Factorization of scattering matrices due to partitioning of potentials in one-dimensional Schrödinger-type equations

Tuncay Aktosun
Department of Mathematics, North Dakota State University, Fargo, North Dakota 58105

Martin Klaus
Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

Cornelis van der Mee
Department of Mathematics, University of Cagliari, Cagliari, Italy

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The one-dimensional Schrödinger equation and two of its generalizations are considered, as they arise in quantum mechanics, wave propagation in a nonhomogeneous medium, and wave propagation in a nonconservative medium where energy may be absorbed or generated. Generically, the zero-energy transmission coefficient vanishes when the potential is nontrivial, but in the exceptional case this coefficient is nonzero, resulting in tunneling through the potential. It is shown that any nontrivial exceptional potential can always be fragmented into two generic pieces. Furthermore, any nontrivial potential, generic or exceptional, can be fragmented into generic pieces in infinitely many ways. The results remain valid when Dirac delta functions are included in the potential and other coefficients are added to the Schrödinger equation. For such Schrödinger equations, factorization formulas are obtained that relate the scattering matrices of the fragments to the scattering matrix of the full problem. © *1996 American Institute of Physics.*
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I. INTRODUCTION

In this paper we consider the one-dimensional Schrödinger equation and two of its generalizations. The Schrödinger equation (2.1) describes the quantum mechanical behavior of a particle interacting with the potential $V(x)$. From the corresponding transmission coefficient $T(k)$ we obtain the probability $|T(k)|^2$ that a particle of energy k^2 can tunnel through this potential. Generically, the zero-energy transmission coefficient is zero and hence a zero-energy particle cannot tunnel through a nontrivial potential. However, in the exceptional case, the transmission coefficient does not vanish at zero energy. In this paper, we analyze certain aspects of this exceptional case. With the help of a factorization formula, we show that a nontrivial exceptional potential can always be fragmented into generic pieces; i.e., a nontrivial potential allowing tunneling at zero energy can always be decomposed into pieces none of which allow such tunneling. The factorization formula (2.17) used to obtain this result allows us to express the scattering coefficients corresponding to a potential in terms of the scattering coefficients corresponding to its fragments. We show that similar factorization formulas hold for certain generalized Schrödinger equations describing the wave propagation in one-dimensional nonhomogeneous or nonconservative media. For such generalized Schrödinger equations, the generic and exceptional cases are again determined by the zero-energy behavior of the transmission coefficients.

The generalized Schrödinger equation (3.3) can be analyzed by locally transforming it into a finite number of Schrödinger equations; the results obtained in Sec. II show that each of these Schrödinger equations can be chosen to have generic potentials. In Sec. III we obtain the corresponding factorization formula for Eq. (3.3). This formula, Eq. (3.15), brings insight to the analysis of wave scattering in a one-dimensional nonhomogeneous medium and allows us to see how the scattering process can be viewed as resulting both from ''soft scatterers''[1] (responsible for continuous changes in the medium parameters) and from ''hard scatterers''[1] (responsible for discontinuous changes in the medium parameters). This formula also explains how the total scattering matrix can be obtained in terms of the scattering matrices of the individual fragments localized in space.

In Sec. IV, we generalize the factorization formula (2.17) in a different way to analyze how the scattering process takes place in a one-dimensional nonconservative medium governed by the generalized Schrödinger equation (4.1), where energy absorption or generation may occur. Although the scattering matrix is no longer unitary when energy absorption or generation is present, we still have a factorization formula, namely Eq. (4.5), showing how the scattering resulting from the fragments is superposed to give the total scattering.

The small-energy analysis of the exceptional case for these three equations usually requires elaborate calculations. In addition to giving insight into the scattering process, the factorization formulas associated with these equations are expected to simplify the small-energy analysis of the wavefunctions and scattering coefficients.

II. SCHRÖDINGER EQUATION

Consider the one-dimensional Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2}+k^2\psi(k,x)=V(x)\psi(k,x), \tag{2.1}$$

where k^2 is energy, x is the space coordinate, and $V(x)$ is a real-valued potential belonging to $L_1^1(\mathbf{R})$, i.e., $\int_{-\infty}^{\infty} dx\,(1+|x|)|V(x)|$ is finite. The scattering solutions of Eq. (2.1) are those that behave like $e^{\pm ikx}$ as $x\to+\infty$ and $x\to-\infty$. There are two linearly independent scattering solutions $f_l(k,x)$ and $f_r(k,x)$ of Eq. (2.1), known as the Jost solutions from the left and from the right, respectively, satisfying the boundary conditions

$$f_l(k,x)=\begin{cases} e^{ikx}+o(1), & x\to+\infty,\\ \dfrac{1}{T(k)}\,e^{ikx}+\dfrac{L(k)}{T(k)}\,e^{-ikx}+o(1), & x\to-\infty, \end{cases} \tag{2.2}$$

$$f_r(k,x)=\begin{cases} \dfrac{1}{T(k)}\,e^{-ikx}+\dfrac{R(k)}{T(k)}\,e^{ikx}+o(1), & x\to+\infty,\\ e^{-ikx}+o(1), & x\to-\infty, \end{cases} \tag{2.3}$$

where $T(k)$ is the transmission coefficient and $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix associated with Eq. (2.1) is defined as

$$\mathbf{S}(k)=\begin{bmatrix} T(k) & R(k)\\ L(k) & T(k)\end{bmatrix}, \tag{2.4}$$

and it satisfies

$$\mathbf{S}(-k)=\overline{\mathbf{S}(k)}, \quad k\in\mathbf{R}, \tag{2.5}$$

where the overline denotes complex conjugation. The scattering matrix is unitary; thus,

$$|T(k)|^2+|R(k)|^2=|T(k)|^2+|L(k)|^2=1, \quad k\in\mathbf{R}, \tag{2.6}$$

and from Eq. (2.5) we see that

$$R(k)T(-k)+L(-k)T(k)=0, \quad k\in\mathbf{R}. \tag{2.7}$$

It is also known that the determinant of $\mathbf{S}(k)$ is given by

$$T(k)^2-R(k)L(k)=\frac{T(k)}{T(-k)}, \quad k\in\mathbf{R}. \tag{2.8}$$

For a potential in $L_1^1(\mathbf{R})$, the corresponding scattering matrix is well understood. Generically, the transmission coefficient vanishes linearly as $k\to 0$ and $R(0)=L(0)=-1$. In the exceptional case, we have $T(0)\neq 0$ and hence $|R(0)|=|L(0)|<1$. There are other characterizations of these two cases. For example, the potential $V(x)$ is exceptional if and only if $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent. Equivalently, $V(x)$ is exceptional if and only if at least one of $f_l(0,x)$ and $f_r(0,x)$ is bounded; in that case both of these functions are bounded for $x\in\mathbf{R}$. Furthermore, the potential $V(x)$ is exceptional if and only if

$$\int_{-\infty}^{\infty} dx\ V(x) f_l(0,x)=0, \tag{2.9}$$

which is equivalent to $\int_{-\infty}^{\infty} dx\ V(x) f_r(0,x)=0$ because $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent in the exceptional case. Moreover, the exceptional case occurs if and only if $f_l'(0,-\infty)=0$ or $f_r'(0,+\infty)=0$. Here and throughout the paper the prime denotes the spatial derivative and we interpret $f_l'(0,-\infty)$ as $\lim_{x\to-\infty} f_l'(0,x)$ and interpret $f_r'(0,+\infty)$ as $\lim_{x\to+\infty} f_r'(0,x)$.

If the potential has support on a half-line, i.e., if $V(x)=0$ for $x>b$ or $x<a$ for some constants a and b, we have the exceptional case if and only if $f_r'(0,x)=0$ for all $x\geqslant b$ or $f_l'(0,x)=0$ for all $x\leqslant a$, respectively. For example, when $V(x)=0$ for $x>b$, the linear dependence of $f_l(0,x)$ and $f_r(0,x)$ in the exceptional case requires that $f_r(0,x)$ is a constant for $x\geqslant b$ and hence $f_r'(0,b)=0$; in the generic case, since $f_r(0,x)$ is linear for $x\geqslant b$ and linearly independent of $f_l(0,x)$, it follows that $f_r'(0,b)\neq 0$. Note that $f_l(0,x)$ and $f_l'(0,x)$ cannot simultaneously vanish at the same x value; otherwise, we would have $f_l(0,x)=0$ for $x\in\mathbf{R}$ contradicting $f_l(0,+\infty)=1$. Similarly, $f_r(0,x)$ and $f_r'(0,x)$ cannot simultaneously vanish at the same x value. Thus, if $V(x)=0$ for $x<a$ and if $f_l(0,a)=0$, then $V(x)$ must be generic. Similarly, if $V(x)=0$ for $x>b$ and $f_r(0,b)=0$, then $V(x)$ must be generic.

In the exceptional case, let γ denote the constant

$$\gamma=\frac{f_l(0,x)}{f_r(0,x)}. \tag{2.10}$$

We have[2]

$$\begin{bmatrix} f_l(-k,x) \\ f_r(-k,x)\end{bmatrix}=\begin{bmatrix} T(k) & -R(k) \\ -L(k) & T(k)\end{bmatrix}\begin{bmatrix} f_r(k,x) \\ f_l(k,x)\end{bmatrix}, \quad k\in\mathbf{R}, \tag{2.11}$$

and hence from Eqs. (2.10) and (2.11) at $k=0$ we get

$$\gamma=\frac{T(0)}{1+R(0)}=\frac{1+L(0)}{T(0)}. \tag{2.12}$$

Using Eqs. (2.7), (2.8), and (2.12), we obtain

$$T(0)=\frac{2\gamma}{\gamma^2+1},\quad L(0)=-R(0)=\frac{\gamma^2-1}{\gamma^2+1}. \tag{2.13}$$

Further information on the generic and exceptional cases can be found in Refs. 2–6. For later reference, we summarize some of the necessary and sufficient conditions for the exceptional case.

Proposition 2.1: A potential $V\in L_1^1(\mathbf{R})$ is exceptional if and only if $f_l'(0,-\infty)=0$ or equivalently if and only if $f_r'(0,+\infty)=0$. If $V(x)$ vanishes for $x>b$, it is exceptional if and only if $f_r'(0,b)=0$. Similarly, if $V(x)$ vanishes for $x<a$, it is exceptional if and only if $f_l'(0,a)=0$.

The trivial potential $V(x)=0$ is exceptional. If $V(x)$ is nontrivial and $V(x)\geqslant 0$, then $V(x)$ is generic. The exceptional case is unstable in the sense that a small change in the potential usually makes the case generic. As an example, consider the square-well potential: the exceptional case occurs at the exact depths when a bound state is added to the potential; at any other depth the square-well potential is generic.

The distinction between the generic and exceptional cases becomes relevant when the small-energy behavior of the scattering coefficients and of the wavefunctions is considered. In many instances one has to deal with quantities involving the factor $T(k)/k$. In the generic case this factor remains bounded and continuous as $k\to 0$, but in the exceptional case it behaves as $T(0)/k$ with $T(0)\neq 0$. In some applications the factor $T(k)/k$ is multiplied by a continuous function $g(k)$ and one has to prove, for example, the integrability of the product $g(k)T(k)/k$ as $k\to 0$. In the generic case this integrability holds automatically, but in the exceptional case one has to prove, for instance, that $g(k)$ is of order $|k|^\gamma$ for some $\gamma\in(0,1]$ as $k\to 0$. This is one of the reasons why proofs tend to be more elaborate in the exceptional case than in the generic case. In this Section we show among other things that an exceptional potential can always be ‘‘fragmented’’ into two generic pieces and that a matrix closely related to the scattering matrix can be written as a product of factors, where each factor carries the information pertaining to one fragment. The term ‘‘fragment’’ will be made precise below. We expect our results to offer simplifications in dealing with exceptional potentials.

We now consider Eq. (2.1) and first explain the term fragment used in this paper. Choose a partition $-\infty<x_1<x_2<\cdots<x_n<+\infty$ of the real line $\mathbf{R}$ and define

$$V_{j,j+1}(x)=\begin{cases} V(x), & x\in(x_j,x_{j+1}),\\ 0, & x\notin(x_j,x_{j+1}),\end{cases}$$

so that

$$V(x)=\sum_{j=0}^{N} V_{j,j+1}(x), \tag{2.14}$$

where in Eq. (2.14) and below we use the convention that $x_0=-\infty$ and $x_{N+1}=+\infty$. We call $V_{j,j+1}(x)$ a fragment of $V(x)$. In analogy to Eq. (2.4) we let

$$\mathbf{S}_{j,j+1}(k)=\begin{bmatrix} T_{j,j+1}(k) & R_{j,j+1}(k)\\ L_{j,j+1}(k) & T_{j,j+1}(k)\end{bmatrix}$$

denote the scattering matrix associated with the potential $V_{j,j+1}(x)$, where each matrix $\mathbf{S}_{j,j+1}(k)$ only carries the information pertaining to the fragment $V_{j,j+1}(x)$. Using the scattering coefficients, we introduce the matrices

$$\Lambda(k)=\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(-k)}\end{bmatrix},\quad \Lambda_{j,j+1}(k)=\begin{bmatrix}\dfrac{1}{T_{j,j+1}(k)} & -\dfrac{R_{j,j+1}(k)}{T_{j,j+1}(k)}\\ \dfrac{L_{j,j+1}(k)}{T_{j,j+1}(k)} & \dfrac{1}{T_{j,j+1}(-k)}\end{bmatrix}. \tag{2.15}$$

Note that each matrix in Eq. (2.15) can be written as the product of two matrices in the following way:

$$\Lambda(k)=\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(-k)}\end{bmatrix}=\begin{bmatrix}1 & 0\\ L(k) & T(k)\end{bmatrix}\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ 0 & 1\end{bmatrix}. \tag{2.16}$$

Note also that using Eq. (2.7) it is possible to express the entries of each matrix in Eq. (2.15) in terms of the transmission coefficient and only one of the reflection coefficients; for example, we have

$$\Lambda(k)=\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ -\dfrac{R(-k)}{T(-k)} & \dfrac{1}{T(-k)}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{T(k)} & \dfrac{L(-k)}{T(-k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(-k)}\end{bmatrix}.$$

It is known[7] that $\Lambda(k)$ can be written as the product

$$\Lambda(k)=\Lambda_{0,1}(k)\Lambda_{1,2}(k)\cdots\Lambda_{N,N+1}(k). \tag{2.17}$$

It can be proved that Eq. (2.17) remains valid if we allow the potential $V(x)$ to contain a finite number of Dirac delta functions. When delta functions are included, the proof of Eq. (2.17) can be obtained from Eqs. (3.15) and (3.16) in the special case $H(x)\equiv 1$. If all the fragments in Eq. (2.14) are delta-function potentials, Eq. (2.17) reduces to Eq. (3.17). In Sec. III we will elaborate on the inclusion of delta functions.

The matrices $\Lambda(k)$ and $\Lambda_{j,j+1}(k)$ are usually called transition matrices. The reason for this terminology is as follows, which at the same time proves Eq. (2.17). Any scattering solution $\psi(k,x)$ of (2.1) obeys $\psi(k,x)=c_1e^{ikx}+c_2e^{-ikx}+o(1)$ as $x\to+\infty$ and $\psi(k,x)=d_1e^{ikx}+d_2e^{-ikx}+o(1)$ as $x\to-\infty$, where c_1,c_2,d_1,d_2 are function of k alone. By using Eqs. (2.2), (2.3), and (2.8), we can relate the vectors $[c_1\ c_2]$ and $[d_1\ d_2]$ corresponding to each of the Jost solutions $f_l(k,x)$ and $f_r(k,x)$, and hence we obtain $[d_1\ d_2]^t=\Lambda(k)[c_1\ c_2]^t$. We use the superscript t to denote the transpose. Hence $\Lambda(k)$ provides the link between the asymptotics of the solutions of Eq. (2.1) at $+\infty$ and those at $-\infty$ when the functions $e^{\pm ikx}$ are chosen as an (asymptotic) basis. Now let $N=1$, i.e., the partition is simply $-\infty<x_1<+\infty$. Let $\psi_{1,2}(k,x)$ be the solution of Eq. (2.1) with the potential $V_{1,2}(x)$ that satisfies $\psi_{1,2}(k,x)=\psi(k,x)$ for $x\geqslant x_1$, and let $\psi_{0,1}(k,x)$ be the solution of Eq. (2.1) with the potential $V_{0,1}(x)$ such that $\psi_{0,1}(k,x)=\psi(k,x)$ for $x\leqslant x_1$. Then $\psi_{1,2}(k,x)=\tilde d_1e^{ikx}+\tilde d_2e^{-ikx}$ for $x\leqslant x_1$, where $[\tilde d_1\ \tilde d_2]^t=\Lambda_{1,2}(k)[c_1\ c_2]^t$. Since $\psi(k,x_1)=\psi_{0,1}(k,x_1)=\psi_{1,2}(k,x_1)$ and $\psi'(k,x_1)=\psi'_{0,1}(k,x_1)=\psi'_{1,2}(k,x_1)$, it follows that $\psi_{0,1}(k,x)=\tilde d_1e^{ikx}+\tilde d_2e^{-ikx}$ for $x\geqslant x_1$. So $\Lambda_{0,1}(k)[\tilde d_1\ \tilde d_2]^t=[d_1\ d_2]^t$, and hence $[d_1\ d_2]^t=\Lambda_{0,1}(k)\Lambda_{1,2}(k)[c_1\ c_2]^t$. Thus, $\Lambda(k)=\Lambda_{0,1}(k)\Lambda_{1,2}(k)$, proving Eq. (2.17) when $N=1$. For $N\geqslant 2$ the result follows by induction.

For later use we note that when $N=1$, from Eqs. (2.15) and (2.17) we obtain

$$\frac{1}{T(k)}=\frac{1-R_{0,1}(k)L_{1,2}(k)}{T_{0,1}(k)T_{1,2}(k)}. \tag{2.18}$$

Now we return to Eq. (2.1) and discuss some consequences of Eqs. (2.17) and (2.18). The first result concerns resonant energies. These are energies at which the potential is perfectly transparent; in other words, energies $k_i^2 \geqslant 0$ where $|T(k_i)|=1$. Because of Eq. (2.5), $T(-k) = \overline{T(k)}$ for real k, and hence it is sufficient to consider the resonant frequencies only for $k_i \geqslant 0$. Typically, if $V(x)$ is a square-well potential, the existence of such resonant energies is well known (p. 94 of Ref. 8). There are also some general existence results[9] concerning resonances for potentials that are symmetric with respect to the midpoint of the potential barrier. The resonant energies play an important role in tunneling spectroscopy.[10] Here we consider a related but somewhat different problem. We consider the one-parameter family of potentials

$$V_\xi(x)=V_{0,1}(x+\xi)+V_{1,2}(x-\xi), \tag{2.19}$$

where $\xi>0$ is a real parameter. In other words, we take a potential $V(x)$ consisting of two fragments $V_{0,1}(x)$ and $V_{1,2}(x)$ and vary the distance between them by changing ξ. The goal is to adjust the distance between the fragments so that the transmission coefficient has magnitude 1. Let $T_\xi(k)$ denote the transmission coefficient for $V_\xi(x)$, and fix any $k=k_0 \geqslant 0$. Then we ask: are there any values of ξ for which $|T_\xi(k_0)|=1$? The answer when $k_0>0$ is contained in the next theorem. The analysis for $k_0=0$ will be given at the end of this section.

Theorem 2.2: Consider the potential $V_\xi(x)$ defined in Eq. (2.19) with the corresponding transmission coefficient $T_\xi(k)$. For any fixed $k_0>0$, there are three possibilities: (i) $|T_\xi(k_0)|=1$ for all $\xi>0$, (ii) there is no $\xi>0$ for which $|T_\xi(k_0)|=1$, (iii) the values $\xi>0$ for which $|T_\xi(k_0)|=1$ form an infinite sequence tending to $+\infty$.

Proof: Before starting the proof we remark that case (i) occurs when both of the fragments have a common resonant energy, that is when $|T_{0,1}(k_0)|=|T_{1,2}(k_0)|=1$; case (ii) occurs when $|T_{0,1}(k_0)| \neq |T_{1,2}(k_0)|$; case (iii) occurs when $|T_{0,1}(k_0)|=|T_{1,2}(k_0)| \neq 1$. For example, if $V(x)$ is symmetric about $x=x_1$ and hence $V_{0,1}(x_1-x)=V_{1,2}(x_1+x)$, then we are either in case (i) or case (iii); the same is true if $V_{1,2}(x)$ is a translate of $V_{0,1}(x)$.

The reflection coefficients from the right and left associated with the potentials $V_{0,1}(x+\xi)$ and $V_{1,2}(x-\xi)$ are given by $R_{0,1}(k)e^{2ik\xi}$ and $L_{1,2}(k)e^{2ik\xi}$, respectively. The transmission coefficients of the individual fragments are not affected by the shifts $\pm\xi$. Thus, by Eq. (2.18), $|T_\xi(k_0)|=1$ if and only if

$$|T_{0,1}(k_0)||T_{1,2}(k_0)|=|1-R_{0,1}(k_0)L_{1,2}(k_0)e^{4ik_0\xi}|. \tag{2.20}$$

Clearly, if $R_{0,1}(k_0)=L_{1,2}(k_0)=0$, then, by Eq. (2.6), $|T_{0,1}(k_0)|=|T_{1,2}(k_0)|=1$, and Eq. (2.20) holds independently of ξ, which is case (i). If $R_{0,1}(k_0)=0$ but $L_{1,2}(k_0)\neq 0$ (or vice versa), then $|T_{0,1}(k_0)|=1$ and $|T_{1,2}(k_0)|<1$ (or vice versa). Then Eq. (2.20) does not hold for any ξ. This is a special case of case (ii). Now suppose that $R_{0,1}(k_0)$ and $L_{1,2}(k_0)$ are both nonzero. Note the inequality

$$1-ab \geqslant (1-a^2)^{1/2}(1-b^2)^{1/2}, \quad a,b\in[0,1],$$

with the equality holding if and only if $a=b$. Using this inequality with $a=|R_{0,1}(k_0)|$ and $b=|L_{1,2}(k_0)|$, we see that Eq. (2.20) holds if and only if $|R_{0,1}(k_0)|=|L_{1,2}(k_0)|$ and

$$R_{0,1}(k_0)L_{1,2}(k_0)e^{4ik_0\xi}=|R_{0,1}(k_0)||L_{1,2}(k_0)|.$$

Hence, if $|R_{0,1}(k_0)|\neq|L_{1,2}(k_0)|$, then we are in case (ii). If $|R_{0,1}(k_0)|=|L_{1,2}(k_0)|$, then we set

$$R_{0,1}(k_0)L_{1,2}(k_0)=|R_{0,1}(k_0)||L_{1,2}(k_0)|e^{i\varphi(k_0)},$$

and we see that the values ξ are given by $4k_0\xi+\varphi(k_0)=2\pi n$, where n is any integer large enough to ensure $\xi>0$. Hence $\xi_n=\pi n/(2k_0)-\varphi(k_0)/(4k_0)$ is the desired sequence in case (iii). ■

Next we give some results concerning the nature of the point $k=0$. Let $f_{l;j,j+1}(k,x)$ and $f_{r;j,j+1}(k,x)$ denote the Jost solutions from the left and from the right, respectively, for the potentials $V_{j,j+1}(x)$. Since the potentials $V_{j,j+1}(x)$ have compact support for $j=1,\ldots,N-1$, using Proposition 2.1 we can conclude that $V_{j,j+1}(x)$ is generic if and only if $f'_{l;j,j+1}(k,x_j)\neq 0$ or if and only if $f'_{r;j,j+1}(k,x_{j+1})\neq 0$. Equivalently, $V_{j,j+1}(x)$ is exceptional if and only if $f'_{l;j,j+1}(k,x_j)=0$ or if and only if $f'_{r;j,j+1}(k,x_{j+1})=0$. This characterization also applies to the fragments $V_{0,1}(x)$ and $V_{N,N+1}(x)$ if we use $f'_{l;0,1}(k,x_0)$ and $f'_{r;N,N+1}(k,x_{N+1})$, respectively.

Theorem 2.3: Consider a potential $V(x)$ given in Eq. (2.14) with $N\geqslant 1$. Then:

(i) $\Rightarrow$ If all $N+1$ of the fragments are exceptional, then $V(x)$ is exceptional.

(ii) $\Rightarrow$ If exactly one fragment is generic, then $V(x)$ is generic.

Proof: (i) We give two proofs of (i) illustrating different aspects of the problem. First let $N=1$. Then, from Eq. (2.18) we see that if both $T_{0,1}(0)$ and $T_{1,2}(0)$ are nonzero, then the transmission coefficient $T(k)$ corresponding to $V(x)$ cannot vanish at $k=0$. Using induction, it then follows from Eq. (2.18) that if none of the transmission coefficients $T_{j,j+1}(k)$ vanish at $k=0$, then $T(k)$ cannot vanish at $k=0$. Hence (i) is proved. Alternatively, one can argue by using the zero-energy Jost solutions. Let $M_{j,j+1}$ denote the transfer matrix such that

$$\begin{bmatrix}\psi(0,\,x_j)\\ \psi'(0,\,x_j)\end{bmatrix}=M_{j,j+1}\begin{bmatrix}\psi(0,\,x_{j+1})\\ \psi'(0,\,x_{j+1})\end{bmatrix},\quad j=1,\ldots,N-1,$$

for any zero-energy solution of Eq. (2.1). Notice that

$$f_{l;j,j+1}(0,\,x_{j+1})=1,\quad f'_{l;j,j+1}(0,\,x_{j+1})=0.$$

Hence, if $V_{j,j+1}(x)$ is exceptional, then $[1\ 0]^t$ is an eigenvector of $M_{j,j+1}$ corresponding to the eigenvalue $f_{l;j,j+1}(0,\,x_j)$; if $V_{j,j+1}(x)$ is generic, then $[1\ 0]^t$ is not an eigenvector of $M_{j,j+1}$, since in that case $f'_{l;j,j+1}(0,\,x_j)\neq 0$ and $f'_{l;j,j+1}(0,\,x_{j+1})=0$. Furthermore, we have $f_l(0,\,x_N)=f_{l;N,N+1}(0,\,x_N)$ for $x\in[x_N,+\infty)$ and hence $f'_l(0,x_N)=0$ whenever $V_{N,N+1}(x)$ is exceptional. Since all fragments are assumed exceptional, and hence $[1\ 0]^t$ is a common eigenvector of all matrices $M_{j,j+1}$, it follows that

$$\begin{bmatrix}f_l(0,\,x_1)\\ f'_l(0,\,x_1)\end{bmatrix}=M_{1,2}\cdots M_{N-1,\,N}\begin{bmatrix}f_l(0,\,x_N)\\ 0\end{bmatrix}=c\begin{bmatrix}1\\ 0\end{bmatrix},$$

where $c=\Pi_{n=1}^{N}f_{l;n,n+1}(0,x_n)$. Now notice that $f_l(0,x)$ satisfies $f''_l(0,x)=V_{0,1}(x)f_l(0,\,x)$ with the boundary conditions $f_l(0,\,x_1)=c$ and $f'_l(0,\,x_1)=0$; since $V_{0,1}(x)$ is exceptional, $f_l(0,\,x)$ must be a constant multiple of $f_r(0,\,x)$ in the interval $(-\infty,\,x_1]$. Hence $V(x)$ is exceptional.

(ii) When $N=1$ and exactly one of the two fragments is generic, then the assertion immediately follows from Eq. (2.18). Indeed, from Eqs. (2.15) and (2.17) we have

$$\frac{1}{T_{1,2}(k)}=\frac{1-L_{0,1}(-k)L(k)}{T_{0,1}(-k)T(k)},$$

and hence if $T(0)\neq 0$ and $T_{0,1}(0)\neq 0$, we must have $T_{1,2}(0)\neq 0$. Consequently, if both $V(x)$ and $V_{0,1}(x)$ are exceptional, $V_{1,2}(x)$ has to be exceptional. A similar argument shows that if $T(0)\neq 0$ and $T_{1,2}(0)\neq 0$, we must have $T_{0,1}(0)\neq 0$. When $N\geqslant 2$, assume that the generic fragment is $V_{j_0,j_0+1}(x)$. Multiply Eq. (2.17) by $T_{j_0,j_0+1}(k)$ so that

$$T_{j_0,j_0+1}(k)\Lambda(k)=\Lambda_{0,1}(k)\cdots[T_{j_0,j_0+1}(k)\Lambda_{j_0,j_0+1}(k)]\cdots\Lambda_{N,N+1}(k). \tag{2.21}$$

Now let $k\rightarrow 0$ in Eq. (2.21). Since in the generic case, $\lim_{k\rightarrow 0} T(k)/k = ic_0$ for some real, nonzero constant c_0 (p. 303 of Ref. 5), we have $T_{j_0,j_0+1}(0)/\overline{T_{j_0,j_0+1}(0)} = -1$. Also, $R(0)=L(0)=-1$ in the generic case. Thus on the right-hand side of Eq. (2.21) we get

$$\lim_{k\rightarrow 0}[T_{j_0,j_0+1}(k)\Lambda(k)] = \Lambda_{0,1}(0)\Lambda_{1,2}(0)\cdots\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\cdots\Lambda_{N,N+1}(0).$$

Since $\det \Lambda_{j,j+1}(k)=1$, the matrices $\Lambda_{j,j+1}(0)$ are invertible and hence it follows that the matrix product in Eq. (2.21) is nonzero as $k\rightarrow 0$. This implies that $\lim_{k\rightarrow 0}[k\Lambda(k)]\neq 0$ and hence $V(x)$ must be generic. As in (i), one could also use the transfer matrices to give an alternate proof of part (ii). ■

Theorem 2.4: Any nontrivial potential, generic or exceptional, can be fragmented into at least two generic pieces. There are infinitely many different ways of fragmenting a nontrivial potential into generic pieces.

Proof: If suffices to show that if a given portion contains an exceptional piece that is not identically zero, then that piece can further be partitioned into infinitely many generic pieces. Suppose that $V_{j,j+1}(x)$ is exceptional and not identically zero. Then there is a subinterval of (x_j, x_{j+1}) on which $f_l'(0, x) \neq 0$. Choosing any point in this subinterval to partition $V_{j,J+1}(x)$ yields two fragments that are both generic.

An alternate proof can be given as follows. Let $f_{l;j,j+1}(k,x)$ be the corresponding Jost solution from the left for the potential $V_{j,j+1}(x)$. From Eq. (2.9) we have

$$\int_{x_j}^{x_{j+1}} dx\ V_{j,j+1}(x) f_{l;j,j+1}(0, x) = 0. \tag{2.22}$$

Then for any $z\in \mathbf{R}$, consider the fragmentation of $V_{j,j+1}(x)$ given by

$$V_{j,j+1}(x) = \theta(z-x)V_{j,j+1}(x) + \theta(x-z)V_{j,j+1}(x), \tag{2.23}$$

where $\theta(x)$ is the Heaviside function, i.e., $\theta(x)=1$ if $x>0$ and $\theta(x)=0$ if $x<0$. The fragments given in Eq. (2.23) have to be generic for an infinite number of values $z\in(x_j, x_{j+1})$, because $V_{j,j+1}(x)$ is nontrivial and so the integral obtained by replacing the lower limit in (2.22) by z has to be nonzero for some z, and hence, by continuity, for infinitely many z. ■

One can also consider fragmentations that contain exceptional pieces. From Theorem 2.3 we already know that a generic potential cannot be divided into two exceptional fragments. A generic potential can be divided into one generic and one exceptional piece if and only if there is a point x_1 where either $f_l'(0, x_1) = 0$ or $f_r'(0, x_1) = 0$. In the first case, the piece to the right of x_1 is exceptional while the piece to the left of x_1 is generic. In the second case, the types of the pieces are reversed. We may or may not be able to fragment a nontrivial exceptional potential into two nontrivial exceptional pieces. For example, the square-well potential supported on $0<x<a$ becomes exceptional at the depths $-j^2\pi^2/a^2$ with $j=1, 2, 3,...$, and hence the square-well potential given by

$$V(x) = \begin{cases} -\pi^2, & x\in(0,1), \\ 0, & \text{elsewhere}, \end{cases}$$

cannot be fragmented into two nontrivial exceptional pieces. A nontrivial exceptional potential can be cut into two nontrivial exceptional pieces if and only if there is a point x_1 where $f_l'(0, x_1) = 0$. If we have an exceptional potential we can choose each zero of $f_l'(0, x)$ as a separation point.

This will give the partition into the largest possible number of exceptional pieces, and that number may be finite or infinite. Example 3.1 demonstrates that an exceptional potential can be fragmented into an infinite number of exceptional pieces. If $V(x)$ is generic, then choosing the zeros of $f_l'(0,x)$ [respectively, $f_r'(0,x)$] as separation points, we obtain a partition where all pieces are exceptional except one, namely $V_{0,1}(x)$ [respectively, $V_{N,N+1}(x)$].

We note that if more than one fragment of $V(x)$ is generic, then $V(x)$ may be generic or exceptional. The following example illustrates this point.

Example 2.5: Assume

$$V_{0,1}(x)=\frac{-4e^{\sqrt{2}x}}{(1+e^{\sqrt{2}x})^2}\,\theta(-x),\quad V_{1,2}(x)=\frac{-4e^{-\sqrt{2}x}}{(1+e^{-\sqrt{2}x})^2}\,\theta(x).$$

Both $V_{0,1}(x)$ and $V_{1,2}(x)$ are generic, and in fact we have

$$T_{0,1}(k)=T_{1,2}(k)=\frac{k(k+i/\sqrt{2})}{k^2+1/4},\quad R_{0,1}(k)=L_{1,2}(k)=\frac{-1}{4k^2+1}.$$

Note that corresponding to $V(x)=V_{0,1}(x)+V_{1,2}(x)$ we have

$$T(k)=\frac{k+i/\sqrt{2}}{k-i/\sqrt{2}},\quad R(k)=0,$$

which is the exceptional case.

On the other hand, in terms of $u(x)$ and $v(x)$ given by

$$u(x)=8[4(3+2\sqrt{2})e^{\sqrt{2}x}-64e^{2x}+8e^{(2+\sqrt{2})x}-e^{(2+2\sqrt{2})x}+4(3-2\sqrt{2})e^{(4+\sqrt{2})x}],$$

$$v(x)=8+8e^{2x}-(3+2\sqrt{2})e^{\sqrt{2}x}-(3-2\sqrt{2})e^{(2+2\sqrt{2})x},$$

let us define

$$V_{0,1}(x)=\frac{u(x)}{v(x)^2}\,\theta(-x),\quad V_{1,2}(x)=\frac{-e^{-\sqrt{2}x}}{(1+e^{-\sqrt{2}x}/4)^2}\,\theta(x),$$

both of which are generic with the corresponding transmission coefficients

$$T_{0,1}(k)=\frac{50k(k+i)(\sqrt{2}k+i)}{50\sqrt{2}k^3+70ik^2+13\sqrt{2}k+31i},\quad T_{1,2}(k)=\frac{25k(\sqrt{2}k+i)}{25\sqrt{2}k^2+15ik+4\sqrt{2}}.$$

The sum $V(x)=V_{0,1}(x)+V_{1,2}(x)$ is a generic potential with the scattering coefficients given by

$$T(k)=\frac{2k(k+i)}{2k^2+1},\quad R(k)=\frac{-1}{2k^2+1}.$$

Finally, we analyze $T_\xi(0)$ corresponding to the potential $V_\xi(x)$ given in Eq. (2.19), as the analysis of $T_\xi(k)$ for $k=0$ was omitted from Theorem 2.2. In order to have $|T_\xi(0)|=1$, it is necessary that $V_\xi(x)$ is exceptional, and hence we first analyze the conditions for which $V_\xi(x)$ is exceptional. Let $F_l(k,x)$ and $F_r(k,x)$ denote the Jost solutions from the left and from the right, respectively, for the potential $V_\xi(x)$. Let us also use $f_l(k,x)$ and $f_r(k,x)$ to denote the Jost solutions from the left and from the right, respectively, for the potential $V(x)$. Note that $V_\xi(x)=0$ for $x\in(x_1-\xi,x_1+\xi)$, and hence we have

$$F_l(0,x)=\begin{cases} f_l(0,x+\xi)\left[1-2\xi\,\dfrac{f_l'(0,x_1)}{f_l(0,x_1)}-2\xi f_l'(0,x_1)^2\displaystyle\int_{x+\xi}^{x_1}\frac{dt}{f_l(0,t)^2}\right], & x\leqslant x_1-\xi,\\ (x-x_1-\xi)f_l'(0,x_1)+f_l(0,x_1), & x\in[x_1-\xi,\,x_1+\xi],\\ f_l(0,x-\xi), & x\geqslant x_1+\xi, \end{cases}\tag{2.24}$$

$$F_r(0,x)=\begin{cases} f_r(0,x+\xi), & x\leqslant x_1-\xi,\\ (x-x_1+\xi)f_r'(0,x_1)+f_r(0,x_1), & x\in[x_1-\xi,\,x_1+\xi],\\ f_r(0,x-\xi)\left[1+2\xi\,\dfrac{f_r'(0,x_1)}{f_r(0,x_1)}-2\xi f_r'(0,x_1)^2\displaystyle\int_{x_1}^{x-\xi}\frac{dt}{f_r(0,t)^2}\right], & x\geqslant x_1+\xi. \end{cases}\tag{2.25}$$

From Eqs. (2.24) and (2.25) we see that $V_\xi(x)$ is exceptional if and only if the ratio $F_r(0,x)/F_l(0,x)$ is independent of x; since $F_l(0,x)$ and $F_r(0,x)$ are linear functions in the interval $x\in[x_1-\xi,x_1+\xi]$, we can conclude that $V_\xi(x)$ is exceptional if and only if

$$\frac{F_r(0,x_1+\xi)}{F_l(0,x_1+\xi)}=\frac{F_r(0,x_1-\xi)}{F_l(0,x_1-\xi)},$$

from which we obtain

$$\xi=\frac{[f_l(0,x_1);f_r(0,x_1)]}{2\,f_l'(0,x_1)f_r'(0,x_1)}=\frac{d_r-d_l}{2d_rd_l},\tag{2.26}$$

where $[f;g]=fg'-f'g$ denotes the Wronskian and we have defined

$$d_l=\frac{f_l'(0,\ x_1)}{f_l(0,\ x_1)},\quad d_r=\frac{f_r'(0,\ x_1)}{f_r(0,\ x_1)}.$$

The cases in which $f_l(0,x_1)=0$ and $f_r(0,x_1)=0$ are included by setting $d_l=\infty$ and $d_r=\infty$, respectively.

(a) If $d_l\neq 0$ and $d_r\neq 0$, then there is exactly one value of ξ given by Eq. (2.26) for which $V_\xi(x)$ is exceptional provided the right-hand side of Eq. (2.26) is positive. Otherwise, $V_\xi(x)$ is generic.

(b) If $d_l=d_r=0$, i.e., if $f_l'(0,x_1)=f_r'(0,x_1)=0$, then both fragments and hence also $V(x)$ are exceptional. Thus, $V_\xi(x)$ is exceptional for all $\xi\geqslant 0$.

(c) If $d_l\neq 0$ and $d_r=0$, then $V_{0,1}(x)$ is exceptional and $V_{1,2}(x)$ is generic. Thus, $T_{0,1}(0)\neq 0$, $T_{1,2}(0)=0$, $R_{1,2}(0)=-1$, and $|L_{0,1}(0)|<1$, and Eq. (2.18) shows that $T_\xi(0)=0$ and hence we are in the generic case for all $\xi\geqslant 0$. This is also in agreement with Theorem 2.3 (ii).

(d) If $d_l=0$ and $d_r\neq 0$, then the analysis is similar to case (c); thus $V_\xi(x)$ is generic for all $\xi\geqslant 0$.

(e) If $d_l\neq 0$ and $d_r=\infty$, then $f_l'(0,x_1)\neq 0$ and $f_r(0,x_1)=0$; both fragments are generic. From Eq. (2.26) we see that $V_\xi(x)$ is exceptional only when $\xi=1/(2d_l)$ provided that $d_l>0$. Otherwise $V_\xi(x)$ is generic, and in particular $V(x)$ is generic.

(f) If $d_l=\infty$ and $d_r\neq 0$, the analysis is similar to case (e). Then, from Eq. (2.26) we see that $V_\xi(x)$ is exceptional only when $\xi=-1/(2d_r)$ provided that $d_r<0$. Otherwise $V_\xi(x)$ is generic, and in particular $V(x)$ is generic.

(g) If $d_l=0$ and $d_r=\infty$, from (2.26) in the limiting case it is seen that no ξ exists for which $V_\xi(x)$ is exceptional. Similarly, if $d_l=\infty$ and $d_r=0$, $V_\xi(x)$ is always generic.

(h) If $d_l=d_r=\infty$, we have $f_l(0,x_1)=f_r(0,x_1)=0$ and hence $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent. Thus, $V(x)$ is exceptional. However, as seen from Eq. (2.26), $V_\xi(x)$ is generic for every $\xi>0$. In other words, $T_\xi(0)\neq 0$ for $\xi=0$ but $T_\xi(0)=0$ for all $\xi>0$.

Once all the ξ values are obtained in cases (a), (b), (e), and (f) for which V_ξ is exceptional, one needs to determine which of these ξ values correspond to $|T_\xi(0)|=1$. For example, in case (b), we can proceed as follows. From Eq. (2.18) we have

$$\frac{1}{T_\xi(0)}=\frac{1-R_{0,1}(0)L_{1,2}(0)}{T_{0,1}(0)T_{1,2}(0)}, \tag{2.27}$$

and hence $T_\xi(0)$ is independent of ξ. Let $\gamma_{0,1}$ be the constant defined as in Eq. (2.10) giving the ratio of the zero-energy Jost solutions for the potential $V_{0,1}(x)$, and let $\gamma_{1,2}$ be defined similarly for the potential $V_{1,2}(x)$. As in Eq. (2.13), we have

$$R_{0,1}(0)=\frac{1-\gamma_{0,1}^2}{1+\gamma_{0,1}^2}, \quad L_{1,2}(0)=\frac{\gamma_{1,2}^2-1}{1+\gamma_{1,2}^2}. \tag{2.28}$$

Using Eq. (2.28) in Eq. (2.27) we obtain

$$T_\xi(0)=\frac{2\gamma_{0,1}\gamma_{1,2}}{1+\gamma_{0,1}^2\gamma_{1,2}^2},$$

from which we see that $|T_\xi(0)|=1$ if and only if $\gamma_{0,1}\gamma_{1,2}=\pm 1$.

III. WAVE PROPAGATION IN A NONHOMOGENEOUS MEDIUM

The fragmentation of an exceptional potential into two generic pieces has important consequences in direct and inverse scattering problems associated with wave propagation, where the governing equations are related to the Schrödinger equation or its variants. One such differential equation is given by

$$\frac{d^2\psi(k,x)}{dx^2}+\frac{k^2}{c(x)^2}\,\psi(k,x)=Q(x)\psi(k,x), \tag{3.1}$$

or by its time domain equivalent

$$\frac{\partial^2\phi(t,x)}{\partial x^2}-\frac{1}{c(x)^2}\,\frac{\partial^2\phi(t,x)}{\partial t^2}=Q(x)\phi(t,x). \tag{3.2}$$

Equation (3.1) describes the quantum mechanical behavior of a particle when the potential also depends on its energy. Equations (3.1) and (3.2) describe the propagation of waves in a one-dimensional nonhomogeneous, nonabsorptive medium where the wavespeed is $c(x)$ and the restoring force density is $Q(x)$. These equations can be analyzed by transforming them into Schrödinger equations by using local Liouville transformations.[11] In the special (but still significant) case $Q(x)=0$, the potential in the transformed Schrödinger equation is always exceptional. One important outcome of Theorem 2.4 is that it is possible to choose the local Liouville transformations in such a way that all the resulting fragments of the transformed Schrödinger equations are either generic or pertain to a potential vanishing identically. This leads to considerable simplifications in the small-k analysis of Eqs. (3.1) and (3.2). For example, consider Eq. (3.25) of Ref. 11 where the Jost solutions and their space derivatives are expressed as a product of matrices, each of which is expressed in terms of the quantities related to one fragment only. The matrices in Eq. (3.25) of Ref. 11 contain the factor $t_{j-1,j}(k)/k$, where $t_{j-1,j}(k)$ is the transmission coefficient corresponding to the jth fragment; that factor remains continuous as $k\to 0$ if the jth piece is generic and it is singular if the jth piece is exceptional. Hence, by fragmenting the exceptional pieces into generic ones, it becomes obvious that the Jost solutions and their space derivatives are continuous at $k=0$.

Let us write Eq. (3.1) as

$$\psi''(k,x)+k^2H(x)^2\psi(k,x)=Q(x)\psi(k,x),\quad x\in\mathbf{R}.\tag{3.3}$$

Our assumptions on $Q(x)$ and $H(x)$ are as follows:

(H1) $H(x)$ is strictly positive, piecewise continuous with possible discontinuities in $H(x)$ or $H'(x)$ occurring at the N points $x_1<\cdots<x_N$.

(H2) $H(x)\to H_\pm$ as $x\to\pm\infty$, where $H_\pm$ are positive constants.

(H3) $H-H_\pm\in L^1(\mathbf{R}^\pm)$, where $\mathbf{R}^-=(-\infty,0)$ and $\mathbf{R}^+=(0,+\infty)$.

(H4) H' is absolutely continuous on (x_n,x_{n+1}) and $2H''H-3(H')^2\in L_1^1(x_n,x_{n+1})$ for $n=0,\ldots,N$, where $x_0=-\infty$ and $x_{N+1}=+\infty$.

(H5) $Q(x)$ is real valued and of the form $Q(x)=W(x)+\Sigma_{j=1}^N c_j\delta(x-x_j)$, where $W\in L_1^1(\mathbf{R})$ and $\delta(x)$ is the Dirac delta function.

Conditions (H1)–(H5), without the delta-function terms in (H5), were introduced in Ref. 11, where the inverse scattering problem for Eq. (3.3), namely the recovery of the coefficient $H(x)$ from an appropriate set of scattering data, was studied. Hypothesis (H1) allows for abrupt changes in the material properties of the medium in which the wave propagates. In (H5) we have now included delta functions because they are often useful in working out explicitly solvable examples. Moreover, it is of interest to see how some of the results are affected by delta functions superimposed on discontinuities in $H(x)$ and $H'(x)$. The delta-function potential $V(x)=\alpha\delta(x-a)$ corresponds to

$$T(k)=\frac{k}{k+i\alpha/2},\quad R(k)=\frac{-i\alpha/2}{k+i\alpha/2}\,e^{2ika},\quad L(k)=\frac{-i\alpha/2}{k+i\alpha/2}\,e^{-2ika},\tag{3.4}$$

from which we see that it is a generic potential.

As for Eq. (2.1), Eq. (3.3) also has two linearly independent scattering solutions, namely the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ satisfying the boundary conditions

$$f_l(k,x)=\begin{cases} e^{ikH_+x}+o(1), & x\to+\infty,\\ \dfrac{1}{T_l(k)}\,e^{ikH_-x}+\dfrac{L(k)}{T_l(k)}\,e^{-ikH_-x}+o(1), & x\to-\infty,\end{cases}$$

$$f_r(k,x)=\begin{cases} \dfrac{1}{T_r(k)}\,e^{-ikH_+x}+\dfrac{R(k)}{T_r(k)}\,e^{ikH_+x}+o(1), & x\to+\infty,\\ e^{-ikH_-x}+o(1), & x\to-\infty.\end{cases}$$

Here, $T_l(k)$ and $T_r(k)$ are the transmission coefficients from the left and from the right, respectively, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively. Associated with Eq. (3.3) is the scattering matrix

$$\mathbf{S}(k)=\begin{bmatrix} T_l(k) & R(k)\\ L(k) & T_r(k)\end{bmatrix}.\tag{3.5}$$

The matrix $\mathbf{S}(k)$ is not unitary unless $H_+=H_-$; we have $\mathbf{S}(-k)=\overline{\mathbf{S}(k)}$ for real k, and

$$H_+T_l(k)=H_-T_r(k),\quad \mathrm{Im}\ k\geqslant 0,$$

$$T_r(-k)T_l(k)+|R(k)|^2=T_r(k)T_l(-k)+|L(k)|^2=1,\quad k\in\mathbf{R},$$

$$R(k)T_r(-k)+L(-k)T_r(k)=0,\quad k\in\mathbf{R}.$$

In the study of the scattering matrix $\mathbf{S}(k)$ given in Eq. (3.5), one again has to distinguish between the generic case and the exceptional case. As in Sec. II, in the generic case the transmission coefficients vanish linearly as $k\to 0$, whereas in the exceptional case we have $T_l(0)\neq 0$ and $T_r(0)\neq 0$. Furthermore, in the generic case $R(0)=L(0)=-1$, while in the exceptional case $|R(0)|=|L(0)|<1$. The coefficient $H(x)$ in Eq. (3.3) has no influence on the leading behavior of the transmission coefficients as $k\to 0$, and hence the generic and exceptional cases are determined by the potential $Q(x)$ only. All the characterizations of the two cases for the Schrödinger equation hold verbatim also for Eq. (3.3). If $Q(x)=0$ in Eq. (3.3), we have the exceptional case. If $Q(x)$ is nontrivial and $Q(x)\geqslant 0$ in Eq. (3.3), then we have the generic case. All the differences between the two cases as $k\to 0$ outlined in Sec. II also exist[11–13] in the wave propagation problem associated with Eq. (3.3).

Let us generalize the factorization formula (2.17) to the scattering problem for Eq. (3.3). Under the Liouville transformation

$$y=y(x)=\int_0^x ds\ H(s), \quad \psi(k,x)=\frac{1}{\sqrt{H(x)}}\ \phi(k,y), \tag{3.6}$$

Eq. (3.3) is transformed into

$$\frac{d^2\phi(k,y)}{dy^2}+k^2\phi(k,y)=V(y)\phi(k,y), \tag{3.7}$$

where

$$V(y)=V(y(x))=\frac{H''(x)}{2H(x)^3}-\frac{3}{4}\frac{H'(x)^2}{H(x)^4}+\frac{Q(x)}{H(x)^2}. \tag{3.8}$$

Since, by (H1), $H(x)$ and $H'(x)$ are allowed to have jump discontinuities at x_j for $j=1,\ldots,N$, the function $V(y)$ is undefined at $y_j=y(x_j)$ for $j=1,\ldots,N$. In agreement with Eq. (3.6), we set $y_0=y(x_0)=-\infty$ and $y_{N+1}=y(x_{N+1})=+\infty$. Then $V(y)$ is well defined in each of the intervals (y_j,y_{j+1}) for $j=0,\ldots,N$, and (H4) ensures that $V\in L_1^1$ on these intervals. In view of (H5), the solutions of Eq. (3.3) satisfy the conditions

$$\psi(k,x_n-0)=\psi(k,x_n+0), \quad \psi'(k,x_n+0)-\psi'(k,x_n-0)=c_n\psi(k,x_n). \tag{3.9}$$

As a result, by using Eqs. (3.6) and (3.9), we deduce that the solutions of Eq. (3.7) satisfy the self-adjoint boundary conditions

$$\phi(k,y_n-0)=\sqrt{q_n}\ \phi(k,y_n+0), \tag{3.10}$$

$$\frac{d\phi(k,y_n-0)}{dy}=\nu_n\phi(k,y_n+0)+\frac{1}{\sqrt{q_n}}\frac{d\phi(k,y_n+0)}{dy}, \tag{3.11}$$

where

$$q_n=\frac{H(x_n-0)}{H(x_n+0)},$$

$$\nu_n=\frac{1}{2\sqrt{H(x_n-0)H(x_n+0)}}\left[\frac{H'(x_n-0)}{H(x_n-0)}-\frac{H'(x_n+0)}{H(x_n+0)}-2c_n\right]. \tag{3.12}$$

The scattering matrix corresponding to Eq. (3.7) equipped with these boundary conditions is known as the ‘‘reduced scattering matrix’’[11] and is given by

$$\sigma(k)=\begin{bmatrix}\tau(k) & \rho(k)\\ \ell(k) & \tau(k)\end{bmatrix},$$

where $\tau(k)$ is the reduced transmission coefficient and $\rho(k)$ and $\ell(k)$ are the reduced reflection coefficients from the right and from the left, respectively. The reduced scattering matrix is unitary and its entries are related to the scattering matrix $\mathbf{S}(k)$ given in Eq. (3.5) as follows:[11]

$$\tau(k)=\sqrt{\frac{H_+}{H_-}}\,T_l(k)e^{ikA}=\sqrt{\frac{H_-}{H_+}}\,T_r(k)e^{ikA}, \tag{3.13}$$

$$\rho(k)=R(k)e^{2ikA_+},\quad \ell(k)=L(k)e^{2ikA_-},$$

where

$$A_\pm=\pm\int_0^{\pm\infty}ds\,[H_\pm-H(s)],\quad A=A_++A_-.$$

The points y_j generate a partition of the real line, and so we define

$$V_{j,j+1}(y)=\begin{cases}V(y), & y\in(y_j,y_{j+1}),\\ 0, & \text{elsewhere.}\end{cases}$$

We let $\tau_{j,j+1}(k)$, $\rho_{j,j+1}(k)$, and $\ell_{j,j+1}(k)$ denote the transmission coefficient and the reflection coefficients from the right and from the left, respectively, for the potential $V_{j,j+1}(y)$, and, as in Eq. (2.15), we define

$$\Lambda(k)=\begin{bmatrix}\dfrac{1}{\tau(k)} & -\dfrac{\rho(k)}{\tau(k)}\\ \dfrac{\ell(k)}{\tau(k)} & \dfrac{1}{\tau(-k)}\end{bmatrix},\quad \Lambda_{j,j+1}(k)=\begin{bmatrix}\dfrac{1}{\tau_{j,j+1}(k)} & -\dfrac{\rho_{j,j+1}(k)}{\tau_{j,j+1}(k)}\\ \dfrac{\ell_{j,j+1}(k)}{\tau_{j,j+1}(k)} & \dfrac{1}{\tau_{j,j+1}(-k)}\end{bmatrix}. \tag{3.14}$$

By suppressing the k-dependence of the transition matrices in Eq. (3.14), we have the generalization of Eq. (2.17) in the case of Eq. (3.3) given by[13]

$$\Lambda=\Lambda_{0,1}F_1\Lambda_{1,2}F_2\Lambda_{2,3}\cdots F_N\Lambda_{N,N+1}, \tag{3.15}$$

where F_j for $j=1,\dots,N$ are the matrices defined by

$$F_j(k)=\begin{bmatrix}\alpha_n+\dfrac{\nu_n}{2ik} & \left(\beta_n+\dfrac{\nu_n}{2ik}\right)e^{-2iky}\\ \left(\beta_n-\dfrac{\nu_n}{2ik}\right)e^{2iky_n} & \alpha_n-\dfrac{\nu_n}{2ik}\end{bmatrix},$$

with

$$\alpha_n=\frac{1}{2}\left[\sqrt{\frac{H(x_n-0)}{H(x_n+0)}}+\sqrt{\frac{H(x+0)}{H(x_n-0)}}\right],\quad \beta_n=\frac{1}{2}\left[\sqrt{\frac{H(x_n-0)}{H(x_n+0)}}-\sqrt{\frac{H(x_n+0)}{H(x_n-0)}}\right],$$

and where the constants ν_n are given in Eq. (3.12).

The matrices F_j account for the internal boundary conditions (3.10) and (3.11). In order to justify Eq. (3.15), again consider the case $N=1$ first. Using notations similar to those used below Eq. (2.17), we let $\phi(k,y)$ be a solution of Eq. (3.7) such that $\phi(k,y)=c_1e^{iky}+c_2e^{-iky}$ as $y\to+\infty$, and we define $\phi_{1,2}(k,y)$ and $\phi_{0,1}(k,y)$ as solutions of Eq. (3.7) for the fragments $V_{0,1}(y)$ and $V_{1,2}(y)$ such that $\phi_{1,2}(k,y)=\phi(k,y)$ for $y>y_1$ and $\phi_{0,1}(k,y)=\phi(k,y)$ for $y<y_1$. Then, $\phi_{1,2}(k,y)=\tilde{d}_1e^{iky}+\tilde{d}_2e^{-iky}$ for $y<y_1$ and $\phi_{0,1}(k,y)=\tilde{c}_1e^{iky}+\tilde{c}_2e^{-iky}$ for $y>y_1$, with suitable constants $\tilde{d}_1$, $\tilde{d}_2$, $\tilde{c}_1$, and $\tilde{c}_2$. Now the coefficients $\tilde{d}_1$ and $\tilde{d}_2$ are related to the coefficients $\tilde{c}_1$ and $\tilde{c}_2$ through the boundary conditions (3.10) and (3.11) by setting $\phi(k,y_1-0)=\phi_{0,1}(k,y_1)$, $\phi'(k,y_1-0)=\phi'_{0,1}(k,y_1)$, and $\phi(k,y_1+0)=\phi_{1,2}(k,y_1)$, $\phi'(k,y_1+0)=\phi'_{1,2}(k,y_1)$. This-yields

$$\begin{bmatrix} e^{iky_1} & e^{-iky_1} \\ ike^{iky_1} & -ike^{-iky_1} \end{bmatrix}\begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix}=\begin{bmatrix} \sqrt{q_1} & 0 \\ \nu_1 & \dfrac{1}{\sqrt{q_1}} \end{bmatrix}\begin{bmatrix} e^{iky_1} & e^{-iky_1} \\ ike^{iky_1} & -ike^{-iky_1} \end{bmatrix}\begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix},$$

from which we obtain $[\tilde{c}_1\ \tilde{c}_2]^t=F_1[\tilde{d}_1\ \tilde{d}_2]^t$. This proves Eq. (3.15) when $N=1$, and the general case follows by induction. Note that F_n can be written as a product of three matrices, namely

$$F_n=\Lambda(x_n-0,\ x_n)\Lambda[x_n,x_n]\Lambda(x_n,x_n+0), \tag{3.16}$$

where

$$\Lambda(x_n-0,\ x_n)=\begin{bmatrix} \alpha_n^-+\dfrac{\nu_n^-}{2ik} & \left(\beta_n^-+\dfrac{\nu_n^-}{2ik}\right)e^{-2iky_n} \\ \left(\beta_n^--\dfrac{\nu_n^-}{2ik}\right)e^{2iky_n} & \alpha_n^--\dfrac{\nu_n^-}{2ik} \end{bmatrix},$$

$$\Lambda[x_n,x_n]=\begin{bmatrix} 1-\dfrac{c_n}{2ik} & \dfrac{c_n}{2ik}\,e^{-2iky_n} \\ -\dfrac{c_n}{2ik}\,e^{2iky_n} & 1+\dfrac{c_n}{2ik} \end{bmatrix},$$

$$\Lambda(x_n,x_n+0)=\begin{bmatrix} \alpha_n^++\dfrac{\nu_n^+}{2ik} & \left(\beta_n^++\dfrac{\nu_n^+}{2ik}\right)e^{-2iky_n} \\ \left(\beta_n^+-\dfrac{\nu_n^+}{2ik}\right)e^{2iky_n} & \alpha_n^+-\dfrac{\nu_n^+}{2ik} \end{bmatrix},$$

with

$$\alpha_n^{\mp}=\frac{1}{2}\left[\sqrt{H(x_n\mp0)}+\frac{1}{\sqrt{H(x_n\mp0)}}\right],\quad \beta_n^{\mp}=\pm\frac{1}{2}\left[\sqrt{H(x_n\mp0)}-\frac{1}{\sqrt{H(x_n\mp0)}}\right],$$

$$\nu_n^{\mp}=\frac{\pm1}{2\sqrt{H(x_n\mp0)}}\,\frac{H'(x_n\mp0)}{H(x_n\mp0)}.$$

We remark that the transition matrix $\Lambda(x_n-0,\ x_n)$ is due to the hard scatterer caused by a jump in $H(x)$ from $H(x_n-0)$ to 1 and a jump in $H'(x)$ from $H'(x_n-0)$ to 0. The transition matrix $\Lambda[x_n,x_n]$ is due to the hard scatterer $c_n\delta(x-x_n)$, as seen from Eq. (3.4). The transition matrix $\Lambda(x_n,x_n+0)$ is due to the hard scatterer caused by a jump in $H(x)$ from 1 to $H(x_n+0)$ and a jump in $H'(x)$ from 0 to $H'(x_n+0)$. The transition matrices $\Lambda_{n,n+1}(k)$ in Eq. (3.15) are due to the soft scatterers $V_{n,n+1}(y)$. In the special case when $H(x)=1$ and $W(x)=0$ in (H5), Eq. (3.15) takes the form

$$\Lambda=\Lambda[x_1,x_1]\cdots\Lambda[x_N,x_N], \tag{3.17}$$

which describes scattering by a superposition of delta functions located at $x_1,\ldots,x_N$.

We mention one application of the factorization formula (3.15) in the inverse scattering problem for Eq. (3.3) concerning the large-k asymptotics of $\tau(k)$, $\rho(k)$, and $\ell(k)$; we refer the reader to Refs. 11–13 for details: it is known that from the large-k asymptotics of a reduced reflection coefficient one can recover the ratios q_n and ν_n (cf. Ref. 13, where the case $c_n=0$ was studied). It is seen from Eq. (3.12) that the coefficients c_n affect the large-k asymptotics through the constants ν_n and thus contribute in the same manner as the jumps in the derivative of $H(x)$. We also see that c_n can be chosen suitably to cancel the contribution from a jump in $H'(x)$.

In the recovery of $H(x)$ in Eq. (3.3), the distinction between the exceptional and generic cases is important. For example, in the absence of bound states, given the scattering data consisting of a reduced reflection coefficient and $Q(x)$, one obtains a one-parameter family of $H(x)$ in the exceptional case and a unique $H(x)$ in the generic case.[11–13] Therefore, in the exceptional case one must include either H_+ or H_- in the scattering data for the unique recovery of $H(x)$; however, in the generic case, H_+ or H_- cannot be specified in the scattering data and instead these constants are themselves recovered during the inversion procedure.

Finally in this section we give an example of an exceptional potential that can be fragmented into an infinite number of only exceptional pieces.

Example 3.1: In Eq. (3.3) choose $Q(x)=0$ and

$$H(x)=\begin{cases}1+\left(\dfrac{\sin x}{x}\right)^3, & x\neq 0,\\ 2, & x=0.\end{cases} \tag{3.18}$$

Note that $H(x)$ is strictly positive and bounded, $H_\pm=1$, and

$$H'(x)=\begin{cases}\dfrac{3\sin^2 x}{x^4}[x\cos x-\sin x], & x\neq 0,\\ 0, & x=0,\end{cases}$$

$$H''(x)=\begin{cases}\dfrac{3\sin x}{x^5}[x^2(3\cos^2 x-1)-6x\cos x\sin x+4\sin^2 x], & x\neq 0,\\ 0, & x=0,\end{cases}$$

and hence H, H', and H'' are all continuous on $\mathbf{R}$. Since $Q(x)=0$, we are in the exceptional case, and hence the transmission coefficients $T_l(k)$ and $T_r(k)$ cannot vanish at $k=0$. Note that $H(n\pi)=1$, $H'(n\pi)=0$, and $H''(n\pi)=0$ for any integer n. Using Eq. (3.6) let us define $y_n=y(n\pi)$. Now consider the potential $V(y)$ obtained by using Eq. (3.18) and $Q(x)=0$ in Eq. (3.8). That potential must be exceptional, and in fact from Eq. (3.13) it can be seen that the transmission coefficient $\tau(k)$ corresponding to the potential $V(y)$ cannot vanish at $k=0$. Now let us fragment $V(y)$ as $V(y)=\Sigma_{n=-\infty}^{\infty}V_{n,n+1}(y)$, where we have defined

$$V_{n,n+1}(y)=\begin{cases}V(y), & y\in(y_n,y_{n+1}),\\ 0, & \text{elsewhere.}\end{cases} \tag{3.19}$$

The following argument shows that each $V_{n,n+1}(y)$ is exceptional. Since $Q(x)=0$ in Eq. (3.3), the corresponding zero-energy Jost solution is given by $f_l(0,x)=1$ for $x\in\mathbf{R}$. Using Eq. (5.1) of Ref. 11, we see that the zero-energy Jost solution from the left of Eq. (3.7) is given by

$$g_l(0,y)=g_l(0,y(x))=\sqrt{H(x)}.$$

Hence, we obtain

$$g_l'(0,y)=\frac{dg_l(0,y)}{dy}=\frac{dx}{dy}\,\frac{d\sqrt{H(x)}}{dx}=\frac{H'(x)}{2H(x)^{3/2}}. \tag{3.20}$$

Since $H'(n\pi)=0$, from Eq. (3.20) we see that $g_l'(0,y_n)=0$, and hence we can choose y_n as the separation points to fragment $V(y)$ into only exceptional pieces, which are given by Eq. (3.19).

IV. WAVE PROPAGATION IN A NONCONSERVATIVE MEDIUM

The wave propagation in a one-dimensional nonconservative medium is described, in the frequency domain, by the generalized Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=[ikP(x)+Q(x)]\psi(k,x),\quad x\in\mathbf{R}, \tag{4.1}$$

where k is the wave number, $P(x)$ represents the joint effect of energy absorption and generation, and $Q(x)$ stands for the restoring force density. In the time domain Eq. (4.1) corresponds to

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}-P(x)\,\frac{\partial u}{\partial t}=Q(x)u,\quad t,x\in\mathbf{R},$$

where the wavespeed is equal to one. We will assume that $Q(x)$ is real valued and belongs to $L_1^1(\mathbf{R})$, and that $P(x)$ is real valued and belongs to $L^1(\mathbf{R})$. We have energy absorption when $P(x)\leqslant 0$ and energy generation when $P(x)\geqslant 0$; however, our results in this section are valid without assuming that $P(x)$ is positive or negative.

The scattering solutions of Eq. (4.1) are those behaving like e^{ikx} or e^{-ikx} as $x\to\pm\infty$, and such solutions occur when $k^2>0$. Among the scattering solutions are the Jost solution from the left $f_l(k,x)$ and the Jost solution from the right $f_r(k,x)$ satisfying the boundary conditions (2.2) and (2.3), respectively. The scattering matrix $\mathbf{S}(k)$ associated with Eq. (4.1) has the form (2.4). When $P(x)$ is purely imaginary, the inverse scattering problem for Eq. (4.1) was analyzed by Jaulent and Jean;[14–17] in this case the scattering matrix $\mathbf{S}(k)$ is unitary and hence the reflection coefficients cannot exceed one in absolute value. An incomplete study of the same problem when $P(x)$ is real was outlined in Ref. 18. In that case the differential equation (4.1) is no longer self-adjoint and the scattering matrix $\mathbf{S}(k)$ is no longer unitary. Consequently, the analysis of the direct and inverse scattering problems for real $P(x)$ is much more complicated than for imaginary $P(x)$.

We are interested in the analog of the factorization formula (2.17). As in Sec. II, let us partition the real axis $\mathbf{R}$ into $x_0<x_1<x_2<\cdots<x_N<x_{N+1}$ with $x_0=-\infty$ and $x_{N+1}=+\infty$. Consider the analog of Eq. (4.1) given by

$$\psi''(k,x)+k^2\psi(k,x)=[ikP_{j,j+1}(x)+Q_{j,j+1}(x)]\psi(k,x), \tag{4.2}$$

where we have defined the fragments

$$P_{j,j+1}(x)=\begin{cases}P(x), & x\in(x_j,x_{j+1}),\\ 0, & \text{elsewhere,}\end{cases} \tag{4.3}$$

$$Q_{j,j+1}(x)=\begin{cases} Q(x), & x\in(x_j,x_{j+1}),\\ 0, & \text{elsewhere}.\end{cases} \tag{4.4}$$

Let the scattering matrix associated with Eq. (4.2) be given by

$$\mathbf{s}_{j,j+1}(k)=\begin{bmatrix} t_{j,j+1}(k) & r_{j,j+1}(k)\\ l_{j,j+1}(k) & t_{j,j+1}(k)\end{bmatrix}.$$

Proceeding as in the previous sections or as in Ref. 7 or Ref. 13 we obtain

$$\Lambda(k)=\Lambda_{0,1}(k)\Lambda_{1,2}(k)\cdots\Lambda_{N,N+1}(k), \tag{4.5}$$

where we have defined the transition matrices

$$\Lambda(k)=\begin{bmatrix} \dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{T(k)^2-L(k)R(k)}{T(k)}\end{bmatrix}, \tag{4.6}$$

$$\Lambda_{j,j+1}(k)=\begin{bmatrix} \dfrac{1}{t_{j,j+1}(k)} & -\dfrac{r_{j,j+1}(k)}{t_{j,j+1}(k)}\\ \dfrac{l_{j,j+1}(k)}{t_{j,j+1}(k)} & \dfrac{t_{j,j+1}(k)^2-l_{j,j+1}(k)r_{j,j+1}(k)}{t_{j,j+1}(k)}\end{bmatrix}. \tag{4.7}$$

As in the previous sections, the transition matrix given in Eq. (4.6) provides the link between the asymptotics of the scattering solutions of Eq. (4.1) at $+\infty$ and those at $-\infty$ when $e^{\pm ikx}$ are chosen as an asymptotic basis; the transition matrices in Eq. (4.7) have similar interpretations. Again, each of the matrices in Eqs. (4.6) and (4.7) can be decomposed as in Eq. (2.16). Note that the (2,2) entry in Eq. (4.6) is analytic in the lower-half complex plane $\mathbf{C}^-$ and in general cannot be replaced by $1/T(-k)$; however, it is known that[14] this entry is equal to $1/\mathscr{T}(-k)$, where $\mathscr{T}(k)$ is the transmission coefficient associated with the differential equation obtained from Eq. (4.1) by changing the sign of $P(x)$.

Again one has to distinguish between the generic and exceptional cases in studying the scattering and inverse scattering problems for Eq. (4.1). As for Eq. (3.3), the potential $Q(x)$ alone determines whether we have the generic case or the exceptional case. The difficulties arising in proofs in the exceptional case outlined in the previous sections remain true also for Eq. (4.1), and by choosing each fragment in the partitioning (4.3) and (4.4) to be either generic or identically zero we expect simplifications in the small k-analysis of the direct and inverse scattering problems for Eq. (4.1).

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