as described by*

$$C \mapsto Ce^{4iA^2 t}, \quad \bar{C} \mapsto \bar{C}e^{-4i\bar{A}^2 t}, \tag{2.35}$$

$$C_j \mapsto C_j e^{4i(A_j)^2 t}, \quad \bar{C}_j \mapsto \bar{C}_j e^{-4i(\bar{A}_j)^2 t}. \tag{2.36}$$

Proof. The proofs of all parts of the theorem can be found in the doctoral thesis[21] of the second author. As an alternative proof, we remark that the results presented in the theorem for each fixed $t \in \mathbb{R}$ have similar proofs when $t = 0$. Hence, for the proofs of (a) and (b), we refer the reader to Theorem 2.2 of Ref. 11; for the proofs of (c)–(e), we refer the reader to Theorem 2.5 of Ref. 11; for the proofs of (h)–(k), we refer the reader to Sec. 3 of Ref. 11; and for the proofs of (2.16), (2.23), and (2.24) and (e), (f), (j), and (l), we refer the reader to Ref. 21. □

The scattering dataset $\mathbf{S}(\zeta, 0)$ at $t = 0$ consists of the six scattering coefficients and the relevant bound-state information. As seen from Theorem 2.1(g), we can suppress the left reflection coefficients from the specification of $\mathbf{S}(\zeta, 0)$. Furthermore, as seen from (j) and (k) of Theorem 2.1, the knowledge of the bound-state information contained in the two sets in (2.27) is equivalent to the knowledge of the pair of matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$. Hence, we can define the scattering dataset $\mathbf{S}(\zeta, 0)$ at $t = 0$ for the unperturbed linear system (2.1) as

$$\mathbf{S}(\zeta, 0) := \{T(\zeta, 0), \bar{T}(\zeta, 0), R(\zeta, 0), \bar{R}(\zeta, 0), (A, B, C), (\bar{A}, \bar{B}, \bar{C})\}. \tag{2.37}$$

Then, the time-evolved scattering dataset $\mathbf{S}(\zeta, t)$ is obtained from $\mathbf{S}(\zeta, 0)$ by using (2.16), (2.23), and (2.35) with the understanding that the matrices A, B, $\bar{A}$, and $\bar{B}$ are unchanged in time. Thus, $\mathbf{S}(\zeta, t)$ for (2.1) can equivalently be described as

$$\mathbf{S}(\zeta, t) = \{T(\zeta, t), \bar{T}(\zeta, t), R(\zeta, t), \bar{R}(\zeta, t), (A, B, C\, e^{4iA^2 t}), (\bar{A}, \bar{B}, \bar{C}\, e^{-4i\bar{A}^2 t})\}, \tag{2.38}$$

where we recall that λ in (2.23) is related to ζ as in (2.12).

Let us remark on the simplicity and elegance of the use of the matrix triplet pair (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ rather than the two sets in (2.27) when the bound states have multiplicities. As seen from (2.36), the time evolutions of the norming constants c_{jk} and $\bar{c}_{jk}$ appearing in (2.27) are too complicated to express individually although the time evolutions of the vectors C_j and $\bar{C}_j$ are very simple when expressed in terms of matrix exponentials. Similarly, as seen from (2.35), the time evolutions of the vectors C and $\bar{C}$ are also very simple when expressed in terms of matrix exponentials. Thus, the use of the matrix triplet pair allows us to describe in a simple and elegant manner the bound states and the time evolutions of the bound-state norming constants no matter how many bound states we have and no matter what their multiplicities are.

III. THE MARCHENKO METHOD FOR THE UNPERTURBED SYSTEM

In Ref. 11, we have developed the Marchenko method for the linear system (2.1) when the potentials $q(x, t)$ and $r(x, t)$ are independent of the parameter t. In this section, we present the extension of the Marchenko method developed in Ref. 11 from the time-independent case to the time-evolved case, and this is done by providing the appropriate time evolution of the scattering data for (2.1). In the Marchenko method for (2.1), the potentials $q(x,t)$ and $r(x,t)$ are recovered from the scattering dataset consisting of the time-evolved scattering coefficients and the time-evolved bound-state information. The input is used to construct the kernel in the Marchenko system of linear integral equations as well as the nonhomogeneous term in the Marchenko system. The potentials and all other relevant quantities associated with (2.1) are then recovered from the solution to the Marchenko system.

In the next theorem, the Marchenko system of linear integral equations for (2.1) is presented, and the resulting Marchenko system is shown to be equivalent to an uncoupled system of Marchenko integral equations.

Theorem 3.1. *Assume that the potentials $q(x,t)$ and $r(x,t)$ in (2.1) belong to the Schwartz class for each fixed $t \in \mathbb{R}$. Let $R(\zeta,t)$ and $\bar{R}(\zeta,t)$ be the corresponding time-evolved reflection coefficients appearing in (2.23). Let (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ be the pair of matrix triplets representing the bound-state information for (2.1), where the matrices are described in Sec. II and contained in the initial scattering dataset in (2.37). We have the following:*

(a) *The Marchenko system of linear integral equations for (2.1) is given by*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y,t) & K_1(x,y,t) \\ \bar{K}_2(x,y,t) & K_2(x,y,t) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y,t) \\ \Omega(x+y,t) & 0 \end{bmatrix} + \int_x^\infty dz \begin{bmatrix} -iK_1(x,z,t)\,\Omega'(z+y,t) & \bar{K}_1(x,z,t)\,\bar{\Omega}(z+y,t) \\ K_2(x,z,t)\,\Omega(z+y,t) & i\bar{K}_2(x,z,t)\,\bar{\Omega}'(z+y,t) \end{bmatrix}, \qquad x<y, \tag{3.1}$$

where $\Omega(y,t)$ and $\bar{\Omega}(y,t)$ are the quantities defined as

$$\Omega(y,t) := \hat{R}(y,t) + C\,e^{4iA^2 t}\,e^{iAy}\,B, \quad \bar{\Omega}(y,t) := \hat{\bar{R}}(y,t) + \bar{C}\,e^{-4i\bar{A}^2 t}\,e^{-i\bar{A}\,y}\,\bar{B}, \tag{3.2}$$

with the prime in $\Omega'(y,t)$ and $\bar{\Omega}'(y,t)$ denoting the y-derivatives, and where we have

$$\hat{R}(y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \frac{R(\zeta,t)}{\zeta}\, e^{i\lambda y}, \quad \hat{\bar{R}}(y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \frac{\bar{R}(\zeta,t)}{\zeta}\, e^{-i\lambda y}, \tag{3.3}$$

$$K_1(x,y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left[\frac{e^{i\mu/2}\,\psi_1(\zeta,x,t)}{\zeta\,E(x,t)}\right] e^{-i\lambda y}, \tag{3.4}$$

$$K_2(x,y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left[e^{-i\mu/2}\,E(x,t)\,\psi_2(\zeta,x,t) - e^{i\lambda x}\right] e^{-i\lambda y}, \tag{3.5}$$

$$\bar{K}_1(x,y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left[\frac{e^{i\mu/2}\,\bar{\psi}_1(\zeta,x,t)}{E(x,t)} - e^{-i\lambda x}\right] e^{i\lambda y}, \tag{3.6}$$

$$\bar{K}_2(x,y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left[\frac{e^{-i\mu/2}\,E(x,t)\,\bar{\psi}_2(\zeta,x,t)}{\zeta}\right] e^{i\lambda y}, \tag{3.7}$$

with λ being related to ζ as in (2.12), $E(x,t)$ and μ being the quantities defined in (1.19) and (2.14), respectively, and $\psi_1(\zeta,x,t)$, $\psi_2(\zeta,x,t)$, $\bar{\psi}_1(\zeta,x,t)$, and $\bar{\psi}_2(\zeta,x,t)$ being the components of the Jost solutions given in (2.10).

(b) *The coupled Marchenko system (3.1) is equivalent to the uncoupled system of equations*

$$\begin{cases} K_1(x,y,t) + \bar{\Omega}(x+y,t) + i\displaystyle\int_x^\infty dz \int_x^\infty ds\, K_1(x,z,t)\,\Omega'(z+s,t)\,\bar{\Omega}(s+y,t) = 0, \\ \bar{K}_2(x,y,t) + \Omega(x+y,t) - i\displaystyle\int_x^\infty dz \int_x^\infty ds\, \bar{K}_2(x,z,t)\,\bar{\Omega}'(z+s,t)\,\Omega(s+y,t) = 0, \end{cases} \tag{3.8}$$

where $x<y$, with the auxiliary equations given by

$$\begin{cases} \bar{K}_1(x,y,t) = i\displaystyle\int_x^\infty dz\, K_1(x,z,t)\,\Omega'(z+y,t), & x<y, \\ K_2(x,y,t) = -i\displaystyle\int_x^\infty dz\, \bar{K}_2(x,z,t)\,\bar{\Omega}'(z+y,t), & x<y. \end{cases} \tag{3.9}$$

Proof. The time evolutions $\Omega(y,0) \mapsto \Omega(y,t)$ and $\bar{\Omega}(y,0) \mapsto \bar{\Omega}(y,t)$ directly follow from (2.23) and (2.35). Since the results are stated for each fixed $t \in \mathbb{R}$, the result in (a) then follows from Theorem 4.2 of Ref. 11 and the result in (b) follows from (4.41) and (4.42) of Ref. 11. □

In the next theorem, we describe the recovery of the relevant quantities for (2.1) from the solution to the Marchenko system (3.1), which uses the time-evolved scattering dataset $\mathbf{S}(\zeta,t)$ of (2.38) as input. Those relevant quantities consist of the key quantity $E(x,t)$ in (1.19), the constant μ in (2.14), the two potentials $q(x,t)$ and $r(x,t)$ in (2.1), and the four Jost solutions to (2.1).

Theorem 3.2. *Let the potentials $q(x,t)$ and $r(x,t)$ in (2.1) belong to the Schwartz class for each fixed $t \in \mathbb{R}$. The relevant quantities are recovered from the solution to the Marchenko system (3.1) or equivalently from the uncoupled counterpart given in (3.8) and (3.9) as follows:*

(a) *The scalar quantity $E(x,t)$ defined in (1.19) is recovered from the solution to the Marchenko system by using*

$$E(x,t) = \exp\left(2\int_{-\infty}^{x} dz\, Q(z,t)\right), \tag{3.10}$$

where $Q(x,t)$ is the auxiliary scalar quantity constructed from $\bar{K}_1(x,x,t)$ and $K_2(x,x,t)$ as

$$Q(x,t) := \bar{K}_1(x,x,t) - K_2(x,x,t). \tag{3.11}$$

(b) *The complex-valued scalar constant μ defined in (2.14) is obtained from the solution to the Marchenko system as*

$$\mu = -4i\int_{-\infty}^{\infty} dz\, Q(z,t). \tag{3.12}$$

Since the value of μ is independent of t, (3.12) is equivalent to

$$\mu = -4i\int_{-\infty}^{\infty} dz\, Q(z,0). \tag{3.13}$$

(c) *The potentials $q(x,t)$ and $r(x,t)$ are recovered from the solution to the Marchenko system as*

$$q(x,t) = -2K_1(x,x,t)\exp\left(-4\int_{x}^{\infty} dz\, Q(z,t)\right), \tag{3.14}$$

$$r(x,t) = -2\bar{K}_2(x,x,t)\exp\left(4\int_{x}^{\infty} dz\, Q(z,t)\right). \tag{3.15}$$

(d) *The Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ to (2.1) are recovered from the solution to the Marchenko system as*

$$\psi_1(\zeta,x,t) = \zeta\left(\int_{x}^{\infty} dy\, K_1(x,y,t)\, e^{i\zeta^2 y}\right)\exp\left(-2\int_{x}^{\infty} dz\, Q(z,t)\right), \tag{3.16}$$

$$\psi_2(\zeta,x,t) = \left(e^{i\zeta^2 x} + \int_{x}^{\infty} dy\, K_2(x,y,t)\, e^{i\zeta^2 y}\right)\exp\left(2\int_{x}^{\infty} dz\, Q(z,t)\right), \tag{3.17}$$

$$\bar{\psi}_1(\zeta,x,t) = \left(e^{-i\zeta^2 x} + \int_{x}^{\infty} dy\, \bar{K}_1(x,y,t)\, e^{-i\zeta^2 y}\right)\exp\left(-2\int_{x}^{\infty} dz\, Q(z,t)\right), \tag{3.18}$$

$$\bar{\psi}_2(\zeta,x,t) = \zeta\left(\int_{x}^{\infty} dy\, \bar{K}_2(x,y,t)\, e^{-i\zeta^2 y}\right)\exp\left(2\int_{x}^{\infty} dz\, Q(z,t)\right), \tag{3.19}$$

where $\psi_1(\zeta,x,t)$, $\psi_2(\zeta,x,t)$, $\bar{\psi}_1(\zeta,x,t)$, and $\bar{\psi}_2(\zeta,x,t)$ are the components of the Jost solutions appearing in (2.10).

(e) *The Jost solutions $\phi(\zeta,x,t)$ and $\bar{\phi}(\zeta,x,t)$ to (2.1) are recovered with the help of*

$$\phi(\zeta,x,t) = \frac{1}{T(\zeta,t)}\,\bar{\psi}(\zeta,x,t) + \frac{R(\zeta,t)}{T(\zeta,t)}\,\psi(\zeta,x,t), \tag{3.20}$$

$$\bar{\phi}(\zeta,x,t) = \frac{\bar{R}(\zeta,t)}{\bar{T}(\zeta,t)}\,\bar{\psi}(\zeta,x,t) + \frac{1}{\bar{T}(\zeta,t)}\,\psi(\zeta,x,t), \tag{3.21}$$

where on the right-hand sides we use $T(\zeta,t)$, $\bar{T}(\zeta,t)$, $R(\zeta,t)$, and $\bar{R}(\zeta,t)$ from the scattering dataset described in (2.38) and use the Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ with components expressed in (3.16)–(3.19), respectively.

Proof. Since the results pertaining to the Marchenko system are obtained for each fixed $t \in \mathbb{R}$, those results in (a)–(d) directly follow from Theorem 4.4 of Ref. 11, where the Marchenko theory has been developed in the time-independent case. In particular, from (4.51) and (4.53) of Ref. 11, we get (3.10) and (3.12), respectively. It is already known from (4.52) of Ref. 11 that the right-hand side of (3.11) yields

$$\bar{K}_1(x,x,t) - K_2(x,x,t) = \frac{i}{4}\, q(x,t)\, r(x,t), \tag{3.22}$$

and hence (3.10) and (3.12) are compatible with (1.19) and (2.14), respectively. From the time-independent formulas (4.43) and (4.46) of Ref. 11, we have

$$q(x,t) = -2\, K_1(x,x,t)\, e^{-i\mu}\, E(x,t)^2, \quad r(x,t) = -2\, \bar{K}_2(x,x,t)\, e^{i\mu}\, E(x,t)^{-2}. \tag{3.23}$$

In addition, from (3.10) and (3.12), we get

$$e^{-i\mu}\, E(x,t)^2 = \exp\left(-4\int_x^\infty dz\, Q(z,t)\right), \quad e^{i\mu}\, E(x,t)^{-2} = \exp\left(4\int_x^\infty dz\, Q(z,t)\right). \tag{3.24}$$

Consequently, using (3.23) and (3.24), we obtain (3.14) and (3.15). The derivation of the equalities in (3.16)–(3.19) is established in a similar manner. The equalities in (3.20) and (3.21) follow from the fact that the two Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ form a fundamental set for (2.1) and the other two Jost solutions $\phi(\zeta,x,t)$ and $\bar{\phi}(\zeta,x,t)$ can be obtained as linear combinations of the fundamental set of solutions with the help of (2.8) and (2.9). □

From (3.14) and (3.15), it follows that

$$q(x,t)\, r(x,t) = 4\, K_1(x,x,t)\, \bar{K}_2(x,x,t). \tag{3.25}$$

Let us define the auxiliary quantity $P(x,t)$ as

$$P(x,t) := K_1(x,x,t)\, \bar{K}_2(x,x,t). \tag{3.26}$$

From (3.11), (3.22), (3.25), and (3.26), we see that the quantity $Q(x,t)$ defined in (3.11) is related to $P(x,t)$ as

$$Q(x,t) = i\, P(x,t). \tag{3.27}$$

In the next corollary, we state the results of Theorem 3.2 in terms of $P(x,t)$. Let us emphasize that the quantity $Q(x,t)$ is constructed from $\bar{K}_1(x,x,t)$ and $K_2(x,x,t)$, whereas the quantity $P(x,t)$ is constructed from $K_1(x,x,t)$ and $\bar{K}_2(x,x,t)$. The corollary provides an alternate way to recover the relevant quantities for (2.1) from the solution to the corresponding Marchenko system (3.1).

Corollary 3.3. Let the potentials $q(x,t)$ and $r(x,t)$ in (2.1) belong to the Schwartz class for each fixed $t \in \mathbb{R}$. The relevant quantities for (2.1) are recovered from the solution to the Marchenko system (3.1), or equivalently from the uncoupled counterpart given in (3.8) and (3.9), as follows:

(a) *The scalar quantity $E(x,t)$ defined in (1.19) is recovered from the solution to the Marchenko system by using*

$$E(x,t) = \exp\left(2i\int_{-\infty}^x dz\, P(z,t)\right), \tag{3.28}$$

where $P(x,t)$ is the auxiliary scalar quantity constructed from $K_1(x,x,t)$ and $\bar{K}_2(x,x,t)$ as in (3.26).

(b) *The complex-valued scalar constant μ defined in (2.14) is obtained from the solution to the Marchenko system as*

$$\mu = 4\int_{-\infty}^\infty dz\, P(z,t).$$

(c) *The potentials $q(x,t)$ and $r(x,t)$ are recovered from the solution to the Marchenko system as*

$$q(x,t) = -2K_1(x,x,t)\exp\left(-4i\int_x^\infty dz\, P(z,t)\right), \tag{3.29}$$

$$r(x,t) = -2\bar{K}_2(x,x,t)\exp\left(4i\int_x^\infty dz\, P(z,t)\right). \tag{3.30}$$

(d) *The Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ to (2.1) are recovered from the solution to the Marchenko system as*

$$\psi_1(\zeta,x,t) = \zeta\left(\int_x^\infty dy\, K_1(x,y,t)\, e^{i\zeta^2 y}\right)\exp\left(-2i\int_x^\infty dz\, P(z,t)\right), \tag{3.31}$$

$$\psi_2(\zeta,x,t) = \left(e^{i\zeta^2 x} + \int_x^\infty dy\, K_2(x,y,t)\, e^{i\zeta^2 y}\right)\exp\left(2i\int_x^\infty dz\, P(z,t)\right), \tag{3.32}$$

$$\bar{\psi}_1(\zeta,x,t) = \left(e^{-i\zeta^2 x} + \int_x^\infty dy\, \bar{K}_1(x,y,t)\, e^{-i\zeta^2 y}\right) \exp\left(-2i\int_x^\infty dz\, P(z,t)\right), \tag{3.33}$$

$$\bar{\psi}_2(\zeta,x,t) = \zeta\left(\int_x^\infty dy\, \bar{K}_2(x,y,t)\, e^{-i\zeta^2 y}\right) \exp\left(2i\int_x^\infty dz\, P(z,t)\right), \tag{3.34}$$

where $\psi_1(\zeta,x,t)$, $\psi_2(\zeta,x,t)$, $\bar{\psi}_1(\zeta,x,t)$, and $\bar{\psi}_2(\zeta,x,t)$ are the components of the Jost solutions appearing in (2.10).

(e) *The Jost solutions $\phi(\zeta,x,t)$ and $\bar{\phi}(\zeta,x,t)$ to (2.1) are recovered with the help of (3.20) and (3.21) by using $T(\zeta,t)$, $\bar{T}(\zeta,t)$, $R(\zeta,t)$, and $\bar{R}(\zeta,t)$ from the scattering dataset described in (2.38) and also using the Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ with components expressed as in (3.31)–(3.34), respectively.*

We have the following remarks for Corollary 3.3. At first sight, it might seem as if the results in Corollary 3.3 are trivial because they are obtained from Theorem 3.2 simply by replacing $Q(x,t)$ by $iP(x,t)$. However, Corollary 3.3 presents an alternate way to recover the relevant quantities from the solution to the uncoupled Marchenko system (3.8). In fact, when we solve (3.8), we obtain $K_1(x,y,t)$ and $\bar{K}_2(x,y,t)$, from which we can construct $P(x,t)$ by using (3.26) without having to solve the auxiliary system (3.9). As seen from (3.29) and (3.30), we can then also construct the quantities $q(x,t)$ and $r(x,t)$ without having to solve the auxiliary system (3.9). The recovery of the quantities $E(x,t)$, μ, $q(x,t)$, and $r(x,t)$ at times may be more efficient by using the procedure of Corollary 3.3 rather than the procedure of Theorem 3.2 because the latter requires the solution to (3.9). On the other hand, even though we have the equality in (3.27), the evaluation of the integral of $Q(x,t)$ may be easier than the evaluation of the integral of $P(x,t)$ because the former integrand is expressed as a difference and the latter integrand as a product, as seen from (3.11) and (3.26), respectively.

An advantage of using Corollary 3.3 instead of Theorem 3.2 becomes apparent in Sec. VIII when the two potentials $q(x,t)$ and $r(x,t)$ are related to each other via complex conjugation as in (8.1). In that case, the potential $q(x,t)$ can be recovered directly from the solution to the scalar Marchenko equation (8.13). As we see from the recovery formula given in (8.14), no other quantities are needed in the recovery besides the solution to the scalar Marchenko equation (8.13).

IV. EXPLICIT SOLUTION FORMULAS FOR THE UNPERTURBED SYSTEM

When the potentials $q(x,t)$ and $r(x,t)$ at $t=0$ in the unperturbed linear system (2.1) are reflectionless, i.e., when the reflection coefficients at $t=0$ are all zero, we see from (2.23) and (2.24) that the time-evolved reflection coefficients remain zero for all t. In that case, the quantities $\Omega(y,t)$ and $\bar{\Omega}(y,t)$ used as input to the Marchenko system (3.1) yield separable integral kernels. This results in explicit solutions to the Marchenko system (3.1) and also in explicit solutions to the unperturbed linear system (2.1), where all those solutions are expressed in terms of the matrix triplet pair (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ appearing in (3.2). In this section, we provide the corresponding explicit solution formulas for the unperturbed linear system (2.1). Certainly, having the solution formulas for the potentials $q(x,t)$ and $r(x,t)$ for the linear system (2.1), we also have the solution formulas for the corresponding nonlinear system (1.2).

When the Marchenko kernels $\Omega(y,t)$ and $\bar{\Omega}(y,t)$ correspond to a reflectionless scattering dataset, from (3.2) we get

$$\Omega(y,t) = C\, e^{iAy+4iA^2 t} B, \quad \bar{\Omega}(y,t) = \bar{C}\, e^{-i\bar{A}y-4i\bar{A}^2 t} \bar{B}, \tag{4.1}$$

$$\Omega'(y,t) = i\, C\, A\, e^{iAy+4iA^2 t} B, \quad \bar{\Omega}'(y,t) = -i\, \bar{C}\bar{A}\, e^{-i\bar{A}y-4i\bar{A}^2 t} \bar{B}, \tag{4.2}$$

which are all explicitly expressed in terms of the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ appearing in (2.38). In the next theorem, we obtain the solution to the Marchenko system (3.1) with the input given in (4.1).

Theorem 4.1. *When the time-evolved reflectionless quantities $\Omega(y,t)$ and $\bar{\Omega}(y,t)$ appearing in (4.1) are used as input to the Marchenko system (3.1), the resulting system of integral equations is solvable in a closed form and has the solution explicitly expressed in terms of the matrix triplet pair (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ as*

$$K_1(x,y,t) = -\bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}y-4i\bar{A}^2 t}\, \bar{B}, \tag{4.3}$$

$$K_2(x,y,t) = C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAx+4iA^2 t} M\bar{A} e^{-i\bar{A}(x+y)-4i\bar{A}^2 t}\, \bar{B}, \tag{4.4}$$

$$\bar{K}_1(x,y,t) = \bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}x-4i\bar{A}^2 t} \bar{M}A\, e^{iA(x+y)+4iA^2 t}\, B, \tag{4.5}$$

$$\bar{K}_2(x,y,t) = -C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAy+4iA^2 t}\, B, \tag{4.6}$$

where $\Gamma(x,t)$ and $\bar{\Gamma}(x,t)$ are the matrix-valued functions of x and t defined as

$$\Gamma(x,t) := I - e^{iAx+4iA^2 t} M\bar{A} e^{-2i\bar{A}x-4i\bar{A}^2 t} \bar{M} e^{iAx}, \tag{4.7}$$

$$\bar{\Gamma}(x,t) := I - e^{-i\bar{A}x-4i\bar{A}^2 t} \bar{M}A e^{2iAx+4iA^2 t} M e^{-i\bar{A}x}, \tag{4.8}$$

with M and $\bar{M}$ being the matrix-valued constants defined as

$$M := \int_0^\infty dz\, e^{iAz}\, B\, \bar{C}\, e^{-i\bar{A}z}, \quad \bar{M} := \int_0^\infty dz\, e^{-i\bar{A}z}\, \bar{B}\, C\, e^{iAz}. \tag{4.9}$$

The constant matrices M and $\bar{M}$ can alternatively be obtained as the unique solutions to the respective linear systems

$$AM - M\bar{A} = iB\bar{C}, \quad \bar{M}A - \bar{A}\bar{M} = i\bar{B}C. \tag{4.10}$$

Proof. From Theorem 3.1, we know that the Marchenko system (3.1) is equivalent to the combination of the uncoupled system (3.8) and the auxiliary system (3.9). To obtain (4.3), we proceed as follows. Using the second equality of (4.1) and the first equality of (4.2) as input to the first line of (3.8), we get

$$K_1(x,y,t) + \bar{C}\, e^{-i\bar{A}(x+y)-4i\bar{A}^2 t}\, \bar{B} + i\int_x^\infty dz \int_x^\infty ds\, K_1(x,z,t)\Big(i\, C\, A\, e^{iA(z+s)+4iA^2 t}\, B\Big)\bar{C}\, e^{-i\bar{A}(s+y)-4i\bar{A}^2 t}\, \bar{B} = 0. \tag{4.11}$$

Note that A, e^{iAy}, and e^{4iA^2t} commute with each other, and also $\bar{A}$, $e^{i\bar{A}y}$, and $e^{4i\bar{A}^2t}$ commute with each other. From (4.11), we see that $K_1(x,y,t)$ has the form

$$K_1(x,y,t) = H_1(x,t)\, e^{-i\bar{A}y-4i\bar{A}^2 t}\, \bar{B}, \tag{4.12}$$

where $H_1(x,t)$ satisfies

$$H_1(x,t)\Big(I - \int_x^\infty dz \int_x^\infty ds\, e^{-i\bar{A}z-4i\bar{A}^2 t}\, \bar{B}\, C\, e^{iAz}\, A\, e^{iAs+4iA^2 t}\, B\bar{C}\, e^{-i\bar{A}s}\Big) = -\bar{C}\, e^{-i\bar{A}x}. \tag{4.13}$$

Using (4.9) on the left-hand side of (4.13), we obtain

$$H_1(x,t)\Big(I - e^{-i\bar{A}x-4i\bar{A}^2 t}\, \bar{M}\, e^{iAx}\, A\, e^{iAx+4iA^2 t}\, M\, e^{-i\bar{A}x}\Big) = -\bar{C}\, e^{-i\bar{A}x},$$

or equivalently,

$$H_1(x,t)\, \bar{\Gamma}(x,t) = -\bar{C}\, e^{-i\bar{A}x}, \tag{4.14}$$

where $\bar{\Gamma}(x,t)$ is the matrix defined in (4.8). From (4.14), we have

$$H_1(x,t) = -\bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1}, \tag{4.15}$$

and using (4.15) in (4.12), we get (4.3). The solution formula for $\bar{K}_2(x,y,t)$ appearing in (4.6) is obtained in a similar manner from the second line of (3.8). Then, using the first equality of (4.1) and the second equality of (4.2) in the second line of (3.8), we have

$$\bar{K}_2(x,y,t) + C\, e^{iA(x+y)+4iA^2 t}\, B - i\int_x^\infty dz \int_x^\infty ds\, \bar{K}_2(x,z,t)\Big(-i\, \bar{C}\, \bar{A}\, e^{-i\bar{A}(z+s)-4i\bar{A}^2 t}\, \bar{B}\Big)C\, e^{iA(s+y)+4iA^2 t}\, B = 0.$$

From (4.16), we see that $\bar{K}_2(x,y,t)$ has the form

$$\bar{K}_2(x,y,t) = H_2(x,t)\, e^{iAy+4iA^2 t}\, B, \tag{4.16}$$

where $H_2(x,t)$ satisfies

$$H_2(x,t)\Big(I - \int_x^\infty dz \int_x^\infty ds\, e^{iAz+4iA^2 t}\, B\, \bar{C}\, e^{-i\bar{A}z}\, \bar{A}\, e^{-i\bar{A}s-4i\bar{A}^2 t}\, \bar{B}\, C\, e^{iAs}\Big) = -C\, e^{iAx}. \tag{4.17}$$

Using again (4.9) on the left-hand side of (4.17), we write (4.17) as

$$H_2(x,t)\Big(I - e^{iAx+4iA^2 t}\, M\, e^{-i\bar{A}x}\, \bar{A}\, e^{-i\bar{A}x-4i\bar{A}^2 t}\, \bar{M}\, e^{iAx}\Big) = -C\, e^{iAx},$$

which can be written as

$$H_2(x,t)\, \Gamma(x,t) = -C\, e^{iAx}, \tag{4.18}$$

where we remark that $\Gamma(x,t)$ is the matrix defined in (4.7). From (4.18), we get

$$H_2(x,t) = -C\, e^{iAx}\, \Gamma(x,t)^{-1}, \tag{4.19}$$

and using (4.19) in (4.16), we obtain (4.6). Next, we present the proof of (4.4). Using (4.6) and the second equality of (4.2) in the second line of (3.9), we have

$$K_2(x,y,t) = \int_x^\infty dz\, C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAz+4iA^2t}\, B\, \bar{C}\, e^{-i\bar{A}z}\, \bar{A}\, e^{-i\bar{A}y-4i\bar{A}^2t}\, \bar{B}. \tag{4.20}$$

The use of the first equality of (4.9) in (4.20) results in

$$K_2(x,y,t) = C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAx+4iA^2t}\, M\, e^{-i\bar{A}x}\, \bar{A}\, e^{-i\bar{A}y-4i\bar{A}^2t}\, \bar{B},$$

which establishes (4.4). To obtain (4.5), we use (4.3) and the first equality of (4.2) in the first line of (3.9). This yields

$$\bar{K}_1(x,y,t) = \int_x^\infty dz\, \bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}z-4i\bar{A}^2t}\, \bar{B}\, C\, e^{iAz}\, A\, e^{iAy+4iA^2t}\, B. \tag{4.21}$$

The use of the second equality of (4.9) in (4.21) gives us

$$\bar{K}_1(x,y,t) = \bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}x-4i\bar{A}^2t}\, \bar{M}\, e^{iAx}\, A\, e^{iAy+4iA^2t}\, B,$$

which yields (4.5). Let us finally prove that the constant matrices M and $\bar{M}$ in (4.9) can also be obtained as the unique solutions to (4.10). From (4.9), we observe that

$$iA\,M - M\,i\bar{A} = \int_0^\infty dz\, \frac{d}{dz}\Big(e^{iAz}\, B\, \bar{C}\, e^{-i\bar{A}z}\Big), \tag{4.22}$$

$$-\,i\bar{A}\,\bar{M} + \bar{M}\,iA = \int_0^\infty dz\, \frac{d}{dz}\Big(e^{-i\bar{A}z}\, \bar{B}\, C\, e^{iAz}\Big). \tag{4.23}$$

From (2.30), (2.32), and Theorem 2.1(i), we know that the eigenvalues of A are all in $\mathbb{C}^+$ and the eigenvalues of $\bar{A}$ are all in $\mathbb{C}^-$. Hence, the integrals in (4.22) and (4.23) both exist, and furthermore, the right-hand side of (4.22) is equal to $-B\bar{C}$ and the right-hand side of (4.23) is equal to $-\bar{B}C$. Thus, we have

$$i(AM - M\bar{A}) = -B\,\bar{C}, \quad i(\bar{M}A - \bar{A}\,\bar{M}) = -\bar{B}\,C, \tag{4.24}$$

which are equivalent to (4.10). The unique solvability of the linear systems in (4.24) is assured[20] because the eigenvalues of A are located in $\mathbb{C}^+$ and the eigenvalues of $\bar{A}$ are in $\mathbb{C}^-$. □

In the next theorem, in the reflectionless case, we present some formulas for the potentials $q(x,t)$ and $r(x,t)$ in (2.1) and for the related key quantity $E(x,t)$, where those formulas are expressed explicitly in terms of the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$.

Theorem 4.2. *Assume that the time-evolved reflectionless Marchenko kernels $\Omega(y,t)$ and $\bar{\Omega}(y,t)$ in (4.1) are used as input to the Marchenko system (3.1). Then, we have the following:*

(a) *The corresponding key quantity $E(x,t)$ defined in (1.19) is expressed explicitly in terms of the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$, and we have*

$$E(x,t) = \exp\left(2\int_{-\infty}^x dz\, Q(z,t)\right), \tag{4.25}$$

where $Q(x,t)$ is the scalar-valued function of x and t defined in (3.11) with $\bar{K}_1(x,x,t)$ and $K_2(x,x,t)$ explicitly expressed in terms of the matrix triplets as

$$\bar{K}_1(x,x,t) := \bar{C}e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1} e^{-i\bar{A}x-4i\bar{A}^2t} \bar{M}A\, e^{2iAx+4iA^2t} B, \tag{4.26}$$

$$K_2(x,x,t) := Ce^{iAx}\, \Gamma(x,t)^{-1} e^{iAx+4iA^2t} M\bar{A}\, e^{-2i\bar{A}x-4i\bar{A}^2t} \bar{B}, \tag{4.27}$$

with M and $\bar{M}$ being the constant matrices in (4.9) and $\Gamma(x,t)$ and $\bar{\Gamma}(x,t)$ being the matrix-valued functions of x and t defined in (4.7) and (4.8), respectively. Alternatively, we have

$$E(x,t) = \exp\left(4i\int_{-\infty}^x dz\, P(z,t)\right), \tag{4.28}$$

where $P(x,t)$ is the scalar-valued function of x and t defined in (3.26) with $K_1(x,x,t)$ and $\bar{K}_2(x,x,t)$ explicitly expressed in terms of the matrix triplets as

$$K_1(x,x,t) = -\bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}x-4i\bar{A}^2t}\, \bar{B}, \tag{4.29}$$

$$\bar{K}_2(x,x,t) = -C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAx+4iA^2t}\, B. \tag{4.30}$$

(b) *The corresponding potentials $q(x,t)$ and $r(x,t)$ in the unperturbed linear system (2.1) are expressed explicitly in terms of the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$, and we have*

$$q(x,t) = \left(2\bar{C}\, e^{-i\bar{A}\,x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}\,x-4i\bar{A}^2 t}\, \bar{B}\right) \exp\left(-4\int_x^\infty dz\, Q(z,t)\right), \tag{4.31}$$

$$r(x,t) = \left(2C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAx+4iA^2 t}\, B\right) \exp\left(4\int_x^\infty dz\, Q(z,t)\right), \tag{4.32}$$

or alternatively, we have

$$q(x,t) = \left(2\bar{C}\, e^{-i\bar{A}\,x}\, \bar{\Gamma}(x,t)^{-1}\, e^{-i\bar{A}\,x-4i\bar{A}^2 t}\, \bar{B}\right) \exp\left(-4i\int_x^\infty dz\, P(z,t)\right), \tag{4.33}$$

$$r(x,t) = \left(2C\, e^{iAx}\, \Gamma(x,t)^{-1}\, e^{iAx+4iA^2 t}\, B\right) \exp\left(4i\int_x^\infty dz\, P(z,t)\right). \tag{4.34}$$

Proof. We note that (4.26), (4.27), (4.29), and (4.30) are obtained from (4.3)–(4.6) by using $y = x$ there. We obtain (4.25) by using (3.10) and (3.11) with the help of (4.26) and (4.27). Similarly, we obtain (4.28) by using (3.26) and (3.28) with the help of (4.29) and (4.30). Hence, the proof of (a) is complete. We get (4.31) from (3.14) with the help of (3.11), (4.26), (4.27), and (4.29). Similarly, we get (4.32) from (3.15) with the help of (3.11), (4.26), (4.27), and (4.30). The alternate expressions in (4.33) and (4.34) are obtained in a similar manner, and this is done with the help of (3.26), (4.29), and (4.30). □

In the next theorem, we provide the explicit expressions for the transmission coefficients for (2.1) corresponding to the reflectionless quantities in (4.1).

Theorem 4.3. *Assume that the potentials $q(x,t)$ and $r(x,t)$ appearing in (2.1) at $t = 0$ belong to the Schwartz class and that the corresponding reflection coefficients $R(\zeta,0)$ and $\bar{R}(\zeta,0)$ are zero. Let $T(\zeta,0)$ and $\bar{T}(\zeta,0)$ be the transmission coefficients in this reflectionless case. Suppose that the corresponding bound-state information is given by the two sets in (2.27), or equivalently by the pair of matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ described in (2.28)–(2.30). Let the parameter λ be related to the spectral parameter ζ as in (2.12). Then, we have the following:*

(a) *The time-evolved potentials $q(x,t)$ and $r(x,t)$ remain reflectionless for all $t \in \mathbb{R}$.*
(b) *The transmission coefficients do not evolve in time, as indicated in (2.16).*
(c) *The total number of poles of $T(\zeta,t)$ including multiplicities in the upper-half complex λ-plane is equal to the total number of poles of $\bar{T}(\zeta,t)$ including multiplicities in the lower-half complex λ-plane. In other words, we have*

$$\mathcal{N} = \bar{\mathcal{N}}, \tag{4.35}$$

where we have defined

$$\mathcal{N} := \sum_{j=1}^{N} m_j, \quad \bar{\mathcal{N}} := \sum_{k=1}^{\bar{N}} \bar{m}_k,$$

and hence the matrices A and $\bar{A}$ in the two matrix triplets have the same size $\mathcal{N} \times \mathcal{N}$.
(d) *The corresponding complex constant $e^{i\mu/2}$, where μ is the complex constant defined in (2.14), is uniquely determined by the eigenvalues of the matrices A and $\bar{A}$ and their corresponding multiplicities. We have*

$$e^{i\mu/2} = \frac{\prod_{k=1}^{\bar{N}} (\bar{\lambda}_k)^{\bar{m}_k}}{\prod_{j=1}^{N} (\lambda_j)^{m_j}}, \tag{4.36}$$

with the restriction $\mathcal{N} = \bar{\mathcal{N}}$. We note that (4.36) is equivalent to the determinant expression

$$e^{i\mu/2} = \det[\bar{A}\, A^{-1}]. \tag{4.37}$$

(e) *The transmission coefficients $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ are determined by the eigenvalues of the matrices A and $\bar{A}$ and their corresponding multiplicities. We have*

$$T(\zeta,t) = \left(\frac{\prod_{k=1}^{\bar{N}} \left((\lambda/\bar{\lambda}_k) - 1\right)^{\bar{m}_k}}{\prod_{j=1}^{N} \left((\lambda/\lambda_j) - 1\right)^{m_j}}\right), \quad \bar{T}(\zeta,t) = \left(\frac{\prod_{j=1}^{N} \left((\lambda/\lambda_j) - 1\right)^{m_j}}{\prod_{k=1}^{\bar{N}} \left((\lambda/\bar{\lambda}_k) - 1\right)^{\bar{m}_k}}\right), \tag{4.38}$$

with the restriction $\mathcal{N} = \bar{\mathcal{N}}$. We remark that (4.38) is equivalent to the pair of equations given by

$$T(\zeta,t) = \det\left[(\lambda\bar{A}^{-1} - I)(\lambda A^{-1} - I)^{-1}\right], \quad \bar{T}(\zeta,t) = \det\left[(\lambda A^{-1} - I)(\lambda\bar{A}^{-1} - I)^{-1}\right], \tag{4.39}$$

and hence we have $\bar{T}(\zeta,t) = 1/T(\zeta,t)$.

Proof. The proof of (a) is as follows. From (2.23), we see that $R(\zeta,t)$ and $\bar{R}(\zeta,t)$ are both zero for $t \in \mathbb{R}$ whenever their values at $t = 0$ are zero. Then, from (2.26), we see that $L(\zeta,t)$ and $\bar{L}(\zeta,t)$ are also zero for $t \in \mathbb{R}$. The proof of (b) is apparent from (2.16). Since the transmission coefficients are independent of t, the result in (c) directly follows from Theorem 5.2(a) of Ref. 11. For the proofs of the remaining items, we proceed as follows. In the reflectionless case, from (2.25), we have

$$T(\zeta,t)\,\bar{T}(\zeta,t) = 1, \qquad \lambda \in \mathbb{R}. \tag{4.40}$$

Note that (4.40) is equivalent to

$$T(\zeta,t)\,e^{i\mu/2}\left(\frac{\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}{\prod_{k=1}^{\bar{N}}(\lambda-\bar{\lambda}_k)^{\bar{m}_k}}\right) = \frac{1}{\bar{T}(\zeta,t)\,e^{-i\mu/2}}\left(\frac{\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}{\prod_{k=1}^{\bar{N}}(\lambda-\bar{\lambda}_k)^{\bar{m}_k}}\right), \qquad \lambda \in \mathbb{R}. \tag{4.41}$$

From Theorem 2.1, we know that $T(\zeta,t)$ is meromorphic in $\lambda \in \mathbb{C}^+$ with poles at $\lambda = \lambda_j$ for $1 \le j \le N$ each with multiplicity m_j, is continuous in $\lambda \in \mathbb{R}$, and has the large λ-asymptotics given in (2.17). Again, from Theorem 2.1, we know that the quantity $1/\bar{T}(\zeta,t)$ is analytic in $\lambda \in \mathbb{C}^-$ with zeros at $\lambda = \bar{\lambda}_k$ for $1 \le k \le \bar{N}$ each with multiplicity $\bar{m}_k$, is continuous in $\lambda \in \mathbb{R}$, and has the large λ-asymptotics expressed in (2.18). Furthermore, we already know that (4.35) holds. Hence, both sides of (4.41) must be identical to 1, from which we conclude that

$$T(\zeta,t) = e^{-i\mu/2}\left(\frac{\prod_{k=1}^{\bar{N}}(\lambda-\bar{\lambda}_k)^{\bar{m}_k}}{\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}\right), \quad \bar{T}(\zeta,t) = e^{i\mu/2}\left(\frac{\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}{\prod_{k=1}^{\bar{N}}(\lambda-\bar{\lambda}_k)^{\bar{m}_k}}\right), \tag{4.42}$$

where we have the restriction $\mathcal{N} = \bar{\mathcal{N}}$. From (2.19) and (2.20), we know that the left-hand sides of the equalities in (4.42) become equal to 1 when $\lambda = 0$. Thus, evaluating the first equality in (4.42) at $\lambda = 0$, we get

$$1 = e^{-i\mu/2}\left(\frac{\prod_{k=1}^{\bar{N}}(-\bar{\lambda}_k)^{\bar{m}_k}}{\prod_{j=1}^{N}(-\lambda_j)^{m_j}}\right). \tag{4.43}$$

From (4.35), we know that A and $\bar{A}$ have the same number of eigenvalues, including their multiplicities. Hence, we can replace the minus signs in the products on the right-hand side of (4.43) with the plus signs. Then, from the resulting equality, we obtain (4.36). Finally, using (4.36) in (4.42), we get (4.38). □

For reflectionless potentials, from Theorem 4.3(d), we know that the vectors C and $\bar{C}$ appearing in the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$, respectively, have no effect on the value of the complex constant $e^{i\mu/2}$. In the next theorem, we show that C and $\bar{C}$ have only some limited effect on the value of μ itself.

Theorem 4.4. *Suppose that the potentials $q(x,t)$ and $r(x,t)$ appearing in (2.1) at $t = 0$ belong to the Schwartz class and that the corresponding reflection coefficients $R(\zeta,0)$ and $\bar{R}(\zeta,0)$ are zero. Let μ be the complex constant defined in (2.14). Assume that the corresponding bound-state information is given by the two sets in (2.27), or equivalently by the pair of matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ described in (2.28)–(2.30). For this reflectionless scattering dataset, there can correspond a countably infinite number of μ-values, and any two such μ-values differ from each other by a constant integer multiple of 4π.*

Proof. The stated nonuniqueness in the value of μ can be established by taking the complex logarithm of both sides of (4.36) or (4.37). From (4.37), we obtain

$$\frac{i\mu}{2} = \log\left[\det\left[\bar{A}\,A^{-1}\right]\right] + 2\pi i n, \qquad n \in \mathbb{Z}, \tag{4.44}$$

where log denotes the principal branch of the complex logarithm function and n takes any integer values. From (4.44), we get

$$\mu = -2i \log\left[\det\left[\bar{A}\, A^{-1}\right]\right] + 4\pi n, \qquad n \in \mathbb{Z}, \tag{4.45}$$

which completes the proof. □

The restriction on the constant μ in (4.45) in the reflectionless case deserves further investigation. In some examples in Sec. IX, we illustrate (4.45) by displaying two distinct values of μ obtained by varying C and $\bar{C}$, where those two μ-values differ only by 4π. In Example 9.4, we demonstrate that we may have three distinct μ-values that can be obtained by varying C and $\bar{C}$. We pose it as an open problem whether there is an upper limit on the number of distinct μ-values in (4.45) differing by 4π as we vary the bound-state norming constants. It is also an open problem to explain physically the meaning of having distinct values of μ differing from each other by an integer multiple of 4π. In the terminology of integrable systems, the quantity μ is a constant of motion, and hence a physical explanation of some distinct values of μ in (4.45) may help us to understand the constants of motion better.

In the next theorem, in the reflectionless case, we show that the integrals of the potentials $q(x,t)$ and $r(x,t)$ over $x \in \mathbb{R}$ each must vanish.

Theorem 4.5. *Suppose that the potentials $q(x,t)$ and $r(x,t)$ appearing in (2.1) at $t = 0$ belong to the Schwartz class and that the corresponding reflection coefficients $R(\zeta,0)$ and $\bar{R}(\zeta,0)$ are zero. Then, for each fixed $t \in \mathbb{R}$, we have*

$$\int_{-\infty}^{\infty} dz\, q(z,t) = 0, \qquad \int_{-\infty}^{\infty} dz\, r(z,t) = 0. \tag{4.46}$$

Proof. From (2.23), we know that the time-evolved reflection coefficients $R(\zeta,t)$ and $\bar{R}(\zeta,t)$ are zero for all $t \in \mathbb{R}$ whenever their initial values at $t = 0$ vanish. Then, from the leading asymptotics as $\zeta \to 0$ in (2.21) and (2.22), we conclude that (4.46) must hold. □

We observe from (4.46) that, in the reflectionless case, the corresponding solution pair $q(x,t)$ and $r(x,t)$ to the nonlinear system (1.6) is somehow restricted. In fact, with the help of (1.20), we see that the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ satisfy the same restriction as in (4.46) whenever $b = a$. When $b = a$, from (1.6), we obtain the one-parameter family of DNLS systems given by

$$\begin{cases} i\,\tilde{q}_t + \tilde{q}_{xx} - i\kappa\left(\tilde{q}^{\,2}\,\tilde{r}\right)_x = 0, \\ i\,\tilde{r}_t - \tilde{r}_{xx} - i\kappa\left(\tilde{q}\,\tilde{r}^{\,2}\right)_x = 0. \end{cases} \tag{4.47}$$

Let us note that (4.47) reduces to the Kaup–Newell system (2.1) when $\kappa = 1$. We conclude that any solution pair $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ to the nonlinear system (4.47) must satisfy the same restriction as in (4.46) if $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ correspond to reflectionless potentials in the related linear system. This indicates the power of the inverse scattering transform method and the physical intuition it provides to solve some related mathematical problems. Without the use of the inverse scattering transform, i.e., without relating (4.47) to the scattering theory for a corresponding linear system, it may not be so easy to prove that there are infinitely many solution pairs to (4.47), where those solutions have zero integrals over $x \in \mathbb{R}$ for each $t \in \mathbb{R}$. Similarly, without the use of the inverse scattering transform, it may not be so easy to prove that if the initial values $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ have zero integrals over $x \in \mathbb{R}$, then the solution pair $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ must each also have zero integrals over $x \in \mathbb{R}$ for any $t \in \mathbb{R}$. With the use of the inverse scattering transform, such mathematical results are naturally discovered and their proofs are easy. Without the use of the inverse scattering transform, those mathematical results may not be so easy to discover or their proofs may not be so easy.

In the next theorem, in the reflectionless case, we express the Jost solutions to (2.1) explicitly in terms of the matrix triplet pair appearing in the scattering dataset (2.38).

Theorem 4.6. *Assume that the time-evolved reflectionless Marchenko kernels $\Omega(y,t)$ and $\bar{\Omega}(y,t)$ in (4.1) are used as input to the Marchenko system (3.1). Let the parameter λ be related to the spectral parameter ζ as in (2.12). Then, the corresponding four Jost solutions to (2.1) with the respective asymptotics in (2.2)–(2.5) can be expressed explicitly in terms of the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$. In fact, we have the following:*

(a) *The Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ are expressed in terms of the two matrix triplets as*

$$\psi_1(\zeta,x,t) = i\sqrt{\lambda}\, e^{i\lambda x}\left[\bar{C}\, e^{-i\bar{A}\,x}\bar{\Gamma}^{-1}\left(\bar{A} - \lambda I\right)^{-1} e^{-i\bar{A}\,x - 4i\bar{A}^{\,2}t}\bar{B}\right] e^{-2\int_x^\infty dz\, Q(z,t)}, \tag{4.48}$$

$$\psi_2(\zeta,x,t) = e^{i\lambda x}\left[1 - iC\, e^{iAx}\Gamma^{-1}\, e^{iAx + 4iA^2 t} M\bar{A}\left(\bar{A} - \lambda I\right)^{-1} e^{-2i\bar{A}\,x - 4i\bar{A}^{\,2}t}\bar{B}\right] e^{2\int_x^\infty dz\, Q(z,t)}, \tag{4.49}$$

$$\bar{\psi}_1(\zeta,x,t) = e^{-i\lambda x}\left[1 + i\bar{C}\, e^{-i\bar{A}\,x}\bar{\Gamma}^{-1}\, e^{-i\bar{A}\,x - 4i\bar{A}^{\,2}t}\bar{M}A(A - \lambda I)^{-1} e^{2iAx + 4iA^2 t} B\right] e^{-2\int_x^\infty dz\, Q(z,t)}, \tag{4.50}$$

$$\bar{\psi}_2(\zeta,x,t) = -i\sqrt{\lambda}\, e^{-i\lambda x}\left[C\, e^{iAx}\Gamma^{-1}(A - \lambda I)^{-1} e^{iAx + 4iA^2 t} B\right] e^{2\int_x^\infty dz\, Q(z,t)}, \tag{4.51}$$

where M and $\bar{M}$ are the constant matrices defined in (4.9); Γ and $\bar{\Gamma}$ are the matrix-valued functions of x and t appearing in (4.7) and (4.8), respectively; and $Q(x,t)$ is the scalar-valued function of x and t defined in (3.11) with its right-hand side expressed by using (4.26) and (4.27).

(b) *Alternatively, the Jost solutions $\psi(\zeta,x,t)$ and $\bar\psi(\zeta,x,t)$ are expressed in terms of the matrix triplet pair as*

$$\psi_1(\zeta,x,t) = i\sqrt{\lambda}e^{i\lambda x}\Big[\bar C\, e^{-i\bar A\, x}\bar\Gamma^{-1}\big(\bar A-\lambda I\big)^{-1}e^{-i\bar A\, x-4i\bar A^2 t}\bar B\Big]e^{-2i\int_x^\infty dz\, P(z,t)}, \tag{4.52}$$

$$\psi_2(\zeta,x,t) = e^{i\lambda x}\Big[1 - iC\, e^{iAx}\Gamma^{-1}\, e^{iAx+4iA^2 t}M\bar A\big(\bar A-\lambda I\big)^{-1}e^{-2i\bar A\, x-4i\bar A^2 t}\bar B\Big]e^{2i\int_x^\infty dz\, P(z,t)}, \tag{4.53}$$

$$\bar\psi_1(\zeta,x,t) = e^{-i\lambda x}\Big[1 + i\bar C\, e^{-i\bar A\, x}\bar\Gamma^{-1}\, e^{-i\bar A\, x-4i\bar A^2 t}\,\bar M A(A-\lambda I)^{-1}e^{2iAx+4iA^2 t}B\Big]e^{-2i\int_x^\infty dz\, P(z,t)}, \tag{4.54}$$

$$\bar\psi_2(\zeta,x,t) = -i\sqrt{\lambda}\, e^{-i\lambda x}\Big[C\, e^{iAx}\Gamma^{-1}(A-\lambda I)^{-1}e^{iAx+4iA^2 t}B\Big]e^{2i\int_x^\infty dz\, P(z,t)}, \tag{4.55}$$

where $P(x,t)$ is the scalar-valued function of x and t defined in (3.26) with its right-hand side expressed by using (4.29) and (4.30).

(c) *For the Jost solutions $\phi(\zeta,x,t)$ and $\bar\phi(\zeta,x,t)$, we have*

$$\phi(\zeta,x,t) = \bar T(\zeta,t)\,\bar\psi(\zeta,x,t),\quad \bar\phi(\zeta,x,t) = T(\zeta,t)\,\psi(\zeta,x,t), \tag{4.56}$$

where $T(\zeta,t)$ and $\bar T(\zeta,t)$ are expressed in terms of the matrices A and $\bar A$ as in (4.39) and where the Jost solutions $\psi(\zeta,x,t)$ and $\bar\psi(\zeta,x,t)$ are expressed in terms of the matrix triplet pair as in (4.48)–(4.51) or alternatively as in (4.52)–(4.55).

Proof. We obtain (4.48)–(4.51) by using (4.3)–(4.6) in (3.16)–(3.19) and by explicitly evaluating the integrals there related to the Fourier transforms. We remark that the exponential terms in (3.16)–(3.19) are expressed in terms of the quantity $Q(x,t)$ with the help of (3.11). Hence, the proof of (a) is complete. We obtain the alternate expressions in (b) by using (4.3)–(4.6) in (3.31)–(3.34) and again by explicitly evaluating the integrals there related to the Fourier transforms. We mention that the exponential terms in (3.31)–(3.34) are expressed in terms of the quantity $P(x,t)$ with the help of (3.26). Thus, the proof of (b) is also complete. Having established (4.48)–(4.55), we use (3.20) and (3.21) with $R(\zeta,t)\equiv 0$ and $\bar R(\zeta,t)\equiv 0$. We then obtain

$$\phi(\zeta,x,t) = \frac{1}{T(\zeta,t)}\,\bar\psi(\zeta,x,t),\quad \bar\phi(\zeta,x,t) = \frac{1}{\bar T(\zeta,t)}\,\psi(\zeta,x,t). \tag{4.57}$$

As indicated in Theorem 4.3(e), the quantities $T(\zeta,t)$ and $\bar T(\zeta,t)$ are reciprocals of each other in the reflectionless case. Hence, (4.57) yields (4.56), which completes the proof of our theorem. □

V. CONNECTION BETWEEN THE PERTURBED AND UNPERTURBED SYSTEMS

In order to understand the relationship between the perturbed nonlinear system (1.6) and the unperturbed nonlinear system (1.2), we need to understand the relationship between the corresponding linear systems. We recall that we use a tilde to denote the quantities related to the perturbed system and that the linear system corresponding to (1.2) is given in (2.1). The linear system corresponding to (1.6) is given in the first equality of (1.12) and, for convenience, we write it in a format similar to that of (2.1). We have

$$\frac{d}{dx}\begin{bmatrix}\tilde\alpha\\ \tilde\beta\end{bmatrix} = \begin{bmatrix}-i\zeta^2+\dfrac{ib}{2}\,\tilde q(x,t)\,\tilde r(x,t) & \kappa\,\zeta\,\tilde q(x,t)\\ \dfrac{1}{\kappa}\,\zeta\,\tilde r(x,t) & i\zeta^2+\dfrac{ia}{2}\,\tilde q(x,t)\,\tilde r(x,t)\end{bmatrix}\begin{bmatrix}\tilde\alpha\\ \tilde\beta\end{bmatrix}. \tag{5.1}$$

In this section, we analyze the relationship between the relevant quantities for (5.1) and the relevant quantities for (2.1).

Let us recall that the potentials $\tilde q(x,t)$ and $\tilde r(x,t)$ appearing in (5.1) are related the potentials $q(x,t)$ and $r(x,t)$ in (2.1) as in (1.20). The four Jost solutions to (5.1) are those solutions satisfying the asymptotics (2.2)–(2.5), respectively. On the other hand, contrary to the coefficient matrix in (2.1), the coefficient matrix in (5.1) does not have zero trace unless the parameters a and b satisfy $a+b=0$. Consequently, the corresponding left and right transmission coefficients for (5.1) are not equal to each other unless $a+b=0$. Instead of obtaining the scattering coefficients from (2.6)–(2.9), we obtain those coefficients from the spatial asymptotics given by

$$\tilde\psi(\zeta,x,t) = \begin{bmatrix}\dfrac{\tilde L(\zeta,t)}{\tilde T_1(\zeta,t)}\,e^{-i\zeta^2 x}[1+o(1)]\\ \dfrac{1}{\tilde T_1(\zeta,t)}\,e^{i\zeta^2 x}[1+o(1)]\end{bmatrix},\qquad x\to-\infty, \tag{5.2}$$

$$\tilde{\bar\psi}(\zeta,x,t) = \begin{bmatrix}\dfrac{1}{\tilde{\bar T}_1(\zeta,t)}\,e^{-i\zeta^2 x}[1+o(1)]\\ \dfrac{\tilde{\bar L}(\zeta,t)}{\tilde{\bar T}_1(\zeta,t)}\,e^{i\zeta^2 x}[1+o(1)]\end{bmatrix},\qquad x\to-\infty, \tag{5.3}$$

$$\tilde{\phi}(\zeta,x,t)=\begin{bmatrix}\dfrac{1}{\tilde{T}_{\mathrm{r}}(\zeta,t)}\,e^{-i\zeta^2 x}[1+o(1)]\\[2ex]\dfrac{\tilde{R}(\zeta,t)}{\tilde{T}_{\mathrm{r}}(\zeta,t)}\,e^{i\zeta^2 x}[1+o(1)]\end{bmatrix},\qquad x\to+\infty, \tag{5.4}$$

$$\tilde{\bar{\phi}}(\zeta,x,t)=\begin{bmatrix}\dfrac{\tilde{\bar{R}}(\zeta,t)}{\tilde{\bar{T}}_{\mathrm{r}}(\zeta,t)}\,e^{-i\zeta^2 x}[1+o(1)]\\[2ex]\dfrac{1}{\tilde{\bar{T}}_{\mathrm{r}}(\zeta,t)}\,e^{i\zeta^2 x}[1+o(1)]\end{bmatrix},\qquad x\to+\infty. \tag{5.5}$$

Thus, for the perturbed linear system (5.1), we have eight scattering coefficients, namely, the left transmission coefficients $\tilde{T}_{\mathrm{l}}(\zeta,t)$ and $\tilde{\bar{T}}_{\mathrm{l}}(\zeta,t)$, the right transmission coefficients $\tilde{T}_{\mathrm{r}}(\zeta,t)$ and $\tilde{\bar{T}}_{\mathrm{r}}(\zeta,t)$, the left reflection coefficients $\tilde{L}(\zeta,t)$ and $\tilde{\bar{L}}(\zeta,t)$, and the right reflection coefficients $\tilde{R}(\zeta,t)$ and $\tilde{\bar{R}}(\zeta,t)$.

We recall that the solutions to the perturbed linear system (5.1) and the solutions to the unperturbed linear system (2.1) are related to each other as in (1.17). Since the linear systems (2.1) and (5.1) are both homogeneous, any constant multiples of their solutions are also solutions. For the Jost solutions to (2.1) and the Jost solutions to (5.1), such constants can be chosen with the help of (2.13). Thus, we obtain the relationships

$$\tilde{\psi}(\zeta,x,t)=e^{-ia\mu/2}\begin{bmatrix}E(x,t)^b & 0\\ 0 & E(x,t)^a\end{bmatrix}\psi(\zeta,x,t), \tag{5.6}$$

$$\tilde{\bar{\psi}}(\zeta,x,t)=e^{-ib\mu/2}\begin{bmatrix}E(x,t)^b & 0\\ 0 & E(x,t)^a\end{bmatrix}\bar{\psi}(\zeta,x,t), \tag{5.7}$$

$$\tilde{\phi}(\zeta,x,t)=\begin{bmatrix}E(x,t)^b & 0\\ 0 & E(x,t)^a\end{bmatrix}\phi(\zeta,x,t), \tag{5.8}$$

$$\tilde{\bar{\phi}}(\zeta,x,t)=\begin{bmatrix}E(x,t)^b & 0\\ 0 & E(x,t)^a\end{bmatrix}\bar{\phi}(\zeta,x,t). \tag{5.9}$$

Using the spatial asymptotics in (5.6)–(5.9), with the help of (2.6)–(2.9) and (5.2)–(5.5), we relate the eight scattering coefficients for (5.1) to the six scattering coefficients for (2.1) as

$$\tilde{T}_{\mathrm{l}}(\zeta,t)=e^{ia\mu/2}\,T(\zeta,t),\quad \tilde{\bar{T}}_{\mathrm{l}}(\zeta,t)=e^{ib\mu/2}\,\bar{T}(\zeta,t), \tag{5.10}$$

$$\tilde{T}_{\mathrm{r}}(\zeta,t)=e^{-ib\mu/2}\,T(\zeta,t),\quad \tilde{\bar{T}}_{\mathrm{r}}(\zeta,t)=e^{-ia\mu/2}\,\bar{T}(\zeta,t), \tag{5.11}$$

$$\tilde{R}(\zeta,t)=e^{i(a-b)\mu/2}\,R(\zeta,t),\quad \tilde{\bar{R}}(\zeta,t)=e^{i(b-a)\mu/2}\,\bar{R}(\zeta,t), \tag{5.12}$$

$$\tilde{L}(\zeta,t)=L(\zeta,t),\quad \tilde{\bar{L}}(\zeta,t)=\bar{L}(\zeta,t). \tag{5.13}$$

We recall that the bound-state information for the unperturbed system (2.1) is described by the two sets specified in (2.27) and that it is the most convenient to use the bound-state information not in the form of (2.27) but in the form of the matrix triplet pair (A,B,C) and $(\bar{A},\bar{B},\bar{C})$. Let us use the matrix triplet pair $(\tilde{A},\tilde{B},\tilde{C})$ and $(\tilde{\bar{A}},\tilde{\bar{B}},\tilde{\bar{C}})$ to represent the bound-state information related to the perturbed system (5.1). We then have

$$\tilde{A}=A,\quad \tilde{B}=B,\quad \tilde{C}=e^{i(a-b)\mu/2}\,C, \tag{5.14}$$

$$\tilde{\bar{A}}=\bar{A},\quad \tilde{\bar{B}}=\bar{B},\quad \tilde{\bar{C}}=e^{i(b-a)\mu/2}\,\bar{C}. \tag{5.15}$$

The first two equalities in (5.14) and (5.15) are obtained as follows. As seen from the first equalities in (5.10) and (5.11), the poles of $\tilde{T}_{\mathrm{l}}(\zeta,0)$, $\tilde{T}_{\mathrm{r}}(\zeta,0)$, and $T(\zeta,0)$ in the upper-half complex λ-plane coincide and also the multiplicities of those poles coincide. From Theorem 2.1, we know that such poles and multiplicities are the only ingredients to construct the matrices A and B with the help of (2.31). Hence, the first two equalities in (5.14) are justified. Similarly, from the second equalities in (5.10) and (5.11), it follows that the poles of $\tilde{\bar{T}}_{\rm l}(\zeta,0)$, $\tilde{\bar{T}}_{\rm r}(\zeta,0)$, and $\bar{T}(\zeta,0)$ in the lower-half complex λ-plane coincide and also the multiplicities of those poles coincide. Again, from Theorem 2.1, we know that such poles and multiplicities are the only ingredients to construct the matrices $\bar{A}$ and $\bar{B}$ with the help of (2.33). Hence, the first two equalities in (5.15) are also justified. The justification of the third equalities in (5.14) and (5.15) follows from the construction of the norming constants c_{jk} appearing in (2.32) and of the norming constants $\bar{c}_{jk}$ appearing in (2.34). The details of those constructions can be found in Sec. 3 of Ref. 11 and Examples 6.1 and 6.2 of that reference. From those constructions, it follows that the norming constant c_{jk} is directly proportional to $T(\zeta,0)$, directly proportional to the Jost solution $\phi(\zeta,x,0)$, and inversely proportional to the Jost solution $\psi(\zeta,x,0)$; the norming constant $\bar{c}_{jk}$ is directly proportional to $\bar{T}(\zeta,0)$, directly proportional to the Jost solution $\bar{\phi}(\zeta,x,0)$, and inversely proportional to the Jost solution $\bar{\psi}(\zeta,x,0)$; the norming constant $\tilde{c}_{jk}$ is directly proportional to $\tilde{T}_{\rm r}(\zeta,0)$, directly proportional to the Jost solution $\tilde{\phi}(\zeta,x,0)$, and inversely proportional to the Jost solution $\tilde{\psi}(\zeta,x,0)$; and the norming constant $\tilde{\bar{c}}_{jk}$ is directly proportional to $\tilde{\bar{T}}_{\rm r}(\zeta,0)$, directly proportional to the Jost solution $\tilde{\bar{\phi}}(\zeta,x,0)$, and inversely proportional to the Jost solution $\tilde{\bar{\psi}}(\zeta,x,0)$. Then, using (5.6)–(5.9) and (5.11), we justify the third equalities in (5.14) and (5.15).

VI. THE SOLUTION TO THE PERTURBED NONLINEAR SYSTEM

In this section, we describe the use of our Marchenko method to obtain the solution to the initial-value problem for (1.1) or equivalently for (1.6). Thus, we are given $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ and we would like to determine $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ satisfying (1.6) at any time t. The following are the steps to obtain $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ with the help of the Marchenko method described in Sec. III.

(a) From the coefficients of the third and fourth terms in the first line of (1.6), we determine the values of the parameters κ and $a-b$. Let us use Δ_1 and Δ_2 to denote the respective coefficients, i.e., let us use

$$\Delta_1 := i\kappa(a-b-2), \quad \Delta_2 := i\kappa(a-b-1). \tag{6.1}$$

From (6.1), we uniquely determine κ and $a-b$ as

$$\kappa = i(\Delta_1-\Delta_2), \quad a-b = \frac{\Delta_1-2\Delta_2}{\Delta_1-\Delta_2}.$$

(b) Next, we use $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ as input to (5.1) at $t=0$, and we obtain the corresponding Jost solutions $\tilde{\psi}(\zeta,x,0)$, $\tilde{\bar{\psi}}(\zeta,x,0)$, $\tilde{\phi}(\zeta,x,0)$, and $\tilde{\bar{\phi}}(\zeta,x,0)$ to (5.1) at $t=0$.

(c) From the four Jost solutions at $t=0$ constructed in step (b), by using the spatial asymptotics (5.2)–(5.5), we get the corresponding eight scattering coefficients $\tilde{T}_{\rm l}(\zeta,0)$, $\tilde{\bar{T}}_{\rm l}(\zeta,0)$, $\tilde{T}_{\rm r}(\zeta,0)$, $\tilde{\bar{T}}_{\rm r}(\zeta,0)$, $\tilde{R}(\zeta,0)$, $\tilde{\bar{R}}(\zeta,0)$, $\tilde{L}(\zeta,0)$, and $\tilde{\bar{L}}(\zeta,0)$.

(d) We remark that, using (2.19), (2.20), and (5.11), we obtain

$$\tilde{T}_{\rm r}(\zeta,0) = e^{-ib\mu/2}, \quad \tilde{\bar{T}}_{\rm r}(\zeta,0) = e^{-ia\mu/2}. \tag{6.2}$$

Since we know the left-hand sides of the two equalities in (6.2) from step (c), we have $e^{ia\mu/2}$ and $e^{ib\mu/2}$ both at hand. Alternatively, we can use the left transmission coefficients to obtain $e^{ia\mu/2}$ and $e^{ib\mu/2}$, and this can be achieved with the help of (2.19), (2.20), and (5.10).

(e) Having the scattering coefficients at $t=0$ from step (c) and the values of $e^{ia\mu/2}$ and $e^{ib\mu/2}$ from step (d), we use (5.10)–(5.13) at $t=0$ and construct the six scattering coefficients at $t=0$ for the unperturbed system (2.1). We have

$$T(\zeta,0) = e^{ib\mu/2}\,\tilde{T}_{\rm r}(\zeta,0), \quad \bar{T}(\zeta,0) = e^{ia\mu/2}\,\tilde{\bar{T}}_{\rm r}(\zeta,0),$$

$$R(\zeta,0) = e^{i(b-a)\mu/2}\,\tilde{R}(\zeta,0), \quad \bar{R}(\zeta,0) = e^{i(a-b)\mu/2}\,\tilde{\bar{R}}(\zeta,0),$$

$$L(\zeta,0) = \tilde{L}(\zeta,0), \quad \bar{L}(\zeta,0) = \tilde{\bar{L}}(\zeta,0).$$

(f) We already know the value of $a-b$ from step (a), and we would like to obtain the values of a and b separately. Since the parameters a, b, and μ appearing on the right-hand sides in (6.2) may be complex, the use of the complex logarithm function cannot uniquely determine a and b from (6.2). In order to have unique values for a and b, we proceed as follows. Using the principal branch of the complex logarithm of the already known quantity $e^{ia\mu/2}$, we uniquely obtain the value of a as

$$a = \log\left[e^{-ia\mu/2}\right]. \tag{6.3}$$

Since the value of $a-b$ is already known, we obtain the value of b uniquely with the help of (6.3) as

$$b = -(a-b) + \log\left[e^{-ia\mu/2}\right].$$

(g) Note that (1.20) implies that

$$\tilde{q}(x,t)\,\tilde{r}(x,t) = q(x,t)\,r(x,t), \tag{6.4}$$

and hence the quantity $E(x,t)$ defined in (1.19) can also be expressed in terms of $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ as

$$E(x,t) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} dz\,\tilde{q}(z,t)\,\tilde{r}(z,t)\right). \tag{6.5}$$

Since we already have $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$ at hand, using (6.5), we construct $E(x,0)$ as

$$E(x,0) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} dz\,\tilde{q}(z,0)\,\tilde{r}(z,0)\right).$$

(h) We have the Jost solutions at $t = 0$ to the perturbed problem (5.1) from step (b), the quantities $e^{ia\mu/2}$ and $e^{ib\mu/2}$ from step (d), the values of a and b from step (f), and the quantity $E(x,0)$ from step (g). Hence, using (5.6)–(5.9), we construct at $t = 0$ the Jost solutions to the unperturbed problem (2.1) as

$$\psi(\zeta,x,0) = e^{ia\mu/2}\begin{bmatrix}\dfrac{1}{E(x,0)^b} & 0\\ 0 & \dfrac{1}{E(x,0)^a}\end{bmatrix}\tilde{\psi}(\zeta,x,0),$$

$$\bar{\psi}(\zeta,x,0) = e^{ib\mu/2}\begin{bmatrix}\dfrac{1}{E(x,0)^b} & 0\\ 0 & \dfrac{1}{E(x,0)^a}\end{bmatrix}\tilde{\bar{\psi}}(\zeta,x,0),$$

$$\phi(\zeta,x,0) = \begin{bmatrix}\dfrac{1}{E(x,0)^b} & 0\\ 0 & \dfrac{1}{E(x,0)^a}\end{bmatrix}\tilde{\phi}(\zeta,x,0),$$

$$\bar{\phi}(\zeta,x,0) = \begin{bmatrix}\dfrac{1}{E(x,0)^b} & 0\\ 0 & \dfrac{1}{E(x,0)^a}\end{bmatrix}\tilde{\bar{\phi}}(\zeta,x,0).$$

(i) Having the unperturbed transmission coefficients $T(\zeta,0)$ and $\bar{T}(\zeta,0)$ and the four unperturbed Jost solutions $\psi(\zeta,x,0)$, $\bar{\psi}(\zeta,x,0)$, $\phi(\zeta,x,0)$, and $\bar{\phi}(\zeta,x,0)$ to (2.1) at $t = 0$, we construct the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ as described in (h), (i), (j), and (k) of Theorem 2.1. The details of the construction can be found in Sec. 3 of Ref. 11. As far as our Marchenko method is concerned, this amounts to including the effects of the bound states in the Marchenko kernels by using the "recipe"

$$\hat{R}(y,0) \mapsto \hat{R}(y,0) + C\,e^{iAy}\,B, \quad \hat{\bar{R}}(y,0) \mapsto \hat{\bar{R}}(y,0) + \bar{C}\,e^{-i\bar{A}\,y}\,\bar{B}. \tag{6.6}$$

In fact, the simple and elegant way of including the bound-state information stated in (6.6) holds, in general, also for other linear systems for which a Marchenko method is available. This is one of the strengths of the Marchenko method in the sense that any number of bound states with any multiplicities can be handled in a simple and elegant manner by using (6.6). We now have the scattering dataset $\mathbf{S}(\zeta,0)$ at $t = 0$ for the unperturbed linear system (2.1).

(j) Having the unperturbed scattering dataset $\mathbf{S}(\zeta,0)$ at $t = 0$ at hand, we use (2.16), (2.23), (2.24), (2.34), and (2.35) in order to obtain the time-evolved scattering dataset $\mathbf{S}(\zeta,t)$ at any time t. Knowing $\mathbf{S}(\zeta,t)$, we also have the time-evolved Marchenko kernels $\Omega(\zeta,t)$ and $\bar{\Omega}(\zeta,t)$ defined in (3.2).

(k) Having at hand the time-evolved Marchenko kernels $\Omega(\zeta,t)$ and $\bar{\Omega}(\zeta,t)$, we use them as input in the Marchenko system (3.1) or equivalently in the uncoupled Marchenko system (3.8) and the auxiliary system (3.9). Then, we obtain the solutions $K_1(x,y,t)$, $K_2(x,y,t)$, $\bar{K}_1(x,y,t)$, and $\bar{K}_2(x,y,t)$. Next, as described in Theorem 3.2 we construct all the relevant quantities associated with (2.1), namely, we get the key quantity $E(x,t)$ defined in (1.19), the constant μ defined in (2.14), the potentials $q(x,t)$ and $r(x,t)$ appearing in (2.1), and the Jost solutions $\psi(\zeta,x,t)$ and $\bar{\psi}(\zeta,x,t)$ to (2.1) satisfying (2.2) and (2.3), respectively. We also obtain the Jost solutions $\phi(\zeta,x,t)$ and $\bar{\phi}(\zeta,x,t)$ to (2.1) satisfying (2.4) and (2.5), respectively, by using (3.20) and (3.21).

(l) Finally, we transform the relevant quantities obtained in step (k) for the unperturbed system (2.1), and we obtain the corresponding relevant quantities for the perturbed system (5.1). This is accomplished as follows. We recover the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ by using (1.20), where we remark that we already know the value of the parameter κ from step (a) and we know the values of the parameters a and b from step (f). We also recover the four Jost solutions to (5.1), and this is done with the help of (5.6)–(5.9) and by using the four Jost solutions $\psi(\zeta,x,t)$, $\bar{\psi}(\zeta,x,t)$, $\phi(\zeta,x,t)$, and $\bar{\phi}(\zeta,x,t)$ to (2.1) already constructed in step (k). Based on the inverse scattering transform method, it is known that $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ are the solutions to the initial-value problem for (1.1).

VII. EXPLICIT SOLUTION FORMULAS FOR THE PERTURBED SYSTEM

In this section, we present some explicit solution formulas for the general DNLS system (1.6) and some explicit solution formulas for the corresponding linear system (5.1). This is done by providing the solution formulas in a closed form for $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ satisfying (1.6), where the formulas are explicitly expressed in terms of the two matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ appearing in (4.1) and the two complex-valued parameters $(a-b)$ and κ appearing in (1.6). When the solution pair $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ is used as potentials in (5.1), we also present the explicit formulas for the corresponding Jost solutions $\tilde{\psi}(\zeta,x,t)$, $\tilde{\phi}(\zeta,x,t)$, $\bar{\tilde{\psi}}(\zeta,x,t)$, and $\bar{\tilde{\phi}}(\zeta,x,t)$, which are all expressed in terms of the two matrix triplets and the three parameters a, b, and κ.

The formulas presented for $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ represent the time-evolved, reflectionless potentials in the linear system (5.1). The formulas we have for the potentials and for the corresponding Jost solutions contain matrix exponentials, and those formulas are valid when each matrix triplet has an arbitrary size. On the other hand, as indicated in Theorem 4.3(c), the corresponding potentials cannot both belong to the Schwartz class unless the triplet sizes for (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ are equal to each other. The use of matrix exponentials allows the formulas presented to have a compact form, independent of the number of bound states and of multiplicities of those bound states. The matrix exponentials can certainly be explicitly expressed in terms of elementary functions, but the resulting expressions, as the matrix size gets large, become extremely lengthy and not practical to display. In such cases, a symbolic software, such as Mathematica, may be used to display the solution formulas by expressing the matrix exponentials in terms of elementary functions.

In the next theorem, we present some explicit formulas for the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ and the Jost solutions for (5.1) expressed in terms of a matrix triplet pair corresponding to reflectionless scattering data. We remark that the formulas presented for $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ make up an explicit solution to the general DNLS system (1.6).

Theorem 7.1. *Suppose that the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ appearing in (5.1) at $t=0$ belong to the Schwartz class and that the corresponding reflection coefficients $\tilde{R}(\zeta,0)$ and $\bar{\tilde{R}}(\zeta,0)$ are zero. Let a, b, and κ be the complex parameters appearing in (5.1). We have the following:*

(a) *The formulas*

$$\tilde{q}(x,t) = \left(\frac{2}{\kappa}\,\bar{C}\,e^{-i\bar{A}\,x}\,\bar{\Gamma}(x,t)^{-1}\,e^{-i\bar{A}\,x-4i\bar{A}^2 t}\,\bar{B}\right)\exp\left(2(b-a)\int_{-\infty}^{x} dz\,Q(z,t)\right), \tag{7.1}$$

$$\tilde{r}(x,t) = \left(2\kappa\,C\,e^{iAx}\,\Gamma(x,t)^{-1}\,e^{iAx+4iA^2 t}\,B\right)\exp\left(2(a-b)\int_{-\infty}^{x} dz\,Q(z,t)\right), \tag{7.2}$$

yield an explicit solution pair for (1.6) with the initial values $\tilde{q}(x,0)$ and $\tilde{r}(x,0)$. Here, (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ are the matrix triplets appearing in (2.28)–(2.30) with equal matrix triplet sizes and all eigenvalues of A located in $\mathbb{C}^+$ and all eigenvalues of $\bar{A}$ located in $\mathbb{C}^-$; Γ and $\bar{\Gamma}$ are the matrix-valued functions of x and t appearing in (4.7) and (4.8), respectively; and $Q(x,t)$ is the scalar-valued function of x and t defined in (3.11) with $\bar{K}_1(x,x,t)$ and $K_2(x,x,t)$ explicitly expressed in terms of the matrix triplets as in (4.26) and (4.27). Alternatively, the formulas in (7.1) and (7.2) can be expressed as

$$\tilde{q}(x,t) = \left(\frac{2}{\kappa}\,\bar{C}\,e^{-i\bar{A}\,x}\,\bar{\Gamma}(x,t)^{-1}\,e^{-i\bar{A}\,x-4i\bar{A}^2 t}\,\bar{B}\right)\exp\left(2i(b-a)\int_{-\infty}^{x} dz\,P(z,t)\right), \tag{7.3}$$

$$\tilde{r}(x,t) = \left(2\kappa\,C\,e^{iAx}\,\Gamma(x,t)^{-1}\,e^{iAx+4iA^2 t}\,B\right)\exp\left(2i(a-b)\int_{-\infty}^{x} dz\,P(z,t)\right), \tag{7.4}$$

where $P(x,t)$ is the scalar-valued function of x and t defined in (3.26) with $K_1(x,x,t)$ and $\bar{K}_2(x,x,t)$ explicitly expressed in terms of the matrix triplets as in (4.29) and (4.30), respectively.

(b) *If the expressions in (7.1) and (7.2), or equivalently (7.3) and (7.4), are used as the potentials in (5.1), then the corresponding Jost solutions $\tilde{\psi}(\zeta,x,t)$ and $\bar{\tilde{\psi}}(\zeta,x,t)$ with components similarly defined as in (2.10) are explicitly expressed by the formulas*

$$\tilde{\psi}_1(\zeta,x,t) = i\sqrt{\lambda}e^{i\lambda x}\left[\bar{C}\,e^{-i\bar{A}\,x}\bar{\Gamma}^{-1}\left(\bar{A}-\lambda I\right)^{-1}e^{-i\bar{A}\,x-4i\bar{A}^2 t}\bar{B}\right]\Upsilon(-a-1,b-a), \tag{7.5}$$

$$\tilde{\psi}_2(\zeta,x,t) = e^{i\lambda x}\left[1-iCe^{iAx}\Gamma^{-1}e^{iAx+4iA^2 t}M\bar{A}\left(\bar{A}-\lambda I\right)^{-1}e^{-2i\bar{A}\,x-4i\bar{A}^2 t}\bar{B}\right]\Upsilon(1-a,0), \tag{7.6}$$

$$\bar{\tilde{\psi}}_1(\zeta,x,t) = e^{-i\lambda x}\left[1+i\bar{C}e^{-i\bar{A}\,x}\bar{\Gamma}^{-1}e^{-i\bar{A}\,x-4i\bar{A}^2 t}\bar{M}A(A-\lambda I)^{-1}e^{2iAx+4iA^2 t}B\right]\Upsilon(-b-1,b-1), \tag{7.7}$$

$$\bar{\tilde{\psi}}_2(\zeta,x,t) = -i\sqrt{\lambda}\,e^{-i\lambda x}\left[C\,e^{iAx}\Gamma^{-1}(A-\lambda I)^{-1}e^{iAx+4iA^2 t}B\right]\Upsilon(-b-1,b-1), \tag{7.8}$$

where M and $\bar{M}$ are the constant matrices defined in (4.9), the parameter λ is related to the spectral parameter ζ as in (2.12), and we have defined the scalar quantity $\Upsilon(a,b)$ as

$$\Upsilon(a,b) := \exp\left(2a\int_x^\infty dz\, Q(z,t) + 2b\int_{-\infty}^x dz\, Q(z,t)\right). \tag{7.9}$$

Note that $\Upsilon(a,b)$ can alternatively be evaluated by replacing $Q(z,t)$ in (7.9) by $iP(z,t)$, as indicated in (3.27).

(c) *If (7.1) and (7.2), or equivalently (7.3) and (7.4), are used as the potentials in (5.1), then the corresponding Jost solutions $\tilde{\phi}(\zeta,x,t)$ and $\tilde{\bar{\phi}}(\zeta,x,t)$ with components similarly defined as in (2.10) are explicitly expressed by the formulas*

$$\tilde{\phi}_1(\zeta,x,t) = e^{-i\lambda x}\bar{T}(\zeta,t)\Big[1 + i\bar{C}e^{-i\bar{A}x}\bar{\Gamma}^{-1}\, e^{-i\bar{A}x-4i\bar{A}^2t}\bar{M}A(A-\lambda I)^{-1}e^{2iAx+4iA^2t}B\Big]\Upsilon(-1,b), \tag{7.10}$$

$$\tilde{\phi}_2(\zeta,x,t) = -i\sqrt{\lambda}e^{-i\lambda x}\bar{T}(\zeta,t)\Big[Ce^{iAx}\Gamma^{-1}(A-\lambda I)^{-1}e^{iAx+4iA^2t}B\Big]\Upsilon(1,a), \tag{7.11}$$

$$\tilde{\bar{\phi}}_1(\zeta,x,t) = i\sqrt{\lambda}\, e^{i\lambda x}\, T(\zeta,t)\Big[\bar{C}\, e^{-i\bar{A}x}\bar{\Gamma}^{-1}\big(\bar{A}-\lambda I\big)^{-1}e^{-i\bar{A}x-4i\bar{A}^2t}\bar{B}\Big]\Upsilon(-1,b), \tag{7.12}$$

$$\tilde{\bar{\phi}}_2(\zeta,x,t) = e^{i\lambda x}\, T(\zeta,t)\Big[1 - iC\, e^{iAx}\Gamma^{-1}\, e^{iAx+4iA^2t}M\bar{A}\big(\bar{A}-\lambda I\big)^{-1}e^{-2i\bar{A}x-4i\bar{A}^2t}\bar{B}\Big]\Upsilon(1,a), \tag{7.13}$$

where $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ are expressed in terms of the matrices A and $\bar{A}$ as in (4.38) or equivalently as in (4.39).

Proof. We get (7.1) from (1.20) with the help of (4.25) and (4.31). Similarly, we get (7.2) from (1.20) with the help of (4.25) and (4.32). The alternate expressions in (7.3) and (7.4) are obtained from (7.1) and (7.2), respectively, by using (3.27). Hence, the proof of (a) is complete. We obtain (7.5)–(7.8) from (5.6) and (5.7) with the help of (4.25) and (4.48)–4.51). Thus, the proof of (b) is complete. We obtain (7.10)–(7.13) from (5.8) and (5.9) with the help of (4.25) and (4.56) with $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ expressed in terms of the matrices A and $\bar{A}$ as in (4.39). Hence, the proof of (c) is also complete. □

VIII. THE REDUCTIONS

When the dependent variables $q(x,t)$ and $r(x,t)$ in the integrable nonlinear system (1.2) are related to each other in some way, the nonlinear system (1.2) consisting of two equations may be reduced to a single equation in one dependent variable. In this section, we consider the two common types of reductions, namely,

$$r(x,t) = q(x,t)^*, \quad r(x,t) = -q(x,t)^*, \tag{8.1}$$

where we recall that we use an asterisk to denote complex conjugation. We treat the two cases simultaneously by writing (8.1) as $r(x,t) = \pm q(x,t)^*$, and we analyze the corresponding reduced equations simultaneously by keeping in mind that, in our analysis in this section, the upper signs in ± and ∓ refer to the first case in (8.1) and the lower signs refer to the second case.

Using (8.1) in (1.2), we see that, in each of the two cases, the nonlinear system (1.2) reduces to the single nonlinear equation given by

$$i\,q_t + q_{xx} \mp i\big(|q|^2 q\big)_x = 0, \qquad x\in\mathbb{R}, \quad t\in\mathbb{R}. \tag{8.2}$$

The two equations in (8.2) are usually called the Kaup–Newell equations.[26] Using (8.1) in (2.1), we obtain the corresponding linear system as

$$\frac{d}{dx}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}-i\zeta^2 & \zeta\, q(x,t)\\ \pm\zeta\, q(x,t)^* & i\zeta^2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \qquad x\in\mathbb{R}. \tag{8.3}$$

In the next theorem, we describe the effect of the reductions of (8.1) on the solution to the direct scattering problem for (8.3). We recall that the direct scattering problem for (8.3) involves, when the potential $q(x,t)$ is given, the determination of the four Jost solutions, the six scattering coefficients, and the bound-state information described by a pair of matrix triplets.

Theorem 8.1. *Assume that the potential $q(x,t)$ appearing in the first-order system (8.3) belongs to the Schwartz class for each fixed $t\in\mathbb{R}$. We have the following:*

(a) *The Jost solutions $\bar{\psi}(\zeta,x,t)$ and $\bar{\phi}(\zeta,x,t)$ to (8.3) with the respective asymptotics (2.3) and (2.5) are related to the Jost solutions $\psi(\zeta,x,t)$ and $\phi(\zeta,x,t)$ to (8.3) with the respective asymptotics (2.2) and (2.4), and we have*

$$\begin{bmatrix}\bar{\psi}_1(\zeta,x,t)\\ \bar{\psi}_2(\zeta,x,t)\end{bmatrix} = \begin{bmatrix}\psi_2(\zeta,x,t)^*\\ \pm\psi_1(\zeta,x,t)^*\end{bmatrix}, \quad \begin{bmatrix}\bar{\phi}_1(\zeta,x,t)\\ \bar{\phi}_2(\zeta,x,t)\end{bmatrix} = \begin{bmatrix}\pm\phi_2(\zeta,x,t)^*\\ \phi_1(\zeta,x,t)^*\end{bmatrix}, \tag{8.4}$$

where the subscripts are used to denote the components as in (2.10) and (2.11).

(b) *The scattering coefficients* $\bar T(\zeta,t)$, $\bar R(\zeta,t)$, *and* $\bar L(\zeta,t)$ *for* (8.3) *appearing in* (2.7) *and* (2.9) *are related to the scattering coefficients* $T(\zeta,t)$, $R(\zeta,t)$, *and* $L(\zeta,t)$ *for* (8.3) *appearing in* (2.6) *and* (2.8) *as*

$$\bar T(\zeta,t) = T(\zeta,t)^*, \quad \bar R(\zeta,t) = \pm R(\zeta,t)^*, \quad \bar L(\zeta,t) = \pm L(\zeta,t)^*. \tag{8.5}$$

(c) *The matrix triplets* (A,B,C) *and* $(\bar A,\bar B,\bar C)$ *appearing in the scattering dataset* (2.38) *and describing the bound-state information for* (8.3) *are related to each other as*

$$(\bar A,\bar B,\bar C) = (A^*,B^*,\pm C^*). \tag{8.6}$$

In fact, since B *and* $\bar B$ *appearing in* (8.6) *are both real, we actually have* $\bar B = B$. *We note that* (8.6) *implies that the two sets in* (2.27) *containing the bound-state information for* (8.3) *are related to each other as*

$$\bar\lambda_j = \lambda_j^*, \quad \bar m_j = m_j, \quad \bar c_{jk} = \pm c_{jk}^*, \tag{8.7}$$

where we have $1\le j\le N$ *and* $0\le k\le m_j-1$.

Proof. The first equality in (8.4) is proved by showing that the vector on the right-hand side of that equality satisfies (8.3) with the spatial asymptotics given in (2.3). Similarly, the second equality in (8.4) is proved by showing that the vector on the right-hand side of that equality satisfies (8.3) with the spatial asymptotics given in (2.5). Hence, the proof of (a) is complete. The proof of (b) is obtained by using (8.4) and the fact that the two vectors on the right-hand sides of the two equalities in (8.4) satisfy the spatial asymptotics given in (2.7) and (2.9), respectively. The proof of (c) is as follows. From the first equality of (8.5), we get

$$\bar\lambda_j = \lambda_j^*, \quad \bar m_j = m_j, \quad \bar A_j = A_j^*, \quad \bar N = N, \qquad 1\le j\le N, \tag{8.8}$$

and using (8.8) in (2.28), we obtain $\bar A = A^*$. Next, we use the second equality of (8.8) and determine that the number of entries in the two column vectors B_j and $\bar B_j$ must be equal. Then, from the first equalities of (2.32) and (2.34), we see that $\bar B_j = B_j$. Using $\bar B_j = B_j$ and the fourth equality of (8.8) in (2.29), we obtain $\bar B = B$. Since B and $\bar B$ are real, we also conclude $\bar B = B^*$. Next, in order to prove that $\bar c_{jk} = \pm c_{jk}^*$, we examine the procedure to construct c_{jk} and $\bar c_{jk}$ from the dataset containing the two transmission coefficients $T(\zeta,0)$ and $\bar T(\zeta,0)$ and the four Jost solutions $\psi(\zeta,x,0)$, $\bar\psi(\zeta,x,0)$, $\phi(\zeta,x,0)$, $\bar\phi(\zeta,x,0)$. That procedure is summarized in Sec. 3 of Ref. 11 with the details provided in Ref. 10. We briefly indicate how that procedure yields $\bar c_{jk} = \pm c_{jk}^*$. From (3.1) and (3.2) of Ref. 11, we know that the transmission coefficients have the expansions

$$T(\zeta) = \frac{t_{jm_j}}{(\lambda-\lambda_j)^{m_j}} + \frac{t_{j(m_j-1)}}{(\lambda-\lambda_j)^{m_j-1}} + \cdots + \frac{t_{j1}}{(\lambda-\lambda_j)} + O(1), \qquad \lambda\to\lambda_j, \tag{8.9}$$

$$\bar T(\zeta) = \frac{\bar t_{j\bar m_j}}{(\lambda-\bar\lambda_j)^{\bar m_j}} + \frac{\bar t_{j(\bar m_j-1)}}{(\lambda-\bar\lambda_j)^{\bar m_j-1}} + \cdots + \frac{\bar t_{j1}}{(\lambda-\bar\lambda_j)} + O(1), \qquad \lambda\to\bar\lambda_j, \tag{8.10}$$

where we refer to the double-indexed quantities t_{jk} and $\bar t_{jk}$ as the "residues." By taking the complex conjugate in (8.9) and using the first two equalities of (8.8) in (8.10), we determine that the residues satisfy $\bar t_{jk} = t_{jk}^*$. Using (3.3) and (3.5) of Ref. 11, we construct the double-indexed dependency constants γ_{jk} and $\bar\gamma_{jk}$. Next, using (8.4) in (3.3) and (3.5) of Ref. 11, we establish that $\bar\gamma_{jk} = \pm\gamma_{jk}^*$. As described in Sec. 3 of Ref. 11, each norming constant c_{jk} consists of a summation of terms containing the products of t_{jl} and γ_{jp}, where for each fixed j, we have $1\le l\le m_j$ and $0\le p\le m_j-1$. Similarly, each norming constant $\bar c_{jk}$ consists of a summation of terms containing the products of $\bar t_{jl}$ and $\bar\gamma_{jp}$, where for each fixed j, we have $1\le l\le \bar m_j$ and $0\le p\le \bar m_j-1$. Hence, as a result of $\bar t_{jk} = t_{jk}^*$ and $\bar\gamma_{jk} = \pm\gamma_{jk}^*$, we get $\bar c_{jk} = \pm c_{jk}^*$. Thus, the third equality of (8.7) is also established. Next, using that third equality in the second equalities of (2.32) and (2.34), we obtain $\bar C_j = \pm C_j^*$. Finally, using $\bar C_j = \pm C_j^*$ in (2.30), we establish $\bar C = \pm C^*$. Thus, the proof is complete. □

We recall that the inverse scattering problem for (8.3) consists of the determination of the potential $q(x,t)$ when the corresponding scattering dataset is known. In order to solve the inverse scattering problem for (8.3), we can use a Marchenko method, which can be obtained by reducing the Marchenko system (3.1) appropriately. The reduction of the Marchenko system (3.1) consisting of four integral equations to the reduced system of a single Marchenko integral equation can be accomplished as follows:

(a) By using the reduction for the right reflection coefficient given in the middle equation in (8.5) and the reduction for the bound-state information given in (8.6), we observe that the two Marchenko kernels $\Omega(y,t)$ and $\bar\Omega(y,t)$ defined in (3.2) are related to each other as

$$\bar\Omega(y,t) = \pm\Omega(y,t)^*, \tag{8.11}$$

where we have also used (3.3) in establishing (8.11).

(b) By using the reduction for the two Jost solutions $\psi(\zeta,x,t)$ and $\bar\psi(\zeta,x,t)$ given in the first equality in (8.4), we determine that the four quantities $K_1(x,y,t)$, $K_2(x,y,t)$, $\bar K_1(x,y,t)$, and $\bar K_2(x,y,t)$ defined in (3.4)–(3.7), respectively, are related to each other as

$$\bar K_1(x,y,t) = K_2(x,y,t)^*, \quad \bar K_2(x,y,t) = \pm K_1(x,y,t)^*. \tag{8.12}$$

(c) Next, we use the reductions (8.11) and (8.12) in the Marchenko system (3.1) or equivalently in the uncoupled Marchenko system (3.8) and the auxiliary system (3.9). In fact, it is the best to use the reductions (8.11) and (8.12) only in the first line of (3.8), and this yields the reduced Marchenko integral equation

$$K_1(x,y,t) \pm \Omega(x+y,t) \pm i\int_x^\infty dz\, K_1(x,z,t)\, \Omega'(z+s,t)\, \Omega(s+y,t)^* = 0, \qquad y > x, \tag{8.13}$$

where the only unknown is the quantity $K_1(x,y,t)$.

We have derived the Marchenko equation for the linear system (8.3), and it is given in (8.13). We remark that (8.13) uses as input the quantity $\Omega(y,t)$ defined in the first equality in (3.2), and the quantity $\Omega(y,t)$ itself is constructed from the right reflection coefficient $R(\zeta,t)$ and the matrix triplet (A,B,C) alone.

In the next theorem, we describe the recovery of the potential $q(x,t)$ from the solution to (8.13).

Theorem 8.2. *Assume that the potential $q(x,t)$ appearing in the first-order system (8.3) belongs to the Schwartz class for each fixed $t\in\mathbb{R}$. Then, $q(x,t)$ can be recovered from the solution $K_1(x,y,t)$ to the reduced Marchenko integral equation (8.13) via*

$$q(x,t) = -2K_1(x,x,t)\, \exp\left(\mp 4i\int_x^\infty dz\, |K_1(z,z,t)|^2\right). \tag{8.14}$$

Proof. We already know how to construct $q(x,t)$ from the solution to the Marchenko system (3.1). It turns out that it is possible to construct the potential $q(x,t)$ by using only the solution $K_1(x,y,t)$ to the reduced Marchenko integral equation (8.13). The construction takes place as follows. Using the second equality of (8.12) in (3.26), we obtain the corresponding quantity $P(x,t)$ as

$$P(x,t) = \pm|K_1(x,x,t)|^2. \tag{8.15}$$

Then, using (8.15) in (3.28), we get the key quantity $E(x,t)$ as

$$E(x,t) = \exp\left(\pm 2i\int_{-\infty}^x dz\, |K_1(z,z,t)|^2\right).$$

Finally, with the help of (8.15), we use (3.29) to construct $q(x,t)$ and we get (8.14). □

In the next theorem, we describe the construction of $q(x,t)$ appearing in (8.3) in the reflectionless case. We provide an explicit expression for $q(x,t)$ in terms of the matrix triplet (A,B,C) alone.

Theorem 8.3. *Suppose that the potential $q(x,t)$ appearing in (8.3) at $t=0$ belongs to the Schwartz class and that the corresponding reflection coefficient $R(\zeta,0)$ is zero. Then, $q(x,t)$ can be constructed explicitly in terms of the matrix triplet (A,B,C) alone, and this can be done by using (8.14), where $K_1(x,x,t)$ is explicitly constructed in terms of the matrix triplet (A,B,C).*

Proof. Since (8.3) corresponds to having $r=\pm q^*$ in (2.1), as seen from (8.14), it is enough to construct $K_1(x,x,t)$ in terms of the matrix triplet (A,B,C), where $K_1(x,x,t)$ is listed in (4.29). From Theorem 8.1, we know that $(\bar A,\bar B,\bar C)$ can be expressed in terms of (A,B,C) as in (8.6). Then, using (8.6) in (4.29), we get

$$K_1(x,x,t) = -C^*\, e^{-iA^*x}\left(\Gamma(x,t)^*\right)^{-1} e^{-iA^*x-4i(A^*)^2t}\, B, \tag{8.16}$$

where we have used $\bar B = B$ and $\bar\Gamma = \Gamma^*$. The equality $\bar B = B$ is proved in Theorem 8.1(c). The proof of $\bar\Gamma=\Gamma^*$ can be given as follows. Using (8.6) in (4.9), we obtain $\bar M = M^*$. Then, using (8.6) in (4.8), we confirm that $\bar\Gamma=\Gamma^*$. Next, using (8.6) in (4.7) and in the first equality of (4.9), we see that Γ can be explicitly constructed in terms of the matrix triplet (A,B,C). Thus, we also see that the right-hand side of (8.16) can be explicitly constructed in terms of (A,B,C). Hence, the proof is complete. □

In the following proposition, we show that if the unperturbed potentials $q(x,t)$ and $r(x,t)$ are related to each other as $r(x,t)=\pm q(x,t)^*$, then the perturbed potentials $\tilde q(x,t)$ and $\tilde r(x,t)$ are also related to each other in almost the same manner.

Proposition 8.4. Assume that the solution pair $q(x,t)$ and $r(x,t)$ to (1.2) belongs to the Schwartz class for each fixed $t\in\mathbb{R}$. Let us also assume that $q(x,t)$ and $r(x,t)$ are related to each other as in (8.1), i.e., $r(x,t)=\pm q(x,t)^*$. Then, the solution pair $\tilde q(x,t)$ and $\tilde r(x,t)$ to (1.6) is related to each other as

$$\tilde r(x,t) = \pm\frac{1}{|\kappa|^2}\,\tilde q(x,t)^*, \tag{8.17}$$

where κ is the complex parameter appearing in (1.6).

Proof. When (8.1) holds, we observe that the quantity $E(x,t)$ defined in (1.19) satisfies

$$E(x,t)^* = \frac{1}{E(x,t)}. \tag{8.18}$$

Using (8.18) in (1.20), we obtain (8.17). □

IX. EXPLICIT EXAMPLES

In this section, we provide some explicit examples to illustrate the theory presented in Secs. I–VIII. We remark that the results in Sec. IV already contain the explicit solution formulas in the reflectionless case both for the unperturbed linear problem (2.1) and the unperturbed nonlinear problem (1.2), respectively, where those formulas are expressed in terms of matrix exponentials. Similarly, as described in Secs. V and VII, we also have the explicit solution formulas in the reflectionless case both for the perturbed linear problem (5.1) and the perturbed nonlinear problem (1.6), where again those formulas are expressed in terms of matrix exponentials. In general, a matrix exponential function of x and t consists of algebraic combinations of polynomials, exponential functions, and trigonometric functions in those two independent variables. As the size in the matrix exponential becomes large, the "unpacking" of a matrix exponential, i.e., expressing it in terms of elementary functions consisting of polynomials, exponential functions, and trigonometric functions, becomes impractical in the sense that the resulting expressions become extremely lengthy as the matrix size increases. Nevertheless, in this section, in order to demonstrate the power of our method, we present a few examples where we unpack the matrix exponentials and present the solutions in terms of elementary functions not containing any matrix exponentials.

We mention that, in the reflectionless case, we have prepared a Mathematica notebook, available[12] from the first author, containing explicit solutions for the perturbed nonlinear system (1.6) and for the linear system (5.1), where the user can specify the input by providing the three parameters a, b, κ appearing in (1.5) as well as the two matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ appearing in (2.28)–(2.30). Certainly, the special choice $(a, b, \kappa) = (0, 0, 1)$ in the input yields explicit solutions for the unperturbed nonlinear system (1.2) and for the linear system (2.1). Our Mathematica notebook not only unpacks all the matrix exponentials in the solution formulas but also verifies that the expressions for $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ not involving matrix exponentials indeed satisfy the corresponding nonlinear system (1.6).

In the reflectionless case, we have the explicit expressions for all the relevant quantities both for the unperturbed linear system in (2.1) and the perturbed linear system (5.1). The relevant quantities include the Jost solutions, the potentials, the transmission coefficients, the key quantity $E(x,t)$ defined in (1.19), and the complex constant μ defined in (2.14). When the input set consisting of (a, b, κ), (A, B, C), and $(\bar{A}, \bar{B}, \bar{C})$ is specified, our Mathematica notebook also displays the aforementioned relevant quantities explicitly in terms of elementary functions and without any matrix exponentials, and it verifies that the Jost solutions satisfy the corresponding linear systems.

In the first example below, we elaborate on Theorem 4.3 by choosing our reflectionless input data containing two matrix triplets of unequal sizes.

Example 9.1. From Theorem 4.3(c), in the reflectionless case, we know that the potentials $q(x,t)$ and $r(x,t)$ cannot both belong to the Schwartz class for all fixed $t \in \mathbb{R}$ unless the sizes of the two matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are equal. To illustrate this, we choose our input data as

$$a = 0, \quad b = 0, \quad \kappa = 1, \tag{9.1}$$

$$A = \begin{bmatrix} i \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 0 \\ 0 & -2i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 3 & 1 \end{bmatrix}, \tag{9.2}$$

where we recall that a, b, and κ are the parameters appearing in (1.5). With the help of (4.31) and (4.32), after unpacking all the matrix exponentials, we obtain the corresponding potentials $q(x,t)$ and $r(x,t)$ expressed in terms of elementary functions as

$$q(x,t) = \frac{6e^{-2x+4i\,t}\left(18e^{6x} - 27i\,e^{2x} - 4i\,e^{12i\,t}\right)\left(18e^{6x} + 6e^{4x+12i\,t} - ie^{12i\,t}\right)}{\left(18e^{6x} + 27i\,e^{2x} + 8i\,e^{12i\,t}\right)^2}, \tag{9.3}$$

$$r(x,t) = -\frac{72e^{4x-4i\,t}\left(18e^{6x} + 27i\,e^{2x} + 8i\,e^{12i\,t}\right)}{\left(18i\,e^{6x} + 27e^{2x} + 4e^{12i\,t}\right)^2}. \tag{9.4}$$

With the help of (3.11), (3.13), and (4.25)–(4.27), we also get

$$E(x,t) = \frac{54e^{2x} + 8e^{12i\,t} + 36i\,e^{6x}}{27e^{2x} + 8e^{12i\,t} - 18i\,e^{6x}}, \quad \mu = 2\pi - 2i\ln(2), \tag{9.5}$$

where we recall that the scalar function $E(x,t)$ and the constant μ are the quantities defined in (1.19) and (2.14), respectively. One can directly verify that (1.2) is satisfied by the quantities $q(x,t)$ and $r(x,t)$ appearing in (9.3) and (9.4), respectively. In this example, the matrix A and the matrix $\bar{A}$ have unequal sizes, and hence we cannot expect both $q(x,t)$ and $r(x,t)$ to belong to the Schwartz class for all fixed $t \in \mathbb{R}$. In fact, from (9.3) and (9.4), we conclude that

$$q(x,t) = \begin{cases} \frac{3}{8}\, e^{-2x+4it}[1+o(1)], & x \to -\infty, \\ 6\, e^{-2x+4it}[1+o(1)], & x \to +\infty, \end{cases} \tag{9.6}$$

$$r(x,t) = \begin{cases} -36i\, e^{4x-24it}[1+o(1)], & x \to -\infty, \\ 72\, e^{-2x+4it}[1+o(1)], & x \to +\infty, \end{cases}$$

and hence from the first line of (9.6), we observe that the function $q(x,t)$ does not belong to the Schwartz class at any fixed value of t. In fact, from the denominator in (9.3), it follows that $|q(x,t)|$ has singularities when we have

$$\left[27e^{2x} + 8\cos(12t)\right]^2 + \left[18e^{6x} - 8\sin(12t)\right]^2 = 0. \tag{9.7}$$

Similarly, from the denominator in (9.4), it follows that $|r(x,t)|$ has singularities when we have

$$\left[27e^{2x} + 4\cos(12t)\right]^2 + \left[18e^{6x} + 4\sin(12t)\right]^2 = 0. \tag{9.8}$$

Let us first analyze the singularities of $|q(x,t)|$. Using $e^{6x} = (e^{2x})^3$, we can eliminate x in (9.7), and hence we see that (9.7) is satisfied if and only if we have

$$128\cos^3(12t) + 2187\sin(12t) = 0. \tag{9.9}$$

Note that (9.9) is equivalent to

$$\frac{\sin(12t)}{\cos(12t)}\sec^2(12t) = -\frac{128}{2187},$$

which, in turn, is equivalent to

$$\tan(12t)\left[1 + \tan^2(12t)\right] = -\frac{128}{2187}. \tag{9.10}$$

Since (9.10) is a cubic equation in $\tan(12t)$, it can be solved explicitly by using algebra. Consequently, all real solutions to (9.10) are obtained as

$$t = \frac{1}{12}\left[\pi n - \tan^{-1}(0.058\,263\bar{2})\right], \qquad n \in \mathbb{Z}, \tag{9.11}$$

where an overbar on a digit indicates a roundoff on that digit. Using the values of t given in (9.11), with the help of (9.7), we evaluate the corresponding x-values. It turns out that the even integer values of n in (9.11) yield complex x-values, and hence they should be excluded. As a result, we determine that the singularities of $|q(x,t)|$ occur in a periodic fashion when we have

$$(x,t) = \left(-0.609\,04\bar{7}, -0.004\,855\,2\bar{7} + \frac{\pi}{12}(2n-1)\right), \qquad n \in \mathbb{Z}. \tag{9.12}$$

In fact, from (9.12), we conclude that the singularity when $t > 0$ occurs the first time at $n = 1$, which corresponds to $t = 0.256\,94\bar{4}$. In Fig. 1, we display the behavior of $|q(x,t)|$ during one period, where from (9.7) we see that the period is equal to $\pi/6$. A similar analysis can be used to determine the singularities of $|r(x,t)|$ with the help of (9.8), from which we determine that those singularities occur periodically when we have

$$(x,t) = \left(-0.954\,82\bar{5}, 0.001\,218\,9\bar{8} + \frac{\pi}{12}(2n-1)\right), \qquad n \in \mathbb{Z}. \tag{9.13}$$

From (9.13), we conclude that the singularity when $t > 0$ occurs the first time at $n = 1$, which corresponds to $t = 0.263\,01\bar{8}$. In Fig. 2, we display the behavior of $|r(x,t)|$ during one period, where from (9.13) we see that the period is equal to $\pi/6$. With the help of our prepared Mathematica notebook, we can observe the animations for each of $|q(x,t)|$ and $|r(x,t)|$. Let us remark that, in our input of (9.1) and (9.2), if we change C and $\bar{C}$ without changing the rest of the input, we may get a different value of the constant μ than that given in the second equality of (9.5). In fact, if we use

$$C = \begin{bmatrix} i \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} i & i \end{bmatrix},$$

we then get $\mu = -2\pi - 2i\ln(2)$, which differs by 4π from the μ-value given in (9.5). We note that a difference of 4π in the two μ-values is consistent with the result stated in Theorem 4.4, even though the potentials do not belong to the Schwartz class. We add the cautious remark that the evaluation of integrals involving the inverse tangent function by using Mathematica may not yield correct values when the argument

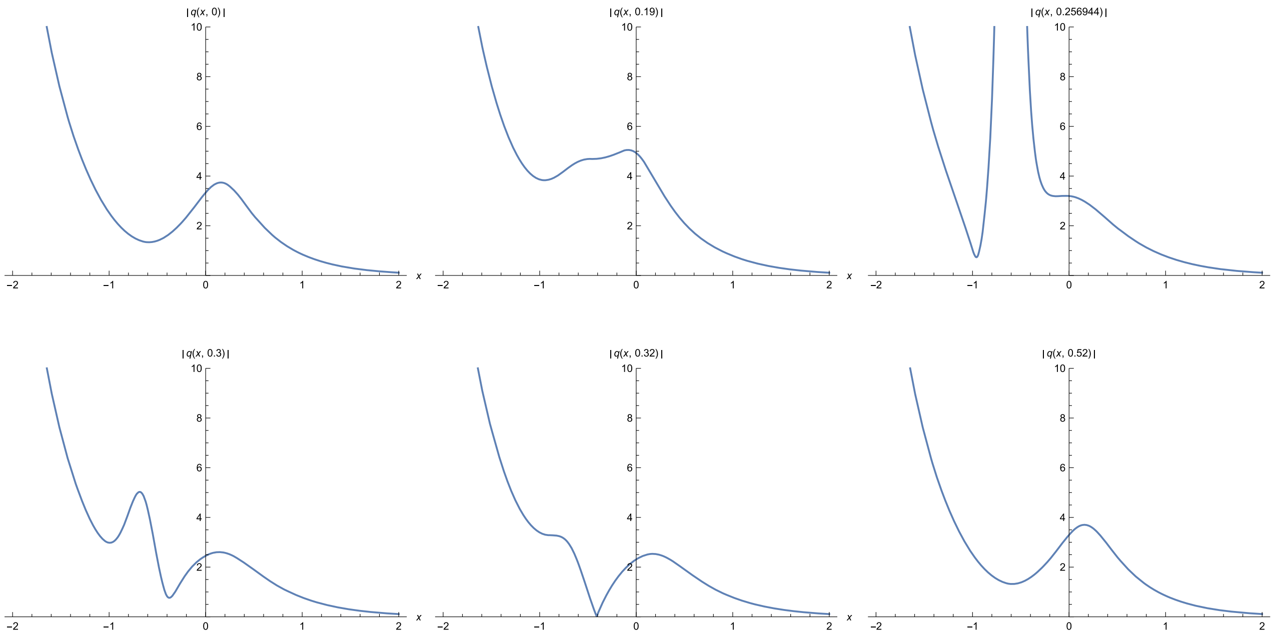

FIG. 1. The snapshots for $|q(x,t)|$ of (9.3) at several t-values in Example 9.1.

of that function is complex valued. For example, the use of (3.13) in Mathematica may not always yield the correct value of μ, and thus, it is better to use (2.15) in Mathematica in the evaluation of μ. In this example, from (4.39), we obtain the two transmission coefficients $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ as

$$T(\zeta,t)=\frac{-i(\lambda+i)(\lambda+2i)}{2(\lambda-i)},\quad \bar{T}(\zeta,t)=\frac{-2(\lambda-i)}{i(\lambda+i)(\lambda+2i)},\tag{9.14}$$

where we recall that $\lambda=\zeta^2$ as stated in (2.12). Note that the properties of the transmission coefficients listed in (9.14) do not agree with (2.17)–(2.20) because the potentials $q(x,t)$ and $r(x,t)$ do not belong to the Schwartz class for each fixed $t\in\mathbb{R}$. Finally, let us remark that, in this example, the potential $r(x,t)$ given in (9.4) satisfies the second equality in (4.46) when the integral there is interpreted as a Cauchy principal value. We determine that the imaginary part of $q(x,t)$ displayed in (9.3) satisfies the first equality in (4.46), i.e., the Cauchy principal value of the integral of the imaginary part of $q(x,t)$ over $x\in\mathbb{R}$ is zero for each fixed $t\in\mathbb{R}$. However, the Cauchy principal value of the integral of the real part of $q(x,t)$ over $x\in\mathbb{R}$ is equal to $+\infty$, and hence the first equality in (4.46) does not hold.

In the next example, we illustrate Theorem 4.4 with the potentials belonging to the Schwartz class and demonstrate that a change in the matrices C and $\bar{C}$ may result in a change in the value of the constant μ, as stated in (4.45).

Example 9.2. Using a, b, κ, and the matrix triplet (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ given by

$$a=0,\quad b=0,\quad \kappa=1,\tag{9.15}$$

$$A=\big[5i\big],\quad B=\big[1\big],\quad C=\big[2\big],\quad \bar{A}=\big[-3i\big],\quad \bar{B}=\big[1\big],\quad \bar{C}=\big[3\big],\tag{9.16}$$

as input in (4.31) and (4.32), after unpacking all the matrix exponentials, we obtain the corresponding potentials $q(x,t)$ and $r(x,t)$ as

$$q(x,t)=\frac{192\,e^{10(x+10i\,t)}\left(32\,e^{16(x+4i\,t)}-15i\right)}{\left(32\,e^{16(x+4i\,t)}+9i\right)^2},\tag{9.17}$$

$$r(x,t)=\frac{128\,e^{6(x-6i\,t)}\left(32\,e^{16(x+4i\,t)}+9i\right)}{\left(32\,e^{16(x+4i\,t)}-15i\right)^2},\tag{9.18}$$

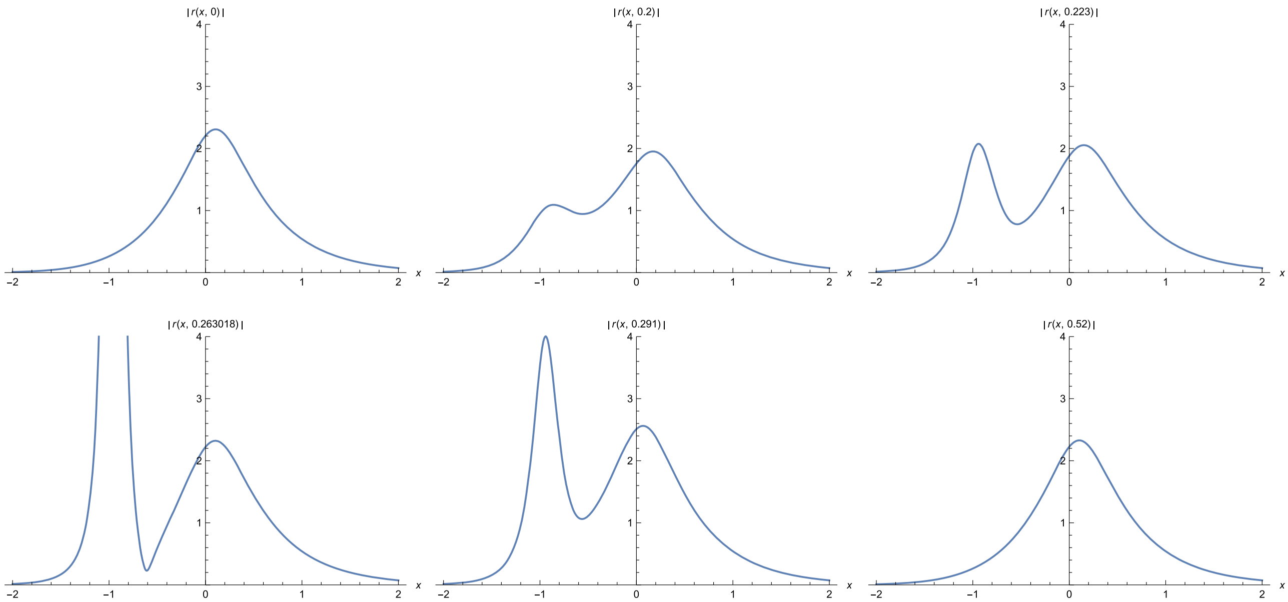

FIG. 2. The snapshots for $|r(x,t)|$ of (9.4) at several t-values in Example 9.1.

which satisfy (1.2). In this example, we obtain the scalar quantity $E(x,t)$, the constant μ, and the two transmission coefficients $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ defined in (1.19), (2.14), and (4.39), respectively, as

$$E(x,t) = \frac{-96e^{16(x+4it)} + 45i}{160e^{16(x+4it)} + 45i}, \quad \mu = 2\pi + 2i \ln(5/3), \tag{9.19}$$

$$T(\zeta,t) = -\frac{5}{3}\left(\frac{\lambda+3i}{\lambda-5i}\right), \quad \bar{T}(\zeta,t) = -\frac{3}{5}\left(\frac{\lambda-5i}{\lambda+3i}\right), \tag{9.20}$$

where we recall that $\lambda = \zeta^2$. From the denominators in (9.20), we see that there are two simple bound states occurring at the λ-values corresponding to the poles of $T(\zeta,t)$ and $\bar{T}(\zeta,t)$, respectively. Our prepared Mathematica notebook provides the animations for $|q(x,t)|$ and $|r(x,t)|$. In Fig. 3, we illustrate the periodic behavior of $|q(x,t)|$ during one period. As observed from the snapshots in Fig. 3, at $t = 0$ there are two overlapping solitons, and as time progresses, they first separate from each other and then they overlap again. With further progress in time, their combined amplitude increases to a finite peak value, and then that combined amplitude decreases when one period is completed. In time, the behavior described during one period keeps repeating itself. We remark that during the first period, the combined amplitude reaches a finite peak value when $t = 4.000\overline{1}$ at $x = -0.79\overline{3}$. The period for the evolution of $|q(x,t)|$ is equal to $5.\overline{6}$. The behavior of $|r(x,t)|$ is very similar to the behavior of $|q(x,t)|$, and it is also periodic with the same period. There is only a minor difference in the behaviors $|q(x,t)|$ and $|r(x,t)|$, and that is why we do not include any snapshots for $|r(x,t)|$. The minor difference is that, when the two solitons in $|q(x,t)|$ separate, the left soliton has a higher amplitude, whereas the right soliton in $|r(x,t)|$ has a higher amplitude. In this example, if we change the matrices C and $\bar{C}$ in (9.16) and instead use

$$C = \begin{bmatrix} i \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} i \end{bmatrix},$$

without changing the rest of the input in (9.15) and (9.16), we get a different pair of potentials $q(x,t)$ and $r(x,t)$, which are given by

$$q(x,t) = \frac{128i\, e^{10(x+10it)}\left(64\, e^{16(x+4it)} + 5i\right)}{\left(64\, e^{16(x+4it)} - 3i\right)^2}, \tag{9.21}$$

$$r(x,t) = \frac{128i\, e^{6(x-6it)}\left(64\, e^{16(x+4it)} - 3i\right)}{\left(64\, e^{16(x+4it)} + 5i\right)^2}. \tag{9.22}$$

The value of the scalar quantity $E(x,t)$ also changes, and the modified value is given by

$$E(x,t) = -\frac{192e^{16(x+4it)} + 15i}{320e^{16(x+4it)} - 15i}.$$

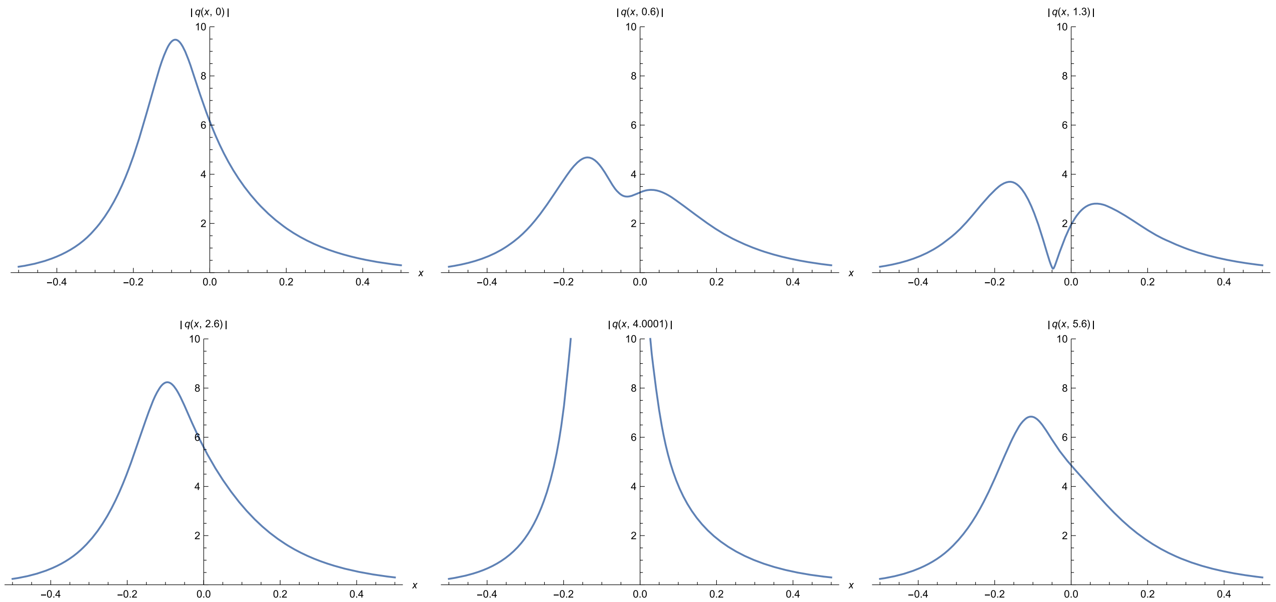

FIG. 3. The snapshots for $|q(x,t)|$ of (9.17) at several t-values in Example 9.2.

The value of the complex constant μ also changes, and the modified value is given by

$$\mu = -2\pi + 2i\,\ln(5/3). \tag{9.23}$$

On the other hand, as seen from (4.39), the transmission coefficients $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ given in (9.20) are not affected by the change in C and $\bar{C}$. We note that the μ-values in (9.19) and (9.23) differ from each other by 4π, which is compatible with (4.45). Let us finally remark on the effect of modifying the parameters a and b appearing (9.15) without changing their difference. For example, in the input of (9.15) and (9.16), let us use

$$a = 5, \quad b = 5, \quad \kappa = 1,$$

without changing (9.16). In that case, as seen from (1.20), the potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ appearing in the nonlinear problem are not affected, and they agree with $q(x,t)$ and $r(x,t)$, respectively. However, the remaining quantities appearing in the associated linear problem are affected. In particular, the Jost solutions are affected as indicated in (5.6)–(5.9), the AKNS pair $(\tilde{\mathcal{X}},\tilde{\mathcal{T}})$ is affected as indicated in (1.9) and (1.12), and the transmission coefficients are affected as indicated in (5.10) and (5.11). For example, the transmission coefficients $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ in (9.20) are transformed into $\tilde{T}(\zeta,t)$ and $\tilde{\bar{T}}(\zeta,t)$, respectively, which are given by

$$\tilde{T}(\zeta,t) = e^{5i\mu/2}\,T(\zeta,t), \quad \tilde{\bar{T}}(\zeta,t) = e^{-5i\mu/2}\,\bar{T}(\zeta,t). \tag{9.24}$$

From the second equality in (9.19), we get $e^{5i\mu/2} = -(3/5)^5$, and hence using (9.20) in (9.24), we obtain

$$\tilde{T}(\zeta,t) = \frac{81}{625}\left(\frac{\lambda+3i}{\lambda-5i}\right), \quad \tilde{\bar{T}}(\zeta,t) = \frac{625}{81}\left(\frac{\lambda-5i}{\lambda+3i}\right).$$

In this example, the potential pair $q(x,t)$ and $r(x,t)$ displayed in (9.17) and (9.18) satisfies (4.46). Similarly, the potential pair $q(x,t)$ and $r(x,t)$ displayed in (9.21) and (9.22) also satisfies (4.46).

In the next example, we illustrate the soliton solutions to (1.2) when the input matrix triplets each has size 2. This example also illustrates the fact that the use of matrix exponentials in expressing solitons solutions is crucial when the number of bound states is high. As seen from (4.31) and (4.32), those solutions are expressed in a compact form using matrix exponentials. As demonstrated in the next example, expressing the solutions in elementary functions after unpacking the matrix exponentials, we obtain explicit but lengthy expressions without gaining much physical insights.

Example 9.3. In this example, we use the reflectionless input dataset consisting of (a,b,κ) and the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$, where we have let

$$a = 0, \quad b = 0, \quad \kappa = 1, \tag{9.25}$$

$$A = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3i & 2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 3i \end{bmatrix}, \tag{9.26}$$

as input in (4.31) and (4.32), and we obtain the corresponding potentials $q(x,t)$ and $r(x,t)$ as

$$q(x,t) = \frac{w_5 + w_6}{w_7^2}, \tag{9.27}$$

$$r(x,t) = \frac{w_8 + w_9}{w_{10}^2}, \tag{9.28}$$

which satisfy (1.2) and where w_5, w_6, w_7, w_8, w_9, and w_{10} are defined as

$$w_5 := -32e^{-6x+4it}\big[24t + 4e^{4x}(-3i + 16t + 4ix) - i(1+6x)\big],$$

$$w_6 := 9 + 16e^{4x}\big[5 - 14x + 24x^2 + 4e^{4x} + 8t(i + 48t)\big],$$

$$w_7 := -32\big[-7x + 84x^2 + 4t(48t - 11i)\big] + 73\cosh(4x) + 55\sinh(4x),$$

$$w_8 := 8e^{-6x-4it}\big[9(1 - 4x + 16it) - 32e^{4x}(-1 + 3x + 12it)\big],$$

$$w_9 := 9 + 32e^{4x}\big[7x - 12x^2 + 2e^{4x} + 4t(11i - 48t)\big],$$

$$w_{10} := 16\big[5 - 14x + 84x^2 + 8t(48t + i)\big] + 73\cosh(4x) + 55\sinh(4x).$$

In this example, we obtain the scalar quantity $E(x,t)$, the constant μ, and the transmission coefficients $T(\zeta,t)$ and $\bar{T}(\zeta,t)$ defined in (1.19), (2.14), and (4.39), respectively, as

$$E(x,t) = \left(\frac{w_{11} + w_{12}}{w_{13} + w_{14}}\right)^{1/2} \exp\left(i\tan^{-1}(128t\, e^{4x}/w_{15}) - i\tan^{-1}(1408t\, e^{4x}/w_{16})\right),$$

$$\mu = 0,$$

$$T(\zeta,t) = \frac{(\lambda+i)^2}{(\lambda-i)^2}, \quad \bar{T}(\zeta,t) = \frac{(\lambda-i)^2}{(\lambda+i)^2}, \tag{9.29}$$

where we recall that $\lambda = \zeta^2$ and we have defined

$$w_{11} := 81 + 4096e^{16x} + \left(288e^{4x} + 2048e^{12x}\right)\left(5 - 14x + 24x^2 + 384t^2\right),$$

$$w_{12} := 128e^{8x}\big[59 - 280x + 872x^2 - 1344x^3 + 1152x^4 + 128t^2(61 - 168x + 288x^2) + 294912t^4\big],$$

$$w_{13} := 81 + 4096e^{16x} - \left(576e^{4x} + 4096e^{12x}\right)\left(-7x + 12x^2 + 192t^2\right),$$

$$w_{14} := 128e^{8x}\big[9 + 392x^2 - 1344x^3 + 1152x^4 + 128t^2(121 - 168x + 288x^2) + 294912t^4\big],$$

$$w_{15} := 9 + 64e^{8x} + 16e^{4x}\left(5 - 14x + 24x^2 + 384t^2\right),$$

$$w_{16} := 9 + 64e^{8x} - 32e^{4x}\left(-7x + 12x^2 + 192t^2\right).$$

As seen from the denominators in (9.29), there are two bound states corresponding to the poles of the transmission coefficients, each with multiplicity two. In this example, we have the even symmetry in time for the absolute values of the potentials $q(x,t)$ and $r(x,t)$ given in (9.27) and (9.28), respectively, i.e., we have

$$|q(x,-t)| = |q(x,t)|, \quad |r(x,-t)| = |r(x,t)|, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}.$$

In Figs. 4 and 5, we present some snapshots of $|q(x,t)|$ and $|r(x,t)|$, respectively. Our prepared Mathematica notebook provides the animations illustrating the time evolutions of $|q(x,t)|$ and $|r(x,t)|$. An analysis on the denominators of $|q(x,t)|$ and $|r(x,t)|$ can be carried as in Example 9.1. We determine that $|q(x,t)|$ has singularities occurring at certain discrete times at which $|q(x,t)|$ becomes equal to $+\infty$ at one particular x-value. On the other hand, $|r(x,t)|$ has no singularities when $x \in \mathbb{R}$ and $t \in \mathbb{R}$. The time evolution of $|q(x,t)|$ depicted in Fig. 4 is as

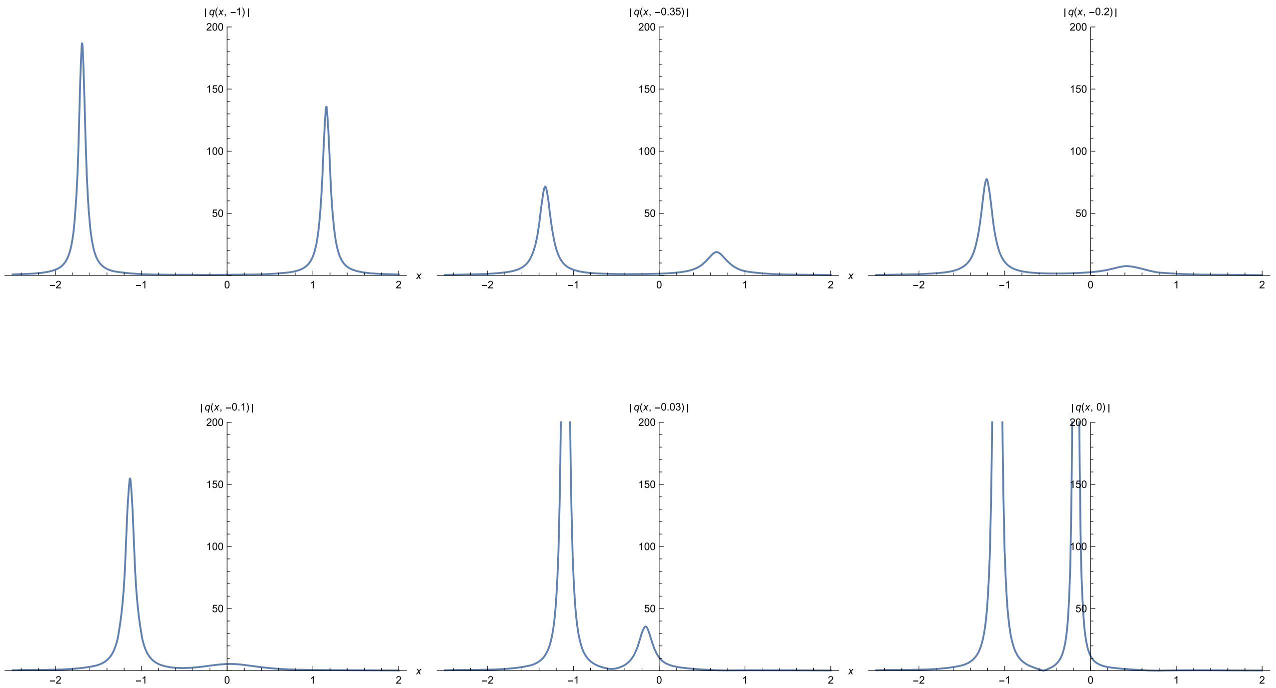

FIG. 4. The snapshots for $|q(x,t)|$ of (9.27) at several t-values in Example 9.3.

follows. First, the two solitons are far apart from each other. As they approach each other, their amplitudes keep changing, and at certain times, they develop singularities. While approaching each other, also their speeds increase. As shown in Fig. 4, the two solitons interact nonlinearly with each other, and during the interaction, their amplitudes get smaller and their widths get larger, but they do not overlap. Then, they are repelled from each other and move away from each other. As they move away from each other, their amplitudes keep changing, and at some particular discrete times they develop singularities where $|q(x,t)|$ becomes equal to $+\infty$ at one particular x-value. The evolution of $|r(x,t)|$ is depicted in Fig. 5 and is as follows. At first, there are two soliton pairs moving toward each other and their speeds increase as the soliton pairs get closer. In each double soliton, the distance between the two peaks remains unchanged. The nonlinear interactions take place roughly during the time interval $t \in (-0.3, 0.3)$. The soliton on the left of the left soliton pair does not interact with the rest. After the soliton pairs complete their collision, the pairs move backward and their speeds decrease as they move away from each other. Our prepared Mathematica notebook provides the animations of $|q(x,t)|$ and $|r(x,t)|$, expresses all the relevant quantities by unpacking the matrix exponentials, and verifies that both the linear and nonlinear systems given in (2.1) and (1.2), respectively, are satisfied.

In the next example, again in the reflectionless case, we use the same input dataset used in Example 9.3, but for three different sets for the matrices C and $\bar{C}$. In all the three cases, the corresponding potentials $q(x,t)$ and $r(x,t)$ have no singularities and they all belong to the Schwartz class for each fixed $t \in \mathbb{R}$.

Example 9.4. In the reflectionless case, let us use

$$a = 0, \quad b = 0, \quad \kappa = 1, \tag{9.30}$$

$$A = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} i & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -i & 1 \end{bmatrix}, \tag{9.31}$$

which agrees with (9.25) and differs from (9.26) only by the values in the matrices C and $\bar{C}$. Using (9.30) and (9.31) as input to (4.31) and (4.32), after unpacking the matrix exponentials, we obtain the corresponding potentials $q(x,t)$ and $r(x,t)$ as

$$q(x,t) = \frac{64\, w_{17}}{w_{18}} \exp\left(2x + 4i\,t - 4i \tan^{-1}(w_{19}/w_{20})\right), \quad r(x,t) = q(x,t)^*, \tag{9.32}$$

where we recall that we use an asterisk to denote complex conjugation and we have defined

$$w_{17} := x + 4i\,t - 8\,e^{4x}(-i + 8t + 2i\,x), \tag{9.33}$$

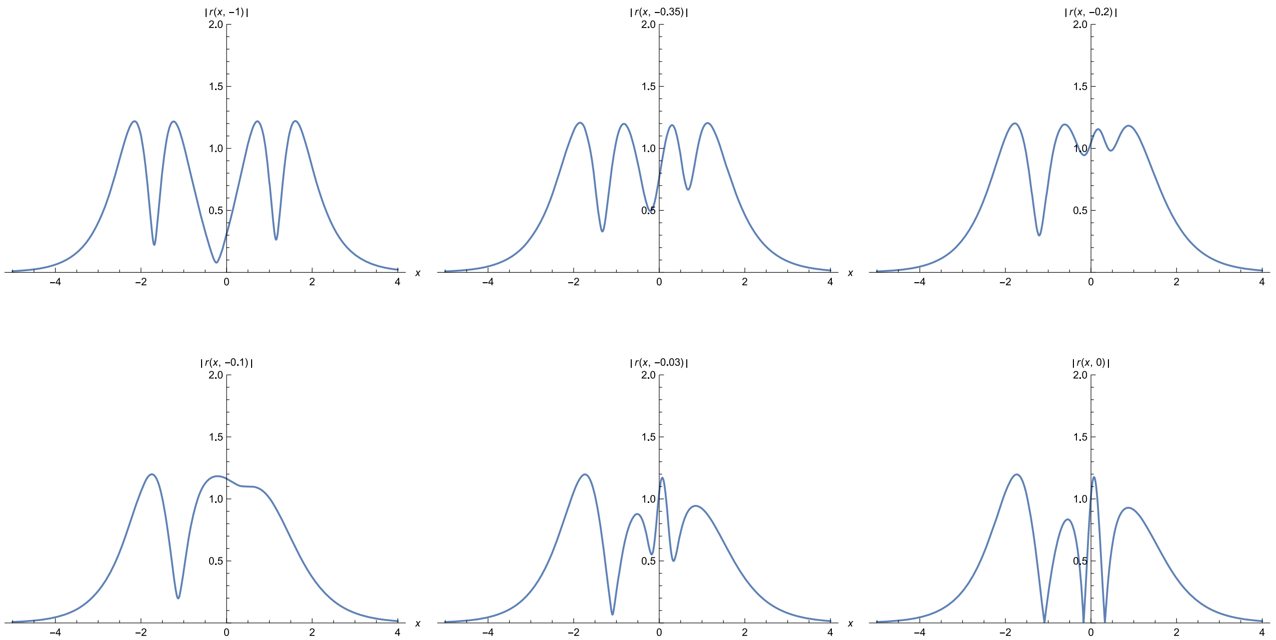

FIG. 5. The snapshots for $|r(x,t)|$ of (9.28) at several t-values in Example 9.3.

$$w_{18} := -i + 256i\, e^{8x} + 32e^{4x}\big(1 - 4x + 8x^2 + 16it + 128t^2\big), \tag{9.34}$$

$$w_{19} := 32e^{4x}\big(1 - 4x + 8x^2 + 128t^2\big), \quad w_{20} := -1 + 256e^{8x} + 512e^{4x}\, t. \tag{9.35}$$

Contrary to the potentials $q(x,t)$ and $r(x,t)$ given in (9.27) and (9.28), the potentials $q(x,t)$ and $r(x,t)$ given in (9.32) have no singularities and they belong to the Schwartz class for each $t \in \mathbb{R}$. For the input data given in (9.30) and (9.31), we obtain the scalar quantity $E(x,t)$ defined in (1.19) and the constant μ appearing in (2.14) as

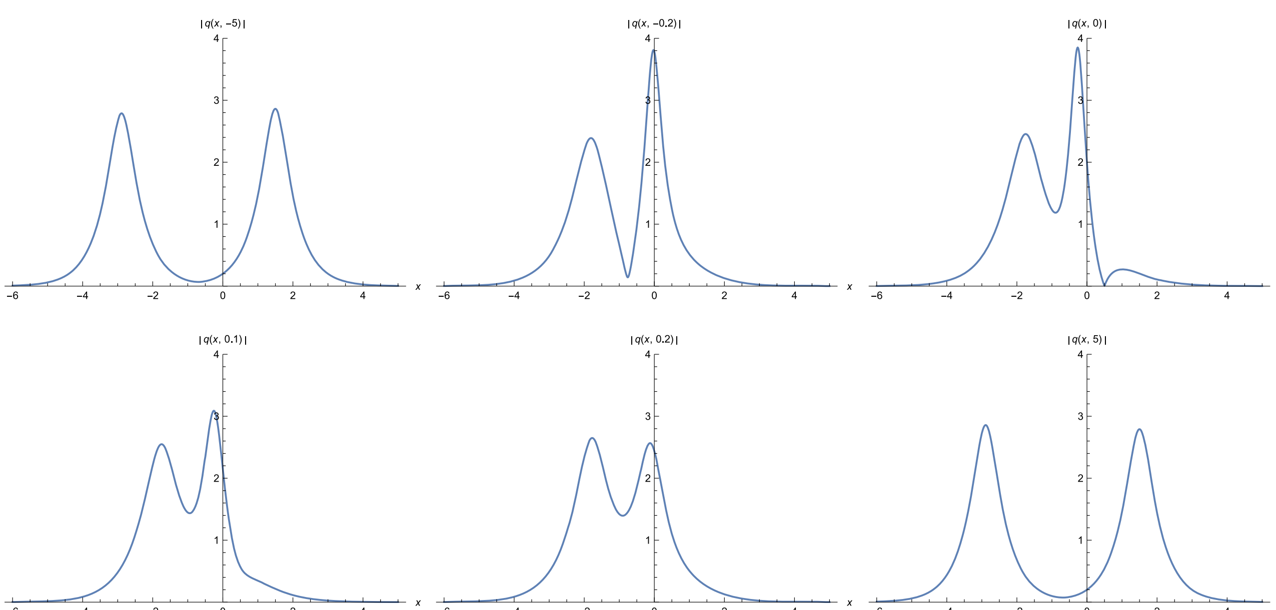

FIG. 6. The snapshots for $|q(x,t)|$ of (9.32) at several t-values in Example 9.4.

$$E(x,t) = \exp\left(-2i\tan^{-1}(w_{19}/w_{20})\right), \quad \mu = 4\pi. \tag{9.36}$$

Since the transmission coefficients are unaffected when we only change C and $\bar{C}$, the transmission coefficients corresponding to the input in (9.30) and (9.31) are still given by (9.52). Because of the symmetry expressed in the second equality in (9.32), we have $|r(x,t)| = |q(x,t)|$, and hence we only discuss the time evolution for $|q(x,t)|$. As seen from Fig. 6, there are two solitons that are initially far apart. They move toward each other, and their speeds increase as they get closer. Then they interact with each other nonlinearly, and then they move away from each other. As they move away from each other, they regain their individual shapes. Our Mathematica notebook provides all the quantities related to the linear and nonlinear problems by unpacking the matrix exponentials, and it verifies that the corresponding linear and nonlinear equations are satisfied. It also confirms that the integrals of $q(x,t)$ and $r(x,t)$ over all x-values at each fixed $t \in \mathbb{R}$ are each zero, as stated in Theorem 4.6. Let us slightly modify the input data given in (9.30) and (9.31) by only changing the matrices C and $\bar{C}$ to the new values given as

$$C = \begin{bmatrix} -i & -1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -i & 1 \end{bmatrix}. \tag{9.37}$$

Then, with the help of (4.31) and (4.32), we obtain the corresponding potentials $q(x,t)$ and $r(x,t)$ as

$$q(x,t) = \frac{64\, w_{22}}{w_{23}} \exp\left(2x + 4i\,t + 4i\tan^{-1}(w_{19}/w_{21})\right), \tag{9.38}$$
$$r(x,t) = -q(x,t)^*,$$

and the quantities $E(x,t)$ and μ are given by

$$E(x,t) = \exp\left(2i\tan^{-1}(w_{19}/w_{21})\right), \quad \mu = -4\pi, \tag{9.39}$$

where we have defined

$$w_{21} := -1 + 256\,e^{8x} - 512\,e^{4x}\,t,$$

$$w_{22} := x + 4i\,t + 8\,e^{4x}(-i + 8t + 2i\,x),$$

$$w_{23} := i - 256i\,e^{8x} + 32\,e^{4x}\left(1 - 4x + 8x^2 + 16it + 128t^2\right).$$

Our Mathematica notebook provides an animation of $|q(x,t)|$ of (9.38), from which we observe that the time evolution of $|q(x,t)|$ in this modified case is similar to the time evolution of $|q(x,t)|$ of (9.32) described earlier and illustrated in Fig. 6. We remark that the μ-value in (9.39) differs from the μ-value in (9.36) by 8π, which agrees with the result in Theorem 4.4. Let us again modify the input dataset given in (9.30) and (9.31) by only changing the matrices C and $\bar{C}$ to the new values given by

$$C = \begin{bmatrix} -i & -1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -i & -1 \end{bmatrix}. \tag{9.40}$$

In this case, the corresponding potentials $q(x,t)$ and $r(x,t)$ and the corresponding quantities $E(x,t)$ and μ are again explicitly determined, and we have

$$q(x,t) = \frac{64\, w_{27}\, w_{28}}{w_{29} + w_{30}} \exp\left(2x + 4i\,t + 2i\tan^{-1}(w_{24}/w_{25}) + 2i\tan^{-1}(w_{26}/w_{20})\right), \tag{9.41}$$

$$r(x,t) = \frac{64\, w_{31}\, w_{32}}{w_{33} + w_{34}} \exp\left(2x - 4i\,t - 2i\tan^{-1}(w_{24}/w_{25}) - 2i\tan^{-1}(w_{26}/w_{20})\right), \tag{9.42}$$

$$E(x,t) = \frac{\sqrt{w_{35} + w_{36}}}{\sqrt{w_{29} + w_{30}}} \exp\left(i\tan^{-1}(w_{24}/w_{25}) + i\tan^{-1}(w_{26}/w_{20})\right), \quad \mu = 0, \tag{9.43}$$

where we have defined

$$w_{24} := 32e^{4x}\left(-3 + 4x + 8x^2 + 128t^2\right),$$

$$w_{25} := -1 + 256e^{8x} - 1536e^{4x}\,t,$$

$$w_{26} := 32e^{4x}\left(1 + 4x + 8x^2 + 128t^2\right),$$

$$w_{27} := x + 4i\,t + 8\,e^{4x}(i + 8t + 2i\,x),$$

$$w_{28} := -i + 256i\,e^{8x} + 32e^{4x}\left(-3 + 4x + 8x^2 - 48it + 128t^2\right),$$

$$w_{29} := 1 + 65\,536\,e^{16x} - 1024\,e^{4x}t + 262\,144\,e^{12x}t,$$

$$w_{30} := 512\, e^{8x}\big[1 + 32\,768t^4 + 16x + 64x^2 + 128x^3 + 128x^4 + 1024t^2(1 + 2x + 4x^2)\big],$$

$$w_{31} := 1 + x - 4i\,t + 8\,e^{4x}(i + 8t - 2i\,x),$$

$$w_{32} := -i + 256i\,e^{8x} - 32e^{4x}\big(1 + 4x + 8x^2 - 16it + 128t^2\big),$$

$$w_{33} := 1 + 65\,536\,e^{16x} + 3072\,e^{4x}t - 786\,432\,e^{12x}t,$$

$$w_{34} := 512\, e^{8x}\big[17 + 32\,768t^4 - 48x - 64x^2 + 128x^3 + 128x^4 + 1024t^2(3 + 2x + 4x^2)\big],$$

$$w_{35} := 1 + 65\,536\,e^{16x} + 3072\,e^{4x}t - 786\,432\,e^{12x}t,$$

$$w_{36} := 512\, e^{8x}\big[17 + 32\,768t^4 - 48x - 64x^2 + 128x^3 + 128x^4 + 1024t^2(3 + 2x + 4x^2)\big].$$

Let us remark that C and $\bar{C}$ appearing in (9.31) are related to each other as $\bar{C} = C^*$ and that C and $\bar{C}$ appearing in (9.37) are related to each other as $\bar{C} = -C^*$. Hence, as indicated in Theorem 8.1, the corresponding potentials are related to each other as $r(x,t) = q(x,t)^*$ and $r(x,t) = -q(x,t)^*$, respectively, for those two cases. However, C and $\bar{C}$ appearing in (9.40) are not related to each other as $\bar{C} = C^*$ or as $\bar{C} = -C^*$, and hence we do not have the reduction $r(x,t) = q(x,t)^*$ and $r(x,t) = -q(x,t)^*$. We also remark that the μ-value in this case given in (9.43) differs by 4π from each of the two μ-values given in (9.36) and (9.39), and those differences are compatible with the result of Theorem 4.4. An analysis of the potentials $q(x,t)$ and $r(x,t)$ appearing in (9.41) and (9.42), respectively, reveals that $q(x,t)$ belongs to the Schwartz class for each fixed $t \in \mathbb{R}$, whereas the potential $r(x,t)$ becomes singular at one particular t-value around $t = -0.2$ and at one corresponding particular x-value, but before and after that singularity moment, the potential $r(x,t)$ belongs to the Schwartz class.

In Example 9.4, we observe that there is no singularity in the potentials if the norming constants are chosen as in (9.31) or as in (9.37), whereas there is a singularity if the norming constants are chosen as in (9.40). When one deals with an inverse problem associated with a non-self-adjoint linear differential operator, it may happen that the eigenvalues are not necessarily real, those eigenvalues are not necessarily simple, and the norming constants are not necessarily positive. In the presence of complex eigenvalues with multiplicities, we may have potentials without developing a singularity, but we may also have potentials developing a singularity at a finite time.

In the next example, we consider a stationary soliton in the reduced case, and we interpret the height and width of that soliton in terms of the bound-state eigenvalue and the norming constant.

Example 9.5. Let us consider the reflectionless case with $(a,b,\kappa) = (0,0,1)$ and by choosing the two matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ as

$$A = [i\lambda_1],\quad B = [1],\quad C = [c_1],\quad \bar{A} = [-i\lambda_1],\quad \bar{B} = [1],\quad \bar{C} = [c_1^*], \tag{9.44}$$

where λ_1 is a positive parameter and c_1 is a complex parameter. We note that the dataset presented in (9.44) satisfies (8.6) with the upper sign used there. Hence, this corresponds to the reduced case with $r(x,t) = q(x,t)^*$. The transmission coefficients corresponding to the input (9.44) can be obtained by using (4.39), and we have

$$T(\zeta,t) = -\frac{\lambda + i\lambda_1}{\lambda - i\lambda_1},\quad \bar{T}(\zeta,t) = -\frac{\lambda - i\lambda_1}{\lambda + i\lambda_1}.$$

We can obtain $q(x,t)$ via (8.14), where the quantity $K_1(x,x,t)$ is constructed by using the input (9.44) in (8.16). Using (9.44) in (4.7)–(4.9), we get

$$M = \left[\frac{c_1^*}{2\lambda_1}\right],\quad \Gamma(x,t) = \left[1 + \frac{i\,|c_1|^2\,e^{-4\lambda_1 x}}{4\lambda_1}\right],\quad \bar{M} = M^*,\quad \bar{\Gamma}(x,t) = \Gamma(x,t)^*. \tag{9.45}$$

With the help of (9.44) and (9.45), from (8.16), we obtain

$$K_1(x,x,t) = \frac{4c_1^*\,e^{2\lambda_1 x + 4i\lambda_1^2 t}}{i\,|c_1|^2 - 4\lambda_1\,e^{4\lambda_1 x}}. \tag{9.46}$$

From (9.46), we see that $|K_1(x,x,t)|$ is independent of t even though $K_1(x,x,t)$ contains t. Using (9.46) in (8.14), after some simplification, we get the stationary soliton given by

$$|q(x,t)| = \frac{\sqrt{2\,\lambda_1}}{\sqrt{\cosh\left(4\lambda_1\left(x - \frac{1}{4\lambda_1}\,\ln\left(\frac{|c_1|^2}{4\lambda_1}\right)\right)\right)}},\qquad x \in \mathbb{R}, \tag{9.47}$$

where ln denotes the natural logarithm. The stationary soliton in (9.47) has the peak amplitude $\sqrt{2\lambda_1}$, and that occurs when the argument of the hyperbolic cosine in the denominator there is zero. Thus, the peak occurs when we have

$$x = \frac{1}{4\lambda_1}\ \ln\left(\frac{|c_1|^2}{4\lambda_1}\right).$$

We can define the width of the stationary soliton as the distance between the two x-values where the amplitude becomes half of the peak amplitude. Hence, the width of the stationary soliton is equal to $[\cosh^{-1}(4)]/(2\lambda_1)$. Thus, the height of the stationary soliton is proportional to $\sqrt{\lambda_1}$ and its width is inversely proportional to λ_1. The norming constant $|c_1|$ only affects the location of the peak, but it does not affect the height or the width of the stationary soliton. In the three plots in Fig. 7, we display the plot of $|q(x,t)|$ when $|c_1| = 1$ and λ_1 takes the values 1, 4, and 9, respectively.

In the next example, we consider the reflectionless case with the reduction in the presence of two eigenvalues without multiplicities.

Example 9.6. In the reflectionless case, let us use the input

$$a = 0, \quad b = 0, \quad \kappa = 1,$$

$$A = \begin{bmatrix} i & 0 \\ 0 & 2i \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 0 \\ 0 & -2i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{9.48}$$

We note that the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ in (9.48) satisfy (8.6) with the upper sign used there. Hence, this corresponds to the reduced case with $r(x,t) = q(x,t)^*$. The transmission coefficients corresponding to the input (9.48) can be obtained by using (4.39), and we get

$$T(\zeta,t) = \frac{(\lambda+i)(\lambda+2i)}{(\lambda-i)(\lambda-2i)}, \quad \bar{T}(\zeta,t) = \frac{(\lambda-i)(\lambda-2i)}{(\lambda+i)(\lambda+2i)}.$$

The potential $q(x,t)$ is constructed as in Example 9.5 with the help of (8.14). We obtain

$$q(x,t) = \frac{72\, w_{39}\, w_{40}}{w_{41} + w_{42}}\ \exp\left(2x + 16i\,t + 2i\tan^{-1}(w_{37}/w_{38}) + 2i\tan^{-1}(w_{37}^*/w_{38})\right), \tag{9.49}$$

where we have defined

$$w_{37} := 36\, e^{4x}\left(9 + 18\, e^{4x} + 8\, e^{2x+12it} + 16\, e^{2x-12it}\right),$$

$$w_{38} := 1 - 2592\, e^{12x},$$

$$w_{39} := 1 + 72i\, e^{8x} + 2\, e^{2x+12it} + 72i\, e^{6x+12it},$$

$$w_{40} := 576\, e^{6x} + 324\, e^{4x+12it} + 288\, e^{6x+24it} + 648\, e^{8x+12it} - 2592i\, e^{12x+12it} + i\, e^{12it},$$

$$w_{41} := e^{24it} + 104\,976\, e^{8x+24it} + 186\,624\, e^{10x+12it} + 373\,248\, e^{10x+36it} + 82\,944\, e^{12x} + 746\,496\, e^{12x+24it},$$

$$w_{42} := 331\,776\, e^{12x+48it} + 6718\,464\, e^{24x+24it} + 373\,248\, e^{14x+12it} + 746\,496\, e^{14x+36it} + 419\,904\, e^{16x+24it}.$$

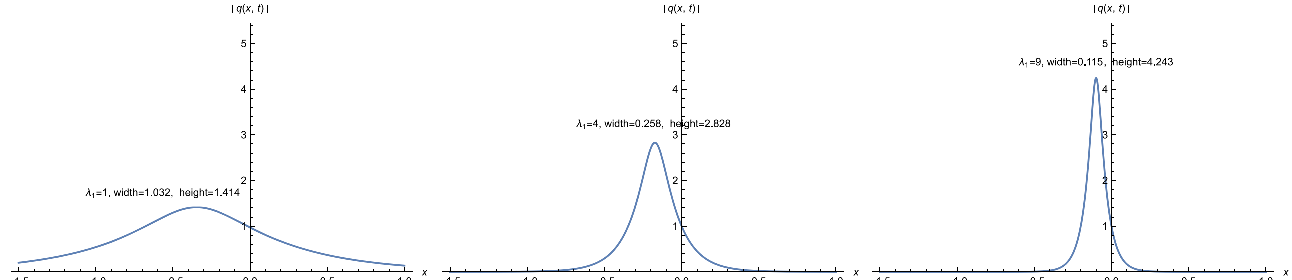

FIG. 7. The stationary soliton $|q(x,t)|$ of (9.47) with $|c_1| = 1$ and three different λ_1 values.

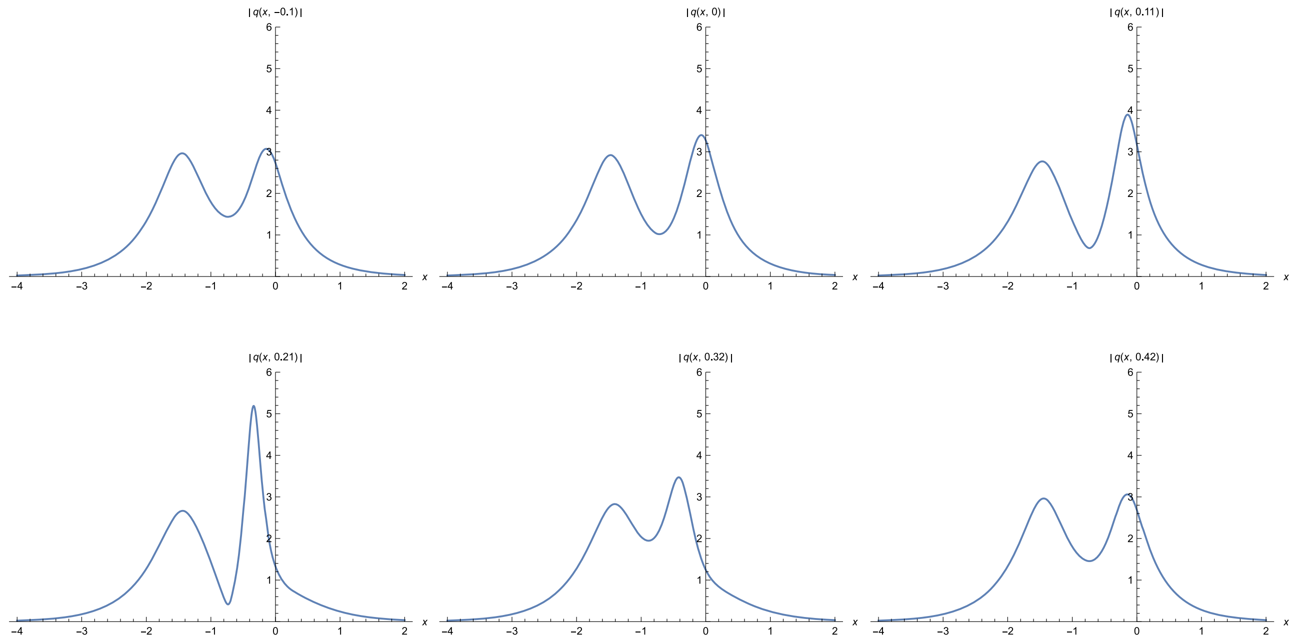

FIG. 8. The snapshots for $|q(x,t)|$ of (9.49) in Example 9.6 during one period.

From the explicit time dependence of $q(x,t)$ given in (9.49), we observe that $q(x,t)$ is periodic in t with a period of $\pi/6$. As shown in the six plots of $|q(x,t)|$ in Fig. 8, the soliton on the left barely moves, while the soliton on the right moves toward the left soliton by first gaining some amplitude and then losing some amplitude, and then that right soliton starts moving away from the left soliton. This process is repeated periodically.

In the next example, we demonstrate that, instead of choosing (1.2) as the unperturbed nonlinear problem, we could choose any particular case of (1.6) as the unperturbed problem.

Example 9.7. As far as the nonlinear problem is concerned, in order to demonstrate that any particular case of (1.6) can be chosen as the unperturbed problem and the rest as the perturbed problem, it is sufficient to show that we can express $q(x,t)$ and $r(x,t)$ in terms of $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ in a way similar to that given in (1.20). We proceed as follows. From (1.20), we observe that (6.4) holds. Similar to (1.19), let us define the quantity $\tilde{E}(x,t)$ as

$$\tilde{E}(x,t) := \exp\left(\frac{i}{2}\int_{-\infty}^{x} dz\, \tilde{q}(z,t)\, \tilde{r}(z,t)\right). \tag{9.50}$$

Note that the use of (6.4) in (9.50) implies that $\tilde{E}(x,t) = E(x,t)$, where $E(x,t)$ is the quantity defined in (1.19) in terms of the unperturbed potentials $q(x,t)$ and $r(x,t)$. Thus, we obtain the inverses of the transformations given in (1.20), and we have

$$q(x,t) := \kappa\, \tilde{q}(x,t)\, \tilde{E}(x,t)^{a-b}, \quad r(x,t) := \frac{1}{\kappa}\, \tilde{r}(x,t)\, \tilde{E}(x,t)^{b-a},$$

which proves our claim that any particular case of (1.6) can be chosen as the unperturbed problem. Nevertheless, as far as the linear problem is concerned, a comparison of (2.1) and (5.1) indicates that the choice of (2.1) as the unperturbed problem is the simplest. Since the analysis of the unperturbed and perturbed linear problems is the crucial part in our paper, we have chosen the particular case with $(a,b,\kappa) = (0,0,1)$ as our unperturbed problem.

In the following example, in the reflectionless case, we present some explicit solutions to the nonlinear system (1.3), which is also called the Chen–Lee–Liu system, and to the nonlinear system (1.4), which is also called the Gerdjikov–Ivanov system.

Example 9.8. Let us recall that (1.3) is obtained from (1.6) when the three parameters in (1.6) are chosen as $(a,b,\kappa) = (1,0,1)$. In order to illustrate some explicit solution to (1.3), we choose our input as

$$a = 1, \quad b = 0, \quad \kappa = 1, \tag{9.51}$$

$$A = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} i & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -i & 1 \end{bmatrix}, \tag{9.52}$$

where we remark that the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ in (9.52) coincide with the triplets in (9.31). Using the unperturbed potentials $q(x,t)$ and $r(x,t)$ appearing in (9.32), the quantity $E(x,t)$ given in the first equality of (9.36), and the three parameters listed in (9.51), we obtain the explicit solution to (1.3) as

$$\tilde{q}(x,t) = \frac{64\, w_{17}}{w_{18}} \exp\left(2x + 4i\,t - 2i \tan^{-1}(w_{19}/w_{20})\right), \quad \tilde{r}(x,t) = \tilde{q}(x,t)^*, \tag{9.53}$$

where w_{17}, w_{18}, w_{19}, and w_{20} are the quantities defined in (9.33)–(9.35), respectively. We remark that the second equality in (9.53) is compatible with (8.17) with $\kappa = 1$ because, as seen from the second equality of (9.32), the unperturbed potentials $q(x,t)$ and $r(x,t)$ are complex conjugates of each other. In Fig. 9, we provide some snapshots for $|q(x,t)|$ corresponding to (9.53). As seen from those snapshots, the two solitons move toward each other and they interact nonlinearly. The soliton on the right gains some amplitude and then loses some amplitude as the nonlinear interactions take place, whereas the soliton on the left is less affected from the interactions. Then, the two solitons start to move away from each other. Let us finally illustrate the explicit solutions to (1.4), which corresponds to choosing the three parameters in (1.6) as $(a, b, \kappa) = (1, -1, 1)$. As our input, let us use the same matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ appearing in (9.52) and replace (9.51) with

$$a = 1, \quad b = -1, \quad \kappa = 1. \tag{9.54}$$

Using (9.54), the unperturbed potentials $q(x,t)$ and $r(x,t)$ appearing in (9.32), and the quantity $E(x,t)$ given in the first equality in (9.36), we obtain the explicit solution to (1.4) as

$$\tilde{q}(x,t) = \frac{64\, w_{17}}{w_{18}} \exp(2x + 4i\,t), \quad \tilde{r}(x,t) = \tilde{q}(x,t)^*, \tag{9.55}$$

where we remark that $\tilde{q}(x,t)$ appearing in (9.55) only differs from $\tilde{q}(x,t)$ listed in (9.53) by the absence of the inverse tangent function and that the second equality in (9.55) is compatible with (8.17) using the parameter $\kappa = 1$. Since w_{19} and w_{20} are real valued, we conclude that the absolute values of the quantities appearing in (9.53) and (9.55), respectively, are equal to each other even though the complex-valued quantities themselves are unequal. Thus, Fig. 9 also represents the behavior of $|q(x,t)|$ corresponding to (9.55).

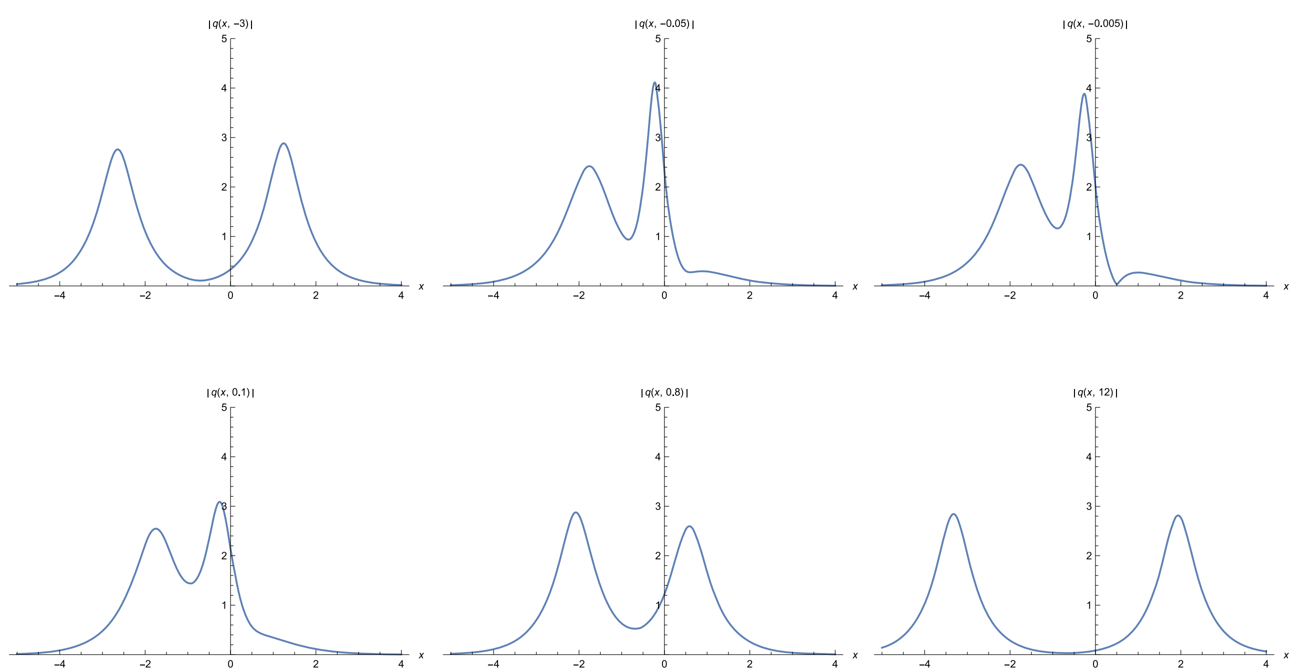

FIG. 9. The snapshots for $|q(x,t)|$ of (9.53) at several t-values in Example 9.8.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Tuncay Aktosun: Formal analysis (equal). **Ramazan Ercan**: Formal analysis (equal). **Mehmet Unlu**: Formal analysis (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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