# Scattering and inverse scattering for the 1-D Schrödinger equation with energy-dependent potentials

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The one-dimensional Schrödinger equation with a potential $k^2V(x)$ proportional to energy is studied. This equation is equivalent to the wave equation with variable speed. When $V(x) < 1$, is bounded below, and satisfies two integrability conditions, the scattering matrix is obtained and its asymptotics for small and large energies are established. The inverse scattering problem of recovering $V(x)$ when the scattering matrix is known is also solved. By proving that all the solutions of a key Riemann–Hilbert problem have the same asymptotics for large energy, it is shown that the potential obtained is unique.

## I. INTRODUCTION

Consider the one-dimensional Schrödinger equation

$$\psi''(k,x) + k^2\psi(k,x) = k^2V(x)\psi(k,x), \qquad (1.1)$$

where $x\in\mathbf{R}$ is the space coordinate, $k^2\in\mathbf{R}$ is energy, and the potential $k^2V(x)$ is proportional to energy. Note that throughout the paper we use the prime to denote the derivative with respect to $x$. For convenience we will call $V(x)$ the potential; $V(x)$ is assumed to decrease to zero as $o(1/x)$ as $x\to\pm\infty$. The Fourier transformation from the frequency $k$ domain into the time $t$ domain changes (1.1) into the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad (1.2)$$

where $c(x) = 1/\sqrt{1-V(x)}$ is the wave speed. The equation in (1.2) describes the propagation of waves (e.g., sound, electromagnetic, or elastic waves) in nondispersive media where the wave speed depends on position. The direct scattering problem for (1.1) consists of finding the scattering matrix when the potential is known; the inverse scattering problem is to recover the potential $V(x)$ when the scattering matrix is known. The inverse scattering problem for (1.1) is important because this problem is equivalent to the determination of the wave speed $c(x)$ from the scattering data, and this has many important applications in acoustic imaging, nondestructive evaluation, and various fields of geophysics such as seismology.

One can define the travel-time coordinate[1]

$$y = \int_0^x d\xi\,\sqrt{1-V(\xi)}$$

and the new wavefunction

$$\phi(k,y) = [1-V(x)]^{1/4}\psi(k,x), \qquad (1.3)$$

and transform (1.1) into the regular Schrödinger equation given by

$$\frac{d^2\phi}{dy^2} + k^2\phi = Q(y)\phi, \qquad (1.4)$$

where the new potential $Q(y)$ is related to the potential of (1.1) as

$$Q(y) = -\frac{5}{16}\frac{V'(x)^2}{[1-V(x)]^3} - \frac{1}{4}\frac{V''(x)}{[1-V(x)]^2}. \qquad (1.5)$$

Ware and Aki[2] proposed to solve the inverse scattering problem for (1.1) using the travel-time coordinate. In our analysis we use the spatial coordinate rather than the travel-time coordinate; this is because the solution of the inverse scattering problem using the travel-time coordinate is not achieved unless the potential $V(x)$ of (1.1) is obtained from $Q(y)$ by inverting (1.5). However, the recovery of $V(x)$ from $Q(y)$ presupposes the knowledge of $V(x)$; hence, switching to the travel-time coordinate does not solve the inverse scattering problem. A method based on the iterative technique of Jost and Kohn[3] was proposed by Razavy[4]; this method uses the spatial coordinate, but it is more suited to find the potential approximately. In this method the potential is expressed as an infinite series; however, even the second term in the series is fairly complicated and no convergence is assured.

The Schrödinger equation (1.1) has no bound state solutions for potentials considered in this paper.[5,6] In this respect the direct and inverse scattering problems for (1.1) are simpler than the corresponding problems for the regular Schrödinger equation $\varphi'' + k^2\varphi = V(x)\varphi$. However, the regular Schrödinger equation is an eigenvalue problem for the Hamiltonian operator $-d^2/dx^2 + V(x)$, whereas (1.1) is not an eigenvalue problem and hence the techniques from the spectral theory of self-adjoint operators are not directly applicable to (1.1). Another important difference between (1.1) and the regular Schrödinger equation is the following. In the regular Schrödinger equation the asymptotics of the solutions as $k\to\pm\infty$ are easy to obtain because one can interchange the limits as $k\to\pm\infty$ and $x\to\pm\infty$, whereas these limits cannot be interchanged in (1.1). In the regular Schrödinger equation, the solutions with the appropriate asymptotics as $k\to\pm\infty$ and the solutions with the appropriate asymptotics as $x\to\pm\infty$ are related to each other in a simple manner, whereas for (1.1) this is not apparent. Informally speaking, when $k \to \pm\infty$, in the regular Schrödinger equation the term proportional to $V(x)$ can be neglected compared to the other terms, whereas in (1.1) we cannot neglect that term. These are some of the main reasons why the direct and inverse scattering problems for (1.1) are more difficult. Here in this paper we overcome these difficulties by explicitly computing the asymptotics of the scattering solutions of (1.1) as $k \to \pm\infty$ and by establishing some analyticity properties of these solutions when $k$ is extended to complex values.

The assumption $V(x) < 1$ guarantees that the wave speed $c(x) = 1/\sqrt{1 - V(x)}$ has meaning. The results given in this paper, with the exception of those in Secs. IV–IX, hold for bounded potentials satisfying the conditions $V(x) < 1$, $V \in L^1_\alpha(\mathbf{R})$, and $G \in L^1_\alpha(\mathbf{R})$ for some $\alpha \in (0,1]$, where

$$G(x) = \frac{1}{4} \frac{V''(x)}{[1 - V(x)]^{3/2}} + \frac{5}{16} \frac{V'(x)^2}{[1 - V(x)]^{5/2}}. \quad (1.6)$$

In Secs. VI–IX we further assume that $V(x) \geqslant 0$, and in Sec. X we generalize the results of Secs. IV–IX to the case where $V(x)$ is no longer assumed non-negative. Throughout the paper we use the notation $H(x) = \sqrt{1 - V(x)}$. Note that whenever $V \in L^1(\mathbf{R})$, we have $1 - H \in L^1(\mathbf{R})$ because $|1 - H| = |V|/(1 + H) \leqslant |V|$. In the beginning of each section we specify the sufficient conditions on the potential for which the results there hold. Note that throughout the paper, $L^1_j(\mathbf{R})$ denotes the space of measurable functions $f(y)$ such that the Lebesgue integral

$$\int_{-\infty}^{\infty} dy (1 + |y|)^j |f(y)|$$

is finite and $L^n(\mathbf{R})$ denotes the space such that

$$\int_{-\infty}^{\infty} dy |f(y)|^n$$

is finite.

This paper is organized as follows. In Sec. II we define the scattering solutions of (1.1), study their properties and establish their asymptotics for small $k$. In Sec. III we study the properties of the scattering matrix and establish its asymptotics for small $k$. In Sec. IV we solve two important integral equations (4.13) and (4.14), obtain the analyticity properties of their solutions, relate these solutions to the scattering solutions of (1.1), and study the large $k$ asymptotics of the scattering solutions of (1.1). In Sec. V we establish the large $k$ asymptotics of the scattering matrix. In Sec. VI we formulate a key Riemann–Hilbert problem whose solution will lead to the recovery of the potential if it is already known that $0 \leqslant V(x) < 1$. In Sec. VII we establish the existence of the canonical Wiener–Hopf factorization of the matrix appearing in the Riemann–Hilbert problem. In Sec. VIII we give the general solution of the Riemann–Hilbert problem and show how a unique potential can be recovered from that solution. Through a Marchenko procedure, in Sec. IX we obtain the canonical factors of the matrix that appears in the Riemann–Hilbert problem. Finally, in Sec. X we generalize the method of recovery of the potential to the case where $V(x)$ is no longer assumed to be non-negative.

## II. SCATTERING SOLUTIONS

In this section we study the properties of the scattering solutions of (1.1) and establish their asymptotics for small $k$. The sufficient assumption on the potential in this section is $V \in L^1_\alpha(\mathbf{R})$ for some $\alpha \in (0,1]$. In fact, we use $V \in L^1_\alpha(\mathbf{R})$ only in Proposition 2.2; otherwise $V \in L^1(\mathbf{R})$ is sufficient.

The physical solutions $\psi_l$ from the left and $\psi_r$ from the right satisfy

$$\psi_l(k,x) = \begin{cases} T_l(k) e^{ikx} + o(1), & x \to \infty, \\ e^{ikx} + L(k) e^{-ikx} + o(1), & x \to -\infty, \end{cases} \quad (2.1)$$

$$\psi_r(k,x) = \begin{cases} e^{-ikx} + R(k) e^{ikx} + o(1), & x \to \infty, \\ T_r(k) e^{-ikx} + o(1), & x \to -\infty. \end{cases} \quad (2.2)$$

Here, $T_l$ and $T_r$ are the transmission coefficients from the left and from the right, respectively, and $L$ and $R$ are the reflection coefficients from the left and from the right, respectively. The scattering matrix $S(k)$ is defined as

$$S(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}. \quad (2.3)$$

We will establish the properties of $S(k)$ in Secs. III and V. The physical solutions $\psi_l$ and $\psi_r$ satisfy the Lippmann–Schwinger equation

$$\begin{bmatrix} \psi_l(k,x) \\ \psi_r(k,x) \end{bmatrix} = \begin{bmatrix} e^{ikx} \\ e^{-ikx} \end{bmatrix} + \frac{k}{2i} \int_{-\infty}^{\infty} dy\, e^{ik|x-y|} V(y) \times \begin{bmatrix} \psi_l(k,y) \\ \psi_r(k,y) \end{bmatrix}. \quad (2.4)$$

The Jost solutions of (1.1), $f_l$ from the left and $f_r$ from the right, are defined as

$$f_l(k,x) = \frac{1}{T_l(k)} \psi_l(k,x)$$
$$f_r(k,x) = \frac{1}{T_r(k)} \psi_r(k,x). \quad (2.5)$$

They satisfy the integral equations

$$f_l(k,x) = e^{ikx} - k \int_x^{\infty} dy \sin k(x-y) V(y) f_l(k,y),$$

$$f_r(k,x) = e^{-ikx} + k \int_{-\infty}^{x} dy \sin k(x-y) V(y) f_r(k,y),$$

and the boundary conditions

$$f_l(k,x) = \begin{cases} e^{ikx} + o(1), & x \to \infty, \\ \dfrac{1}{T_l(k)} e^{ikx} + \dfrac{L(k)}{T_l(k)} e^{-ikx} + o(1), & x \to -\infty, \end{cases} \quad (2.6)$$

$f_r(k,x)$

$$=\begin{cases}\frac{1}{T_r(k)}e^{-ikx}+\frac{R(k)}{T_r(k)}e^{ikx}+o(1), & x\to\infty,\\ e^{-ikx}+o(1), & x\to-\infty.\end{cases} \tag{2.7}$$

Let us also define

$$m_l(k,x)=\frac{1}{T_l(k)}e^{-ikx}\psi_l(k,x)$$
$$m_r(k,x)=\frac{1}{T_r(k)}e^{ikx}\psi_r(k,x). \tag{2.8}$$

Then from (1.1) and (2.8) it is seen that $m_l$ and $m_r$ satisfy the equations

$$m_l''(k,x)+2ikm_l'(k,x)=k^2V(x)m_l(k,x), \tag{2.9}$$

$$m_r''(k,x)-2ikm_r'(k,x)=k^2V(x)m_r(k,x). \tag{2.10}$$

We will call $m_l$ and $m_r$ the Faddeev solutions from the left and right, respectively; they satisfy the integral equations

$$m_l(k,x)=1-\frac{k}{2i}\int_x^\infty dy[1-e^{2ik(y-x)}]V(y)m_l(k,y), \tag{2.11}$$

$$m_r(k,x)=1-\frac{k}{2i}\int_{-\infty}^x dy[1-e^{2ik(x-y)}]V(y)m_r(k,y), \tag{2.12}$$

and the boundary conditions

$$m_l(k,x)=1+o(1) \text{ and } m_l'(k,x)=o(1), \quad x\to\infty,$$

$$m_r(k,x)=1+o(1) \text{ and } m_r'(k,x)=o(1), \quad x\to-\infty.$$

Next we show that the Faddeev solutions defined in (2.8) can be extended analytically in $k$ to the upper half complex plane $\mathbf{C}^+$ if $V\in L^1(\mathbf{R})$. We will use the notation $\mathbf{C}^-$ for the lower half complex plane and $\overline{\mathbf{C}^\pm}$ to denote $\mathbf{C}^\pm\cup\mathbf{R}$.

**Theorem 2.1:** When $V\in L^1(\mathbf{R})$, the Faddeev solutions $m_l(k,x)$ and $m_r(k,x)$ are analytic in $k$ for $k\in\mathbf{C}^+$ and continuous in $k$ for $k\in\overline{\mathbf{C}^+}$.

*Proof:* From (2.11) we have $m_l(k,x)=\Sigma_{j=0}^\infty n_j(k,x)$ where $n_0(k,x)=1$ and

$$n_j(k,x)=-\frac{k}{2i}\int_x^\infty dy[1-e^{2ik(y-x)}]\times V(y)n_{j-1}(k,y), \quad j\geqslant 1.$$

Using $|1-e^{2ik(y-x)}|\leqslant 2$ when $y\geqslant x$ and $k\in\overline{\mathbf{C}^+}$, we obtain

$$|n_j(k,x)|\leqslant\frac{|k|^j}{j!}\left[\int_x^\infty dy|V(y)|\right]^j,$$

so that

$$|m_l(k,x)|\leqslant\exp\left(|k|\int_x^\infty dy|V(y)|\right)$$
$$\leqslant\exp\left(|k|\int_{-\infty}^\infty dy|V(y)|\right), \quad k\in\overline{\mathbf{C}^+}. \tag{2.13}$$

Furthermore, each $n_j(k,x)$ is analytic in $k$ for $k\in\mathbf{C}^+$ and continuous in $k$ for $k\in\overline{\mathbf{C}^+}$, and thus by the Weierstrass theorem, $m_l(k,x)$, being the limit of a uniformly convergent sequence of analytic functions in compact subsets in $\mathbf{C}^+$, is analytic in $k$ for $k\in\mathbf{C}^+$ and continuous in $k$ for $k\in\overline{\mathbf{C}^+}$. From (2.12) we obtain in a similar way

$$|m_r(k,x)|\leqslant\exp\left(|k|\int_{-\infty}^x dy|V(y)|\right)$$
$$\leqslant\exp\left(|k|\int_{-\infty}^\infty dy|V(y)|\right), \quad k\in\overline{\mathbf{C}^+}, \tag{2.14}$$

and that $m_r(k,x)$ is analytic in $k$ for $k\in\mathbf{C}^+$ and continuous in $k$ for $k\in\overline{\mathbf{C}^+}$. ■

We remark, however, that (2.13) should not be interpreted as an indication that $m_l(k,x)$ may be unbounded as $k\to\pm\infty$. The Lippmann–Schwinger equation given in (2.11) is not suitable to study the large $k$ asymptotics of $m_l(k,x)$. We will study these asymptotics in Sec. IV and show that $m_l(k,x)$ remains bounded as $k\to\pm\infty$ under the assumption $G\in L^1(\mathbf{R})$, where $G(x)$ is the function defined in (1.6). Note that when $V(x)<1$, the only possible solutions of (1.1) are oscillatory in nature when $k\in\mathbf{R}$; hence, we should expect $\psi_l(k,x)$ and $m_l(k,x)$ to remain bounded for all $k\in\mathbf{R}$ even when $k\to\pm\infty$. We remark that whenever we write $k\to\pm\infty$, it is understood that the limit is taken through the real values of $k$. The remarks made above concerning the boundedness of $m_l(k,x)$ as $k\to\pm\infty$ also apply to $m_r(k,x)$.

Note that from the analysis leading to (2.13) and (2.14), it follows that for each $k\in\overline{\mathbf{C}^+}$, both $m_l(k,\cdot)$ and $m_r(k,\cdot)$ are bounded continuous functions of $x$ for $x\in\mathbf{R}$.

From (2.11) and (2.12), we obtain

$$m_l'(k,x)=-k^2\int_x^\infty dy\, e^{2ik(y-x)}V(y)m_l(k,y),$$

$$m_r'(k,x)=k^2\int_{-\infty}^x dy\, e^{2ik(x-y)}V(y)m_r(k,y).$$

Hence, using $m_l(k,x)=\Sigma_{j=0}^\infty n_j(k,x)$ and the properties of $m_l(k,x)$ obtained above, we have

$$|m_l'(k,x)|\leqslant|k|\left[\exp\left(|k|\int_x^\infty d\xi|V(\xi)|\right)-1\right]$$
$$\leqslant|k|\left[\exp\left(|k|\int_{-\infty}^\infty d\xi|V(\xi)|\right)-1\right], \quad k\in\overline{\mathbf{C}^+}, \tag{2.15}$$

and similarly

$$|m_r'(k,x)| < |k| \left[\exp\left(|k| \int_{-\infty}^{x} d\xi |V(\xi)|\right) - 1\right]$$

$$< |k| \left[\exp\left(|k| \int_{-\infty}^{\infty} d\xi |V(\xi)|\right) - 1\right], \quad k \in \overline{\mathbf{C}^+}. \tag{2.16}$$

Thus, if $V \in L^1(\mathbf{R})$, the functions $m_l'(k,x)$ and $m_r'(k,x)$ are analytic in $k \in \mathbf{C}^+$ and continuous in $k \in \overline{\mathbf{C}^+}$ for each $x \in \mathbf{R}$.

From (2.11) and (2.12) it is seen that as $k \to 0$ in $\overline{\mathbf{C}^+}$

$$m_l(k,x) = 1 - \frac{k}{2i} \int_x^{\infty} dy [1 - e^{2ik(y-x)}] V(y) + O(k^2), \tag{2.17}$$

$$m_r(k,x) = 1 - \frac{k}{2i} \int_{-\infty}^{x} dy [1 - e^{2ik(x-y)}] V(y) + O(k^2), \tag{2.18}$$

and from (2.15) and (2.16), as $k \to 0$ in $\overline{\mathbf{C}^+}$,

$$m_l'(k,x) = O(k^2) \text{ and } m_r'(k,x) = O(k^2).$$

*Proposition 2.2:* If $V \in L^1_\alpha(\mathbf{R})$ for some $\alpha \in (0,1]$, then for $k_1, k_2 \in \overline{\mathbf{C}^+}$, the quantities

$$\frac{m_l(k_1,x) - m_l(k_2,x)}{|k_1 - k_2|^\alpha} \text{ and } \frac{m_r(k_1,x) - m_r(k_2,x)}{|k_1 - k_2|^\alpha}$$

are bounded in absolute value by $(1 + |x|)^\alpha C_1(\alpha,K)$, where

$$C_1(\alpha,K) = \max\{2^{1-\alpha}K^{1-\alpha}, 2^{5\alpha/2}K\} \times \exp\left(K \int_{-\infty}^{\infty} dz [1 + (1+|z|)^\alpha] |V(z)|\right) \times \int_{-\infty}^{\infty} dy (1+|y|)^\alpha |V(y)|, \tag{2.19}$$

with $K = \max\{|k_1|, |k_2|\}$.

*Proof:* We will give the proof for

$$\Delta(k_1,k_2;x) = \frac{m_l(k_1,x) - m_l(k_2,x)}{|k_1 - k_2|^\alpha}$$

only; the proof for

$$\frac{m_r(k_1,x) - m_r(k_2,x)}{|k_1 - k_2|^\alpha}$$

is similar. Letting

$$A(k;x,y) = -\frac{k}{2i} [1 - e^{2ik(y-x)}] V(y),$$

from (2.11) we obtain

$$\Delta(k_1,k_2;x) = \int_x^{\infty} dy \frac{A(k_1;x,y) - A(k_2;x,y)}{|k_1 - k_2|^\alpha} m_l(k_1,y) + \int_x^{\infty} dy\, A(k_2;x,y) \Delta(k_1,k_2;y). \tag{2.20}$$

We have

$$\frac{|A(k_1;x,y) - A(k_2;x,y)|}{|k_1 - k_2|^\alpha}$$
$$\leqslant [|k_1 - k_2|^{1-\alpha} + 2^{\alpha/2} |k_2| (y-x)^\alpha] |V(y)|$$
$$\leqslant 2^{1-\alpha}K^{1-\alpha}[1 + 2^{3\alpha/2-1}K^\alpha(|y| + |x|)^\alpha] |V(y)|$$
$$\leqslant 2^{1-\alpha}K^{1-\alpha}[1 + 2^{5\alpha/2-1}K^\alpha(|y|^\alpha + |x|^\alpha)] |V(y)|$$
$$\leqslant \max\{2^{1-\alpha}K^{1-\alpha}, 2^{5\alpha/2}K\} \times (1+|x|)^\alpha (1+|y|)^\alpha |V(y)|.$$

Using (2.13) and $|A(k_2;x,y)| \leqslant K|V(y)|$, from (2.20) we obtain

$$\frac{|\Delta(k_1,k_2;x)|}{C_1(\alpha,K)(1+|x|)^\alpha} \leqslant 1 + K \int_x^{\infty} dy (1+|y|)^\alpha |V(y)| \times \frac{|\Delta(k_1,k_2;y)|}{C_1(\alpha,K)(1+|y|)^\alpha}. \tag{2.21}$$

Solving (2.21) iteratively the proof is completed. ■

## III. SCATTERING MATRIX

In this section we study the properties of the scattering matrix $S(k)$ and establish its asymptotics as $k \to 0$. In this section the sufficient assumption is $V \in L^1_\alpha(\mathbf{R})$ for some $\alpha \in (0,1]$. In fact, $V \in L^1_\alpha(\mathbf{R})$ is used only in Propositions 3.1 and 3.2; otherwise, $V \in L^1(\mathbf{R})$ is sufficient.

From (2.1), (2.2), and (2.4) we obtain the expressions for the transmission coefficients

$$T_l(k) = 1 + \frac{k}{2i} \int_{-\infty}^{\infty} dy\, e^{-iky} V(y) \psi_l(k,y), \tag{3.1}$$

$$T_r(k) = 1 + \frac{k}{2i} \int_{-\infty}^{\infty} dy\, e^{iky} V(y) \psi_r(k,y), \tag{3.2}$$

and the reflection coefficients

$$L(k) = \frac{k}{2i} \int_{-\infty}^{\infty} dy\, e^{iky} V(y) \psi_l(k,y), \tag{3.3}$$

$$R(k) = \frac{k}{2i} \int_{-\infty}^{\infty} dy\, e^{-iky} V(y) \psi_r(k,y). \tag{3.4}$$

Using the derivative of (2.4) with respect to $x$, we obtain

$$\psi_l'(k,x) = \begin{cases} ikT_l(k)e^{ikx} + o(1), & x \to \infty, \\ ike^{ikx} - ikL(k)e^{-ikx} + o(1), & x \to -\infty, \end{cases}$$

$$\psi_r'(k,x) = \begin{cases} -ike^{-ikx} + ikR(k)e^{ikx} + o(1), & x \to \infty, \\ -ikT_r(k)e^{-ikx} + o(1), & x \to -\infty, \end{cases}$$

which are exactly the expressions obtained by differentiating (2.1) and (2.2) termwise.

Let $[f;g] = fg' - f'g$ denote the Wronskian of $f$ and $g$. It can be shown that the Wronskian of any two solutions of (1.1) is independent of $x$. Hence, as $x \to \pm\infty$, from $[\psi_l(-k,x);\psi_l(k,x)]$ we obtain

$$T_l(k)T_l(-k) + L(k)L(-k) = 1, \quad k \in \mathbf{R}. \tag{3.5}$$

From the Wronskian $[\psi_r(-k,x);\psi_r(k,x)]$ we obtain

$$T_r(k)T_r(-k)+R(k)R(-k)=1, \quad k\in\mathbf{R}, \tag{3.6}$$

and from the Wronskian $[\psi_l(k,x);\psi_r(-k,x)]$ we find

$$T_l(k)R(-k)+L(k)T_r(-k)=0, \quad k\in\mathbf{R}. \tag{3.7}$$

Since $k$ appears as $ik$ in (2.11) and (2.12), it follows that

$$m_l(-k,x)=\overline{m_l(k,x)} \text{ and } m_r(-k,x)=\overline{m_r(k,x)}, \quad k\in\mathbf{R}, \tag{3.8}$$

where the bar denotes complex conjugation. Hence, from (3.5), (3.6), and (3.7), it is seen that the scattering matrix $S(k)$ defined in (2.3) is unitary and that we have

$$S(-k)^t=\overline{S(k)}^t=S(k)^{-1}, \quad k\in\mathbf{R}, \tag{3.9}$$

where $S(k)^t$ denotes the transpose and $S(k)^{-1}$ the inverse of the matrix $S(k)$. As a consequence, the transmission and reflection coefficients cannot exceed 1 in absolute value for $k\in\mathbf{R}$.

Using (2.1) and (2.2), we obtain

$$[\psi_l(k,x);\psi_r(k,x)]=-2ikT_l(k)=-2ikT_r(k).$$

Therefore, the transmission coefficients from the right and left coincide, and this common value will be denoted by $T(k)$:

$$T(k)=T_l(k)=T_r(k). \tag{3.10}$$

Let us now study the asymptotics of $S(k)$ as $k\to 0$. Using (2.8), from (3.1) and (3.2) we have

$$1-\frac{1}{T(k)}=\frac{k}{2i}\int_{-\infty}^{\infty} dy\, V(y)m_l(k,y)=\frac{k}{2i}\int_{-\infty}^{\infty} dy\, V(y)m_r(k,y), \tag{3.11}$$

and from (3.3) and (3.4) we have

$$\frac{L(k)}{T(k)}=\frac{k}{2i}\int_{-\infty}^{\infty} dy\, e^{2iky}V(y)m_l(k,y), \tag{3.12}$$

$$\frac{R(k)}{T(k)}=\frac{k}{2i}\int_{-\infty}^{\infty} dy\, e^{-2iky}V(y)m_r(k,y). \tag{3.13}$$

Notice that using (2.13) in (3.11), (3.12), and (3.13), from the Weierstrass theorem we conclude that $L(k)$ and $R(k)$ are continuous for $k\in\mathbf{R}$.

Using (2.17) and (2.18), from (3.11), (3.12), and (3.13) we obtain as $k\to 0$

$$T(k)=1+\frac{k}{2i}\int_{-\infty}^{\infty} dy\, V(y)+O(k^2), \quad k\in\overline{\mathbf{C}^+}, \tag{3.14}$$

$$L(k)=\frac{k}{2i}\int_{-\infty}^{\infty} dy\, e^{2iky}V(y)+O(k^2), \quad k\in\mathbf{R}, \tag{3.15}$$

$$R(k)=\frac{k}{2i}\int_{-\infty}^{\infty} dy\, e^{-2iky}V(y)+O(k^2), \quad k\in\mathbf{R}. \tag{3.16}$$

Using (2.9) and (2.10), we obtain the Wronskian

$$[m_r(k,x);m_l(k,x)]=-2ikm_r(k,x)m_l(k,x)+2ik/T(k). \tag{3.17}$$

In Sec. II, we have shown that $m_l(k,x)$, $m_r(k,x)$, $m_l'(k,x)$, and $m_r'(k,x)$ are continuous in $k$ for $k\in\overline{\mathbf{C}^+}$ and analytic in $k$ for $k\in\mathbf{C}^+$. Thus $1/T(k)$ is continuous in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$. It is known[5,6] that $1/T(k)$ has no zeros in $\overline{\mathbf{C}^+}$. Writing (3.17) as

$$T(k)=\frac{2ik}{2ikm_l(k,x)m_r(k,x)+[m_r(k,x);m_l(k,x)]},$$

and using (3.14), we see that $T(k)$ is continuous in $\overline{\mathbf{C}^+}$, is analytic in $\mathbf{C}^+$, and has no zeros in $\overline{\mathbf{C}^+}$. Moreover, because of the unitarity of $S(k)$ and $T(k)\neq 0$ for $k\in\mathbf{R}$, the reflection coefficients $R(k)$ and $L(k)$ cannot take the value 1 in absolute value when $k\in\mathbf{R}$. We will study the asymptotics of $S(k)$ as $k\to\pm\infty$ in Sec. V.

*Proposition 3.1:* If $V\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$, then $|T(k_1)-T(k_2)|/|k_1-k_2|^\alpha$ is uniformly bounded for $k_1\neq k_2$ in any compact subset of $\overline{\mathbf{C}^+}$.

*Proof:* Let $\Upsilon=\max\{|T(k_1)|,|T(k_2)|\}$. Using $K$ and $\Delta(k_1,k_2;x)$ defined in Proposition 2.2, from (3.11) we obtain

$$\begin{aligned}\frac{|T(k_1)-T(k_2)|}{|k_1-k_2|^\alpha}&\leqslant\Upsilon^2\frac{|1/T(k_1)-1/T(k_2)|}{|k_1-k_2|^\alpha}\\ &\leqslant 2^{-1}|k_1-k_2|^{1-\alpha}\Upsilon^2\int_{-\infty}^{\infty} dy|V(y)||m_l(k_1,y)|\\ &\quad+2^{-1}|k_2|\Upsilon^2\int_{-\infty}^{\infty} dy|V(y)||\Delta(k_1,k_2;y)|\\ &\leqslant\Upsilon^2\max\left\{2^{-\alpha}K^{1-\alpha}\exp\left(K\int_{-\infty}^{\infty} dz|V(z)|\right),\right.\\ &\quad\left.2^{-1}KC_1(\alpha,K)\right\}\int_{-\infty}^{\infty} dy(1+|y|)^\alpha|V(y)|,\end{aligned}$$

where $C_1(\alpha,K)$ is the quantity in (2.19). ∎

*Proposition 3.2:* If $V\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$, then

$$\frac{|L(k_1)-L(k_2)|}{|k_1-k_2|^\alpha} \text{ and } \frac{|R(k_1)-R(k_2)|}{|k_1-k_2|^\alpha}$$

are uniformly bounded for $k_1\neq k_2$ in any compact subset of $\mathbf{R}$.

*Proof:* In view of Proposition 3.1, $T(k)\neq 0$, and the continuity of $T(k)$, it is sufficient to consider the functions $L(k)/T(k)$ and $R(k)/T(k)$. We will give the proof only for $L(k)/T(k)$. Using $K$ and $\Delta(k_1,k_2;x)$ defined in Proposition 2.2, from (3.12) we have

$$\frac{|L(k_1)/T(k_1)-L(k_2)/T(k_2)|}{|k_1-k_2|^\alpha}$$

$$<\int_{-\infty}^{\infty} dy[2^{-\alpha}K^{1-\alpha}+2^{\alpha/2+1}K|y|^\alpha]|V(y)|\exp\left(K\int_{-\infty}^{\infty} dz\left|V(z)\right|\right)+2^{-1}K\int_{-\infty}^{\infty} dy|V(y)||\Delta(k_1,k_2;y)|$$

$$<\left[\max\{2^{1+\alpha/2}K,2^{-\alpha}K^{1-\alpha}\}\exp\left(K\int_{-\infty}^{\infty} dz\left|V(z)\right|\right)+2^{-1}KC_1(\alpha,K)\right]\int_{-\infty}^{\infty} dy(1+|y|)^\alpha|V(y)|,$$

where $C_1(\alpha,K)$ is the quantity in (2.19). ■

## IV. LARGE $k$ ASYMPTOTICS OF THE SCATTERING SOLUTIONS

In this section, the sufficient conditions are $V\in L^1(\mathbf{R})$, $V(x)<1$, $1-H\in L^1(\mathbf{R})$, *and* $G\in L^1_\alpha(\mathbf{R})$ *for some* $\alpha\in(0,1]$, *where* $G(x)$ is the quantity defined in (1.6) and $H(x)=\sqrt{1-V(x)}$; we also assume that $V(x)$ is bounded below and hence $\sup_{x\in\mathbf{R}}H(x)<\infty$. The condition $G\in L^1_\alpha(\mathbf{R})$ is needed only in Proposition 4.1; otherwise, $G\in L^1(\mathbf{R})$ is sufficient. First, using techniques similar to those used in Ref. 7, we show the existence of two linearly independent solutions of the Schrödinger equation (1.1) and establish their large $k$ asymptotics. Then, we relate these solutions to the scattering solutions $\psi_l$ and $\psi_r$ of (1.1) and establish the large $k$ asymptotics of $\psi_l$ and $\psi_r$.

Assume a solution of (1.1) of the form

$$\psi(k,x)=e^{ik\beta(x)}\sum_{j=0}^{\infty}\frac{a_j(x)}{k^j}=Y(k,x)Z(k,x), \tag{4.1}$$

where $Y(k,x)=a_0(x)e^{ik\beta(x)}$ and

$$Z(k,x)=\frac{1}{a_0(x)}\sum_{j=0}^{\infty}\frac{a_j(x)}{k^j}.$$

The functions $\beta(x)$ and $a_0(x)$ are to be determined; substituting (4.1) into (1.1), we obtain

$$k^2(-\beta'^2+1-V)a_0+k[i\beta''a_0+2i\beta'a_0'+(-\beta'^2+1-V)a_1]+\sum_{j=0}^{\infty}\frac{1}{k^j}[i\beta''a_{j+1}+2i\beta'a'_{j+1}+a''_j+(-\beta'^2+1-V)a_{j+2}]=0.$$

Thus we have

$$\beta'(x)=\pm H(x), \tag{4.2}$$

$$2i\beta'a_0'+i\beta''a_0=0, \tag{4.3}$$

$$2i\beta'a'_{j+1}+i\beta''a_{j+1}+a''_j=0,\quad j\geqslant 0.$$

From (4.2) it is seen that there are two linearly independent solutions corresponding to $\beta'=H$ and $\beta'=-H$, respectively. Substituting these values into (4.3), we obtain $a_0=H^{-1/2}$ apart from a multiplicative constant. Hence, we obtain two functions for $Y(k,x)$ which we will call

$$Y_l(k,x)=\exp\left(ik\int_0^x dt\,H(t)\right)(\sqrt{H(x)})^{-1} \tag{4.4}$$

and

$$Y_r(k,x)=\exp\left(-ik\int_0^x dt\,H(t)\right)(\sqrt{H(x)})^{-1}. \tag{4.5}$$

In general, the series in (4.1) does not converge, and hence it is not very useful. Therefore, in order to compute $Z(k,x)$, instead of using the series given in (4.1), we proceed as follows. Once $Y(k,x)$ is known, substituting $\psi=YZ$ into the Schrödinger equation (1.1), we obtain

$$YZ''+2Y'Z'+[Y''+k^2(1-V)Y]Z=0.$$

Multiplying the above equation by $Y$ and rearranging terms, we have

$$(Y^2Z')'+Y^2[Y''/Y+k^2(1-V)]Z=0. \tag{4.6}$$

Note that from (4.4) and (4.5) we have in terms of the function $G(x)$ defined in (1.6)

$$\frac{Y''}{Y}+k^2(1-V)=\frac{1}{4}\frac{V''}{1-V}+\frac{5}{16}\frac{(V')^2}{(1-V)^2}=G(x)H(x), \tag{4.7}$$

which is independent of $k$. Integrating (4.6) with the boundary condition $Z'(k,x_0)=0$, we obtain

$$Y^2(k,x)Z'(k,x)=-\int_{x_0}^x dt\,Y^2(k,t)G(t)H(t)Z(k,t),$$

or equivalently

$$Z'(k,x)=-\int_{x_0}^x dt\,\frac{Y^2(k,t)}{Y^2(k,x)}G(t)H(t)Z(k,t). \tag{4.8}$$

Integrating (4.8) with the boundary condition $Z(k,x_0)=1$ and changing the order of integration in the resulting equation, we obtain

$$Z(k,x)=1-\int_{x_0}^x dt\,\mathscr{L}(k;x,t)Z(k,t), \tag{4.9}$$

where

$$\mathscr{L}(k;x,t)=G(t)H(t)\int_t^x d\xi\,\frac{Y^2(k,t)}{Y^2(k,\xi)}. \tag{4.10}$$

Using (4.4) and (4.5) in (4.10), we obtain

$$\mathscr{L}_l(k;x,t)=\frac{G(t)}{2ik}\left[1-\exp\left(2ik\int_x^t d\xi\,H(\xi)\right)\right], \tag{4.11}$$

$$\mathscr{L}_r(k;x,t) = -G(t)/2ik \times\left[1-\exp\left(-2ik\int_x^t d\xi\, H(\xi)\right)\right]. \quad (4.12)$$

From (4.9) choosing $x_0 = \pm\infty$, we obtain two independent solutions denoted by $Z_l$ and $Z_r$, respectively, satisfying

$$Z_l(k,x) = 1 + \int_x^\infty dt\, \mathscr{L}_l(k;x,t) Z_l(k,t), \quad (4.13)$$

$$Z_r(k,x) = 1 - \int_{-\infty}^x dt\, \mathscr{L}_r(k;x,t) Z_r(k,t), \quad (4.14)$$

such that $Z_l'(k,\infty) = 0$ and $Z_l(k,\infty) = 1$. Similarly $Z_r'(k,-\infty) = 0$ and $Z_r(k,-\infty) = 1$. We see from (4.11) and (4.12) that

$$|\mathscr{L}_l(k;x,t)| \leqslant (1/|k|)|G(t)| \text{ and } |\mathscr{L}_r(k;x,t)| \leqslant (1/|k|)|G(t)|, \quad (4.15)$$

for $k\in\overline{\mathbf{C}^+}$ in their domains of integration given in (4.13) and (4.14).

By iterating (4.13) and (4.14) and using (4.15), we obtain

$$|Z_l(k,x)| \leqslant \exp\left(\frac{1}{|k|}\int_x^\infty dt|G(t)|\right) \leqslant \exp\left(\frac{1}{|k|}\int_{-\infty}^\infty dt|G(t)|\right), \quad k\in\overline{\mathbf{C}^+}\setminus\{0\}, \quad (4.16)$$

$$|Z_r(k,x)| \leqslant \exp\left(\frac{1}{|k|}\int_{-\infty}^x dt|G(t)|\right) \leqslant \exp\left(\frac{1}{|k|}\int_{-\infty}^\infty dt|G(t)|\right), \quad k\in\overline{\mathbf{C}^+}\setminus\{0\}.$$

Hence, by the Weierstrass theorem, when $G\in L^1(\mathbf{R})$, for each $x$ both $Z_l(k,x)$ and $Z_r(k,x)$ have continuous extensions in $k$ to $\overline{\mathbf{C}^+}\setminus\{0\}$ which are analytic on $\mathbf{C}^+$. Furthermore, $Z_l(k,x) = 1 + O(1/k)$ and $Z_r(k,x) = 1 + O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$.

From (4.8) we obtain for $k\in\overline{\mathbf{C}^+}\setminus\{0\}$,

$$|Z_l'(k,x)| \leqslant \int_x^\infty dt\, H(x)|G(t)||Z_l(k,t)| \leqslant \sup_{x\in\mathbf{R}} H(x) \times\exp\left(\frac{1}{|k|}\int_{-\infty}^\infty d\xi|G(x)|\right)\left(\int_{-\infty}^\infty dt|G(t)|\right),$$

$$|Z_r'(k,x)| \leqslant \int_{-\infty}^x dt\, H(x)|G(t)||Z_r(k,t)| \leqslant \sup_{x\in\mathbf{R}} H(x) \times\exp\left(\frac{1}{|k|}\int_{-\infty}^\infty d\xi|G(\xi)|\right)\left(\int_{-\infty}^\infty dt|G(t)|\right).$$

Hence, if $G\in L^1(\mathbf{R})$, both $Z_l'(k,x)$ and $Z_r'(k,x)$ have continuous extensions to $k\in\mathbf{C}^+\setminus\{0\}$ which are analytic on $\mathbf{C}^+$, and $Z_l'(k,x) = O(1)$ and $Z_r'(k,x) = O(1)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$.

Next we will show that the physical solutions $\psi_l$ and $\psi_r$ are related to $Z_l$ and $Z_r$ in a simple manner. From (4.1) we know that $Y_l(k,x)Z_l(k,x)$ is a solution of (1.1), and we have

$$Y_l(k,x)Z_l(k,x) = \exp\left(ikx - ik\int_0^\infty [1-H]\right) + o(1), \quad x\to\infty.$$

Thus the Jost solution from the left defined in (2.5) is given by

$$f_l(k,x) = \exp\left(ik\int_0^\infty [1-H]\right) Y_l(k,x)Z_l(k,x). \quad (4.17)$$

In the same way we obtain

$$Y_r(k,x)Z_r(k,x) = \exp\left(-ikx - ik\int_{-\infty}^0 [1-H]\right) + o(1), \quad x\to-\infty,$$

and hence the Jost solution from the right is given by

$$f_r(k,x) = \exp\left(ik\int_{-\infty}^0 [1-H]\right) Y_r(k,x)Z_r(k,x). \quad (4.18)$$

Thus from (2.5) it is seen that the physical solutions of (1.1) are given by

$$\psi_l(k,x) = T(k)\exp\left(ik\int_0^\infty [1-H]\right) Y_l(k,x)Z_l(k,x),$$

$$\psi_r(k,x) = T(k)\exp\left(ik\int_{-\infty}^0 [1-H]\right) Y_r(k,x)Z_r(k,x),$$

and from (2.8) we obtain

$$m_l(k,x) = \frac{1}{\sqrt{H(x)}}\exp\left(ik\int_x^\infty [1-H]\right) Z_l(k,x), \quad (4.19)$$

$$m_r(k,x) = \frac{1}{\sqrt{H(x)}}\exp\left(ik\int_{-\infty}^x [1-H]\right) Z_r(k,x), \quad (4.20)$$

and hence, as $k\to\infty$ we obtain

$$m_l(k,x) = \frac{\exp(ik\int_x^\infty [1-H])}{\sqrt{H(x)}}[1+O(1/k)], \quad k\in\overline{\mathbf{C}^+}, \quad (4.21)$$

$$m_r(k,x) = \frac{\exp(ik\int_{-\infty}^x [1-H])}{\sqrt{H(x)}}[1+O(1/k)], \quad k\in\overline{\mathbf{C}^+}. \quad (4.22)$$

Since $m_l(k,x)$ and $m_r(k,x)$ are continuous at $k=0$ when $V\in L^1(\mathbf{R})$, from (4.19) and (4.20), it follows that $Z_l(k,x)$ and $Z_r(k,x)$ are also continuous at $k=0$.

*Proposition 4.1:* If $G\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$ and $V(x)$ is bounded below, then for all $k_1\neq k_2\in\overline{\mathbf{C}^+}\setminus\{0\}$, the quantities

$$\frac{Z_l(k_1,x)-Z_l(k_2,x)}{|k_1-k_2|^\alpha} \text{ and } \frac{Z_r(k_1,x)-Z_r(k_2,x)}{|k_1-k_2|^\alpha}$$

are bounded in absolute value by $(1+|x|)^\alpha C_3(\alpha,\kappa)$, where

$$C_3(\alpha,\kappa)=\max\{2^{3-\alpha}\kappa^{-1-\alpha},2^{3-\alpha/2}\kappa^{-1}\sup_{\xi\in\mathbf{R}} H(\xi)^\alpha\}\times\exp\left(\frac{1}{\kappa}\int_{-\infty}^{\infty} dt[1+(1+|t|)^\alpha]|G(t)|\right)\times\int_{-\infty}^{\infty} dt(1+|t|)^\alpha|G(t)| \tag{4.23}$$

and $\kappa=\min\{|k_1|,|k_2|\}>0$.

*Proof:* We will only give the proof for $Z_l(k,x)$. Put

$$\Xi(k_1,k_2;x)=\frac{Z_l(k_1,x)-Z_l(k_2,x)}{|k_1-k_2|^\alpha}.$$

Then from (4.13) we have

$$\Xi(k_1,k_2;x)=\int_x^\infty dt\,\frac{\mathscr{L}_l(k_1;x,t)-\mathscr{L}_l(k_2;x,t)}{|k_1-k_2|^\alpha}Z_l(k_1,t)+\int_x^\infty dt\,\mathscr{L}_l(k_2;x,t)\Xi_l(k_1,k_2;t). \tag{4.24}$$

Using (4.11) and (4.15), we obtain

$$\frac{|\mathscr{L}_l(k_1;x,t)-\mathscr{L}_l(k_2;x,t)|}{|k_1-k_2|^\alpha} < 2^{1-\alpha}\kappa^{\alpha-1}|G(t)|^{1-\alpha}\left|\frac{\mathscr{L}_l(k_1;x,t)-\mathscr{L}_l(k_2;x,t)}{|k_1-k_2|}\right|^\alpha$$
$$<2\kappa^{\alpha-1}|G(t)|[\kappa^{-2\alpha}+2^{\alpha/2}\kappa^{-\alpha}\times(|x|+|t|)^\alpha\sup_{\xi\in\mathbf{R}}H(\xi)^\alpha]$$
$$<C_2(\alpha,\kappa)|G(t)|(1+|x|)^\alpha(1+|t|)^\alpha,$$

where

$$C_2(\alpha,\kappa)=\max\{2^{4-2\alpha}\kappa^{-1-\alpha},2^{4-\alpha/2}\kappa^{-1}\sup_{\xi\in\mathbf{R}}H(\xi)^\alpha\}.$$

Then, using (4.15) and (4.16), from (4.24) we obtain

$$|\Xi(k_1,k_2;x)|<C_2(\alpha,\kappa)\exp\left(\frac{1}{\kappa}\int_{-\infty}^{\infty}dt|G(t)|\right)\times(1+|x|)^\alpha\int_{-\infty}^{\infty}dt(1+|t|)^\alpha|G(t)|+\int_x^\infty dt\frac{|G(t)|}{\kappa}|\Xi(k_1,k_2;t)|,$$

and through iteration, we obtain $\Xi(k_1,k_2;x)|\leqslant C_3(\alpha,\kappa)\times(1+|x|)^\alpha$. ■

## V. LARGE $k$ ASYMPTOTICS OF THE SCATTERING MATRIX

In this section we obtain the large $k$ asymptotics of the scattering matrix. It is sufficient to assume $G\in L^1(\mathbf{R})$ and $V(x)$ is bounded below for the results to hold in this section, except in Propositions 5.1 and 5.2 where we further assume $G\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$.

From (4.17) and (4.18) we obtain

$$\sqrt{H(x)}f_l(k,x)\exp\left(ik\int_x^\infty[1-H]\right)=e^{ikx}\left(1+\frac{1}{2ik}\int_x^\infty dt\,G(t)Z_l(k,t)\right)+e^{-ikx}\frac{1}{2ik}\int_x^\infty dt\,G(t)Z_l(k,t)\times\exp\left(2ikt-2ik\int_0^t[1-H]\right) \tag{5.1}$$

and

$$\sqrt{H(x)}f_r(k,x)\exp\left(ik\int_{-\infty}^x[1-H]\right)=e^{-ikx}\left(1+\frac{1}{2ik}\int_{-\infty}^x dt\,G(t)Z_r(k,t)\right)-e^{ikx}\frac{1}{2ik}\int_{-\infty}^x dt\,G(t)Z_r(k,t)\times\exp\left(-2ikt+2ik\int_0^t[1-H]\right), \tag{5.2}$$

and hence, using (2.6) and (2.7) we have

$$\frac{1}{T(k)}=\exp\left(ik\int_{-\infty}^\infty[1-H]\right)\times\left[1+\frac{1}{2ik}\int_{-\infty}^\infty dt\,G(t)Z_l(k,t)\right], \tag{5.3}$$

$$\frac{1}{T(k)}=\exp\left(ik\int_{-\infty}^\infty[1-H]\right)\times\left[1+\frac{1}{2ik}\int_{-\infty}^\infty dt\,G(t)Z_r(k,t)\right],$$

$$\frac{L(k)}{T(k)}=-\exp\left(ik\int_0^\infty[1-H]-ik\int_{-\infty}^0[1-H]\right)\times\int_{-\infty}^\infty dt\frac{G(t)Z_l(k,t)}{2ik}\times\exp\left(2ikt-2ik\int_0^t[1-H]\right), \tag{5.4}$$

$$\frac{R(k)}{T(k)}=-\exp\left(ik\int_{-\infty}^0[1-H]-ik\int_0^\infty[1-H]\right)\times\int_{-\infty}^\infty dt\frac{G(t)Z_r(k,t)}{2ik}\times\exp\left(-2ikt+2ik\int_0^t[1-H]\right). \tag{5.5}$$

From (5.3), (5.4), and (5.5), as $|k|\to\infty$ we obtain

$$T(k)=\exp\left(-ik\int_{-\infty}^{\infty}[1-H]\right)\times\left[1-\frac{1}{2ik}\int_{-\infty}^{\infty}dt\,G(t)+O(1/k^2)\right],\quad k\in\overline{\mathbf{C}^+},\tag{5.6}$$

$$L(k)=-\left[\exp\left(-2ik\int_{-\infty}^{0}[1-H]\right)\right]\frac{1}{2ik}\times\int_{-\infty}^{\infty}dt\,G(t)\exp\left(2ikt-2ik\int_0^t[1-H]\right)+O(1/k^2),\quad k\in\mathbf{R},\tag{5.7}$$

$$R(k)=-\left[\exp\left(-2ik\int_0^{\infty}[1-H]\right)\right]\frac{1}{2ik}\int_{-\infty}^{\infty}dt\,G(t)\times\exp\left(-2ikt+2ik\int_0^t[1-H]\right)+O(1/k^2),\quad k\in\mathbf{R}.\tag{5.8}$$

*Proposition 5.1:* Let $\kappa=\min\{|k_1|,|k_2|\}$ and

$$A_0(k)=T(k)\exp\left(ik\int_{-\infty}^{\infty}[1-H]\right).$$

If $G\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$ and $V(x)$ is bounded below, then $|A_0(k_1)-A_0(k_2)|/|k_1-k_2|^\alpha$ is uniformly bounded for $k_1\neq k_2\in\mathbf{C}^+\setminus\{0\}$ with $\kappa\geqslant\delta>0$.

*Proof:* Let $\Xi(k_1,k_2;x)$ be as in the proof of Proposition 4.1 and let $A_1(k)=1/A_0(k)$. Then from (5.3), we have

$$A_1(k)=1+\int_{-\infty}^{\infty}dz\frac{G(z)Z_l(k,z)}{2ik}.$$

Thus

$$\frac{A_1(k_1)-A_1(k_2)}{|k_1-k_2|^\alpha}=\frac{1/k_1-1/k_2}{2i|k_1-k_2|^\alpha}\int_{-\infty}^{\infty}dz\,G(z)Z_l(k_1,z)+\frac{1}{2ik_2}\int_{-\infty}^{\infty}dz\,G(z)\Xi(k_1,k_2;z),$$

and hence using (4.16) and Proposition 4.1, we obtain

$$\frac{|A_1(k_1)-A_1(k_2)|}{|k_1-k_2|^\alpha}\leqslant\frac{\exp(\kappa^{-1}\int_{-\infty}^{\infty}dt|G(t)|)}{2^\alpha\kappa^{1+\alpha}}\int_{-\infty}^{\infty}dz|G(z)|+\frac{C_3(\alpha,\kappa)}{2\kappa}\int_{-\infty}^{\infty}dz(1+|z|)^\alpha|G(z)|,$$

where $C_3(\alpha,\kappa)$ is the quantity in (4.23). Since $|T(k)|$ is bounded away from zero and

$$\frac{|A_0(k_1)-A_0(k_2)|}{|k_1-k_2|^\alpha}\leqslant\frac{1}{|T(k_1)T(k_2)|}\frac{|A_1(k_1)-A_1(k_2)|}{|k_1-k_2|^\alpha},$$

the left-hand side of the last expression is bounded for $k_1\neq k_2$ with $\kappa\geqslant\delta>0$. ■

*Proposition 5.2:* Let $\kappa=\min\{|k_1|,|k_2|\}$ and let $F(k)$ denote either $L(k)\exp(2ik\int_{-\infty}^0[1-H])$ or $R(k)\times\exp(2ik\int_0^\infty[1-H])$. If $G\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$ and $V(x)$ is bounded below, then $F(k_1)-F(k_2)|/|k_1-k_2|^\alpha$ is uniformly bounded for $k_1\neq k_2\in\mathbf{R}\setminus\{0\}$ with $\kappa\geqslant\delta>0$.

*Proof:* Let

$$A_2(k)=-\frac{L(k)}{T(k)}\exp\left(-ik\int_0^{\infty}[1-H]+ik\int_{-\infty}^{0}[1-H]\right).$$

As seen from (5.4), we have

$$A_2(k)=\frac{1}{2ik}\int_{-\infty}^{\infty}dz\,G(z)Z_l(k,z)\exp\left(2ik\int_0^z H\right).$$

Hence using $\Xi(k_1,k_2;x)$ defined in the proof of Proposition 4.1, we have

$$\frac{A_2(k_1)-A_2(k_2)}{|k_1-k_2|^\alpha}=\frac{1/k_1-1/k_2}{2i|k_1-k_2|^\alpha}\int_{-\infty}^{\infty}dz\,G(z)Z_l(k_1,z)\exp\left(2ik_1\int_0^z H\right)+\frac{1}{2ik_2}\int_{-\infty}^{\infty}dz\,G(z)Z_l(k_2,z)\times\left[\exp\left(2ik_1\int_0^z H\right)-\exp\left(2ik_2\int_0^z H\right)\right]+\frac{1}{2ik_2}\int_{-\infty}^{\infty}dz\,G(z)\Xi(k_1,k_2;z)\exp\left(2ik_1\int_0^z H\right),$$

so that

$$\frac{|A_2(k_1)-A_2(k_2)|}{|k_1-k_2|^\alpha}\leqslant 2^{-\alpha}\kappa^{-1-\alpha}\exp\left(\frac{1}{\kappa}\int_{-\infty}^{\infty}dt|G(t)|\right)\int_{-\infty}^{\infty}dz|G(z)|+\left[2^{\alpha/2}\kappa\sup_{\xi\in\mathbf{R}}H(\xi)^\alpha\exp\left(\frac{1}{\kappa}\int_{-\infty}^{\infty}dt|G(t)|\right)+2^{-1}\kappa^{-1}C_3(\alpha,\kappa)\right]\int_{-\infty}^{\infty}dz(1+|z|)^\alpha|G(z)|,$$

where $C_3(\alpha,\kappa)$ is the quantity in (4.23). This estimate and Proposition 5.1 imply the statement of the proposition. ■

## VI. RIEMANN–HILBERT PROBLEM

In this section the sufficient conditions for the results to hold are $0 \leqslant V(x) < 1$, $V \in L^1_\alpha(\mathbf{R})$ and $G \in L^1_\alpha(\mathbf{R})$ for some $\alpha \in (0,1]$. We will show that we can recover $V(x)$ from its scattering matrix uniquely. The condition $V(x) \geqslant 0$ insures us that $m_l(k,x)$ and $m_r(k,x)$ remain bounded as $k \to \infty$ in $\overline{\mathbf{C}^+}$. In order to formulate the inverse scattering problem as a Riemann–Hilbert problem, in this section we assume that $V(x) \geqslant 0$. However, in Sec. X, we will generalize our results so that $V(x) \geqslant 0$ is no longer needed.

Let us denote by $\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^n)$ the Hilbert space of all vector functions $f:\mathbf{R} \to \mathbf{C}^n$ which are analytic on $\mathbf{C}^\pm$ and satisfy

$$\sup_{b>0} \int_{-\infty}^{\infty} dk \| f(k \pm ib) \|^2_{\mathbf{C}^n} < \infty.$$

Then the Hardy spaces $\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^n)$ are closed complementary, mutually orthogonal subspaces of the Hilbert space $L^2(\mathbf{R};\mathbf{C}^n)$ of square integrable vector functions $f:\mathbf{R} \to \mathbf{C}^n$. We write $\Pi_\pm$ for the orthogonal projections of $L^2(\mathbf{R};\mathbf{C}^n)$ onto $\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^n)$ and abbreviate $\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^1)$ by $\mathbf{H}^2_\pm(\mathbf{R})$.

Since $k$ appears as $k^2$ in (1.1), $\psi_l(-k,x)$ and $\psi_r(-k,x)$ are also solutions of (1.1) whenever $\psi_l(k,x)$ and $\psi_r(k,x)$ are the physical solutions. Using (2.1) and (2.2) as well as (3.9) and (3.10), the solution vectors

$$\psi(\pm k,x) = \begin{bmatrix} \psi_l(\pm k,x) \\ \psi_r(\pm k,x) \end{bmatrix}$$

are found to be related to each other as

$$\psi(-k,x) = S(-k)^t q \psi(k,x), \quad k \in \mathbf{R}, \tag{6.1}$$

where $q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Letting

$$m(k,x) = \begin{bmatrix} m_l(k,x) \\ m_r(k,x) \end{bmatrix}$$

and using (2.8) and (3.9), we can write (6.1) as

$$m(-k,x) = \Lambda(k) q m(k,x), \quad k \in \mathbf{R}, \tag{6.2}$$

where we have defined

$$\Lambda(k) = \begin{bmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{-2ikx} & T(k) \end{bmatrix}$$

in such a way that the $x$ dependence is suppressed.

The large $k$-asymptotics of $m(\pm k,x)$, as seen from (4.21) and (4.22), make it unsuitable to solve (6.2) as it is stated. However, the transformation $k \mapsto -1/k$ changes (6.2) into a Riemann–Hilbert problem that can be solved. Under this transformation $\mathbf{C}^+$ and $\mathbf{R} \cup \{\infty\}$ are mapped onto themselves in a one-to-one manner. Let us use a superscript tilde to denote the transformed function under the map $k \mapsto -1/k$; i.e., let us use the notation $\tilde{F}(k) = F(-1/k)$ throughout the paper. The transformation $k \mapsto -1/k$ changes (6.2) into

$$\tilde{m}(-k,x) = \tilde{\Lambda}(k) q \tilde{m}(k,x), \quad k \in \mathbf{R}. \tag{6.3}$$

Let $\hat{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From Sec. II it is known that $\tilde{m}(k,x)$ is continuous in $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, has an analytic extension in $k$ to $\mathbf{C}^+$ for each $x$, and $\tilde{m}(k,x) - \hat{1} = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, which is seen from (2.17) and (2.18). If we assume $0 \leqslant V(x) < 1$, from (4.21) and (4.22) we see that $\tilde{m}(k,x)$ also remains bounded as $k \to 0$ in $\overline{\mathbf{C}^+}$. Similarly, $\tilde{m}(-k,x)$ is continuous in $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, has an analytic extension in $k$ to $\mathbf{C}^-$ for each $x$, and $\tilde{m}(-k,x) - \hat{1} = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^-}$. Hence, when the scattering matrix is known, solving (6.3) for $\tilde{m}(-k,x)$ and $\tilde{m}(k,x)$ constitutes a Riemann–Hilbert problem in which we seek solutions satisfying $\overline{\tilde{m}(-k,x)} = \tilde{m}(k,x)$.

Letting $F(k) = [\tilde{\Lambda}(k) - \mathbf{I}]\hat{1}$ and defining

$$\begin{aligned} n_+(k) &= q[\tilde{m}(k,x) - \hat{1}] \\ n_-(k) &= \tilde{m}(-k,x) - \hat{1}, \end{aligned} \tag{6.4}$$

we can write (6.3) as

$$n_-(k) = \tilde{\Lambda}(k) n_+(k) + F(k), \quad k \in \mathbf{R}, \tag{6.5}$$

where $n_\pm \in \mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^2)$ and $F \in L^2(\mathbf{R};\mathbf{C}^2)$; we will seek solutions of (6.5) satisfying

$$\overline{n_+(-k)} = n_+(k) \text{ and } \overline{n_-(-k)} = n_-(k), \quad k \in \mathbf{R}, \tag{6.6}$$

$$n_\pm(-k) = q n_\mp(k), \quad k \in \mathbf{R}. \tag{6.7}$$

Here we use the notation $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The Riemann–Hilbert problem stated in (6.5) differs from a conventional Riemann–Hilbert problem[8,9] because the matrix $\tilde{\Lambda}(k)$ has a discontinuity of almost periodic type[10,11] at $k=0$. In order to solve (6.5), we will apply a matrix analog of the methods of Refs. 10 and 11. Using these methods we will show that (6.5), although not a Fredholm problem, is a problem that has solutions for every nonhomogeneous term $F \in L^2(\mathbf{R};\mathbf{C}^2)$ and that the corresponding homogeneous problem has infinitely many linearly independent solutions. In spite of these non-Fredholm characteristics, we will show that the large $k$ asymptotics of different solutions of (6.5) are the same and as a consequence of this fact all solutions lead to the same potential.

Let us now study the matrix $\tilde{\Lambda}(k)$. Let $A = \int_{-\infty}^{\infty}[1 - H]$; note that $A \geqslant 0$ if we assume that $0 \leqslant V(x) < 1$. Define the matrix $M(k)$ as

$$M(k)=e^{-iA/k}\tilde{\Lambda}(k)=\begin{bmatrix} T(-1/k)e^{-iA/k} & -R(-1/k)e^{-iA/k-2ix/k} \\ -L(-1/k)e^{-iA/k+2ix/k} & T(-1/k)e^{-iA/k} \end{bmatrix}. \tag{6.8}$$

From the properties of the scattering matrix $S(k)$, it follows that $\tilde{\Lambda}(k)$ is unitary. We also have

$$\overline{\tilde{\Lambda}(-k)}=\tilde{\Lambda}(k) \text{ and } \overline{F(-k)}=F(k), \quad k\in\mathbf{R},$$

$$\tilde{\Lambda}(k)^{-1}=q\tilde{\Lambda}(-k)q \text{ and}$$

$$\tilde{\Lambda}(k)^{-1}F(k)=-qF(-k), \quad k\in\mathbf{R}.$$

Note that as $k\to 0$, from (3.14), (3.15), and (3.16) we have

$$T(k)=1+O(k), \quad k\in\overline{\mathbf{C}^+},$$

$$R(k)=O(k), \quad k\in\mathbf{R},$$

$$L(k)=O(k), \quad k\in\mathbf{R},$$

and as $|k|\to\infty$, from (5.6), (5.7), and (5.8) we have

$$T(k)e^{iAk}=1+O(1/k), \quad k\in\overline{\mathbf{C}^+},$$

$$R(k)e^{iAk}=O(1/k), \quad k\in\mathbf{R},$$

$$L(k)e^{iAk}=O(1/k), \quad k\in\mathbf{R}.$$

Thus we have $M(k)=\mathbf{I}+O(1/k)$ as $k\to\pm\infty$, and $M(k)$ is continuous on the real axis including the continuity at $k=0$. In fact, as $k\to 0$ in $\mathbf{R}$, we have $M(k)=\mathbf{I}+O(k)$. Since $\tilde{\Lambda}(k)$ is unitary, it follows from (6.8) that $M(k)$ is a unitary matrix.

*Proposition 6.1:* Let $\tilde{\tau}(k)=T(-1/k)e^{-iA/k}$. *If* $V,G\in L^1(\mathbf{R})$, *then* $\tilde{\tau}(k)-1$ belongs to the Hardy space $\mathbf{H}^2_+(\mathbf{R})$.

*Proof:* Using (3.14), (5.6), and the continuity of $T(k)$ in $\overline{\mathbf{C}^+}$ which has been established in Sec. III, we obtain for $k\in\overline{\mathbf{C}^+}$

$$|\tilde{\tau}(k)-1|\leqslant\begin{cases} c_2, & |k|\leqslant 1,\\ \dfrac{c_1}{|k|}, & |k|\geqslant 1,\end{cases}$$

where $c_1$ and $c_2$ are some positive constants. Thus, for $b>0$ we obtain

$$\int_{-\infty}^{\infty} dk|\tilde{\tau}(k+ib)-1|^2$$

$$\leqslant\begin{cases} 2c_2^2\sqrt{1-b^2}+(2c_1^2/b)\arcsin b, & 0<b<1,\\ \pi c_1^2/b, & b\geqslant 1,\end{cases}$$

and, hence, using $\theta/\sin\theta\leqslant 6/5$ for $\theta\in[0,1]$, we obtain

$$\int_{-\infty}^{\infty} dk|\tilde{\tau}(k+ib)-1|^2\leqslant\max\left\{2c_2^2+\frac{12c_1^2}{5},\pi c_1^2\right\}.$$

Thus, $\tilde{\tau}(k)-1\in\mathbf{H}^2_+(\mathbf{R})$. As a consequence, letting

$$u(y)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{iky}[\tilde{\tau}(k)-1], \tag{6.9}$$

we obtain $u(y)=0$ for $y>0$. ■

## VII. WIENER–HOPF FACTORIZATION

The sufficient assumptions in this section are the same as those in Sec. VI. In this section we show that the matrix $M(k)$ given in (6.8) has a unique canonical Wiener–Hopf factorization.

By the Wiener–Hopf factorization of a continuous $n\times n$ matrix function $M(k)$ with $M(\pm\infty)=\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, we mean a representation of $M(k)$ in the form

$$M(k)=M_-(k)D(k)M_+(k), \quad k\in\mathbf{R}, \tag{7.1}$$

where $M_\pm(k)$ are continuous on $\overline{\mathbf{C}^\pm}$ and analytic on $\mathbf{C}^\pm$, $M_\pm(k)$ are boundedly invertible for all $k\in\overline{\mathbf{C}^\pm}$, $\|M_\pm(k)-\mathbf{I}\|\to 0$ as $k\to\infty$ in $\overline{\mathbf{C}^\pm}$, and

$$D(k)=P_0+\sum_{j=1}^{n}\left(\frac{k-i}{k+i}\right)^{\mu_j}P_j$$

is the diagonal factor. Here $P_1,\ldots,P_n$ are mutually disjoint one-dimensional projections, and $P_0=\mathbf{I}-\Sigma_{j=1}^n P_j$. The $\mu_1,\ldots,\mu_n$ are nonzero integers depending only on $M(k)$ and are called the partial indices of $M(k)$. If the partial indices are all zero and hence $D(k)\equiv\mathbf{I}$, the factorization (7.1) is called canonical.

The Möbius transformation $k\mapsto\xi=(k-i)/(k+i)$ maps $\mathbf{R}\cup\{\infty\}$ onto the unit circle $\mathbf{T}$ of the complex plane, $\mathbf{C}^+$ onto the interior of the unit disk, and $\mathbf{C}^-$ onto the exterior of the unit disk. It is known[12] that $M(k)$ has a Wiener–Hopf factorization if $\xi\mapsto M(\mathrm{i}(1+\xi)/(1-\xi))$ is Hölder continuous; i.e., if there exists a constant $\beta\in(0,1]$ such that

$$\left\|M\left(i\frac{1+\xi_1}{1-\xi_1}\right)-M\left(i\frac{1+\xi_2}{1-\xi_2}\right)\right\|\leqslant c|\xi_1-\xi_2|^\beta$$

for all $\xi_1,\xi_2\in\mathbf{T}$ for some constant $c$. The constant $\beta$ is called the exponent. The next theorem is about the Wiener–Hopf factorization of the $2\times 2$ matrix $M(k)$ defined in (6.8).

**Theorem 7.1:** If $V,G\in L^1_\alpha(\mathbf{R})$ for some $\alpha\in(0,1]$, then $M(k)$ defined in (6.8) has a Wiener–Hopf factorization.

*Proof:* To prove the existence of the Wiener–Hopf factorization of $M(k)$, it is sufficient to prove that for $\xi\in\mathbf{T}$, the matrix $M(\mathrm{i}(1+\xi)/(1-\xi))$ is Hölder continuous of exponent $\beta$ for some $\beta\in(0,1]$. From Propositions 3.1, 3.2, 5.1, and 5.2, because $T(k)$ is continuous and never vanishes for $k\in\mathbf{R}$, it follows that, if $V,G\in L^1_\alpha(\mathbf{R})$, $\|M(k+\delta)-M(k)\|\leqslant c\delta^\alpha$ for some constant $c$ and for all $\delta>0$. We have

$$|\xi_1-\xi_2|=\frac{2|k_1-k_2|}{\sqrt{k_1^2+1}\sqrt{k_2^2+1}}.$$

Thus

$$\frac{\|M(i(1+\xi_1)/(1-\xi_1))-M(i(1+\xi_2)/(1-\xi_2))\|}{|\xi_1-\xi_2|^\beta}$$

$$=\frac{\|M(k_1)-M(k_2)\|}{2^\beta|k_1-k_2|^\beta}(k_1^2+1)^{\beta/2}(k_2^2+1)^{\beta/2}. \quad (7.2)$$

We need to show that the quantity in (7.2) is uniformly bounded for some $\beta\in(0,1]$. Letting $k_1 = k$ and $\delta = |k_1 - k_2|$, it is enough to show that

$$\Theta(k,\delta)=\frac{\|M(k+\delta)-M(k)\|}{2^\beta\delta^\beta}(k^2+1)^{\beta/2}\times[(k+\delta)^2+1]^{\beta/2}$$

is uniformly bounded for a suitable $\beta\in(0,1]$. Due to the uniform boundedness of $M(k)$ on the extended real axis, it is enough to give the proof for $\delta<1$. For every $\epsilon\in[0,1]$ we have

$$\Theta(k,\delta)\leqslant\|M(k+\delta)-M(k)\|^{1-\epsilon}c^\epsilon\delta^{\alpha\epsilon-\beta}2^{-\beta}\times(k^2+1)^{\beta/2}[(k+\delta)^2+1]^{\beta/2} \quad (7.3)$$

and also

$$\Theta(k,\delta)\leqslant c\delta^{\alpha-\beta}2^{-\beta}(k^2+1)^{\beta/2}[(k+\delta)^2+1]^{\beta/2}. \quad (7.4)$$

Hence, if $|k|\leqslant1$, assuming $\alpha\geqslant\beta\geqslant0$, from (7.4) we obtain $\Theta(k,\delta)\leqslant c2^{-\beta}2^{\beta/2}5^{\beta/2}$. If $|k|\geqslant1$, from (7.3) we obtain

$$\Theta(k,\delta)\leqslant[\|M(k+\delta)-\mathbf{I}\|+\|M(k)-\mathbf{I}\|]^{1-\epsilon}c^\epsilon\delta^{\alpha\epsilon-\beta}\times2^{-\beta}(10)^{\beta/2}|k|^{2\beta}$$
$$\leqslant c^\epsilon\delta^{\alpha\epsilon-\beta}2^{-\beta}(10)^{\beta/2}\Big[\sup_{k\in\mathbf{R}}|k|^{2\beta}\|M(k)-\mathbf{I}\|^{1-\epsilon}+\sup_{k\in\mathbf{R}}|k|^{2\beta}\|M(k+\delta)-\mathbf{I}\|^{1-\epsilon}\Big],$$

and since the matrix $M(k)$ is continuous and $M(k)=\mathbf{I}+O(1/k)$ as $k\to\pm\infty$, the suprema in the last expression are finite numbers if $1-\epsilon\geqslant2\beta$. Thus if $1-\epsilon\geqslant2\beta$ and $\alpha\epsilon\geqslant\beta$, or equivalently if $0<\beta\leqslant\alpha/(2\alpha+1)$, $\Theta(k,\delta)$ is uniformly bounded. In case $\xi_2 = 1$, we have

$$\frac{\|M(i(1+\xi)/(1-\xi))-M(1)\|}{|\xi-1|^\beta}=\frac{\|M(k)-\mathbf{I}\|}{|(k-i)/(k+i)-1|^\beta}=\frac{(k^2+1)^{\beta/2}}{2^\beta}\|M(k)-\mathbf{I}\|$$

and since $M(k)$ is continuous and $M(k)=\mathbf{I}+O(1/k)$ as $k\to\pm\infty$, the last expression is uniformly bounded for all $\beta\leqslant1$. ■

The next theorem shows that the partial indices of $M(k)$ are both zero, and hence $M(k)$ has a unique canonical factorization $M(k) = M_-(k)M_+(k)$.

**Theorem 7.2:** If $V\in L_\alpha^1(\mathbf{R})$ and $G\in L_\alpha^1(\mathbf{R})$ for some $\alpha\in(0,1]$, then $M(k)$ defined in (6.8) has a unique canonical Wiener–Hopf factorization.

*Proof:* Let

$$\sigma(k)=\begin{bmatrix}\tau(k) & \rho(k)\\ \ell(k) & \tau(k)\end{bmatrix}$$

be the scattering matrix for the Schrödinger equation given in (1.4). Then from the asymptotics of (1.3) as $y\to\pm\infty$, we obtain

$$\tau(k)=T(k)\exp\left(ik\int_{-\infty}^{\infty}[1-H]\right),$$

$$\ell(k)=L(k)\exp\left(2ik\int_{-\infty}^{0}[1-H]\right),$$

$$\rho(k)=R(k)\exp\left(2ik\int_{0}^{\infty}[1-H]\right).$$

Let $\mathbf{J} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Then

$$\mathbf{J}\sigma(k)\mathbf{J}=\begin{bmatrix}\tau(k) & -\rho(k)\\ -\ell(k) & \tau(k)\end{bmatrix}.$$

Let

$$\gamma=y+\frac12\left(\int_{-\infty}^0[1-H]-\int_0^\infty[1-H]\right).$$

From (6.8) we obtain $\tilde{M}(k) = \mathbf{J}\sigma_\gamma(k)\mathbf{J}$, where

$$\sigma_\gamma(k)=e^{ik\gamma\mathbf{J}}\sigma(k)e^{-ik\gamma\mathbf{J}}=\begin{bmatrix}\tau(k) & \rho(k)e^{2ik\gamma}\\ \ell(k)e^{-2ik\gamma} & \tau(k)\end{bmatrix}.$$

It is known[13,14] from the inverse scattering theory for the Schrödinger equation (1.4) that $\sigma_\gamma(k)$ is the scattering matrix for the Schrödinger equation (1.4) that corresponds to the potential shifted by $\gamma$. Due to the fact that $\tau(k)$ never vanishes and there are no bound states, the vector Riemann–Hilbert problems for the matrices $\sigma_\gamma(k)$ and $\mathbf{J}\sigma_\gamma(k)\mathbf{J}$ have unique solutions that can be obtained by the Marchenko procedure.[15,16] As a result the matrix $\mathbf{J}\sigma_\gamma(k)\mathbf{J}$ has a canonical Wiener–Hopf factorization, and its Wiener–Hopf factors can be obtained as follows.[17,18] Let $\begin{bmatrix}\alpha_l(k)\\ \alpha_r(k)\end{bmatrix}$ be the solution of the vector Riemann–Hilbert problem with matrix $\mathbf{J}\sigma_\gamma(k)\mathbf{J}$, and let $\begin{bmatrix}\beta_l(k)\\ \beta_r(k)\end{bmatrix}$ be the solution of the vector Riemann–Hilbert problem with matrix $\sigma_\gamma(k)$. Then upon forming the matrix

$$U(k)=\frac12\begin{bmatrix}\alpha_l+\beta_l & \alpha_l-\beta_l\\ \alpha_r-\beta_r & \alpha_r+\beta_r\end{bmatrix},$$

we obtain the unique solution of the matrix Riemann–Hilbert problem

$$U(-k)=\tilde{M}(k)qU(k)q, \quad k\in\mathbf{R},$$

which leads to the unique canonical Wiener–Hopf factorization $\tilde{M}(k) = \tilde{M}_-(k)\tilde{M}_+(k)$, where

$$\tilde{M}_-(k)=U(-k) \text{ and } \tilde{M}_+(k)=qU(k)^{-1}q.$$

Since $M(0)=\mathbf{I}$ and $M(k)=\tilde{M}(-1/k)$, the matrix $M(k)$ has also a canonical factorization with factors $U(1/k)U(0)^{-1}$ and $qU(0)U(-1/k)^{-1}q$, respectively. ■

## VIII. RECOVERY OF THE POTENTIAL

In this section we solve the key Riemann–Hilbert problem (6.3), using the canonical factorization of the matrix $M(k)$ given in (6.8) This is done by solving (6.5) and restricting its general solution so that (6.6) and (6.7) are satisfied. The sufficient assumptions in this section are the same as those in Sec. VI.

*Proposition 8.1:* Suppose

$$\widetilde{\Lambda}(k)=e^{iA/k}M_-(k)M_+(k), \quad k\in\mathbf{R},$$

where $A>0$, $M_\pm(k)$ are continuous on $\overline{\mathbf{C}^\pm}$ and analytic on $\mathbf{C}^\pm$, satisfy $M_\pm(\infty)=\mathbf{I}$, and are boundedly invertible on $\overline{\mathbf{C}^\pm}$. Then for every $F\in L^2(\mathbf{R};\mathbf{C}^2)$ the Riemann–Hilbert problem (6.5) has at least one solution and the general solution is given by

$$n_+(k)=-e^{-iA/k}M_+(k)^{-1}[\Pi_+\{M_-^{-1}F\}](k)+\frac{i}{k}M_+(k)^{-1}\int_0^A dt\, e^{-it/k}\eta(t), \tag{8.1}$$

$$n_-(k)=M_-(k)[\Pi_-\{M_-^{-1}F\}](k)+\frac{i}{k}M_-(k)\int_0^A dt\, e^{i(A-t)/k}\eta(t), \tag{8.2}$$

where $\eta\in L^2((0,A);\mathbf{C}^2)$ is arbitrary and $\Pi_\pm$ are the orthogonal projection operators onto $\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^2)$.

*Proof:* One can directly verify that (8.1) and (8.2) represent a solution of (6.5) for every $F\in L^2(\mathbf{R};\mathbf{C}^2)$ and every $\eta\in L^2((0,A);\mathbf{C}^2)$. To prove that (8.1) and (8.2) represent all solutions of (6.5), we will compute the general solution of the corresponding homogeneous Riemann–Hilbert problem

$$n_-^{[0]}(k)=\widetilde{\Lambda}(k)n_+^{[0]}(k), \quad k\in\mathbf{R}. \tag{8.3}$$

Letting

$$n_+^{[0]}(k)=M_+(k)^{-1}p_+^{[0]}(k), \qquad n_-^{[0]}(k)=M_-(k)p_-^{[0]}(k), \tag{8.4}$$

we can write (8.3) as

$$p_-^{[0]}(k)=e^{iA/k}p_+^{[0]}(k), \quad k\in\mathbf{R}, \tag{8.5}$$

where $p_\pm^{[0]}\in\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^2)$. Now note that $(\mathscr{U}f)(k)=(i/k)f(-1/k)$ defines a unitary operator on $L^2(\mathbf{R};\mathbf{C}^2)$ which maps $\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^2)$ onto themselves and is its own inverse. Letting $r_\pm^{[0]}=\mathscr{U}p_\pm^{[0]}$, from (8.5) we obtain

$$r_-^{[0]}(k)=e^{-ikA}r_+^{[0]}(k), \quad k\in\mathbf{R}, \tag{8.6}$$

where $r_\pm^{[0]}\in\mathbf{H}^2_\pm(\mathbf{R};\mathbf{C}^2)$. Thus there exist $\eta_+\in L^2((0,\infty);\mathbf{C}^2)$ and $\eta_-\in L^2((-\infty,0);\mathbf{C}^2)$ such that

$$r_+^{[0]}(k)=\int_0^\infty dt\, e^{ikt}\eta_+(t)$$

and

$$r_-^{[0]}(k)=\int_{-\infty}^0 dt\, e^{ikt}\eta_-(t),$$

and using these integral representations in (8.6), we obtain $\eta_+(t)=0$ for $t>A$, $\eta_-(t)=0$ for $t<-A$, and $\eta_-(t))=\eta_+(t+A)$. Defining $\eta(t)\overset{\text{def}}{=}\eta_+(t)=\eta_-(t-A)$ for $0<t<A$, we have

$$p_+^{[0]}(k)=\frac{i}{k}\int_0^A dt\, e^{-it/k}\eta(t)$$

and

$$p_-^{[0]}(k)=\frac{i}{k}\int_{-A}^0 dt\, e^{-it/k}\eta(t+A)=\frac{i}{k}\int_0^A dt\, e^{i(A-t)/k}\eta(t),$$

and using (8.4), we obtain the complementary solutions given in (8.1) and (8.2). ∎

Using $\overline{M(-k)}=qM(-k)^{-1}q=M(k)$, we obtain $\overline{M_\pm(-k)}=M_\pm(k)$ and

$$M_\pm(k)^{-1}=qM_\mp(-k)q. \tag{8.7}$$

Next, using $F(k)=[\widetilde{\Lambda}(k)-\mathbf{I}]\hat{1}$, we obtain

$$M_-(k)^{-1}F(k)=[e^{iA/k}-1]M_+(k)\hat{1}+[M_+(k)-\mathbf{I}]\hat{1}+[\mathbf{I}-M_-(k)^{-1}]\hat{1}.$$

Since $M_-(k)^{-1}$and $M_+(k)$ are bounded, it follows that $M_-(k)^{-1}F(k)$ and $[e^{iA/k}-1]M_+(k)\hat{1}$ belong to $L^2(\mathbf{R};\mathbf{C}^2)$. Thus $g(k)=[M_+(k)-M_-(k)^{-1}]\hat{1}$ belongs to $L^2(\mathbf{R};\mathbf{C}^2)$. Hence, $[\Pi_+g](k)=[M_+(k)-\mathbf{I}]\hat{1}$ belongs to $\mathbf{H}^2_+(\mathbf{R};\mathbf{C}^2)$ and $[\Pi_-g](k)=[\mathbf{I}-M_-(k)^{-1}]\hat{1}$ belongs to $\mathbf{H}^2_-(\mathbf{R};\mathbf{C}^2)$. This conclusion may also be drawn if $\hat{1}$ were an arbitrary two-vector. Hence, there exists a real $2\times2$ matrix function $\Gamma\in L^2((0,\infty);\mathbf{C}^2\times\mathbf{C}^2)$, such that

$$\Gamma(t)=\int_{-\infty}^\infty \frac{dk}{2\pi}e^{-ikt}\frac{i}{k}\left[M_+\left(\frac{-1}{k}\right)-\mathbf{I}\right]. \tag{8.8}$$

Writing

$$\mathscr{U}[M_-^{-1}F](k)=e^{-ikA}\frac{i}{k}\left[M_+\left(\frac{-1}{k}\right)-\mathbf{I}\right]\hat{1}-\frac{i}{k}\left[M_-\left(\frac{-1}{k}\right)^{-1}-\mathbf{I}\right]\hat{1}+\frac{i}{k}[e^{-ikA}-1]\hat{1},$$

we get

$$\mathscr{U}[\Pi_+\{M_-^{-1}F\}](k)=\int_0^\infty dt\, e^{ikt}\Gamma(t+A)\hat{1},$$

$$\mathscr{U}[\Pi_-\{M_-^{-1}F\}](k)=\int_0^A dt\, e^{-ikt}\Gamma(A-t)\hat{1}-\frac{i}{k}\left[M_-\left(\frac{-1}{k}\right)^{-1}-\mathbf{I}\right]\hat{1}+\frac{i}{k}[e^{-ikA}-1]\hat{1}.$$

Hence,

$$[\Pi_+\{M_-^{-1}F\}](k)=\frac{i}{k}\int_0^\infty dt\, e^{it/k}\Gamma(t+A)\hat{1}, \tag{8.9}$$

$$[\Pi_-\{M_-^{-1}F\}](k)$$
$$=\frac{i}{k}\int_0^A dt\, e^{it/k}\Gamma(A-t)\hat{1}-[M_-(k)^{-1}-\mathbf{I}]\hat{1}$$
$$+[e^{iA/k}-1]\hat{1}. \tag{8.10}$$

Among all the solutions of (6.5) we are interested in those satisfying (6.6) and (6.7). In Proposition 8.1 we have obtained the general solution of (6.5) in terms of the canonical factors of $M(k)$. The next theorem shows how to obtain the general solution of (6.2) by restricting the solutions of (6.5) given in (8.1) and (8.2) to those satisfying (6.6) and (6.7).

**Theorem 8.2:** Suppose

$$\tilde{\Lambda}(k)=e^{iA/k}M_-(k)M_+(k), \quad k\in\mathbf{R},$$

where $A>0$, $M_\pm(k)$ are continuous on $\overline{\mathbf{C}^\pm}$ and analytic on $\mathbf{C}^\pm$, satisfy $M_\pm(\infty)=\mathbf{I}$, and are boundedly invertible on $\overline{\mathbf{C}^\pm}$. Then the general solution $m(k,x)$ of (6.2) is given by

$$m(k,x)=\hat{1}+\frac{1}{2}M_-\left(\frac{1}{k}\right)\left(ik\int_A^\infty dt\, e^{ikt}q\Gamma(t)\hat{1}\right.$$
$$+ik\int_0^A dt\, e^{ikt}\Gamma(A-t)\hat{1}$$
$$+\left[e^{ikA}-M_-\left(\frac{1}{k}\right)^{-1}\right]\hat{1}$$
$$\left.+ik\int_0^A dt\, e^{ikt}\begin{bmatrix}\omega(t)\\-\omega(A-t)\end{bmatrix}\right), \tag{811}$$

where $\omega\in L^2(0,A)$ is an arbitrary real function and $\Gamma$ is the matrix defined in (8.8)

*Proof:* Using (8.9) and (8.10) in (8.1) and (8.2), we obtain the general solution of (6.5). If we choose $\eta$ in (8.1) to be real, (6.6) is satisfied. In order to satisfy (6.7), it is sufficient to replace $n_\pm(k)$ by $\frac{1}{2}[n_\pm(k)+qn_\mp(-k)]$. Thus the general solution $m(k,x)$ of (6.2) is given by

$$m(k,x)=\hat{1}+\frac{1}{2}qn_+\left(\frac{-1}{k}\right)+\frac{1}{2}n_-\left(\frac{1}{k}\right). \tag{8.12}$$

Finally, putting $\omega(t)=\eta_1(A-t)-\eta_2(t)$, where $\eta_1(t)$ and $\eta_2(t)$ are the first and second components of the vector $\eta(t)$, and using (8.7) in (8.12) we obtain (8.11).■

Since $\omega$ appearing in the general solution (8.11) belongs to $L^2(0,A)$ and $A$ is finite, it follows that $\omega\in L^1(0,A)$. Hence, the Riemann–Lebesgue lemma implies that

$$\lim_{k\to\pm\infty}\int_0^A dt\, e^{ikt}\omega(t)=\lim_{k\to\pm\infty}\int_0^A dt\, e^{ikt}\omega(A-t)=0. \tag{8.13}$$

Once (6.2) is solved, from (4.19) and (4.20) we have immediately

$$\lim_{k\to\pm\infty}\frac{-i\ln m_l(k,x)}{k}=\int_x^\infty d\xi[1-H(\xi)], \tag{8.14}$$

$$\lim_{k\to\pm\infty}\frac{-i\ln m_r(k,x)}{k}=\int_{-\infty}^x d\xi[1-H(\xi)]. \tag{8.15}$$

Hence, the potential $V(x)$ can be obtained from (8.14) or (8.15) through differentiation.

Although the Riemann–Hilbert problem (6.3) has infinitely many solutions, we will now show that all solutions of (6.3) lead to the same potential $V(x)$, and hence the inverse scattering problem for (1.1) has a unique solution. If we denote by $m(k,x)_{\omega=0}$ and $m(k,x)_\omega$ the solutions (8.11) with $\omega=0$ and real arbitrary $\omega\in L^2(0,A)$, respectively, then from (8.13) it follows that

$$\lim_{k\to\pm\infty}\left(\frac{-i\ln m_l(k,x)_\omega}{k}-\frac{-i\ln m_l(k,x)_{\omega=0}}{k}\right)$$
$$=\lim_{k\to\pm\infty}\frac{-i}{k}\ln\frac{m_l(k,x)_\omega}{m_l(k,x)_{\omega=0}}=0,$$

and similarly for $m_r(k,x)$, so that all solutions of (6.3) lead to the same potential $V(x)$ through the use of (8.14) and (8.15). Hence, the solution of the inverse scattering problem obtained by this method is unique.

## IX. WIENER–HOPF FACTORS OF $M(k)$ VIA THE MARCHENKO METHOD

In this section we use the Marchenko procedure in order to obtain the canonical Wiener–Hopf factors $M_+(k)$ and $M_-(k)$ of the matrix $M(k)$ given in (6.8). The sufficient assumptions in this section are the same as those stated in the beginning of Sec. VI.

Consider the two vector Riemann–Hilbert problems

$$n(-k)=M(k)qn(k), \quad k\in\mathbf{R}, \tag{9.1}$$

$$p(-k)=\mathbf{J}M(k)\mathbf{J}qp(k), \quad k\in\mathbf{R}, \tag{9.2}$$

where the vectors $n(k)$ and $p(k)$ have analytic extensions in $k$ to $\mathbf{C}^+$ for each $x$, and $n(k)-\hat{1}=O(1/k)$ and $p(k)-\hat{1}=O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$.

Below we will solve (9.1) and (9.2) by the Marchenko method. From (9.1) we obtain

$$n(-k)-\hat{1}=[M(k)-\mathbf{I}]qn(k)+q[n(k)-\hat{1}], \tag{9.3}$$

and using the Fourier transformation, we transform (9.3) into

$$B(y)=\int_{-\infty}^\infty\frac{dk}{2\pi}e^{iky}[M(k)-\mathbf{I}]q[n(k)-\hat{1}]$$
$$+\int_{-\infty}^\infty\frac{dk}{2\pi}e^{iky}[M(k)-\mathbf{I}]\hat{1}+qB(-y), \tag{9.4}$$

where we have defined

$$B(y)=\int_{-\infty}^\infty\frac{dk}{2\pi}e^{-iky}[n(k)-\hat{1}].$$

If $n(k)-1$ belongs to the Hardy space $\mathbf{H}^2_+(\mathbf{R};\mathbf{C}^2)$, then $B(y)=\begin{bmatrix}0\\0\end{bmatrix}$ for $y<0$. Let

$$n(k)=\begin{bmatrix}n_l(k)\\n_r(k)\end{bmatrix},$$

and let us define

$$B_l(y)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,[n_l(k)-1]e^{-iky}, \tag{9.5}$$

$$B_r(y)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}[n_r(k)-1]e^{-iky}, \tag{9.6}$$

$$g(y)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{iky}[M(k)-\mathbf{I}],$$

$$g_l(y)=-\int_{-\infty}^{\infty}\frac{dk}{2\pi}R\,(-1/k)\,e^{-2ix/k-iA/k}e^{iky},$$

$$g_r(y)=-\int_{-\infty}^{\infty}\frac{dk}{2\pi}L\,(-1/k)\,e^{2ix/k-iA/k}e^{iky},$$

where we have again suppressed the $x$ dependence. Then from (9.4) we derive the $2\times 1$ system of integral equations

$$B(y)=g(y)\,\hat{1}+qB(-y)+\int_{-\infty}^{\infty}dz\,g(y+z)\times qB(z),\quad y\in\mathbf{R}. \tag{9.7}$$

Note that

$$g(y)\,\hat{1}=\begin{bmatrix}g_l(y)\\g_r(y)\end{bmatrix}+u(y)\,\hat{1},$$

where $u(y)$ is the quantity defined in (6.9). From Proposition 6.1 it follows that $u(y)=0$ for $y>0$, and hence (9.7) uncouples into the two scalar Marchenko equations

$$B_l(y)=g_l(y)+\int_0^{\infty}dz\,g_l(y+z)B_l(z),\quad y>0, \tag{9.8}$$

$$B_r(y)=g_r(y)+\int_0^{\infty}dz\,g_r(y+z)B_r(z),\quad y>0. \tag{9.9}$$

On the other hand, if we replace $y$ by $-y$ in (9.7) and restrict the resulting identity to $y>0$, we obtain the coupled system of equations

$$B_l(y)+g_r(-y)+u(-y)+\int_0^{\infty}dz\,g_r(-y+z)B_r(z)+\int_0^{y}dz\,u(-y+z)B_l(z)=0,\quad y>0, \tag{9.10}$$

$$B_r(y)+g_l(-y)+u(-y)+\int_0^{\infty}dz\,g_l(-y+z)B_l(z)+\int_0^{y}dz\,u(-y+z)B_r(z)=0,\quad y>0. \tag{9.11}$$

Let us write (9.8) and (9.9) in operator form as

$$B=g+\mathscr{G}B. \tag{9.12}$$

We then have the following result concerning the solvability of the Marchenko integral equations (9.8) and (9.9).

**Theorem 9.1:** Suppose $V(x)$ satisfies $1-H\in L^1(\mathbf{R})$, $0\leqslant V(x)<1$, and $G\in L^1(\mathbf{R})$, where $G$ is the quantity defined in (1.6). Then the operator $\mathscr{G}$ in (9.12) defined on $L^2(0,\infty)$ is self-adjoint and its operator norm satisfies $\|\mathscr{G}\|<1$. Thus the Marchenko integral equations (9.8) and (9.9) are uniquely solvable.

*Proof:* Note that the reflection coefficients $R(k)$ and $L(k)$ are strictly less than 1 in absolute value, are continuous for $k\in\mathbf{R}$, and are of $O(1/k)$ as $k\to\pm\infty$. Thus, $\sup_{k\in\mathbf{R}}|R(k)|=\sup_{k\in\mathbf{R}}|L(k)|<1$. Let $\langle\cdot,\cdot\rangle$ denote the usual inner product on $L^2(\mathbf{R})$. Then for $B\in L^2(\mathbf{R})$ such that $B(y)=0$ for $y<0$ we have

$$\langle\mathscr{G}B,\mathscr{G}B\rangle=\frac{1}{2\pi}\langle h\hat{B},h\hat{B}\rangle\leqslant\frac{1}{2\pi}[\sup_{k\in\mathbf{R}}|h(k)|]^2\langle\hat{B},\hat{B}\rangle=[\sup_{k\in\mathbf{R}}|h(k)|]^2\langle B,B\rangle,$$

where $h(k)$ denotes $R(-1/k)e^{-2ix/k-iA/k}$ or $L(-1/k)e^{2ix/k-iA/k}$. Hence, the operator norm of $\mathscr{G}$ is bounded above by $\sup_{k\in\mathbf{R}}|h(k)|$. Here $\hat{B}$ denotes the $L^2$-Fourier transform of $B$. Thus, $\|\mathscr{G}\|<1$, where $\|\cdot\|$ denotes the operator norm on $L^2(0,\infty)$. Hence, the integral equation (9.12) is uniquely solvable for $B\in L^2(0,\infty)$ and its solution can be obtained by iteration.

From (3.8) and (3.9) it follows that $g(y)$ is real, and from (9.8) and (9.9) it is seen that $\mathscr{G}$ has a symmetric kernel. Hence, $\mathscr{G}$ is self-adjoint. ■

Once $B_l(y)$ and $B_r(y)$ are obtained from the Marchenko equations (9.8) and (9.9), they lead to a solution of the Riemann–Hilbert problem (9.1) if and only if they also satisfy the ancillary equations (9.10) and (9.11). If this is, indeed, the case, then by using the inverse Fourier transform on (9.5) and (9.6), we obtain $n_l(k)$ and $n_r(k)$. The Riemann–Hilbert problem (9.2) can be solved the same way. In fact, since the only difference between (9.1) and (9.2) is the sign of the off-diagonal entries of the matrix coefficient, (9.2) is uniquely solvable if and only if (9.1) is uniquely solvable.

From

$$n(k,x)=\begin{bmatrix}n_l(k)\\n_r(k)\end{bmatrix},$$

the solution of the vector Riemann–Hilbert problem (9.1), and

$$p(k)=\begin{bmatrix}p_l(k)\\p_r(k)\end{bmatrix},$$

the solution of (9.2), we obtain the matrix

$$N(k)=\frac{1}{2}\begin{bmatrix}n_l+p_l & n_l-p_l\\ n_r-p_r & n_r+p_r\end{bmatrix},$$

which is the unique solution of the matrix Riemann–Hilbert problem[17,18]

$$N(-k)=M(k)qN(k)q,\quad k\in\mathbf{R}.$$

Thus we obtain the canonical Wiener–Hopf factorization $M(k)=M_-(k)M_+(k)$ with the Wiener–Hopf factors

$$M_-(k)=N(-k)\ \text{ and }\ M_+(k)=qN(k)^{-1}q.$$

## X. GENERALIZATION WHEN $V(x)$ HAS MIXED SIGN

In this section we generalize the results of Secs. VI–IX to the case where $V(x)$ is no longer assumed non-negative. The sufficient assumptions in this section are the same as those stated in the beginning of Sec. IV.

Let $\varphi(x) = \int_{-\infty}^{x}[1 - H]$, and let $A = \int_{-\infty}^{\infty}[1 - H]$ as before. Then $A - \varphi(x) = \int_x^\infty[1 - H]$. As seen from (4.19) and (4.20), if $V(x)$ assumes negative values, then $\varphi(x)$ or $A-\varphi(x)$ may be negative, and as a result, $m_l(k,x)$ or $m_r(k,x)$ may blow up exponentially as $k\to\infty$ in $\mathbf{C}^+$ so that $\widetilde{m}(k,x)$ may no longer belong to $\mathbf{H}^2_+(\mathbf{R};\mathbf{C}^2)$. We will now solve (6.2) in such a case.

Since $1-H\in L^1(\mathbf{R})$, $\varphi(-\infty)=0$, and $\varphi(+\infty)=A$, $\varphi(x)$ is continuous and thus uniformly bounded on the extended real axis. Let $\lambda$ be a constant such that $\lambda<\inf_{x\in\mathbf{R}}\{\varphi(x), A-\varphi(x)\}$. Define $v(k,x)=e^{-ik\lambda}m(k,x)$. Then, from the properties of $m(k,x)$, it follows that $\widetilde{v}(k,x)$ is continuous in $k\in\overline{\mathbf{C}^+}\setminus\{0\}$, has an analytic extension in $k$ to $\mathbf{C}^+$ for each $x$, and $\widetilde{v}(k,x)-\hat{1}=O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Furthermore, for each $x\in\mathbf{R}$, we have $\widetilde{v}(k,x)-\hat{1}\in \mathbf{H}^2_+(\mathbf{R};\mathbf{C}^2)$. Let us write (6.2) in the form

$$v(-k,x)=e^{2ik\lambda}e^{-iAk}\widetilde{M}(k)qv(k,x),\quad k\in\mathbf{R},$$

or equivalently, after the transformation $k\mapsto -1/k$,

$$\widetilde{v}(-k,x)=e^{-2i\lambda/k}e^{iA/k}M(k)q\,\widetilde{v}(k,x),\quad k\in\mathbf{R}. \qquad (10.1)$$

Since $A-2\lambda\geqslant 0$, the Riemann–Hilbert problem (10.1) can be solved in exactly the same way as we solved (6.3) in earlier sections under the assumption $A\geqslant 0$. In fact the solution of (10.1) is obtained from the solution of (6.3) if we replace $A$ in the solution of (6.3) by $A-2\lambda$. Hence, the solution method for the inverse problem of recovering $V(x)$ presented earlier in this paper is easily extended to the case $V(x)<1$ where $V(x)$ is no longer non-negative.


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