

The transformations to remove or add bound states for the half-line matrix Schrödinger operator

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ABSTRACT

We present the transformations to remove or add bound states or to decrease or increase the multiplicities of any existing bound states for the half-line matrix-valued Schrödinger operator with the general selfadjoint boundary condition, without changing the continuous spectrum of the operator. When the matrix-valued potential is selfadjoint, is integrable, and has a finite first moment, the relevant transformations are constructed through the development of the Gel'fand–Levitan method for the corresponding Schrödinger operator. In particular, the bound-state normalization matrices are constructed at each bound state for any multiplicity. The transformations are obtained for all relevant quantities, including the matrix potential, the Jost solution, the regular solution, the Jost matrix, the scattering matrix, and the boundary condition. For each bound state, the corresponding dependency matrix is introduced by connecting the normalization matrix used in the Gel'fand–Levitan method and the normalization matrix used in the Marchenko method of inverse scattering. Various estimates are provided to describe the large spacial asymptotics for the change in the potential when the bound states are removed or added or their multiplicities are modified. An explicit example is provided showing that an asymptotic estimate available in the literature in the scalar case for the potential increment is incorrect.

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I. INTRODUCTION

In this paper we consider the half-line matrix-valued Schrödinger operator with the general selfadjoint boundary condition under the assumption that the matrix-valued potential is selfadjoint, is integrable, and has a finite first moment. We analyze the change in all relevant quantities associated with this Schrödinger operator when we change its discrete spectrum by removing or adding a finite number of eigenvalues or decreasing or increasing the multiplicities of the existing eigenvalues, without changing the continuous spectrum. In other words, we determine the transformations of all relevant quantities associated with this Schrödinger operator when a bound state of any multiplicity is removed from or added to the discrete spectrum or the multiplicity of a bound state is changed. Since the boundary condition directly affects the definition of the Schrödinger operator, in general we expect that the selfadjoint boundary condition also undergoes a change, with the only exception that a purely Dirichlet boundary condition remains a purely Dirichlet boundary condition. By extending the Gel'fand–Levitan method of inverse spectral theory from the scalar case to the matrix case, we are able to obtain the transformations for all relevant quantities when the bound states of any multiplicities are removed or added or when the bound-state multiplicities are modified. The aforementioned relevant quantities include the potential, the regular solution, the Jost solution, the Jost matrix, the scattering matrix, and the boundary matrices used in the description of the selfadjoint boundary condition.

The use of a matrix-valued Schrödinger equation finds its origin in the beginning of the development of quantum mechanics. The matrix Schrödinger operator plays a central role to take into account the internal structure, such as spin, of quantum mechanical particles and also to consider collections of particles such as nuclei, atoms, and molecules. An important example of a matrix-valued Schrödinger equation is the Pauli equation describing particles of spin equal to one half. We refer the reader to Refs. 11 and 24 for the applications of matrix-valued Schrödinger equation in quantum mechanics and in nuclear physics. In the nonlinear case of a matrix-valued time-dependent Schrödinger equation, we refer the reader to Ref. 29 for the analysis of long-time asymptotics of the corresponding solutions. We remark that matrix-valued Schrödinger equations have important applications in the context of quantum graphs^{9,23} in various fields including nanotechnology, quantum wires, and quantum computing. In the analysis of quantum graphs the use of a general selfadjoint boundary condition, rather than a purely Dirichlet boundary condition, is crucial. For example, a star graph, which is a quantum graph with one vertex and a finite number of semi-infinite edges meeting at the vertex, is mathematically described by a matrix Schrödinger operator with the general boundary condition. For a further discussion on applications of matrix-valued Schrödinger equations with the general boundary condition, we refer the reader to Ref. 7.

A simple but useful method to remove or add bound states for scalar Sturm–Liouville equations has a long history dating back to Jacobi²¹ and Darboux,¹³ with later contributions by Crum,¹² Deift,¹⁴ Deift and Trubowitz,¹⁵ and Schminke.³² This method is often called the Darboux transformation, the Darboux–Crum transformation, or the single commutation method. It involves factoring the corresponding Sturm–Liouville operator into two noncommuting operators. However, it has the disadvantage that it requires the positivity of certain eigensolutions to the relevant Sturm–Liouville problem, and consequently it can only be applied to remove the bound state with the lowest energy or to add a bound state at an energy below the existing bound-state energy levels. An alternate method is the double commutation method.^{18–20,33} In this alternate method a second commutation is used, which allows to remove or add bound states at any energy in the spectrum of the Sturm–Liouville operator. It is mentioned in Ref. 20 that the second commutation method can be found in the seminal work of Gel’fand and Levitan.¹⁷ The literature on methods to remove or add bound states is extensive. Besides the references we have quoted already, we mention Refs. 5, 16, 25, and 27 and the references therein. Such methods have applications in others areas as well. For example, they are used to obtain explicit solution to integrable evolution equations.^{25,27,28,31}

The Gel’fand–Levitan method was developed by Gel’fand and Levitan in Ref. 17 (see also Ref. 26) to recover a scalar-valued potential when the corresponding spectral function is used as the input data set. The method was extended to the matrix case by Newton and Jost³⁰ by considering the matrix-valued Schrödinger operator when the boundary condition is purely Dirichlet. In our paper we extend the Gel’fand–Levitan method in the matrix case by allowing the general selfadjoint boundary condition, which includes the purely Dirichlet boundary condition as a specific case. We provide the transformations of all relevant quantities by using the Gel’fand–Levitan method for the matrix-valued Schrödinger operator with the general selfadjoint boundary condition. We construct the bound-state normalization matrices by taking into consideration the multiplicities of the bound states. We refer to the constructed normalization matrices as the Gel’fand–Levitan normalization matrices. This is in contrast with the normalization matrices in the Marchenko method of inverse scattering, which is a method used to recover the matrix-valued potential from the scattering data set consisting of the scattering matrix, the bound-state energies, and the bound-state normalization matrices. We refer to the normalization matrices used in the Marchenko method as the Marchenko normalization matrices. The Marchenko method for the half-line matrix-valued Schrödinger operator with the purely Dirichlet boundary condition was developed by Agranovich and Marchenko in their seminal work.¹ We refer the reader to our own monograph⁷ for the Marchenko method for the half-line matrix-valued Schrödinger operator with the general selfadjoint boundary condition. For some of our results in this paper, the existence of the first moment of the potential is not needed. On the other hand, for some other results we impose a stronger moment condition on the potential than just the first moment condition.

The analysis of the half-line Schrödinger operator in the matrix case is much more involved than in the scalar case. The Gel’fand–Levitan normalization constants in the scalar case are defined in a straightforward manner by normalizing the regular solution, whereas in the matrix case the definition of a Gel’fand–Levitan normalization matrix is more elaborate. This is because each bound state is simple in the scalar case whereas a bound state in the matrix case may have a multiplicity. Furthermore, in the scalar case a boundary condition is either Dirichlet or non-Dirichlet, whereas in the matrix case a boundary condition may be purely Dirichlet, purely non-Dirichlet, or a mixture of Dirichlet and non-Dirichlet. Under a transformation to remove or to add bound states in the scalar case, a Dirichlet boundary condition remains Dirichlet and a non-Dirichlet boundary condition remains non-Dirichlet, although the boundary parameter changes in the latter case. In the matrix case, we observe the following. The boundary matrix A appearing in (2.5) and (2.9) remains unchanged under the transformation. In the matrix case, the purely Dirichlet case occurs when $A = 0$, and hence a purely Dirichlet case remains purely Dirichlet under the transformation, which resembles the scalar case. In the matrix case, the boundary matrix B appearing in (2.5) and (2.9) changes under the transformation unless $A = 0$, and this also has a resemblance in the scalar case. In our paper we obtain the transformations to remove or add bound states or to decrease or increase the multiplicity of a bound state when the potential is matrix valued and the boundary condition is the general selfadjoint boundary condition, and this is done without restricting ourselves to the purely Dirichlet case or to the purely non-Dirichlet case. We deal with bound-state multiplicities by using the appropriate orthogonal projections onto the kernels of the Jost matrix and of the matrix adjoint of the Jost matrix, respectively, evaluated at the bound-state energies. Our analysis also involves the introduction of a dependency matrix at each bound state and the use of a Moore–Penrose inverse. The bound-state dependency matrix connects the normalized matrix-valued bound-state wavefunction used in the Gel’fand–Levitan method to the normalized matrix-valued bound-state wavefunction used in the Marchenko method.

In the half-line scalar case, when we remove or add a bound state, the Gel'fand–Levitan normalization constants for the remaining bound states are not affected.^{25,27} We observe the same in the matrix case even with the general selfadjoint boundary condition. In other words, we find that the Gel'fand–Levitan normalization matrices remain unchanged for the bound states not affected by the transformation. We also observe that, when we change the multiplicity of a bound state, the Gel'fand–Levitan normalization matrices for the remaining bound states are not affected. As a result, in the half-line case, a removal or addition of a bound state or a decrease or increase in the multiplicity of a bound state does not affect the remaining bound states. One important consequence of this result is that the mathematical formulation of the transformation does not require that the bound-states must be ordered in an increasing or a decreasing manner with respect to the bound-state energies.

Our paper is organized as follows. In Sec. II we present the preliminaries needed for our analysis, by introducing the boundary matrices associated with a selfadjoint boundary condition, the Jost solution and the regular solution to the matrix Schrödinger equation on the half line, the Jost matrix, the scattering matrix, and the physical solution. In Sec. III we deal with the bound states, by introducing the orthogonal projections onto the kernels of the Jost matrix and of the adjoint of the Jost matrix, respectively, at each bound state. We then introduce the Gel'fand–Levitan normalization matrices and the matrix-valued Gel'fand–Levitan normalized bound-state solutions. This is done in a similar manner the Marchenko normalization matrices and the matrix-valued Marchenko normalized bound state solutions are introduced⁷ for the corresponding matrix Schrödinger operator with the general selfadjoint boundary condition. In Sec. IV we introduce the dependency matrices that relate the Marchenko normalized bound-state solutions to the Gel'fand–Levitan normalized bound-state solutions, and we present the basic properties of the dependency matrices. We show that the matrix adjoint of a dependency matrix is the same as its Moore–Penrose inverse, and we also show that the dependency matrices help us treat bound states with multiplicities not only in a rather simple and mathematically elegant way but also in a practical way suitable for computations. In Sec. V we introduce the Gel'fand–Levitan system of integral equations and Parseval's equality related to the matrix-valued Schrödinger operator with the general selfadjoint boundary condition. We show how the input to the Gel'fand–Levitan system is related to the change in the spectral measure. We also describe how the change in the potential, the change in the regular solution, and the change in the boundary matrices are expressed in terms of the solution to the Gel'fand–Levitan system of integral equations. In Sec. VI we present the transformations of all relevant quantities when a bound state is removed. In particular, we show how the regular solution, the potential, the boundary matrices, the Jost matrix, the scattering matrix, the Jost solution, and other relevant quantities change. We also show that the Gel'fand–Levitan normalization matrices for the bound states not involved in the transformation remain unchanged. Further, we show that the continuous part of the spectral measure does not change. In Sec. VII we present the transformations of all relevant quantities when the multiplicity of a bound state is decreased, without totally removing the bound state. Note that this can only happen in the matrix case since in the scalar case the multiplicity of each bound state is one. The proofs of the results in this section are similar to the proofs in Sec. VI, and hence the results in Sec. VII are stated without their proofs. In Sec. VIII we present the transformations of all relevant quantities when a new bound state is added to the discrete spectrum of the operator. In (IV.1.30) of Ref. 11 it is claimed that, in the scalar case with the Dirichlet boundary condition when adding a bound state with the energy $-\kappa^2$, the increment of the potential given by $\tilde{V}(x) - V(x)$ decays as $Fe^{-2\kappa x}$ as $x \rightarrow +\infty$, where F is a nonzero constant. In Example 8.9, by expressing the relevant quantities and their transformations explicitly, we show that the result of (IV.1.30) in Ref. 11 is incorrect. Finally, in Sec. IX we present the transformations of all relevant quantities when we increase the multiplicity of a bound state without changing the rest of the spectrum. The results in Sec. IX are presented without their proofs because those proofs are similar to the proofs in Sec. VIII.

II. THE HALF-LINE MATRIX SCHRÖDINGER EQUATION

In this section we present the preliminaries needed for the remaining sections. In particular, we introduce the boundary matrices A and B , the Jost solution $f(k, x)$, the regular solution $\varphi(k, x)$, the Jost matrix $J(k)$, the scattering matrix $S(k)$, and the physical solution $\Psi(k, x)$ associated with the half-line matrix Schrödinger operator. We refer the reader to Ref. 7 for further elaborations on the material presented here.

Consider the half-line Schrödinger equation

$$-\psi'' + V(x)\psi = k^2\psi, \quad x \in \mathbb{R}^+, \quad (2.1)$$

where the prime denotes the x -derivative, we let $\mathbb{R}^+$ denote the interval $(0, +\infty)$, and the potential V is an $n \times n$ selfadjoint matrix-valued function of x , with n being a fixed positive integer. The wavefunction ψ is either an $n \times n$ matrix or a column vector with n components. Note that the selfadjointness is expressed as

$$V(x)^\dagger = V(x), \quad x \in \mathbb{R}^+, \quad (2.2)$$

with the dagger denoting the matrix adjoint (matrix transpose and complex conjugation). We always assume that the potential V is integrable, i.e. we have

$$\int_0^\infty dx |V(x)| < +\infty, \quad (2.3)$$

where $|V(x)|$ denotes the operator norm of the $n \times n$ matrix $V(x)$. At times we impose further restrictions on the potential V so that it satisfies

$$\int_0^\infty dx (1+x)^\epsilon |V(x)| < +\infty, \quad (2.4)$$

for some fixed positive constant ϵ . We say that V belongs to $L^1_\epsilon(\mathbb{R}^+)$ if (2.4) is satisfied. Since all matrix norms are equivalent in finite-dimensional vector spaces, any other matrix norm can be used in (2.3) and (2.4) instead of the operator norm. The adjectives selfadjoint and Hermitian are used interchangeably. We use $\mathbb{C}^+$ to denote the upper-half complex plane, use $\mathbb{R}$ for the real axis, and let $\overline{\mathbb{C}^+} := \mathbb{C}^+ \cup \mathbb{R}$. We use both V and $V(x)$ to refer to the potential V , where the latter usage emphasizes the dependence on x .

We obtain a selfadjoint Schrödinger operator on the half line by supplementing (2.1) with the general selfadjoint boundary condition at $x = 0$, which can be written as

$$-B^\dagger \psi(0) + A^\dagger \psi'(0) = 0, \quad (2.5)$$

where the constant $n \times n$ matrices A and B satisfy

$$-B^\dagger A + A^\dagger B = 0, \quad (2.6)$$

$$A^\dagger A + B^\dagger B > 0, \quad (2.7)$$

and we refer to A and B as the boundary matrices. We use the positivity and the positive definiteness for a matrix interchangeably. Thus, the positivity for the matrix appearing on the left-hand side of (2.7) is the same as the positive definiteness of that matrix. We recall that an $n \times n$ matrix is positive if and only if all its eigenvalues are positive. Similarly, a matrix is nonnegative if and only if its eigenvalues are real and nonnegative. We remark that (2.7) is equivalent⁷ to having the matrix-rank relation given by

$$\text{rank} \begin{bmatrix} A \\ B \end{bmatrix} = n,$$

where $\begin{bmatrix} A \\ B \end{bmatrix}$ is the $2n \times n$ matrix formed from A and B . Multiplying the boundary matrices A and B on the right by an invertible $n \times n$ matrix T does not change (2.5). Thus, even though the boundary condition (2.5) itself is uniquely determined by the boundary-matrix pair (A, B) , the matrix pair (AT, BT) with any invertible T also yields the same boundary condition (2.5). In fact, this is the only freedom we have in choosing the boundary matrices A and B . It is possible to use a single constant $n \times n$ matrix to describe the boundary condition (2.5) instead of a pair of constant $n \times n$ matrices. However, it is more advantageous⁷ to describe (2.5) by using the pair of boundary matrices A and B appearing in the initial conditions (2.9). By using $A = 0$ and $B = I$ in (2.9) we obtain the Dirichlet boundary condition $-\psi(0) = 0$, which can also be written as $\psi(0) = 0$ by letting $T = -I$. At times we use the term purely Dirichlet for emphasis to make a distinction with the boundary condition consisting of a mixture of Dirichlet and non-Dirichlet components. The notation 0 is used for the scalar zero, the zero vector with n components, and the $n \times n$ zero matrix, depending on the context. We use I to denote the $n \times n$ identity matrix.

When the potential V satisfies (2.2) and (2.3), there are two relevant particular $n \times n$ matrix solutions to (2.1). One of them is the Jost solution $f(k, x)$, and it satisfies the spacial asymptotics

$$f(k, x) = e^{ikx} [I + o(1)], \quad f'(k, x) = e^{ikx} [ikI + o(1)], \quad x \rightarrow +\infty, \quad (2.8)$$

for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$. In fact the Jost solution $f(k, x)$ is defined as the unique solution satisfying the first asymptotics in (2.8), and the second asymptotics there is automatically satisfied. If $V \in L^1_1(\mathbb{R}^+)$ then the Jost solution exists also at $k = 0$. The second relevant particular solution to (2.1) is the regular solution $\varphi(k, x)$ satisfying the initial conditions

$$\varphi(k, 0) = A, \quad \varphi'(k, 0) = B, \quad (2.9)$$

where A and B are the boundary matrices appearing in (2.5). When the potential V satisfies (2.2) and (2.3), the regular solution $\varphi(k, x)$ is entire in $k \in \mathbb{C}$ for each fixed $x \in [0, +\infty)$. We remark that in general the Jost solution $f(k, x)$ does not satisfy the boundary condition (2.5). However, the regular solution $\varphi(k, x)$ indeed satisfies (2.5) at any $k \in \mathbb{C}$.

When the potential V satisfies (2.2) and (2.3), the Jost matrix $J(k)$ associated with (2.1) and (2.5) is defined as

$$J(k) := f(-k^*, 0)^\dagger B - f'(-k^*, 0)^\dagger A, \quad k \in \mathbb{R} \setminus \{0\}, \quad (2.10)$$

where the asterisk denotes complex conjugation. If we further have $V \in L^1_1(\mathbb{R}^+)$ then $J(k)$ exists also at $k = 0$. The $n \times n$ matrix-valued $J(k)$ has an extension from $k \in \mathbb{R} \setminus \{0\}$ to $k \in \mathbb{C}^+$, and in fact the asterisk in (2.10) is used to indicate how that extension occurs. The scattering matrix $S(k)$ associated with (2.1) and (2.5) is defined as

$$S(k) := -J(-k) J(k)^{-1}, \quad k \in \mathbb{R} \setminus \{0\}. \quad (2.11)$$

If $V \in L^1_1(\mathbb{R}^+)$ then the domain of the scattering matrix, by continuity, extends also to $k = 0$. We remark that the definition of the scattering matrix given in (2.11) differs by a minus sign from the definition used in Refs. 1 and 30 in the matrix case with the purely Dirichlet boundary condition as well as the traditional definition used in the scalar case^{11,25,27} with the Dirichlet boundary condition. On the other hand, in the scalar case with a non-Dirichlet boundary condition a minus sign as in (2.11) is used^{25,27} to define the scattering matrix. We refer the reader to

Refs. 6 and 7 for the elaboration on the issue of defining the scattering matrix for the half-line matrix Schrödinger operator and the troubling consequences of defining the scattering matrix differently in the Dirichlet and non-Dirichlet cases.

The physical solution $\Psi(k, x)$ associated with (2.1) and (2.5) is defined as

$$\Psi(k, x) := f(-k, x) + f(k, x) S(k). \quad (2.12)$$

For each fixed $x \in \mathbb{R}^+$ the physical solution $\Psi(k, x)$ has a meromorphic extension from $k \in \mathbb{R} \setminus \{0\}$ [or from $k \in \mathbb{R}$ if $V \in L_1^1(\mathbb{R}^+)$] to $k \in \mathbb{C}^+$ and the regular solution $\varphi(k, x)$ is entire in k . The regular solution can be expressed in terms of the physical solution, the Jost solution, and the Jost matrix in various equivalent forms such as

$$\varphi(k, x) = -\frac{1}{2ik} \Psi(k, x) J(k), \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}, \quad (2.13)$$

$$\varphi(k, x) = \frac{1}{2ik} [f(k, x) J(-k) - f(-k, x) J(k)], \quad k \in \mathbb{R} \setminus \{0\}. \quad (2.14)$$

The regular solution is even in k , i.e. we have

$$\varphi(-k, x) = \varphi(k, x), \quad k \in \mathbb{C}.$$

III. THE BOUND STATES OF THE SCHRÖDINGER OPERATOR

In this section we introduce the relevant material related to the bound states of the half-line matrix Schrödinger operator. For the treatment of bound states and the Marchenko normalization matrices, we refer the reader to Ref. 7. In this paper, since we use the Gel'fand–Levitan method to establish the relevant transformations for the half-line matrix Schrödinger operator, we introduce the Gel'fand–Levitan normalization matrices by using a procedure similar to the construction of the Marchenko normalization matrices.

A bound-state solution corresponds to a square-integrable column-vector solution to (2.1) satisfying the boundary condition (2.5). We let $\lambda := k^2$. As stated in Theorem 3.11.1 of Ref. 7, there are no bound states when $\lambda > 0$ but a bound state at $\lambda = 0$ is possible. The bound states when $\lambda < 0$ occur at the k -values on the positive imaginary axis of the complex k -plane corresponding to the zeros of $\det[J(k)]$. We use $\det[J(k)]$ to denote the determinant of the Jost matrix $J(k)$ defined in (2.10). If there are an infinite number of bound states when $\lambda < 0$, the corresponding bound-state k -values only accumulate at $k = 0$. As indicated in Theorem 4.3.3 of Ref. 7, the selfadjoint Schrödinger operator related to (2.1) and (2.5) has no singular continuous spectrum and its absolutely continuous spectrum is the λ -interval $[0, +\infty)$. If $V \in L_1^1(\mathbb{R}^+)$, then from Theorem 3.11.1 of Ref. 7 we conclude that the zero value of λ does not correspond to a bound state and that the number of bound states when $\lambda < 0$ is finite. We assume that there are N zeros of $\det[J(k)]$ occurring when $k = i\kappa_j$ for $1 \leq j \leq N$, with κ_j being distinct positive constants. Note that N is a nonnegative integer or it is equal to $+\infty$, and it represents the number of negative-energy bound states without counting multiplicities. In case $N = 0$, there are no negative-energy bound states. Thus, $\det[J(i\kappa_j)] = 0$ and we use m_j to denote the number of linearly independent vectors in $\mathbb{C}^n$ spanning the kernel $\text{Ker}[J(i\kappa_j)]$. We remark that m_j is a positive integer satisfying $1 \leq m_j \leq n$, and it corresponds to the multiplicity of the bound state at $k = i\kappa_j$. Thus, the total number of negative-energy bound states including the multiplicities, denoted by $\mathcal{N}$, is given by

$$\mathcal{N} := \sum_{j=1}^N m_j.$$

Note that $\mathcal{N}$ can be zero, a positive integer, or infinity. At $k = i\kappa_j$ there are $2n$ linearly independent column-vector solutions to (2.1). We remark that n of such solutions are exponentially decreasing as $x \rightarrow +\infty$ and hence square integrable in x , and n of such solutions are exponentially increasing as $x \rightarrow +\infty$ and hence not square integrable in x . For further elaborations on this issue, we refer the reader to Propositions 3.2.1, 3.2.2, and 3.2.3 of Ref. 7. It turns out that, by using linear combinations of n square-integrable solutions, we can construct exactly m_j square-integrable column-vector solutions that also satisfy the boundary condition. Thus, the multiplicity m_j of the bound state at $k = i\kappa_j$ cannot exceed n . Consequently, there are exactly m_j linearly independent square-integrable column-vector solutions satisfying the boundary condition, and those solutions make up the bound states.

The bound-state solutions to (2.1) can be analyzed either via the Marchenko theory or the Gel'fand–Levitan theory. In the Marchenko theory the normalization matrices for the bound states are obtained by normalizing the Jost solution $f(k, x)$ at the bound states, whereas in the Gel'fand–Levitan theory the normalization matrices for the bound states are obtained by normalizing the regular solution $\varphi(k, x)$ at the bound states. Thus, the Marchenko normalization matrices and the corresponding normalized matrix solutions to the Schrödinger equation are different from the Gel'fand–Levitan normalization matrices and the corresponding normalized matrix solutions. We use M_j and $\Psi_j(x)$ to denote the $n \times n$ Marchenko normalization matrix and the corresponding $n \times n$ normalized matrix solution at the bound state with $k = i\kappa_j$, respectively. Similarly, we use C_j and $\Phi_j(x)$ to denote the Gel'fand–Levitan normalization matrix and the corresponding normalized matrix solution at the bound state with $k = i\kappa_j$, respectively. In this section we introduce the $n \times n$ matrices C_j and $\Phi_j(x)$ in a similar way M_j and $\Psi_j(x)$ are defined.⁷

We remark that the $n \times n$ matrices $J(i\kappa_j)$ and $J(i\kappa_j)^\dagger$ have the same rank, and their common rank is equal to $n - m_j$. We see that when $n = 1$ we must have $m_j = 1$, in which case $J(i\kappa_j)$ and $J(i\kappa_j)^\dagger$ are both equal to the scalar zero. If $n \geq 2$, then the kernels $\text{Ker}[J(i\kappa_j)]$ and

$\text{Ker}[J(ik_j)^\dagger]$ are in general different, but they have the same dimension equal to m_j . We use Q_j and P_j to denote the orthogonal projections onto $\text{Ker}[J(ik_j)]$ and $\text{Ker}[J(ik_j)^\dagger]$, respectively. By the definition of an orthogonal projection, we have

$$P_j^2 = P_j^\dagger = P_j, \quad Q_j^2 = Q_j^\dagger = Q_j. \quad (3.1)$$

Let us use $\mathbf{M}^+$ to denote the Moore–Penrose inverse of a matrix $\mathbf{M}$. Even though the Moore–Penrose inverse is uniquely defined for any matrix,^{8,10} in our paper we only deal with Moore–Penrose inverses of square matrices. As indicated in Definitions 1.12 and 1.13 and Theorem 1.1.1 of Ref. 10, the matrix $\mathbf{M}^+$ is the Moore–Penrose inverse of the matrix $\mathbf{M}$ if the matrices $\mathbf{M}$ and $\mathbf{M}^+$ satisfy the four equalities given by

$$\mathbf{M}\mathbf{M}^+\mathbf{M} = \mathbf{M}, \quad \mathbf{M}^+\mathbf{M}\mathbf{M}^+ = \mathbf{M}^+, \quad (\mathbf{M}\mathbf{M}^+)^\dagger = \mathbf{M}\mathbf{M}^+, \quad (\mathbf{M}^+\mathbf{M})^\dagger = \mathbf{M}^+\mathbf{M}. \quad (3.2)$$

We list the basic properties of the Moore–Penrose inverse of a square matrix in the following theorem. We refer the reader to Theorems 1.1.1 and 1.2.1 of Ref. 10 for a proof.

Theorem 3.1. *The Moore–Penrose inverse $\mathbf{M}^+$ of an $n \times n$ matrix $\mathbf{M}$ has the following properties:*

- (a) *The matrix $\mathbf{M}^+$ exists and is unique.*
- (b) *If $\mathbf{M}$ is invertible, then $\mathbf{M}^+ = \mathbf{M}^{-1}$.*
- (c) *The Moore–Penrose inverse commutes with the matrix adjoint, i.e. we have*

$$(\mathbf{M}^+)^\dagger = (\mathbf{M}^\dagger)^+, \quad (3.3)$$

where we recall that the dagger denotes the matrix adjoint.

- (d) *The matrix $\mathbf{M}^+\mathbf{M}$ is the orthogonal projection onto $\text{Ran}[\mathbf{M}^+]$, where we use Ran to denote the range.*
- (e) *The matrix $\mathbf{M}\mathbf{M}^+$ is the orthogonal projection onto $\text{Ran}[\mathbf{M}]$.*
- (f) *The Moore–Penrose inverse of $\mathbf{M}^\dagger\mathbf{M}$ is expressed in terms of the Moore–Penrose inverse $\mathbf{M}^+$ as in the case of the standard matrix inversion, i.e. we have*

$$(\mathbf{M}^\dagger\mathbf{M})^+ = \mathbf{M}^+(\mathbf{M}^\dagger)^+. \quad (3.4)$$

- (g) *For any $n \times n$ unitary matrices $\mathbf{U}_1$ and $\mathbf{U}_2$, the matrices $\mathbf{M}$ and $\mathbf{M}^+$ satisfy*

$$(\mathbf{U}_1 \mathbf{M} \mathbf{U}_2)^+ = \mathbf{U}_2^\dagger \mathbf{M}^+ \mathbf{U}_1^\dagger.$$

At times we need to use the x -derivative of the Moore–Penrose inverse of a square matrix when that square matrix contains the independent variable x in a special way. The following proposition provides the appropriate derivative formula that we need in our paper in that special case.

Proposition 3.2. *Assume that $\mathbb{C}^n$ has the direct-sum decomposition into any two of its subspaces $\mathbb{D}_1$ and $\mathbb{D}_2$ as*

$$\mathbb{C}^n = \mathbb{D}_1 \oplus \mathbb{D}_2.$$

Let $\mathbf{M}(x)$ be an $n \times n$ matrix-valued differentiable function of $x \in \mathbb{R}^+$. Suppose that $\mathbf{M}(x)$ has the direct-sum decomposition given by

$$\mathbf{M}(x) = \mathbf{M}_1(x) \oplus \mathbf{M}_2, \quad x \in \mathbb{R}^+,$$

where the components $\mathbf{M}_1(x)$ and $\mathbf{M}_2$ belong to $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively, in such a way that $\mathbf{M}_1(x)$ is invertible on $\mathbb{D}_1$ and that $\mathbf{M}_2$ is independent of x . We have the following:

- (a) *The Moore–Penrose inverse $\mathbf{M}(x)^+$ has the direct-sum decomposition given by*

$$\mathbf{M}(x)^+ = \mathbf{M}_1(x)^{-1} \oplus \mathbf{M}_2^+, \quad (3.5)$$

where $\mathbf{M}_1(x)^{-1}$ denotes the inverse of the matrix $\mathbf{M}_1(x)$ on $\mathbb{D}_1$ and $\mathbf{M}_2^+$ denotes the Moore–Penrose inverse of the constant matrix $\mathbf{M}_2$.

- (b) *The x -derivative $[\mathbf{M}(x)^+]'$ is expressed in terms of $\mathbf{M}(x)^+$ and the x -derivative $\mathbf{M}'(x)$ as*

$$[\mathbf{M}(x)^+]' = -\mathbf{M}(x)^+ \mathbf{M}'(x) \mathbf{M}(x)^+, \quad (3.6)$$

which can also be written as

$$[\mathbf{M}(x)^+]' = -[\mathbf{M}_1(x)^{-1} \oplus \mathbf{M}_2^+][\mathbf{M}_1'(x) \oplus 0][\mathbf{M}_1(x)^{-1} \oplus \mathbf{M}_2^+]. \quad (3.7)$$

Proof. The proof of (3.5) is obtained by verifying that the right-hand side of (3.5) satisfies the four equalities in (3.2) used in the definition of the Moore–Penrose inverse of $\mathbf{M}(x)$. Hence, the proof of (a) is complete. The proof of (b) is obtained as follows. Since $\mathbf{M}_1(x)$ is invertible on $\mathbb{D}_1$, we use the standard derivative formula for the matrix inverse of $\mathbf{M}_1(x)$ and get

$$[\mathbf{M}_1(x)^{-1}]' = -\mathbf{M}_1(x)^{-1} \mathbf{M}_1'(x) \mathbf{M}_1(x)^{-1}.$$

Furthermore, since $\mathbf{M}_2$ is independent of x , its x -derivative is equal to the zero matrix 0. Thus, (3.7) holds and also its equivalent (3.6) is valid. Hence, the proof of (b) is also complete. $\square$

Comparing (3.1) and (3.2) we see that P_j is equal to its own Moore–Penrose inverse and that Q_j is also equal to its own Moore–Penrose inverse, i.e. we have

$$P_j^+ = P_j, \quad Q_j^+ = Q_j, \quad 1 \leq j \leq N.$$

We can explicitly construct P_j and Q_j in terms of any orthonormal basis $\{v_j^{(l)}\}_{l=1}^{m_j}$ for $\text{Ker}[J(i\kappa_j)^\dagger]$ and any orthonormal basis $\{w_j^{(l)}\}_{l=1}^{m_j}$ for $\text{Ker}[J(i\kappa_j)]$, respectively. We recall that m_j corresponds to the common dimension of those two kernels. Thus, we have

$$P_j = \sum_{l=1}^{m_j} v_j^{(l)} (v_j^{(l)})^\dagger, \quad Q_j = \sum_{l=1}^{m_j} w_j^{(l)} (w_j^{(l)})^\dagger. \quad (3.8)$$

The kernels $\text{Ker}[J(i\kappa_j)^\dagger]$ and $\text{Ker}[J(i\kappa_j)]$ play a key role in the analysis of bound states for the matrix Schrödinger operator associated with (2.1) and (2.5). Those two kernels satisfy

$$\mathbb{C}^n = \text{Ker}[J(i\kappa_j)^\dagger] \oplus (\text{Ker}[J(i\kappa_j)^\dagger])^\perp, \quad 1 \leq j \leq N, \quad (3.9)$$

$$\mathbb{C}^n = \text{Ker}[J(i\kappa_j)] \oplus (\text{Ker}[J(i\kappa_j)])^\perp, \quad 1 \leq j \leq N, \quad (3.10)$$

$$(\text{Ker}[J(i\kappa_j)])^\perp = \text{Ran}[J(i\kappa_j)^\dagger], \quad 1 \leq j \leq N,$$

$$(\text{Ker}[J(i\kappa_j)^\dagger])^\perp = \text{Ran}[J(i\kappa_j)], \quad 1 \leq j \leq N,$$

where we use the superscript $\perp$ to denote the orthogonal complement.

Let us consider the Schrödinger equation (2.1) at the bound state $k = i\kappa_j$, which is given by

$$-\psi'' + V(x) \psi = -\kappa_j^2 \psi, \quad x \in \mathbb{R}^+, \quad 1 \leq j \leq N. \quad (3.11)$$

We remark that in general the solution $f(i\kappa_j, x)$ to (3.11) does not satisfy the boundary condition (2.5). On the other hand, $f(i\kappa_j, x)$ is exponentially decaying asymptotically, i.e. we have

$$f(i\kappa_j, x) = e^{-\kappa_j x} [I + o(1)], \quad x \rightarrow +\infty, \quad 1 \leq j \leq N, \quad (3.12)$$

which is seen from (2.8). Hence, for large x -values the n columns of $f(i\kappa_j, x)$ are linearly independent exponentially decaying solutions to (3.11) although those columns do not, in general, satisfy the boundary condition (2.5). It is desirable to construct an $n \times n$ matrix solution to (3.11) satisfying the boundary condition (2.5). Such a matrix solution, denoted by $\Psi_j(x)$, is obtained by constructing⁷ the $n \times n$ Marchenko normalization matrix M_j and by letting

$$\Psi_j(x) := f(i\kappa_j, x) M_j, \quad 1 \leq j \leq N, \quad (3.13)$$

where we refer to the $n \times n$ matrix $\Psi_j(x)$ as the Marchenko normalized bound-state solution to the Schrödinger equation (3.11) at $k = i\kappa_j$. The Marchenko normalization matrices M_j are constructed in such a way that the matrices $\Psi_j(x)$ defined in (3.13) for $1 \leq j \leq N$ satisfy the boundary condition (2.5) and also satisfy the respective normalization and orthogonalization conditions given by

$$\int_0^\infty dx \Psi_j(x)^\dagger \Psi_j(x) = P_j, \quad 1 \leq j \leq N, \quad (3.14)$$

$$\int_0^\infty dx \Psi_j(x)^\dagger \Psi_l(x) = 0, \quad j \neq l, \quad (3.15)$$

where we have $1 \leq j \leq N$ and $1 \leq l \leq N$. We will see that each M_j is a nonnegative matrix of rank m_j . In other words, exactly m_j eigenvalues of the matrix M_j are positive and the remaining $n - m_j$ eigenvalues are zero. Hence, although each column of the matrix $\Psi_j(x)$ is a solution to (3.11) and satisfies (2.5), among the n columns of $\Psi_j(x)$ we only have exactly m_j linearly independent column-vector solutions to (3.11)

satisfying the boundary condition (2.5). We observe that any nontrivial square-integrable column-vector solution to (3.11) satisfying (2.5) can be written as $\Psi_j(x)v$ for a constant nonzero column vector $v \in \mathbb{C}^n$.

For the construction of the $n \times n$ Marchenko normalization matrix M_j , we refer the reader to Ref. 1 when the boundary condition (2.5) is purely Dirichlet and to Sec. 3.11 of Ref. 7 when the boundary condition (2.5) is the general selfadjoint boundary condition. We remark that the construction of the matrix M_j in Sec. 3.11 of Ref. 7 is given by assuming that the potential V in (3.11) belongs to $L^1_1(\mathbb{R}^+)$. However, the same proof applies by replacing that assumption by the weaker assumption $V \in L^1(\mathbb{R}^+)$.

The construction of M_j can be summarized as follows. With the help of the orthogonal projection matrix P_j and the Jost solution $f(ik_j, x)$, we form the $n \times n$ matrix A_j defined as

$$A_j := \int_0^\infty dx P_j f(ik_j, x)^\dagger f(ik_j, x) P_j, \quad 1 \leq j \leq N. \quad (3.16)$$

We then define the $n \times n$ matrix B_j as

$$B_j := I - P_j + A_j, \quad 1 \leq j \leq N. \quad (3.17)$$

From (3.1), (3.16), and (3.17) it follows that A_j and B_j are both hermitian. Furthermore, it can be shown⁷ that B_j is a positive matrix. Thus, there exists a unique positive hermitian matrix $B_j^{1/2}$ so that

$$B_j^{1/2} B_j^{1/2} = B_j, \quad 1 \leq j \leq N. \quad (3.18)$$

Since $B_j^{1/2}$ is positive, its inverse exists and is denoted by $B_j^{-1/2}$. It can be shown⁷ that each of B_j , $B_j^{1/2}$, and $B_j^{-1/2}$ commutes with P_j . Then, the Marchenko normalization matrix M_j is defined as

$$M_j := B_j^{-1/2} P_j, \quad 1 \leq j \leq N. \quad (3.19)$$

We construct the $n \times n$ matrix-valued Gel'fand–Levitan normalized bound-state solutions in a manner similar to the construction of the $n \times n$ matrix-valued Marchenko normalized bound-state solutions. In analogy with (3.13) we construct the $n \times n$ Gel'fand–Levitan normalization matrix C_j and let

$$\Phi_j(x) := \varphi(ik_j, x) C_j, \quad 1 \leq j \leq N, \quad (3.20)$$

where we refer to the $n \times n$ matrix $\Phi_j(x)$ as the Gel'fand–Levitan normalized bound-state solution to the Schrödinger equation (3.11) at $k = ik_j$. In the construction of the matrix C_j presented below, we observe that each C_j is a nonnegative matrix of rank m_j . We construct the matrices C_j for $1 \leq j \leq N$ in such a way that the matrices $\Phi_j(x)$ are square integrable and satisfy the respective normalization and orthogonalization conditions given by

$$\int_0^\infty dx \Phi_j(x)^\dagger \Phi_j(x) = Q_j, \quad 1 \leq j \leq N, \quad (3.21)$$

$$\int_0^\infty dx \Phi_j(x)^\dagger \Phi_l(x) = 0, \quad j \neq l, \quad (3.22)$$

which are the analogs of (3.14) and (3.15), respectively.

Although the regular solution $\varphi(ik_j, x)$ to (3.11) satisfies the boundary condition (2.5), unless we have $m_j = n$ that solution becomes unbounded as $x \rightarrow +\infty$, and we have⁷

$$\varphi(ik_j, x) = O(e^{k_j x}), \quad x \rightarrow +\infty. \quad (3.23)$$

We recall that m_j corresponds to the multiplicity of the bound state at $k = ik_j$. Thus, in the scalar case, i.e. when $n = 1$, the regular solution $\varphi(ik_j, x)$ to (3.11) is exponentially decaying like $O(e^{-k_j x})$ as $x \rightarrow +\infty$. In the matrix case, i.e. when $n \geq 2$, that regular solution in general cannot be square integrable in $x \in \mathbb{R}^+$ unless $m_j = n$. We know from Propositions 3.2.1, 3.2.2, and 3.2.3 of Ref. 7 that, in general, each column of $\varphi(ik_j, x)$ consists of two components behaving like $O(e^{k_j x})$ and $O(e^{-k_j x})$, respectively, as $x \rightarrow +\infty$. The normalization matrix C_j allows us to form some appropriate linear combinations of the columns of $\varphi(ik_j, x)$ in such a way that those linear combinations behave like $O(e^{-k_j x})$ as $x \rightarrow +\infty$, and that there are exactly m_j such linearly independent combinations of the columns. Unless $m_j = n$, it is possible that none of the n columns of $\varphi(ik_j, x)$ behave like $O(e^{-k_j x})$ as $x \rightarrow +\infty$. On the other hand, we have

$$\Phi_j(x) = O(e^{-k_j x}), \quad x \rightarrow +\infty.$$

Furthermore, $\Phi_j(x)$ satisfies the boundary condition (2.5) as a consequence of (2.9). We remark that any nontrivial square-integrable column-vector solution to (3.11) satisfying (2.5) can be written as $\Phi_j(x)v$, where v is a constant nonzero column vector in $\mathbb{C}^n$.

We present the construction of the $n \times n$ Gel'fand–Levitan normalization matrix C_j in a similar way to the construction of the Marchenko normalization matrix M_j outlined in (3.16)–(3.19). With the help of the orthogonal projection matrix Q_j and the regular solution $\varphi(ik_j, x)$ to (3.11), we form the $n \times n$ matrix G_j as

$$G_j := \int_0^\infty dx Q_j \varphi(ik_j, x)^\dagger \varphi(ik_j, x) Q_j, \quad 1 \leq j \leq N. \quad (3.24)$$

It follows from Theorem 3.11.1(e) of Ref. 7 that the integral on the right-hand side of (3.24) is finite. Next, we introduce the matrix $\mathbf{H}_j$ as

$$\mathbf{H}_j := I - Q_j + \mathbf{G}_j, \quad 1 \leq j \leq N. \quad (3.25)$$

From (3.1), (3.24), and (3.25) it follows that the matrices $\mathbf{G}_j$ and $\mathbf{H}_j$ are both hermitian. Moreover, $\mathbf{H}_j$ is a positive matrix, and the proof for this is similar to the proof⁷ that $\mathbf{B}_j$ defined in (3.17) is a positive matrix, which is given in the proof of Proposition 3.11.10 of Ref. 7. Thus, there exists a unique positive Hermitian matrix $\mathbf{H}_j^{1/2}$ so that

$$\mathbf{H}_j^{1/2} \mathbf{H}_j^{1/2} = \mathbf{H}_j, \quad 1 \leq j \leq N. \quad (3.26)$$

Since the matrix $\mathbf{H}_j^{1/2}$ is positive, its inverse exists and is denoted by $\mathbf{H}_j^{-1/2}$. We note that each of $\mathbf{H}_j$, $\mathbf{H}_j^{1/2}$, and $\mathbf{H}_j^{-1/2}$ commutes with Q_j , and the proof for this is similar to the proof⁷ regarding the commutation of each of $\mathbf{B}_j$, $\mathbf{B}_j^{1/2}$, and $\mathbf{B}_j^{-1/2}$ with P_j , which is given in the proof of Proposition 3.11.10 of Ref. 7. The Gel'fand–Levitan normalization matrix C_j is defined as

$$C_j := \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N, \quad (3.27)$$

which is analogous to the definition of the Marchenko normalization matrix M_j given in (3.19). From (3.27) it follows that C_j is Hermitian and nonnegative and has rank equal to m_j , which is the same as the rank of Q_j .

We recall that $\varphi(i\kappa_j, x)$ for each j with $1 \leq j \leq N$ satisfies the boundary condition (2.5). It follows from the second equality in (3.8) and Theorem 3.11.1 of Ref. 7 that each column of $\varphi(i\kappa_j, x)Q_j$ decreases exponentially as $O(e^{-\kappa_j x})$ as $x \rightarrow +\infty$. Thus, the integral on the right-hand side of (3.24) is finite. The normalization condition (3.21) follows from (3.24), (3.25), and the fact that Q_j is an orthogonal projection that commutes with $\mathbf{H}_j$, $\mathbf{H}_j^{1/2}$, and $\mathbf{H}_j^{-1/2}$. The orthogonality in (3.22) can be established in a similar way (3.15) is established in the proof of Proposition 5.10.5 in Ref. 7. Since $\varphi(i\kappa_j, x)C_j$ satisfies the boundary condition (2.5) and decays exponentially as $O(e^{-\kappa_j x})$ as $x \rightarrow +\infty$, it is square integrable and hence the column vector $\varphi(i\kappa_j, x)C_j v$, where v is any column vector in $\mathbb{C}^n$, is a bound state with the energy $-\kappa_j^2$. From Theorem 3.11.1 in Ref. 7 we know that any bound state with the energy $-\kappa_j^2$ is of the form $\varphi(i\kappa_j, x)C_j v$, where v is a column vector in $\mathbb{C}^n$. Moreover, Theorem 3.11.1 of Ref. 7 indicates that there are precisely m_j linearly independent column vectors of that type, and hence m_j is the multiplicity of the bound state at $k = i\kappa_j$. We recall that m_j is the positive integer equal to the dimension of the kernel of $J(i\kappa_j)$.

We remark that the integral in (3.16) remains finite even in the absence of P_j there, i.e. we have

$$\int_0^\infty dx f(i\kappa_j, x)^\dagger f(i\kappa_j, x) < +\infty,$$

which can be proved with the help of (3.12). On the other hand, if we remove Q_j in the integral in (3.24) then the resulting integral is in general not convergent when $n \geq 2$, i.e. the integral

$$\int_0^\infty dx \varphi(i\kappa_j, x)^\dagger \varphi(i\kappa_j, x),$$

may not be finite because of (3.23), unless $m_j = n$, in which case we have $Q_j = I$.

With the help of (3.19) and (3.27), let us write (3.13) and (3.20) as

$$\Psi_j(x) = f(i\kappa_j, x) \mathbf{B}_j^{-1/2} P_j, \quad \Phi_j(x) = \varphi(i\kappa_j, x) \mathbf{H}_j^{-1/2} Q_j, \quad (3.28)$$

where we recall that $\mathbf{B}_j^{-1/2}$ and P_j commute with each other and that $\mathbf{H}_j^{-1/2}$ and Q_j commute with each other. We can write the orthonormality relations (3.14), (3.15), (3.21), and (3.22) in a compact form as

$$\int_0^\infty dx \Psi_j(x)^\dagger \Psi_l(x) = \delta_{jl} P_l, \quad \int_0^\infty dx \Phi_j(x)^\dagger \Phi_l(x) = \delta_{jl} Q_l,$$

where δ_{jl} denotes the Kronecker delta, and we have $1 \leq j \leq N$ and $1 \leq l \leq N$.

IV. THE BOUND-STATE DEPENDENCY MATRICES

In this section we assume that the potential $V(x)$ appearing in (2.1) satisfies (2.2) and (2.3) and that the corresponding matrix-valued Schrödinger operator is associated with the selfadjoint boundary condition described in (2.5)–(2.7). Although the Schrödinger operator is selfadjoint, the number of bound states may be infinite. For each bound state, we introduce the concept of the bound-state dependency matrix, which relates the Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$ to the Marchenko normalized bound-state solution $\Psi_j(x)$. We describe the basic properties of the bound-state dependency matrix. In the scalar case, i.e. when $n = 1$, it is appropriate to refer to dependency matrices as dependency constants. We refer the reader to Refs. 2–4 for the use of dependency constants in some scattering problems associated

with nonselfadjoint systems, especially when the bound states are not necessarily simple, and for the use of the time evolutions of dependency constants arising in the analysis of integrable systems.

We recall that a bound-state solution at $k = i\kappa_j$ is a square-integrable column-vector solution to (3.11) satisfying the boundary condition (2.5). We know that there are exactly m_j linearly independent bound-state solutions at $k = i\kappa_j$, where m_j is a positive integer corresponding to the multiplicity of the bound state and we have $1 \leq m_j \leq n$. At the bound state with $k = i\kappa_j$, from (3.13) we know that the $n \times n$ Marchenko normalized bound-state solution $\Psi_j(x)$ has m_j linearly independent columns and that each column of $\Psi_j(x)$ satisfies (3.11) and the boundary condition (2.5). Similarly, we know from (3.20) that the $n \times n$ Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$ has m_j linearly independent columns and that each column of $\Phi_j(x)$ satisfies (3.11) and the boundary condition (2.5). Thus, there exists a constant $n \times n$ matrix relating the columns of the matrix $\Psi_j(x)$ to the columns of the matrix $\Phi_j(x)$. Let us introduce the $n \times n$ matrix D_j in such a way that

$$\Phi_j(x) = \Psi_j(x) D_j, \quad 1 \leq j \leq N, \quad (4.1)$$

where we recall that the number N of bound states may be infinite. Since D_j is an $n \times n$ matrix relating the two $n \times n$ matrices with equal ranks m_j , the matrix D_j in (4.1) cannot be unique unless we have $m_j = n$. When $m_j < n$, we impose a further restriction on it. Without loss of generality, we can choose that additional restriction as

$$D_j = P_j D_j, \quad (4.2)$$

where we recall that the $n \times n$ matrix P_j is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)^\dagger]$ and appearing in (3.1). We refer to D_j appearing in (4.1) and (4.2) as the dependency matrix associated with the bound state at $k = i\kappa_j$.

We construct the dependency matrix D_j satisfying (4.1) and (4.2) as follows. From Theorem 3.11.1 of Ref. 7 we know that there exists a bijection $\alpha \mapsto \beta$ from the kernel of $J(i\kappa_j)$ onto the kernel of $J(i\kappa_j)^\dagger$ in such a way that

$$\varphi(i\kappa_j, x) \alpha = f(i\kappa_j, x) \beta, \quad 1 \leq j \leq N. \quad (4.3)$$

Let $\{w_j^{(l)}\}_{l=1}^{m_j}$ be an orthonormal basis for the subspace $\text{Ker}[J(i\kappa_j)]$, and let $\beta_j^{(l)}$ be the column vector in the subspace $\text{Ker}[J(i\kappa_j)^\dagger]$ corresponding to the column vector $w_j^{(l)}$ under the aforementioned bijection. Thus, (4.3) implies that for $1 \leq j \leq N$ we have

$$\varphi(i\kappa_j, x) w_j^{(l)} = f(i\kappa_j, x) \beta_j^{(l)}, \quad 1 \leq l \leq m_j,$$

from which we get

$$\varphi(i\kappa_j, x) w_j^{(l)} (w_j^{(l)})^\dagger = f(i\kappa_j, x) \beta_j^{(l)} (\beta_j^{(l)})^\dagger, \quad 1 \leq l \leq m_j. \quad (4.4)$$

Summing over l in (4.4), with the help of the second equality of (3.8), we obtain

$$\varphi(i\kappa_j, x) Q_j = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (\beta_j^{(l)})^\dagger, \quad 1 \leq j \leq N. \quad (4.5)$$

Postmultiplying each side of (4.5) by the invertible matrix $\mathbf{H}_j^{-1/2}$ appearing in (3.27), we get

$$\varphi(i\kappa_j, x) Q_j \mathbf{H}_j^{-1/2} = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2}, \quad 1 \leq j \leq N. \quad (4.6)$$

We already know that the matrices Q_j and $\mathbf{H}_j^{-1/2}$ commute, and hence (4.6) is equivalent to

$$\varphi(i\kappa_j, x) \mathbf{H}_j^{-1/2} Q_j = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2}, \quad 1 \leq j \leq N. \quad (4.7)$$

Postmultiplying both sides of (4.7) by Q_j and then using $Q_j^2 = Q_j$, we obtain

$$\varphi(i\kappa_j, x) \mathbf{H}_j^{-1/2} Q_j = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (4.8)$$

Using the second equality of (3.28) on the left-hand side of (4.8), we get

$$\Phi_j(x) = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (4.9)$$

Hence, from (4.1) and (4.9) we have

$$\Psi_j(x) D_j = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (4.10)$$

Using (3.13) and (3.19) on the left-hand side of (4.10), we obtain

$$f(i\kappa_j, x) \mathbf{B}_j^{-1/2} P_j D_j = f(i\kappa_j, x) \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (4.11)$$

From (3.12) we see that the matrix $f(i\kappa_j, x)$ is invertible for large x -values. Hence, (4.11) yields

$$\mathbf{B}_j^{-1/2} P_j D_j = \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (4.12)$$

We recall that the matrix $\mathbf{B}_j^{-1/2}$ is invertible and its inverse is given by $\mathbf{B}_j^{1/2}$. Premultiplying both sides of (4.12) by $P_j \mathbf{B}_j^{1/2}$ and then using (4.2) in the resulting equation, we obtain

$$D_j = P_j \mathbf{B}_j^{1/2} \sum_{l=1}^{m_j} \beta_j^{(l)} (w_j^{(l)})^\dagger \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (4.13)$$

The next theorem shows that the dependency matrix D_j satisfying (4.1) and (4.2) is unique, and it presents some basic properties of D_j . We remark that (4.2) alone cannot uniquely determine D_j because any constant scalar multiple of D_j satisfying (4.2) also satisfies (4.2).

Theorem 4.1. Assume that the potential V appearing in (2.1) satisfies (2.2) and (2.3) and that the corresponding matrix-valued Schrödinger operator is associated with the selfadjoint boundary condition described in (2.5)–(2.7). Let D_j be the $n \times n$ dependency matrix satisfying (4.1) and (4.2) associated with the bound state at $k = i\kappa_j$. We have the following:

- (a) The matrix D_j is uniquely determined.
- (b) The matrix D_j satisfies

$$D_j = D_j Q_j, \quad (4.14)$$

where we recall that the $n \times n$ matrix Q_j is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)]$ and satisfies the second set of equalities in (3.1).

- (c) The matrix product $D_j^\dagger D_j$ is an orthogonal projection, and we have

$$D_j^\dagger D_j = Q_j. \quad (4.15)$$

- (d) The matrix product $D_j D_j^\dagger$ is also an orthogonal projection, and we have

$$D_j D_j^\dagger = P_j, \quad (4.16)$$

where we recall that the $n \times n$ matrix P_j is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)^\dagger]$ and satisfies the first set of equalities in (3.1).

- (e) The matrices D_j and $D_j^\dagger$ each have rank m_j , which is the same as the common rank of the orthogonal projection matrices P_j and Q_j .
- (f) The Moore–Penrose inverse of the matrix D_j is equal to $D_j^\dagger$, i.e. we have

$$D_j^+ = D_j^\dagger. \quad (4.17)$$

- (g) The $n \times n$ Marchenko normalized bound-state solution $\Psi_j(x)$ is related to the $n \times n$ Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$ as

$$\Psi_j(x) = \Phi_j(x) D_j^\dagger. \quad (4.18)$$

- (h) The matrix D_j satisfies

$$D_j D_j^\dagger D_j = D_j, \quad D_j^\dagger D_j D_j^\dagger = D_j^\dagger. \quad (4.19)$$

Proof. The proof of (a) can be given as follows. The existence of D_j follows from the explicit expression given in (4.13). To prove its uniqueness, let us assume that D_j and $\hat{D}_j$ both satisfy (4.1) and (4.2) so that we have

$$\Phi_j(x) = \Psi_j(x) D_j, \quad \Phi_j(x) = \Psi_j(x) \hat{D}_j, \quad (4.20)$$

$$D_j = P_j D_j, \quad \hat{D}_j = P_j \hat{D}_j. \quad (4.21)$$

From (4.20) we get

$$\Psi_j(x) (D_j - \hat{D}_j) = 0. \quad (4.22)$$

Using the first equality of (3.28) in (4.22) we obtain

$$f(ik_j, x) \mathbf{B}_j^{-1/2} P_j (D_j - \hat{D}_j) = 0. \quad (4.23)$$

We already know that the matrix $\mathbf{B}_j^{-1/2}$ is invertible. Furthermore, from (3.12) it follows that the matrix $f(ik_j, x)$ for large x -values is also invertible. Hence, (4.23) yields

$$P_j (D_j - \hat{D}_j) = 0. \quad (4.24)$$

Using (4.21) in (4.24) we get

$$D_j - \hat{D}_j = 0,$$

which yields $D_j = \hat{D}_j$ and hence D_j is uniquely determined by (4.1) and (4.2). Thus, the proof of (a) is complete. For the proof of (b), we proceed as follows. We remark that (4.13) and the uniqueness proved in (a) imply that (b) holds. Note that (b) can alternatively be proved as follows. From the second set of equalities in (3.1) and the second equality in (3.28), we see that $\Phi_j(x) Q_j = \Phi_j(x)$. Hence, postmultiplying (4.1) by Q_j we obtain

$$\Phi_j(x) = \Psi_j(x) D_j Q_j. \quad (4.25)$$

Subtracting (4.25) from (4.1) we get

$$\Psi_j(x) (D_j - D_j Q_j) = 0. \quad (4.26)$$

Proceeding as in the steps involving (4.22)–(4.24), from (4.26) we have

$$P_j (D_j - D_j Q_j) = 0. \quad (4.27)$$

Finally, using (4.2) in (4.27) we obtain (4.14), which completes the proof of (b). For the proof of (c) we proceed as follows. Using (4.1) in (3.21), with the help of (3.14) we get

$$D_j^\dagger P_j D_j = Q_j. \quad (4.28)$$

Using (4.2) in (4.28) we obtain (4.15). Since Q_j is an orthogonal projection, we see that the proof of (c) is complete. To prove (d) we first need to show that the matrix product $D_j D_j^\dagger$ is an orthogonal projection, i.e. we need to show that $D_j D_j^\dagger$ is selfadjoint and it satisfies

$$(D_j D_j^\dagger) (D_j D_j^\dagger) = D_j D_j^\dagger. \quad (4.29)$$

The selfadjointness of $D_j D_j^\dagger$ is self evident, and (4.29) is verified by using the set of equalities

$$D_j (D_j^\dagger D_j) D_j^\dagger = D_j Q_j D_j^\dagger = D_j D_j^\dagger,$$

which are obtained by using (4.15) and then (4.14). Since the matrices $D_j D_j^\dagger$ and P_j are orthogonal projections and have the same domain $\mathbb{C}^n$, in order to complete the proof of (d) we need to show that the matrices $D_j D_j^\dagger$ and P_j have the same range. For this we proceed as follows. With the help of (4.2) we see that

$$P_j D_j D_j^\dagger = D_j D_j^\dagger,$$

from which we obtain the range relations given by

$$\text{Ran}[D_j D_j^\dagger] = \text{Ran}[P_j D_j D_j^\dagger] \subset \text{Ran}[P_j].$$

We have

$$\text{Ker}[D_j D_j^\dagger] = \text{Ker}[D_j^\dagger],$$

and hence by using the rank-nullity theorem we conclude that

$$\text{rank}[D_j D_j^\dagger] = \text{rank}[D_j^\dagger]. \quad (4.30)$$

Since D_j and $D_j^\dagger$ have the same rank, from (4.30) we get

$$\text{rank}[D_j D_j^\dagger] = \text{rank}[D_j]. \quad (4.31)$$

From (4.31) we conclude that the two matrices $D_j D_j^\dagger$ and D_j have the same nullity. On the other hand, (4.15) implies that

$$\text{Ker}[Q_j] = \text{Ker}[D_j^\dagger D_j] = \text{Ker}[D_j], \quad (4.32)$$

and hence we conclude that the rank of D_j is the same as the rank of Q_j , which is also equal to the rank of P_j . Then, from (4.31) and (4.32) we get

$$\text{rank}[D_j D_j^\dagger] = \text{rank}[P_j], \quad (4.33)$$

and hence (4.33) implies (4.16). Thus, the proof of (d) is complete. The proof of (e) directly follows from (4.31)–(4.33). The proof of (f) is obtained by using $\mathbf{M} = D_j$ and $\mathbf{M}^+ = D_j^\dagger$ in (3.2) and by verifying that the four equalities listed there are satisfied. Note that the last two equalities in (3.2) are satisfied because we have $\mathbf{M}^+ = \mathbf{M}^\dagger$ there. The first two equalities in (3.2) are also satisfied, and this can be proved as follows. Using (4.15) and then (4.14), we have

$$D_j(D_j^\dagger D_j) = D_j Q_j = D_j, \quad (4.34)$$

and hence the first equality in (3.2) is satisfied. Similarly, using (4.14), (4.15), and the property $Q_j^\dagger = Q_j$ listed in (3.1), we obtain

$$(D_j^\dagger D_j) D_j^\dagger = Q_j D_j^\dagger = (D_j Q_j)^\dagger = D_j^\dagger, \quad (4.35)$$

which verifies the second equality in (3.2). Consequently, with the help of the uniqueness of the Moore–Penrose inverse, we confirm (4.17). This completes the proof of (f). Note that (4.18) follows from (4.1), (4.16), and the first equality in (3.28), which completes the proof of (g). Note also that (4.19) directly follows from (4.34) and (4.35), and hence the proof of (h) is also complete. $\square$

In the previous theorem we have established the uniqueness of the dependency matrix D_j satisfying (4.1) and (4.2). We have presented an explicit formula for D_j in (4.13). We would like to show that D_j can also be expressed explicitly in terms of the Jost solution $f(i\kappa_j, x)$, the Marchenko normalization matrix M_j appearing in (3.19), and the Gel'fand–Levitan bound-state normalized solution defined in (3.20) as

$$D_j = M_j^+ f(i\kappa_j, x)^{-1} \Phi_j(x), \quad (4.36)$$

where we evaluate the right-hand side at any x -value at which the matrix $f(i\kappa_j, x)$ is invertible and we recall that M_j^+ denotes the Moore–Penrose inverse of M_j . The invertibility of $f(i\kappa_j, x)$ is ensured for large x -values, as seen from (3.12). We can establish (4.36) as follows. As seen from (3.1) the orthogonal projection matrix P_j satisfies $P_j^2 = P_j$, and we already know that P_j commutes with $\mathbf{B}_j^{-1/2}$ appearing in (3.19). Hence, from (3.19) we have

$$M_j = P_j M_j P_j, \quad (4.37)$$

Using (3.9) we obtain the decomposition for M_j given by

$$M_j = [M_j]_1 \oplus 0, \quad (4.38)$$

where $[M_j]_1$ is defined as the restriction of M_j to $\text{Ker}[J(i\kappa_j)^\dagger]$. From (3.19) it follows that the matrix $[M_j]_1$ is invertible in $\text{Ker}[J(i\kappa_j)^\dagger]$. Thus, from (3.5) we obtain

$$M_j^+ = ([M_j]_1)^{-1} \oplus 0. \quad (4.39)$$

Using (3.13) and the invertibility of $f(i\kappa_j, x)$ ensured for large x , from (4.1) we get

$$M_j^+ M_j D_j = M_j^+ f(i\kappa_j, x)^{-1} \Phi_j(x). \quad (4.40)$$

From (4.38) and (4.39) we have

$$M_j^+ M_j = I \oplus 0, \quad (4.41)$$

which indicates that

$$M_j^+ M_j = P_j. \quad (4.42)$$

Using (4.42) in (4.40) we get

$$P_j D_j = M_j^+ f(i\kappa_j, x)^{-1} \Phi_j(x), \quad (4.43)$$

and using (4.2) on the right-hand side of (4.43) we obtain (4.36).

If the matrix $f(i\kappa_j, x)$ is not invertible at a particular x -value but the matrix $f'(i\kappa_j, x)$ is invertible there, then we can derive an expression for D_j in an analogous manner we have obtained (4.36), and we get

$$D_j = M_j^+ f'(i\kappa_j, x)^{-1} \Phi_j'(x). \quad (4.44)$$

The derivation of (4.44) is straightforward and can be established by taking the x -derivative of both sides of (4.1) and then by using the analogs of the steps involving (4.37)–(4.43).

The dependency matrix D_j appearing in (4.1) and (4.2) and its adjoint $D_j^\dagger$ are related to certain partial isometries. In order to see this we proceed as follows. We recall²² that a linear map L on $\mathbb{C}^n$ is an isometry if $|Lv| = |v|$ for all vectors $v \in \mathbb{C}^n$, where we use $|\cdot|$ to denote the standard norm on $\mathbb{C}^n$. On the other hand, L is a partial isometry if $|Lv| = |v|$ for all vectors v in $(\text{Ker}[L])^\perp$. We recall that the superscript $\perp$ denotes the orthogonal complement. It is known²² that the linear map L is a partial isometry if and only if $\langle Lv, Lw \rangle = \langle v, w \rangle$ for all vectors v and w in $(\text{Ker}[L])^\perp$, where we use $\langle \cdot, \cdot \rangle$ to denote the standard inner product on $\mathbb{C}^n$. If the linear map L is a partial isometry, then the subspace $(\text{Ker}[L])^\perp$ is called the initial subspace of $\mathbb{C}^n$ and the subspace $\text{Ran}[L]$ is called the final subspace of $\mathbb{C}^n$. It is also known²² that a linear map L is partially isometric if and only if the adjoint linear map $L^\dagger$ is a partial isometry.

The relevant properties of a partial isometry are known,²² and we list them in the following theorem without a proof.

Theorem 4.2. *Let L be a linear map on $\mathbb{C}^n$. Then, the following are equivalent:*

- (a) *The map L is a partial isometry, with the understanding that the initial subspace is $(\text{Ker}[L])^\perp$ and the final subspace is $\text{Ran}[L]$.*
- (b) *The adjoint map $L^\dagger$ is a partial isometry, with the understanding that the initial subspace is $(\text{Ker}[L^\dagger])^\perp$ and the final subspace is $\text{Ran}[L^\dagger]$.*
- (c) *The map $LL^\dagger$ is the orthogonal projection onto $\text{Ran}[L]$.*
- (d) *The map $L^\dagger L$ is the orthogonal projection onto $(\text{Ker}[L])^\perp$.*
- (e) *We have $LL^\dagger L = L$.*
- (f) *We have $L^\dagger LL^\dagger = L^\dagger$.*

In the next theorem, with the help of Theorem 4.2, we relate the bound-state dependency matrix D_j appearing in (4.1) and (4.2) to a certain partial isometry and to a bijection from $\text{Ker}[J(i\kappa_j)]$ onto $\text{Ker}[J(i\kappa_j)^\dagger]$.

Theorem 4.3. *Assume that the potential V appearing in (2.1) satisfies (2.2) and (2.3) and that the corresponding half-line matrix Schrödinger operator is associated with the selfadjoint boundary condition described in (2.5)–(2.7). Let D_j be the $n \times n$ dependency matrix satisfying (4.1) and (4.2) associated with the bound state at $k = i\kappa_j$. We have the following:*

- (a) *The matrix D_j is a partial isometry with the initial subspace $\text{Ker}[J(i\kappa_j)]$ and the final subspace $\text{Ker}[J(i\kappa_j)^\dagger]$.*
- (b) *The adjoint matrix $D_j^\dagger$ is a partial isometry with the initial subspace $\text{Ker}[J(i\kappa_j)^\dagger]$ and the final subspace $\text{Ker}[J(i\kappa_j)]$.*

Proof. From the first equality of (4.19) we see that Theorem 4.2(e) holds with $L = D_j$. Hence, all items in Theorem 4.2 hold when $L = D_j$. In particular, Theorem 4.2(a) implies that D_j is a partial isometry with the initial subspace $(\text{Ker}[D_j])^\perp$ and the final subspace $\text{Ran}[D_j]$. On the other hand, from (4.15) we know that $D_j^\dagger D_j$ is equal to Q_j . By Theorem 4.2(d) we know that $D_j^\dagger D_j$ is the orthogonal projection onto $(\text{Ker}[D_j])^\perp$ and we already know that Q_j is the orthogonal projection onto the kernel of $J(i\kappa_j)$. Thus, we conclude that

$$(\text{Ker}[D_j])^\perp = \text{Ker}[J(i\kappa_j)].$$

Moreover, by Theorem 4.2(c) we know that $D_j D_j^\dagger$ is the orthogonal projection onto $\text{Ran}[D_j]$, which is equal to $\text{Ker}[D_j^\dagger]^\perp$. On the other hand, (4.16) shows that $D_j D_j^\dagger$ is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)^\dagger]$. Hence, we have

$$\text{Ran}[D_j] = (\text{Ker}[D_j^\dagger])^\perp = \text{Ker}[J(i\kappa_j)^\dagger].$$

Consequently, with the help of Theorem 4.2(a) we conclude that D_j is a partial isometry with the initial subspace $\text{Ker}[J(i\kappa_j)]$ and the final subspace $\text{Ker}[J(i\kappa_j)^\dagger]$, which completes the proof of (a). With the help of the equivalences stated in Theorem 4.2, we see that (b) is equivalent to (a), and hence the proof of (b) is also complete. $\square$

V. THE GEL'FAND-LEVITAN SYSTEM OF INTEGRAL EQUATIONS

In this section we introduce the spectral measure related to the half-line matrix Schrödinger operator described by (2.1) and (2.5), provide Parseval's equality involving the regular solution and the Gel'fand–Levitan normalized bound-state solutions, and derive the corresponding matrix-valued Gel'fand–Levitan system of linear integral equations.

When the potential V in (2.1) satisfies (2.2) and belongs to $L_1^1(\mathbb{R}^+)$, as indicated in Sec. III, the Schrödinger operator associated with (2.1) and (2.5) has at most a finite number of distinct bound states occurring at $k = i\kappa_j$ for $1 \leq j \leq N$. It is understood that $N = 0$ if there are no bound states. The spectral measure $d\rho$ associated with the corresponding selfadjoint Schrödinger operator related to (2.1) and (2.5) is given by

$$d\rho := \begin{cases} \frac{\sqrt{\lambda}}{\pi} \left(J(\sqrt{\lambda})^\dagger J(\sqrt{\lambda}) \right)^{-1} d\lambda, & \lambda \geq 0, \\ \sum_{j=1}^N C_j^2 \delta(\lambda - \lambda_j) d\lambda, & \lambda < 0, \end{cases} \quad (5.1)$$

where $\delta(\cdot)$ denotes the Dirac delta distribution and we recall that $J(k)$ is the Jost matrix given in (2.10), the quantity C_j is the Gel'fand–Levitan normalization matrix defined in (3.27), and we let $\lambda := k^2$ and $\lambda_j := -\kappa_j^2$ for $1 \leq j \leq N$.

In the next theorem we present Parseval's equality for the half-line matrix Schrödinger operator associated with (2.1) and (2.5).

Theorem 5.1. *We assume that the potential V appearing in (2.1) satisfies (2.2) and belongs to $L_1^1(\mathbb{R}^+)$. Associated with the half-line selfadjoint Schrödinger operator described by (2.1) and (2.5), we have the completeness relation known as Parseval's equality, which is given by*

$$\int_{\lambda \in \mathbb{R}} \varphi(k, x) d\rho \varphi(k, y)^\dagger = \delta(x - y) I, \quad (5.2)$$

with the integration from $\lambda = -\infty$ to $\lambda = +\infty$ covering the domain of the spectral measure $d\rho$. Parseval's equality can also be written as

$$\int_{\lambda \in \mathbb{R}^+} \varphi(k, x) d\rho \varphi(k, y)^\dagger + \sum_{j=1}^N \Phi_j(x) \Phi_j(y)^\dagger = \delta(x - y) I, \quad (5.3)$$

where the integration is from $\lambda = 0$ to $\lambda = +\infty$, and we recall that $\varphi(k, x)$ is the regular solution to (2.1) satisfying the initial conditions (2.9), $d\rho$ is the spectral measure appearing in (5.1), $\Phi_j(x)$ is the $n \times n$ Gel'fand–Levitan normalized bound-state solution at $k = i\kappa_j$ and appearing in (3.20), and N is the number of bound states without counting the multiplicities.

Proof. We already know from (5.9.1) of Ref. 7 that Parseval's equality can be expressed in terms of the physical solution $\Psi(k, x)$ appearing in (2.12) and the $n \times n$ Marchenko normalized bound-state solutions $\Psi_j(x)$ appearing in (3.13), and we have

$$\frac{1}{2\pi} \int_0^\infty dk \Psi(k, x) \Psi(k, y)^\dagger + \sum_{j=1}^N \Psi_j(x) \Psi_j(y)^\dagger = \delta(x - y) I. \quad (5.4)$$

From (2.13) we obtain

$$\Psi(k, x) = -2ik \varphi(k, x) J(k)^{-1}, \quad k \in \mathbb{R}, \quad (5.5)$$

where the existence of $J(k)^{-1}$ for $x \in \mathbb{R} \setminus \{0\}$ and the continuity of $k J(k)^{-1}$ at $k = 0$ are ensured⁷ under the stated assumptions on the potential and the selfadjoint boundary condition. From (5.5) we get

$$\Psi(k, x) \Psi(k, y)^\dagger = 4k^2 \varphi(k, x) J(k)^{-1} [J(k)^{-1}]^\dagger \varphi(k, y)^\dagger, \quad k \in \mathbb{R},$$

which yields

$$\frac{dk}{2\pi} \Psi(k, x) \Psi(k, y)^\dagger = \frac{2k^2 dk}{\pi} \varphi(k, x) [J(k)^\dagger J(k)]^{-1} \varphi(k, y)^\dagger, \quad k \in \mathbb{R}. \quad (5.6)$$

Since $\lambda = k^2$, we can write (5.6) in an equivalent form as

$$\frac{d\lambda}{2\pi} \Psi(k, x) \Psi(k, y)^\dagger = \frac{\sqrt{\lambda} d\lambda}{\pi} \varphi(k, x) [J(k)^\dagger J(k)]^{-1} \varphi(k, y)^\dagger, \quad \lambda \geq 0. \quad (5.7)$$

Using the first line of (5.1) on the right-hand side of (5.7) and then integrating both sides of the resulting equality, we obtain

$$\frac{1}{2\pi} \int_0^\infty dk \Psi(k, x) \Psi(k, y)^\dagger = \int_{\lambda \in \mathbb{R}^+} \varphi(k, x) d\rho \varphi(k, y)^\dagger. \quad (5.8)$$

On the other hand, from (4.1) we have

$$\Phi_j(x) \Phi_j(y)^\dagger = \Psi_j(x) D_j D_j^\dagger \Psi_j(y)^\dagger. \quad (5.9)$$

Using (4.16) in (5.9) we obtain

$$\Phi_j(x) \Phi_j(y)^\dagger = \Psi_j(x) P_j \Psi_j(y)^\dagger. \quad (5.10)$$

From the first equality in (3.28) we see that

$$\Psi_j(x) P_j = \Psi_j(x),$$

and hence (5.10) yields

$$\Phi_j(x) \Phi_j(y)^\dagger = \Psi_j(x) \Psi_j(y)^\dagger,$$

which implies that

$$\sum_{j=1}^N \Phi_j(x) \Phi_j(y)^\dagger = \sum_{j=1}^N \Psi_j(x) \Psi_j(y)^\dagger. \quad (5.11)$$

Adding (5.8) and (5.11) we obtain the equivalent of the left-hand side of (5.4), and hence we get (5.3). Using (3.20) and (5.1) on the left-hand side of (5.3), we observe that (5.3) is equivalent to (5.2). $\square$

Let us view $V(x)$, $\varphi(k, x)$, (A, B) , and $d\rho$ as the potential, the regular solution, the pair of boundary matrices, and the spectral measure corresponding to the unperturbed problem. Thus, the regular solution $\varphi(k, x)$ satisfies the unperturbed Schrödinger equation

$$\varphi''(k, x) = [V(x) - k^2] \varphi(k, x), \quad (5.12)$$

and the unperturbed initial conditions (2.9). Let us view $\tilde{V}(x)$, $\tilde{\varphi}(k, x)$, $(\tilde{A}, \tilde{B})$, and $d\tilde{\rho}$ as the potential, the regular solution, the pair of boundary matrices, and the spectral measure corresponding to the perturbed problem. Thus, the perturbed regular solution $\tilde{\varphi}(k, x)$ satisfies the perturbed Schrödinger equation

$$\tilde{\varphi}''(k, x) = [\tilde{V}(x) - k^2] \tilde{\varphi}(k, x), \quad (5.13)$$

and the perturbed initial conditions

$$\tilde{\varphi}(k, 0) = \tilde{A}, \quad \tilde{\varphi}'(k, 0) = \tilde{B}. \quad (5.14)$$

Let us assume that the unperturbed and perturbed potentials satisfy (2.2) and belong to $L^1_1(\mathbb{R}^+)$, and let us also assume that the unperturbed and perturbed boundary conditions expressed by the respective unperturbed and perturbed boundary matrices are compatible with the selfadjoint boundary condition described in (2.5)–(2.7).

We can specify the perturbation relating the unperturbed problem and the perturbed problem in various different ways. In the rest of this section we specify the perturbation by relating the perturbed regular solution $\tilde{\varphi}(k, x)$ to the unperturbed regular solution $\varphi(k, x)$ through a matrix-valued quantity $\mathcal{A}(x, y)$ via the relation given by

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \int_0^x dy \mathcal{A}(x, y) \varphi(k, y). \quad (5.15)$$

We would like to find out how the perturbed potential $\tilde{V}(x)$ is related to the unperturbed potential $V(x)$ through the quantity $\mathcal{A}(x, y)$. Similarly, we would like to relate the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ to the unperturbed boundary matrices A and B through the quantity $\mathcal{A}(x, y)$. In the next theorem we present the appropriate restrictions on $\mathcal{A}(x, y)$ to obtain the aforementioned relations.

Theorem 5.2. Assume that the $n \times n$ matrix-valued potential V appearing in (2.1) satisfies (2.2) and is locally integrable on $x \in [0, +\infty)$. For $k \in \mathbb{C}$, let $\varphi(k, x)$ be the $n \times n$ matrix-valued solution to (2.1) with the initial values specified in (2.9), and assume that $\varphi(k, x)$ is continuously differentiable in $x \in [0, +\infty)$. Suppose that the quantity $\mathcal{A}(x, y)$ appearing in (5.15) is continuously differentiable on the set $\mathbb{D}$ defined as

$$\mathbb{D} := \{(x, y) : 0 \leq y \leq x < +\infty\}, \quad (5.16)$$

and that the second derivatives $\mathcal{A}_{xx}(x, y)$ and $\mathcal{A}_{yy}(x, y)$ are locally integrable on $\mathbb{D}$, where the subscripts denote the appropriate partial derivatives. Further, assume that the quantity $\mathcal{A}(x, y)$ satisfies the second-order matrix-valued partial differential equation

$$\mathcal{A}_{xx}(x, y) - \mathcal{A}_{yy}(x, y) = \tilde{V}(x) \mathcal{A}(x, y) - \mathcal{A}(x, y) V(y), \quad (5.17)$$

with the quantity $\tilde{V}(x)$ given by

$$\tilde{V}(x) := V(x) + 2 \frac{d \mathcal{A}(x, x)}{dx}, \quad (5.18)$$

where we use $\mathcal{A}(x, x)$ to denote $\mathcal{A}(x, x^-)$. Then, we have the following:

(a) The $n \times n$ matrix-valued function $\tilde{\varphi}(k, x)$ appearing in (5.15) satisfies (2.1) with the potential $\tilde{V}(x)$ specified in (5.18).

(b) The $n \times n$ constant matrices A and B appearing in (2.9) are related to each other via the matrix $\mathcal{A}(x, y)$ as

$$-\mathcal{A}_y(x, 0) A + \mathcal{A}(x, 0) B = 0. \quad (5.19)$$

(c) The $n \times n$ constant matrices $\tilde{A}$ and $\tilde{B}$ appearing in (5.14) are related to the constant matrices A and B appearing in (2.9) as

$$\tilde{A} = A, \quad \tilde{B} = B + \mathcal{A}(0, 0) A. \quad (5.20)$$

Proof. Using the quantity $\tilde{\varphi}(k, x)$ specified in (5.15), by taking the first and second x -derivatives, respectively, we obtain

$$\begin{aligned} \tilde{\varphi}'(k, x) &= \varphi'(k, x) + \mathcal{A}(x, x) \varphi(k, x) + \int_0^x dy \mathcal{A}_x(x, y) \varphi(k, y), \\ \tilde{\varphi}''(k, x) &= q_1(k, x), \end{aligned} \quad (5.21)$$

where we have defined

$$\begin{aligned} q_1(k, x) &:= \varphi''(k, x) + \mathcal{A}_x(x, x) \varphi(k, x) + \mathcal{A}_y(x, x) \varphi(k, x) + q_2(x), \\ q_2(x) &:= \mathcal{A}(x, x) \varphi'(k, x) + \mathcal{A}_x(x, x) \varphi(k, x) + \int_0^x dy \mathcal{A}_{xx}(x, y) \varphi(k, y). \end{aligned} \quad (5.22)$$

Using (5.12), we write (5.22) as

$$q_1(k, x) = q_3(x) + q_2(x) - \int_0^x dy \mathcal{A}_{yy}(x, y) \varphi(k, y) + \int_0^x dy \mathcal{A}_{yy}(x, y) \varphi(k, y), \quad (5.23)$$

where we have defined

$$q_3(x) := [V(x) - k^2] \varphi(k, x) + \mathcal{A}_x(x, x) \varphi(k, x) + \mathcal{A}_y(x, x) \varphi(k, x).$$

An integration by parts in the last term of (5.23) yields

$$\int_0^x dy \mathcal{A}_{yy}(x, y) \varphi(k, y) = \mathcal{A}_y(x, x) \varphi(k, x) - \mathcal{A}_y(x, 0) \varphi(k, 0) - \int_0^x dy \mathcal{A}_y(x, y) \varphi'(k, y). \quad (5.24)$$

Using integration by parts in the last integral on the right-hand side of (5.24), we have

$$\int_0^x dy \mathcal{A}_y(x, y) \varphi'(k, y) = \mathcal{A}(x, x) \varphi'(k, x) - \mathcal{A}(x, 0) \varphi'(k, 0) - \int_0^x dy \mathcal{A}(x, y) \varphi''(k, y). \quad (5.25)$$

Using (5.12) in the last term on the right-hand side of (5.25), we can write (5.25) as

$$\int_0^x dy \mathcal{A}_y(x, y) \varphi'(k, y) = q_4(x), \quad (5.26)$$

where we have defined

$$q_4(x) := \mathcal{A}(x, x) \varphi'(k, x) - \mathcal{A}(x, 0) \varphi'(k, 0) - \int_0^x dy \mathcal{A}(x, y) [V(y) - k^2] \varphi(k, y).$$

Then, using (5.24)–(5.26) on the right-hand side of (5.23), we obtain

$$q_1(k, x) = q_3(x) + q_2(x) + q_5(x) - q_4(x), \quad (5.27)$$

where we have defined

$$q_5(x) := - \int_0^x dy \mathcal{A}_{yy}(x, y) \varphi(k, y) + \mathcal{A}_y(x, x) \varphi(k, x) - \mathcal{A}_y(x, 0) \varphi(k, 0).$$

We have the identity

$$\mathcal{A}_x(x, x) + \mathcal{A}_y(x, x) = \frac{d \mathcal{A}(x, x)}{dx}, \quad (5.28)$$

where we recall that the values evaluated at $y = x$ are obtained by using the values at $y = x^-$. Using (5.28) in (5.27), with the help of (5.21) we get

$$\begin{aligned} \tilde{\varphi}''(k, x) &= \left[V(x) + 2 \frac{d \mathcal{A}(x, x)}{dx} - k^2 \right] \varphi(k, x) - \mathcal{A}_y(x, 0) \varphi(k, 0) + \mathcal{A}(x, 0) \varphi'(k, 0) \\ &\quad + \int_0^x dy [\mathcal{A}_{xx}(x, y) - \mathcal{A}_{yy}(x, y) + \mathcal{A}(x, y) [V(y) - k^2]] \varphi(k, y). \end{aligned} \quad (5.29)$$

From (5.15), (5.17), (5.21), and (5.29) we obtain

$$\tilde{\varphi}''(k, x) = \left[V(x) + 2 \frac{d}{dx} \mathcal{A}(x, x) - k^2 \right] \tilde{\varphi}(k, x) - \mathcal{A}_y(x, 0) \varphi(k, 0) + \mathcal{A}(x, 0) \varphi'(k, 0).$$

From (5.29) we see that the proof of (a) is complete provided we have

$$- \mathcal{A}_y(x, 0) \varphi(k, 0) + \mathcal{A}(x, 0) \varphi'(k, 0) = 0. \quad (5.30)$$

We establish (5.30) as follows. Because of (2.9) we observe that (5.30) is equivalent to (5.19). We next show that (5.19) is automatically satisfied. By multiplying (5.15) on the right by $d\rho \varphi(k, z)^\dagger$, we obtain

$$\tilde{\varphi}(k, x) d\rho \varphi(k, z)^\dagger = \varphi(k, x) d\rho \varphi(k, z)^\dagger + \int_0^x dy \mathcal{A}(x, y) \varphi(k, y) d\rho \varphi(k, z)^\dagger. \quad (5.31)$$

Integrating both sides of (5.31) over $\lambda \in \mathbb{R}$, we get

$$\int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho \varphi(k, z)^\dagger = \delta(x - z) I + \int_0^x dy \mathcal{A}(x, y) \delta(y - z) I, \quad (5.32)$$

where we have used Parseval's equality given in (5.2). From (5.32) we have

$$\mathcal{A}(x, y) = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho \varphi(k, y)^\dagger, \quad 0 \leq y < x,$$

which yields

$$\mathcal{A}(x, 0) = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho \varphi(k, 0)^\dagger, \quad (5.33)$$

$$\mathcal{A}_y(x, y) = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho \varphi'(k, y)^\dagger,$$

$$\mathcal{A}_y(x, 0) = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho \varphi'(k, 0)^\dagger. \quad (5.34)$$

Using (2.9) in (5.33) and (5.34), we have

$$\mathcal{A}(x, 0) = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho A^\dagger, \quad (5.35)$$

$$\mathcal{A}_y(x, 0) = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho B^\dagger. \quad (5.36)$$

Hence, from (5.35) and (5.36) we obtain

$$- \mathcal{A}_y(x, 0)A + \mathcal{A}(x, 0)B = \int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\rho (-B^\dagger A + A^\dagger B). \quad (5.37)$$

Using (2.6) in (5.37) we get (5.19). This completes the proofs of (a) and (b). Since $\varphi(k, x)$ satisfies the initial conditions (2.9) and $\tilde{\varphi}(k, x)$ satisfies the initial conditions (5.14), we obtain (5.20) by evaluating (5.15) and its x -derivative at $x = 0$. Thus, the proof of (c) is also complete. $\square$

We have the following remark. Recall that an input data set $\mathbf{D}$ consists of the potential and the boundary condition, or equivalently it is described by the set $\{V, A, B\}$. The perturbation of the potential is expressed in (5.18), and the perturbation of the boundary matrices A and B are expressed in (5.20). The perturbation of the wavefunction is given by (5.15), and as the Gel'fand–Levitan method indicates it is appropriate to describe the perturbation at the wavefunction level by providing the perturbation of the regular solution given in (5.15). This is because the regular solution is the wavefunction directly related to the input data set, as it is related to the potential $V(x)$ through the Schrödinger equation (5.12) and the boundary matrices A and B through the initial conditions (2.9). In particular, in the evaluation of the transformations on the half line it is appropriate to describe how the regular solution $\varphi(k, x)$ is transformed. For the half-line Schrödinger equation, the Jost solution $f(k, x)$ is not the appropriate wavefunction to express the perturbation at the wavefunction level. This is because $f(k, x)$ is only affected by the potential $V(x)$ but not by the boundary matrices A and B .

In the next theorem we show that the quantity $\mathcal{A}(x, y)$ appearing in (5.15), (5.17), and (5.18) satisfies the matrix-valued Gel'fand–Levitan system of integral equations.

Theorem 5.3. Assume that the unperturbed and perturbed potentials V and $\tilde{V}$ appearing in (5.12) and (5.13), respectively, both satisfy (2.2) and belong to $L_1^1(\mathbb{R}^+)$. Furthermore, assume that the unperturbed boundary matrix pair (A, B) and the perturbed boundary matrix pair $(\tilde{A}, \tilde{B})$ appearing in (2.9) and (5.14), respectively, both yield a selfadjoint boundary condition described by (2.5) with the restrictions indicated

in (2.6) and (2.7). Let the perturbed regular solution $\tilde{\varphi}(k, x)$ be related to the unperturbed regular solution $\varphi(k, x)$ through the $n \times n$ matrix $\mathcal{A}(x, y)$ as in (5.15). Then, $\mathcal{A}(x, y)$ satisfies the Gelfand–Levitan system of integral equations given by

$$\mathcal{A}(x, y) + G(x, y) + \int_0^x dz \mathcal{A}(x, z) G(z, y) = 0, \quad 0 \leq y < x, \quad (5.38)$$

where we have defined

$$G(x, y) := \int_{\lambda \in \mathbb{R}} \varphi(k, x) [d\tilde{\rho} - d\rho] \varphi(k, y)^\dagger, \quad (5.39)$$

with $d\rho$ and $d\tilde{\rho}$ denoting the unperturbed and perturbed spectral measures, respectively.

Proof. Let us multiply (5.15) on the right by $d\tilde{\rho} \varphi(k, z)^\dagger$ and integrate over $\lambda \in \mathbb{R}$, where we recall that $\lambda := k^2$. We get

$$\int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\tilde{\rho} \varphi(k, z)^\dagger = \int_{\lambda \in \mathbb{R}} \varphi(k, x) d\tilde{\rho} \varphi(k, z)^\dagger + \int_0^x dy \mathcal{A}(x, y) \int_{\lambda \in \mathbb{R}} \varphi(k, y) d\tilde{\rho} \varphi(k, z)^\dagger. \quad (5.40)$$

We have

$$\int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\tilde{\rho} \varphi(k, z)^\dagger = 0, \quad 0 \leq z < x, \quad (5.41)$$

which is obtained with the help of Parseval's equality given by

$$\int_{\lambda \in \mathbb{R}} \tilde{\varphi}(k, x) d\tilde{\rho} \tilde{\varphi}(k, z)^\dagger = \delta(x - z) I,$$

and by using the fact that for each fixed $k \in \mathbb{C}$ and $x \in \mathbb{R}^+$ we can express $\varphi(k, z)$ in terms of the functions in the set $\{\tilde{\varphi}(k, y) : y \in [0, z]\}$. To prove this we remark that we can consider (5.15) as an integral equation where $\varphi(k, x)$ is the unknown quantity and $\tilde{\varphi}(k, x)$ is the nonhomogeneous term. We further observe that the integral operator on the right-hand side of (5.15) is a Volterra integral operator, and hence the integral Eq. (5.15) can be inverted into an integral equation of a similar form, which is given by

$$\varphi(k, x) = \tilde{\varphi}(k, x) + \int_0^x dy \tilde{\mathcal{A}}(x, y) \tilde{\varphi}(k, y),$$

where the quantity $\tilde{\mathcal{A}}(x, y)$ corresponds to the kernel, $\varphi(k, x)$ is the nonhomogeneous term, and $\tilde{\varphi}(k, x)$ is the unknown quantity. We write the first integral on the right-hand side of (5.40) as

$$\int_{\lambda \in \mathbb{R}} \varphi(k, x) d\tilde{\rho} \varphi(k, z)^\dagger = \int_{\lambda \in \mathbb{R}} \varphi(k, x) [d\tilde{\rho} - d\rho] \varphi(k, z)^\dagger + \int_{\lambda \in \mathbb{R}} \varphi(k, x) d\rho \varphi(k, z)^\dagger. \quad (5.42)$$

Using (5.2) and (5.39) on the right-hand side of (5.42), we obtain

$$\int_{\lambda \in \mathbb{R}} \varphi(k, x) d\tilde{\rho} \varphi(k, z)^\dagger = G(x, z) + \delta(x - z) I. \quad (5.43)$$

Thus, from (5.43) we get

$$\int_{\lambda \in \mathbb{R}} \varphi(k, x) d\tilde{\rho} \varphi(k, z)^\dagger = G(x, z), \quad 0 \leq z < x. \quad (5.44)$$

The second integral on the right-hand side of (5.40) can be written as

$$\int_0^x dy \mathcal{A}(x, y) \int_{\lambda \in \mathbb{R}} \varphi(k, y) d\tilde{\rho} \varphi(k, z)^\dagger = \int_0^x dy \mathcal{A}(x, y) \int_{\lambda \in \mathbb{R}} \varphi(k, y) [d\tilde{\rho} - d\rho] \varphi(k, z)^\dagger + \int_0^x dy \mathcal{A}(x, y) \int_{\lambda \in \mathbb{R}} \varphi(k, y) d\rho \varphi(k, z)^\dagger. \quad (5.45)$$

Using (5.2) in the second integral and using (5.39) in the first integral on the right-hand side of (5.45), we obtain

$$\int_0^x dy \mathcal{A}(x, y) \int_{\lambda \in \mathbb{R}} \varphi(k, y) d\tilde{\rho} \varphi(k, z)^\dagger = \int_0^x dy \mathcal{A}(x, y) G(y, z) + \mathcal{A}(x, z), \quad 0 \leq z < x. \quad (5.46)$$

Thus, using (5.41), (5.44), and (5.46) in (5.40) we get (5.38). $\square$

VI. THE TRANSFORMATION TO REMOVE A BOUND STATE

We recall that we consider the half-line matrix Schrödinger operator described by (2.1) and (2.5), and we change the unperturbed problem into the perturbed problem without changing the continuous spectrum but only by adding or removing a finite number of bound states or by decreasing or increasing the multiplicities of the bound states. We are interested in finding the transformations describing the change in the potential, the boundary matrices, the regular solution, and in other relevant quantities. In this case, it is convenient to use (5.15), (5.18), and (5.20) to describe the transformations for the regular solution, the potential, and the boundary matrices, respectively, when the spectral measures $d\tilde{\rho}$ and $d\rho$ differ from each other only in the bound states. Since those transformations involve the quantity $\mathcal{A}(x, y)$ appearing in the Gel'fand–Levitan system (5.38) with the input given in (5.39), it is natural to use the Gel'fand–Levitan method to determine the relevant transformations.

Since the removal or addition of bound states or changing their multiplicities can be performed in succession, there is no loss of generality in considering only the case when one bound state is removed or added or when its multiplicity is decreased or increased. In this section, we consider the case where a bound state is completely removed from the spectrum of the half-line Schrödinger operator associated with (2.1) and (2.5).

Let us start with the unperturbed Schrödinger operator described by (2.1) and (2.5), where the potential V satisfies (2.2) and belongs to $L_1^1(\mathbb{R}^+)$ and the boundary matrices A and B satisfy (2.5)–(2.7). Thus, for the unperturbed problem, we have the potential $V(x)$, the boundary matrices A and B , the regular solution $\varphi(k, x)$, the spectral measure $d\rho$, the Jost matrix $J(k)$, the scattering matrix $S(k)$, and N bound states with energies $-\kappa_j^2$, the Gel'fand–Levitan normalization matrices C_j , the orthogonal projections Q_j onto $\text{Ker}[J(i\kappa_j)]$, the orthogonal projections P_j onto $\text{Ker}[J(i\kappa_j)]^\dagger$, the Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$, the Marchenko normalized bound-state solutions $\Psi_j(x)$, the dependency matrices D_j , and the multiplicities m_j of the bound states for $1 \leq j \leq N$.

We would like to remove any one of the bound states at $\lambda = \lambda_j$, or equivalently at $k = i\kappa_j$, with the Gel'fand–Levitan normalization matrix C_j . Since we do not order the distinct positive constants κ_j in any increasing or decreasing manner, without loss of generality we can assume that we remove the bound state with $k = i\kappa_N$ and the Gel'fand–Levitan normalization matrix C_N . We choose the Gel'fand–Levitan kernel $G(x, y)$ given in (5.39) as

$$G(x, y) = -\varphi(i\kappa_N, x) C_N^2 \varphi(i\kappa_N, y)^\dagger. \quad (6.1)$$

Next, we use the input (6.1) in the Gel'fand–Levitan system (5.38) and we obtain its solution $\mathcal{A}(x, y)$. With the help of $\mathcal{A}(x, y)$ we form the perturbed Schrödinger operator with the potential $\tilde{V}(x)$, the boundary matrices $\tilde{A}$ and $\tilde{B}$, the regular solution $\tilde{\varphi}(k, x)$, the spectral measure $d\tilde{\rho}$, the Jost matrix $\tilde{J}(k)$, the scattering matrix $\tilde{S}(k)$, and $N - 1$ bound states with energies $-\kappa_j^2$, the Gel'fand–Levitan normalization matrices $\tilde{C}_j$, the orthogonal projections $\tilde{Q}_j$ onto $\text{Ker}[\tilde{J}(i\kappa_j)]$, the orthogonal projections $\tilde{P}_j$ onto $\text{Ker}[\tilde{J}(i\kappa_j)]^\dagger$, the Gel'fand–Levitan normalized bound-state solutions $\tilde{\Phi}_j(x)$, the Marchenko normalized bound-state solutions $\tilde{\Psi}_j(x)$, the dependency matrices $\tilde{D}_j$, and the multiplicities m_j of the bound states for $1 \leq j \leq N - 1$. Our goal is to express the perturbed quantities distinguished with a tilde in their notations in terms of the unperturbed quantities not containing a tilde and the perturbation identified by κ_N and C_N .

In the next theorem, we present the solution to the Gel'fand–Levitan system (5.38) with the input (6.1).

Theorem 6.1. *Consider the unperturbed Schrödinger operator associated with (2.1) and (2.5), where the potential V satisfies (2.2) and belongs to $L_1^1(\mathbb{R}^+)$ and the boundary matrices A and B appearing in (2.5) satisfy (2.6) and (2.7). Let $\varphi(k, x)$ be the corresponding regular solution satisfying the initial conditions (2.9), and each of $k = i\kappa_j$ for $1 \leq j \leq N$ correspond to the bound state with the energy $-\kappa_j^2$, the Gel'fand–Levitan normalization matrix C_j , the orthogonal projection Q_j onto $\text{Ker}[J(i\kappa_j)]$, and the Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$. If the kernel $G(x, y)$ of the matrix-valued Gel'fand–Levitan system (5.38) is chosen as in (6.1), then the solution $\mathcal{A}(x, y)$ to (5.38) is given by*

$$\mathcal{A}(x, y) = \Phi_N(x) W_N(x)^+ \Phi_N(y)^\dagger, \quad 0 \leq y < x, \quad (6.2)$$

with $W_N(x)$ defined as

$$W_N(x) := \int_x^\infty dz \Phi_N(z)^\dagger \Phi_N(z), \quad (6.3)$$

where we recall that $W_N(x)^+$ denotes the Moore–Penrose inverse of $W_N(x)$.

Proof. From (3.20) we see that (6.1) is equivalent to

$$G(x, y) = -\Phi_N(x) \Phi_N(y)^\dagger. \quad (6.4)$$

Using the input (6.4) in (5.38), we write the Gel'fand–Levitan system of integral equations as

$$\mathcal{A}(x, y) - \Phi_N(x) \Phi_N(y)^\dagger - \int_0^x dz \mathcal{A}(x, z) \Phi_N(z) \Phi_N(y)^\dagger = 0, \quad 0 \leq y \leq x. \quad (6.5)$$

The specific y -dependence of the second and third terms in (6.5) indicates that the solution $\mathcal{A}(x, y)$ to (6.5) must have the form

$$\mathcal{A}(x, y) = \alpha_N(x) \Phi_N(y)^\dagger, \quad (6.6)$$

for some $\alpha_N(x)$ satisfying

$$\alpha_N(x) = \alpha_N(x) Q_N. \quad (6.7)$$

Using (6.6) in (6.5), we get

$$\alpha_N(x) - \Phi_N(x) - \alpha_N(x) \int_0^x dz \Phi_N(z)^\dagger \Phi_N(z) = 0,$$

or equivalently

$$\alpha_N(x) \left[Q_N - \int_0^x dz \Phi_N(z)^\dagger \Phi_N(z) \right] = \Phi_N(x). \quad (6.8)$$

From (3.21) and (6.3) we see that we can write $W_N(x)$ also as

$$W_N(x) = Q_N - \int_0^x dz \Phi_N(z)^\dagger \Phi_N(z). \quad (6.9)$$

Hence, (6.8) is equivalent to

$$\alpha_N(x) W_N(x) = \Phi_N(x),$$

from which we get

$$\alpha_N(x) W_N(x) W_N(x)^+ = \Phi_N(x) W_N(x)^+. \quad (6.10)$$

Since Q_N is the orthogonal projection onto $\text{Ker}[J(i\kappa_N)]$, with the help of (3.10) and (3.20) we can express $W_N(x)$ as a direct sum and obtain

$$W_N(x) = [W_N(x)]_1 \oplus 0, \quad (6.11)$$

where we use $[W_N(x)]_1$ to denote the restriction of $W_N(x)$ to $\text{Ker}[J(i\kappa_N)]$. We would like to prove that $[W_N(x)]_1$ is invertible for every $x \geq 0$, and for this we proceed as follows. If $[W_N(x)]_1$ were not invertible, then we would have a nonzero column vector v in $\text{Ker}[J(i\kappa_N)]$ satisfying $W_N(x)v = 0$, which would yield $v^\dagger W_N(x)v = 0$. From (6.3) we would obtain

$$v^\dagger W_N(x) v = \int_x^\infty dy |\Phi_N(y) v|^2, \quad (6.12)$$

and hence we see that the left-hand side of (6.12) would be equal to zero if and only

$$\Phi_N(y) v = 0, \quad y \geq x. \quad (6.13)$$

On the other hand, since $\Phi_N(x)v$ is a column-vector solution to (2.1), the equality in (6.13) would hold if and only if we had

$$\Phi_N(x) v = 0, \quad x \geq 0. \quad (6.14)$$

However, (6.14) would imply that

$$\Phi_N(0) v = 0, \quad \Phi_N'(0) v = 0. \quad (6.15)$$

From (2.9), (3.27), and the second equality of (3.28), we observe that (6.15) would imply

$$A H_N^{-1/2} Q_N v = 0, \quad B H_N^{-1/2} Q_N v = 0,$$

which in turn would yield

$$(A^\dagger A + B^\dagger B) H_N^{-1/2} Q_N v = 0. \quad (6.16)$$

From (2.7) we know that the matrix $A^\dagger A + B^\dagger B$ is invertible and from Sec. III we know that $H_N^{-1/2}$ is invertible. Thus, (6.16) would yield $Q_N v = 0$. Since Q_N is the orthogonal projection onto $\text{Ker}[J(i\kappa_N)]$ the equality $Q_N v = 0$ would imply that $v = 0$. This would contradict the assumption that v is a nonzero vector in $\text{Ker}[J(i\kappa_N)]$. Thus, we have proved that $[W_N(x)]_1$ is invertible for every $x \geq 0$. We note that (3.5) and (6.11) imply that

$$W_N(x)^+ = ([W_N(x)]_1)^{-1} \oplus 0, \quad (6.17)$$

$$W_N(x) W_N(x)^+ = W_N(x)^+ W_N(x) = I \oplus 0, \quad (6.18)$$

and hence from (6.18) we conclude that

$$W_N(x) W_N(x)^+ = W_N(x)^+ W_N(x) = Q_N. \quad (6.19)$$

Using (6.19) in (6.10), with the help of (6.7), we get

$$\alpha_N(x) = \Phi_N(x) W_N(x)^+. \quad (6.20)$$

Finally, using (6.20) in (6.6), we obtain (6.2). $\square$

In the following theorem we present some relevant properties of the quantity $\mathcal{A}(x, y)$ appearing in (6.2).

Theorem 6.2. Consider the unperturbed Schrödinger operator associated with (2.1) and (2.5), where the potential V satisfies (2.2) and belongs to $L^1_1(\mathbb{R}^+)$ and the boundary matrices A and B appearing in (2.5) satisfy (2.6) and (2.7). Let $\varphi(k, x)$ be the corresponding regular solution satisfying the initial conditions (2.9), each of $k = ik_j$ for $1 \leq j \leq N$ correspond to the bound state with the energy $-\kappa_j^2$, the Gel'fand–Levitan normalization matrix C_j , the orthogonal projection Q_j onto $\text{Ker}[J(ik_j)]$, and the Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$. Let $\mathcal{A}(x, y)$ be the matrix-valued quantity given in (6.2). Let the matrix-valued perturbed potential $\tilde{V}(x)$ be given by

$$\tilde{V}(x) := V(x) + 2 \frac{d}{dx} \left[\Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger \right], \quad (6.21)$$

where $W_N(x)$ is the quantity defined in (6.3). We have the following:

- (a) The quantity $\mathcal{A}(x, y)$ is continuously differentiable in the set $\mathbb{D}$ defined in (5.16).
- (b) The second partial derivatives $\mathcal{A}_{xx}(x, y)$ and $\mathcal{A}_{yy}(x, y)$ are locally integrable in $\mathbb{D}$.
- (c) The quantity $\mathcal{A}(x, y)$ satisfies the second-order matrix-valued partial differential equation (5.17).
- (d) The perturbed potential $\tilde{V}$ defined in (6.21) is related to the unperturbed potential V and the quantity $\mathcal{A}(x, y)$ as in (5.18).

Proof. From Proposition 3.2.9 of Ref. 7 and (3.20) we see that $\Phi_N(x)$ is continuously differentiable on $[0, +\infty)$. Furthermore, $\Phi_N''(x)$ is locally integrable on $[0, +\infty)$ because $\Phi_N(x)$ is a solution to (3.11). Using (3.6), (6.11), and (6.17), we obtain the x -derivative of $W_N(x)^+$ as

$$[W_N(x)^+]^\dagger = W_N(x)^+ \Phi_N(x)^\dagger \Phi_N(x) W_N(x)^+. \quad (6.22)$$

With the help of (6.17) and (6.22), we see that $W_N(x)^+$ is continuously differentiable on $[0, +\infty)$. Hence, $\mathcal{A}(x, y)$ given in (6.2) is continuously differentiable on $\mathbb{D}$. This completes the proof of (a). For the proof of (b), we proceed as follows. By taking the derivative of both sides of (6.22) and using (6.22) in the resulting equation, we prove that $[W_N^+(x)]''$ is continuous on $[0, +\infty)$. Consequently, using (6.2) we conclude that the second derivatives $\mathcal{A}_{xx}(x, y)$ and $\mathcal{A}_{yy}(x, y)$ are locally integrable in $\mathbb{D}$. Thus, the proof of (b) is complete. The proof of (c) is obtained as follows. From (6.2), by taking the appropriate derivatives we obtain

$$\mathcal{A}_x(x, y) = [\Phi_N'(x) W_N(x)^+ + \Phi_N(x) (W_N(x)^+)'] \Phi_N(y)^\dagger,$$

$$\mathcal{A}_y(x, y) = \Phi_N(x) W_N(x)^+ \Phi_N'(y)^\dagger,$$

$$\mathcal{A}_{xx}(x, y) = [\Phi_N''(x) W_N(x)^+ + 2 \Phi_N'(x) (W_N(x)^+)'] \Phi_N(y)^\dagger, \quad (6.23)$$

$$\mathcal{A}_{yy}(x, y) = \Phi_N(x) W_N(x)^+ \Phi_N''(y)^\dagger. \quad (6.24)$$

Using (5.12), we can express each of (6.23) and (6.24) in their equivalent forms as

$$\mathcal{A}_{xx}(x, y) = [V(x) + \kappa_N^2] \Phi_N(x) W_N(x)^+ \Phi_N(y)^\dagger + [2 \Phi_N'(x) (W_N(x)^+)'] \Phi_N(y)^\dagger, \quad (6.25)$$

$$\mathcal{A}_{yy}(x, y) = \Phi_N(x) W_N(x)^+ \Phi_N'(y)^\dagger [V(y) + \kappa_N^2]. \quad (6.26)$$

From (6.2) and (6.21) we also get

$$\tilde{V}(x) \mathcal{A}(x, y) = \left[V(x) + 2 \left(\Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger \right)' \right] \Phi_N(x) W_N(x)^+ \Phi_N(y)^\dagger, \quad (6.27)$$

$$\mathcal{A}(x, y) V(y) = \Phi_N(x) W_N(x)^+ \Phi_N(y)^\dagger V(y). \quad (6.28)$$

From (6.26) and (6.28) we obtain

$$-\mathcal{A}_{yy}(x, y) + \mathcal{A}(x, y) V(y) = -\kappa_N^2 \Phi_N(x) W_N(x)^+ \Phi_N(y)^\dagger.$$

Using (6.25), (6.27), and (6.28), we observe that (5.17) holds provided we have

$$2\Phi'_N(x) (W_N(x)^+)' + \Phi_N(x) (W_N(x)^+)'' = 2 [\Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger]' \Phi_N(x) W_N(x)^+. \quad (6.29)$$

We prove (6.29) by using (6.22) and the identity

$$\Phi'_N(x)^\dagger \Phi_N(x) - \Phi_N(x)^\dagger \Phi'_N(x) \equiv 0. \quad (6.30)$$

The proof of (6.30) can be given as follows. From (3.20) and (5.12) we get

$$\Phi''_N(x) = [V(x) + \kappa_N^2] \Phi_N(x). \quad (6.31)$$

By taking the x -derivative of the left-hand side of (6.30) and using (6.31), we establish that the left-hand side of (6.30) is constant, and hence that constant value can be evaluated at $x = 0$. For this, we proceed as follows. With the help of (2.9) and (3.20) we get

$$\Phi_N(0) = A C_N, \quad \Phi'_N(0) = B C_N. \quad (6.32)$$

Next, using (6.32) and the fact that C_N is selfadjoint, we evaluate the left-hand side of (6.30) as

$$\Phi'_N(0)^\dagger \Phi_N(0) - \Phi_N(0)^\dagger \Phi'_N(0) = C_N (B^\dagger A - A^\dagger B) C_N. \quad (6.33)$$

Using (2.6) on the right-hand side of (6.33) we obtain

$$\Phi'_N(0)^\dagger \Phi_N(0) - \Phi_N(0)^\dagger \Phi'_N(0) = 0,$$

which ensures that (6.30) holds. With the help of (6.22) and (6.30), we verify that (6.29) indeed holds, and hence $\mathcal{A}(x, y)$ satisfies (5.17). Thus, the proof of (c) is complete. The proof of (d) follows by comparing (5.18) and (6.21) and by using the fact that the two expressions agree when we evaluate $\mathcal{A}(x, x)$ by letting $y = x^-$ in (6.2). Hence, the proof is complete. $\square$

In the next theorem we show that the quantity $\tilde{\varphi}(k, x)$ in (5.15) with $\mathcal{A}(x, y)$ as in (6.2) is the perturbed regular solution to (2.1) with the potential $\tilde{V}(x)$ in (6.21) and with an appropriate pair of boundary matrices $\tilde{A}$ and $\tilde{B}$.

Theorem 6.3. Consider the unperturbed Schrödinger operator associated with (2.1) and (2.5), where the potential V satisfies (2.2) and belongs to $L^1_+(\mathbb{R}^+)$ and the boundary matrices A and B appearing in (2.5) satisfy (2.6) and (2.7). Let $\varphi(k, x)$ be the corresponding regular solution satisfying the initial conditions (2.9), each of $k = i\kappa_j$ for $1 \leq j \leq N$ correspond to the bound state with the energy $-\kappa_j^2$, the Gel'fand–Levitan normalization matrix C_j , the orthogonal projection Q_j onto $\text{Ker}[J(i\kappa_j)]$, and the Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$. Let $\tilde{\varphi}(k, x)$ be the quantity given in (5.15) with $\mathcal{A}(x, y)$ expressed as in (6.2). Then, we have the following:

- (a) The quantity $\tilde{\varphi}(k, x)$ is a solution to the perturbed Schrödinger equation (2.1) with the potential $\tilde{V}(x)$ given in (6.21). We can express $\tilde{\varphi}(k, x)$ also as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \Phi_N(x) W_N(x)^+ \int_0^x dy \Phi_N(y)^\dagger \varphi(k, y). \quad (6.34)$$

- (b) The perturbed potential $\tilde{V}(x)$ given in (6.21) satisfies (2.2). Moreover, the potential increment $\tilde{V}(x) - V(x)$ has the asymptotic behavior

$$\tilde{V}(x) - V(x) = O(q_6(x)), \quad x \rightarrow +\infty, \quad (6.35)$$

where we have defined $q_6(x)$ as

$$q_6(x) := \int_x^\infty dy |V(y)|. \quad (6.36)$$

If the unperturbed potential V is further restricted to $L^1_{1+\epsilon}(\mathbb{R}^+)$ for some fixed $\epsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L^1_\epsilon(\mathbb{R}^+)$.

- (c) If the unperturbed potential $V(x)$ is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (6.37)$$

for some positive constants α and x_0 and with c denoting a generic constant, then the potential increment $\tilde{V}(x) - V(x)$ also satisfies

$$|\tilde{V}(x) - V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0. \quad (6.38)$$

- (d) If the support of the unperturbed potential V is contained in the interval $[0, x_0]$ for some positive x_0 , then the support of the perturbed potential $\tilde{V}$ is also contained in $[0, x_0]$.

(e) For $k \neq i\kappa_N$, the perturbed quantity $\tilde{\varphi}(k, x)$ can be expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_N(x) W_N(x)^+ \left[\Phi_N'(x)^\dagger \varphi(k, x) - \Phi_N(x)^\dagger \varphi'(k, x) \right]. \quad (6.39)$$

(f) The perturbed quantity $\tilde{\varphi}(k, x)$ satisfies the initial conditions (5.14) where the matrices $\tilde{A}$ and $\tilde{B}$ are expressed in terms of the unperturbed boundary matrices A and B and the Gel'fand–Levitan normalization matrix C_N for the bound state at $k = i\kappa_N$ as

$$\tilde{A} = A, \quad \tilde{B} = B + A C_N^2 A^\dagger A. \quad (6.40)$$

(g) The matrices $\tilde{A}$ and $\tilde{B}$ appearing in (6.40) satisfy (2.6) and (2.7). Hence, as a consequence of (a) and (f), the quantity $\tilde{\varphi}(k, x)$ corresponds to the regular solution to the matrix Schrödinger equation with the potential $\tilde{V}(x)$ in (6.21) and with the selfadjoint boundary condition (2.5) with A and B there replaced with $\tilde{A}$ and $\tilde{B}$, respectively.

Proof. The proof of (a) is obtained by using Theorems 5.2 and 6.2. For the proof of (b) we proceed as follows. We recall that $V(x)$ satisfies (2.2). Furthermore, from (6.3) we know that $W_N(x)$ is Hermitian. Hence, from (6.4) we see that $\tilde{V}(x)$ also satisfies (2.2). The proof of (6.35) is obtained as follows. From (3.2.3) and (3.2.30) of Ref. 7 we conclude that the unperturbed Jost solution $f(k, x)$ satisfies

$$f(k, x) = e^{ikx} [I + O(q_6(x))], \quad x \rightarrow +\infty. \quad (6.41)$$

Similarly, with the help of (3.2.7) of Ref. 7 we obtain

$$f'(k, x) = ik e^{ikx} [I + O(q_6(x))], \quad x \rightarrow +\infty. \quad (6.42)$$

With the help of (4.1), (4.2), (4.14), (6.41), (6.42), and the first equality in (3.28), we get

$$\Phi_N(x) = e^{-\kappa_N x} [I + O(q_6(x))] \mathbf{B}_N^{-1/2} D_N Q_N, \quad x \rightarrow +\infty, \quad (6.43)$$

$$\Phi_N'(x) = -\kappa_N e^{-\kappa_N x} [(I + O(q_6(x))) \mathbf{B}_N^{-1/2} D_N Q_N], \quad x \rightarrow +\infty. \quad (6.44)$$

We have the estimate

$$\int_x^\infty dy e^{-2\kappa_N y} O(q_6(y)) \leq c \int_x^\infty dy e^{-2\kappa_N y} q_6(y), \quad x \rightarrow +\infty, \quad (6.45)$$

for some generic constant c . Using integration by parts on the right-hand side of (6.45), we get

$$\int_x^\infty dy e^{-2\kappa_N y} q_6(y) = \frac{1}{2\kappa_N} \left[e^{-2\kappa_N x} q_6(x) - \int_x^\infty dy e^{-2\kappa_N y} |V(y)| \right]. \quad (6.46)$$

Using (6.45) and (6.46) we conclude that

$$\int_x^\infty dy e^{-2\kappa_N y} O(q_6(y)) \leq \frac{c}{2\kappa_N} e^{-2\kappa_N x} q_6(x), \quad x \rightarrow +\infty. \quad (6.47)$$

With the help of (6.3), (6.11), (6.43), and (6.47), we obtain

$$W_N(x) = \frac{1}{2\kappa_N} e^{-2\kappa_N x} Q_N \left[D_N^\dagger \mathbf{B}_N^{-1} D_N + O(q_6(x)) \right] Q_N \oplus 0, \quad x \rightarrow +\infty, \quad (6.48)$$

which yields

$$W_N(x) = \frac{1}{2\kappa_N} e^{-2\kappa_N x} Q_N \left[D_N^\dagger \mathbf{B}_N^{-1} D_N + O(q_6(x)) \right] Q_N. \quad x \rightarrow +\infty. \quad (6.49)$$

From (4.14) we conclude that the $n \times n$ matrix $D_N^\dagger \mathbf{B}_N^{-1} D_N$ has the decomposition analogous to the decomposition in (3.10). Let us use $[D_N^\dagger \mathbf{B}_N^{-1} D_N]_1$ to denote the restriction of $D_N^\dagger \mathbf{B}_N^{-1} D_N$ to the subspace $Q_N \mathbb{C}^n$. From Theorem 4.3(a) we know that D_N is an isometry from $\text{Ker}[J(i\kappa_N)]$ to $\text{Ker}[J(i\kappa_N)^\dagger]$, and hence it is an isometry from $Q_N \mathbb{C}^n$ to $P_N \mathbb{C}^n$. Similarly, from Theorem 4.3(b) we know that $D_N^\dagger$ is an isometry from $\text{Ker}[J(i\kappa_N)^\dagger]$ to $\text{Ker}[J(i\kappa_N)]$ and hence it is an isometry from $P_N \mathbb{C}^n$ to $Q_N \mathbb{C}^n$. We also already know that $\mathbf{B}_N^{-1}$ is invertible and commutes with P_N . Thus, $D_N^\dagger \mathbf{B}_N^{-1} D_N$ is an invertible map from $Q_N \mathbb{C}^n$ to $Q_N \mathbb{C}^n$. Consequently, $[D_N^\dagger \mathbf{B}_N^{-1} D_N]_1$ is invertible in $Q_N \mathbb{C}^n$. Hence, we have

$$\left(D_N^\dagger \mathbf{B}_N^{-1} D_N \right)^+ = \left([D_N^\dagger \mathbf{B}_N^{-1} D_N]_1 \right)^{-1} \oplus 0. \quad (6.50)$$

With the help of (6.11), (6.17), (6.49), and (6.50), we get

$$W_N(x)^+ = 2\kappa_N e^{2\kappa_N x} Q_N \left[[D_N^\dagger \mathbf{B}^{-1} D_N]_1^{-1} + O(q_6(x)) \right] Q_N \oplus 0, \quad x \rightarrow +\infty. \quad (6.51)$$

Using (6.50) in (6.51), after some simplification we obtain

$$W_N(x)^+ = 2\kappa_N e^{2\kappa_N x} \left[\left(D_N^\dagger \mathbf{B}^{-1} D_N \right)^+ + O(q_6(x)) \right], \quad x \rightarrow +\infty. \quad (6.52)$$

Next, using (6.22), (6.49), (6.52), and the second equality of (3.2), we get

$$[W_N(x)^+]^+ = 4\kappa_N^2 e^{2\kappa_N x} \left[\left(D_N^\dagger \mathbf{B}_N^{-1} D_N \right)^+ + O(q_6(x)) \right], \quad x \rightarrow +\infty. \quad (6.53)$$

Then, with the help of (6.43), (6.44), (6.52), and (6.53), we obtain

$$2 \frac{d}{dx} [\Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger] = O(q_6(x)), \quad x \rightarrow +\infty. \quad (6.54)$$

Using (6.54), from (6.21) we observe that the proof of (6.35) is complete. For any $\epsilon \geq 0$, we have

$$\int_0^\infty dx (1+x)^\epsilon q_6(x) = \frac{1}{1+\epsilon} \int_0^\infty dx ((1+x)^{1+\epsilon} - 1) |V(x)|. \quad (6.55)$$

Using (6.21), (6.54), and (6.55), we conclude that $\tilde{V}$ belongs to $L^1_e(\mathbb{R}^+)$ if V belongs to $L^1_{1+\epsilon}(\mathbb{R}^+)$. This completes the proof of (b). For the proof of (c) we proceed as follows. If (6.37) holds, then (6.43) and (6.44) imply the respective asymptotics

$$\Phi_N(x) = e^{-\kappa_N x} [I + O(e^{-\alpha x})] \mathbf{B}_N^{-1/2} D_N Q_N, \quad x \rightarrow +\infty, \quad (6.56)$$

$$\Phi'_N(x) = -\kappa_N e^{-\kappa_N x} [I + O(e^{-\alpha x})] \mathbf{B}_N^{-1/2} D_N Q_N, \quad x \rightarrow +\infty. \quad (6.57)$$

Comparing (6.56) and (6.57) with (6.43) and (6.44), respectively, we observe that we can repeat the argument in (6.48)–(6.54) by using $O(e^{-\alpha x})$ in place of $O(q_6(x))$. Consequently, in analogy with (6.54) we get

$$\left| 2 \frac{d}{dx} [\Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger] \right| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (6.58)$$

where c is a generic constant. Then, using (6.21), (6.37), and (6.58), we obtain (6.38). Thus, the proof of (c) is complete. To prove (d), we proceed as follows. If $V(x) = 0$ for $x \geq x_0$, then we have

$$\Phi_N(x) = R e^{-\kappa_N x}, \quad x \geq x_0, \quad (6.59)$$

where R is a constant $n \times n$ matrix. Using (6.59) in (6.3) we get

$$W_N(x) = \frac{1}{2\kappa_N} R^\dagger R e^{-2\kappa_N x}, \quad x \geq x_0,$$

which yields

$$W_N(x)^+ = 2\kappa_N (R^\dagger R)^+ e^{2\kappa_N x}, \quad x \geq x_0. \quad (6.60)$$

From (6.59) and (6.60) we have

$$\Phi_N(x)^\dagger W_N(x)^+ \Phi_N(x) = 2\kappa_N R^\dagger (R^\dagger R)^+ R, \quad x \geq x_0. \quad (6.61)$$

Since the right-hand side in (6.61) is independent of x , using (6.61) in (6.21) we obtain

$$\tilde{V}(x) = V(x), \quad x \geq x_0. \quad (6.62)$$

We see that (d) follows from (6.62). For the proof of (e) we proceed as follows. From (5.12) and (6.31), we respectively have

$$-\Phi_N(x)^\dagger \varphi''(k, x) + \Phi_N(x)^\dagger V(x) \varphi(k, x) = k^2 \Phi_N(x)^\dagger \varphi(k, x), \quad (6.63)$$

$$-\Phi''_N(x)^\dagger \varphi(k, x) + \Phi_N(x)^\dagger V(x) \varphi(k, x) = -\kappa_N^2 \Phi_N(x)^\dagger \varphi(k, x). \quad (6.64)$$

From (6.63) and (6.64), we get

$$\frac{d}{dx} [\Phi'_N(x)^\dagger \varphi(k, x) - \Phi_N(x)^\dagger \varphi'(k, x)] = (k^2 + \kappa_N^2) \Phi_N(x)^\dagger \varphi(k, x). \quad (6.65)$$

By integrating (6.65), we obtain

$$\Phi'_N(x)^\dagger \varphi(k, x) - \Phi_N(x)^\dagger \varphi'(k, x) - \Phi'_N(0)^\dagger \varphi(k, 0) + \Phi_N(0)^\dagger \varphi'(k, 0) = (k^2 + \kappa_N^2) \int_0^x dy \Phi_N(y)^\dagger \varphi(k, y). \quad (6.66)$$

Using (2.9) and (3.20), with the help of (2.6) we get

$$\Phi'_N(0)^\dagger \varphi(k, 0) - \Phi_N(0)^\dagger \varphi'(k, 0) = 0. \quad (6.67)$$

From (6.66) and (6.67) we have

$$\int_0^x dy \Phi_N(y)^\dagger \varphi(k, y) = \frac{1}{k^2 + \kappa_N^2} [\Phi'_N(x)^\dagger \varphi(k, x) - \Phi_N(x)^\dagger \varphi'(k, x)]. \quad (6.68)$$

Finally, using (6.68) in (6.34) we obtain (6.39). Thus, the proof of (e) is complete. To prove (f), we proceed as follows. Using (3.21) when $j = N$, from (6.3) we get

$$W_N(0) = Q_N, \quad (6.69)$$

which implies

$$W_N(0)^+ = Q_N, \quad (6.70)$$

based on the fact that Q_N is its own Moore–Penrose inverse. Evaluating both sides of (6.39) at $x = 0$, with the help of the first equality of (2.9), the first equality of (5.14), and (6.67) we establish the first equality of (6.40). Taking the derivative of both sides of (6.39), we get

$$\tilde{\varphi}'(k, x) = \varphi'(k, x) + \Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger \varphi(k, x) + \frac{[\Phi_N(x) W_N(x)^+]'}{k^2 + \kappa_N^2} [\Phi'_N(x)^\dagger \varphi(k, x) - \Phi_N(x)^\dagger \varphi'(k, x)], \quad (6.71)$$

where we have used (6.65). By evaluating both sides of (6.71) at $x = 0$, with the help of the second equality of (2.9), the second equality of (5.14), (6.67), and (6.70), we get

$$\tilde{B} = B + A C_N Q_N C_N^\dagger A. \quad (6.72)$$

Using the properties of the Gel'fand–Levitan normalization matrix C_N given by

$$C_N Q_N = C_N, \quad C_N^\dagger = C_N,$$

on the right-hand side of (6.72), we obtain the second equality of (6.40). Hence, the proof of (f) is complete. For the proof of (g), we proceed as follows. Using (6.40), after some minor simplification we obtain

$$\tilde{A}^\dagger \tilde{B} - \tilde{B}^\dagger \tilde{A} = A^\dagger B - B^\dagger A. \quad (6.73)$$

The right-hand side of (6.73) vanishes as a result of (2.6). Hence, (2.6) is satisfied if we replace A and B by $\tilde{A}$ and $\tilde{B}$, respectively. For any column vector v in $\mathbb{C}^n$, we have

$$v^\dagger (\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B}) v = |Av|^2 + |Bv|^2. \quad (6.74)$$

The right-hand side of (6.74) shows that the matrix $\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B}$ is nonnegative. Hence, in order to prove that $\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B}$ is a positive matrix, it is enough to show that its kernel contains only the zero vector. We prove this as follows. From (6.40) we obtain

$$\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B} = A^\dagger A + B^\dagger B + (AC_N^\dagger A^\dagger A)^\dagger (AC_N^\dagger A^\dagger A) + B^\dagger (AC_N^\dagger A^\dagger A) + (AC_N^\dagger A^\dagger A)^\dagger B. \quad (6.75)$$

For any column vector v in $\mathbb{C}^n$, from (6.75) we get

$$v^\dagger (\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B}) v = |Av|^2 + |Bv|^2 + |(AC_N^\dagger A^\dagger A)v|^2 + 2 \operatorname{Re} [Bv, (AC_N^\dagger A^\dagger A)v], \quad (6.76)$$

where the last term on the right-hand side involves the real part of the standard scalar product. With the help of a Cauchy–Schwarz inequality, we have

$$-2 |Bv| |(AC_N^2 A^\dagger A)v| \leq 2 \operatorname{Re}[\langle Bv, (AC_N^2 A^\dagger A)v \rangle] \leq 2 |Bv| |(AC_N^2 A^\dagger A)v|. \quad (6.77)$$

From (6.77) we obtain

$$-2 \operatorname{Re}[\langle Bv, (AC_N^2 A^\dagger A)v \rangle] \geq -2 |Bv| |(AC_N^2 A^\dagger A)v|. \quad (6.78)$$

If $(\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B})v = 0$, then from (6.76) and (6.78) we get

$$|Av|^2 + \left(|Bv| - |(AC_N^2 A^\dagger A)v| \right)^2 \leq 0,$$

which implies that $Av = 0$ and $Bv = 0$. From (2.7) we know that $A^\dagger A + B^\dagger B > 0$, and hence the simultaneous equalities $Av = 0$ and $Bv = 0$ can hold only when v is the zero column vector. Thus, the proof of (g) is complete. $\square$

In the next theorem, we describe how the Jost matrix, the scattering matrix, and the Jost solution transform when we change the potential, the boundary matrices, and the regular solution as described in Theorem 6.3. The theorem also establishes the fact that such a transformation only removes the bound state with the energy $-\kappa_N^2$, leaves the remaining bound states and their multiplicities unchanged, and introduces no new bound states. We note that, in the theorem, we impose the stronger assumption that the unperturbed potential V belongs to $L_2^1(\mathbb{R}^+)$ rather than $L_1^1(\mathbb{R}^+)$ so that the perturbed potential $\tilde{V}$ belongs to $L_1^1(\mathbb{R}^+)$, which is ensured by Theorem 6.3(b). We remark that this is a sufficiency assumption because the asymptotic estimates in Theorem 6.3(b) on the potentials are not necessarily sharp.

Theorem 6.4. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_2^1(\mathbb{R}^+)$, the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), the regular solution $\varphi(k, x)$ satisfying the initial conditions (2.9), the Jost solution $f(k, x)$ satisfying (2.8), and with N bound states with the energies $-\kappa_j^2$, the Gel'fand–Levitan normalization matrices C_j , the orthogonal projections Q_j onto $\operatorname{Ker}[J(i\kappa_j)]$, the orthogonal projections P_j onto $\operatorname{Ker}[J(i\kappa_j)^\dagger]$, the Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$, the Marchenko normalized bound-state solutions $\Psi_j(x)$, the dependency matrices D_j , and the multiplicities m_j for the bound states for $1 \leq j \leq N$. Let us use a tilde to identify the quantities associated with the perturbed Schrödinger operator with the potential $\tilde{V}(x)$ expressed as in (6.21) and the boundary condition (2.5) where the boundary matrices A and B are replaced with $\tilde{A}$ and $\tilde{B}$, respectively, given in (6.40). Thus, we use $\tilde{f}(k, x)$ for the perturbed Jost solution satisfying the asymptotics (2.8), $\tilde{\varphi}(k, x)$ for the perturbed regular solution satisfying the initial conditions (5.14), $\tilde{J}(k)$ for the perturbed Jost matrix defined as

$$\tilde{J}(k) := \tilde{f}(-k^*, 0)^\dagger \tilde{B} - \tilde{f}'(-k^*, 0)^\dagger \tilde{A}, \quad k \in \overline{\mathbb{C}^+}, \quad (6.79)$$

and we use $\tilde{S}(k)$ for the perturbed scattering matrix defined as

$$\tilde{S}(k) := -\tilde{J}(-k) \tilde{J}(k)^{-1}, \quad k \in \mathbb{R}. \quad (6.80)$$

We then have the following:

- (a) The unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right] J(k), \quad k \in \overline{\mathbb{C}^+}. \quad (6.81)$$

- (b) The perturbation does not change the matrix product $J(k)^\dagger J(k)$ for $k \in \mathbb{R}$, i.e. we have

$$\tilde{J}(k)^\dagger \tilde{J}(k) = J(k)^\dagger J(k), \quad k \in \mathbb{R}, \quad (6.82)$$

and hence the continuous part of the spectral measure dp does not change under the perturbation.

- (c) Under the perturbation, the determinant of the Jost matrix is transformed as

$$\det[\tilde{J}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{m_N} \det[J(k)], \quad k \in \overline{\mathbb{C}^+}, \quad (6.83)$$

where we recall that m_N is the multiplicity of the bound state at $k = i\kappa_N$ for the unperturbed Schrödinger operator.

- (d) Under the perturbation, the bound state with the energy $-\kappa_N^2$ is removed without adding any new bound states in such a way that the remaining bound states with the energies $-\kappa_j^2$ and multiplicities m_j for $1 \leq j \leq N - 1$ are unchanged.

(e) The perturbed scattering matrix $\tilde{S}(k)$ is related to the unperturbed scattering matrix $S(k)$ as

$$\tilde{S}(k) = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_N \right] S(k) \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_N \right], \quad k \in \mathbb{R}. \quad (6.84)$$

(f) Under the perturbation, the determinant of the scattering matrix is transformed as

$$\det [\tilde{S}(k)] = \left(\frac{k - i\kappa_N}{k + i\kappa_N} \right)^{2m_N} \det [S(k)], \quad k \in \mathbb{R}. \quad (6.85)$$

(g) For $k \in \overline{\mathbb{C}^+}$, the perturbed Jost solution $\tilde{f}(k, x)$ is related to the unperturbed Jost solution $f(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_N(x) W_N(x)^+ q_7(k, x) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right], \quad (6.86)$$

where we have defined

$$q_7(k, x) := \Phi_N'(x)^\dagger f(k, x) - \Phi_N(x)^\dagger f'(k, x). \quad (6.87)$$

The transformation in (6.86) can equivalently be expressed as

$$\tilde{f}(k, x) = \left[f(k, x) - \Phi_N(x) W_N(x)^+ \int_x^\infty dy \Phi_N(y)^\dagger f(k, y) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right]. \quad (6.88)$$

Proof. From the x -derivative of (2.14) we have

$$\varphi'(k, x) = \frac{1}{2ik} [f'(k, x) J(-k) - f'(-k, x) J(k)], \quad k \in \mathbb{R}. \quad (6.89)$$

The analog of (2.14) for the perturbed problem yields

$$\tilde{\varphi}(k, x) = \frac{1}{2ik} [\tilde{f}(k, x) \tilde{J}(-k) - \tilde{f}(-k, x) \tilde{J}(k)], \quad k \in \mathbb{R}. \quad (6.90)$$

Using (2.14), (6.89), and (6.90) in (6.39), we obtain

$$\Delta(k, x) = \Delta(-k, x), \quad k \in \mathbb{R}, \quad (6.91)$$

where we have defined

$$\Delta(k, x) := \tilde{f}(k, x) \tilde{J}(-k) - f(k, x) J(-k) - \frac{\Phi_N(x) W_N(x)^+}{k^2 + \kappa_N^2} q_7(k, x) J(-k), \quad k \in \mathbb{R}. \quad (6.92)$$

As we will show below, from (6.92) we get

$$\Delta(k, x) = e^{ikx} q_8(-k) [I + o(1)], \quad x \rightarrow +\infty, \quad k \in \mathbb{R}, \quad (6.93)$$

where the $n \times n$ matrix $q_8(k)$ is defined as

$$q_8(k) := \tilde{J}(k) - J(k) - \frac{2i\kappa_N}{k - i\kappa_N} P_N J(k), \quad k \in \mathbb{R}. \quad (6.94)$$

Note that (6.91) and (6.93) yield

$$e^{ikx} q_8(-k) [I + o(1)] = e^{-ikx} q_8(k) [I + o(1)], \quad x \rightarrow +\infty, \quad k \in \mathbb{R},$$

from which we conclude that $q_8(k) \equiv 0$. Hence, (6.94) yields (6.81) for $k \in \mathbb{R}$. Using the fact⁷ that the Jost matrix $J(k)$ has an analytic extension from $k \in \mathbb{R}$ to the upper-half complex plane $\mathbb{C}^+$, we conclude that (6.81) holds for $k \in \overline{\mathbb{C}^+}$. Thus, the proof of (a) is complete, provided we show that (6.93) holds with $q_8(k)$ as in (6.94). To establish (6.93) we proceed as follows. From (4.1), when $j = N$ we get

$$\Phi_N(x) = \Psi_N(x) D_N, \quad (6.95)$$

and from the first equality of (3.28) we have

$$\Psi_N(x) = f(i\kappa_N, x) \mathbf{B}_N^{-1/2} P_N. \quad (6.96)$$

Using (6.96) in (6.95) we obtain

$$\Phi_N(x) = f(i\kappa_N, x) \mathbf{B}_N^{-1/2} P_N D_N, \quad (6.97)$$

and using (2.8) in (6.97) we get

$$\Phi_N(x) = e^{-\kappa_N x} \mathbf{B}_N^{-1/2} P_N [I + o(1)], \quad x \rightarrow +\infty, \quad (6.98)$$

$$\Phi'_N(x) = e^{-\kappa_N x} \mathbf{B}_N^{-1/2} P_N D_N [-\kappa_N I + o(1)], \quad x \rightarrow +\infty. \quad (6.99)$$

By taking the matrix adjoints, from (6.98) and (6.99) we obtain the respective asymptotics

$$\Phi_N(x)^\dagger = e^{-\kappa_N x} D_N^\dagger P_N \mathbf{B}_N^{-1/2} [I + o(1)], \quad x \rightarrow +\infty, \quad (6.100)$$

$$\Phi'_N(x)^\dagger = e^{-\kappa_N x} D_N^\dagger P_N \mathbf{B}_N^{-1/2} [-\kappa_N I + o(1)], \quad x \rightarrow +\infty, \quad (6.101)$$

where we have used the fact that each of P_N , Q_N , and $\mathbf{B}_N^{-1/2}$ is an $n \times n$ selfadjoint matrix. By taking the x -derivative of both sides of (6.3) and then using (6.98)–(6.101) in the resulting equality, we obtain

$$W'_N(x) = -e^{-2\kappa_N x} D_N^\dagger P_N \mathbf{B}_N^{-1} P_N D_N [I + o(1)], \quad x \rightarrow +\infty. \quad (6.102)$$

Using (4.2) in (6.102) we get

$$W'_N(x) = -e^{-2\kappa_N x} D_N^\dagger \mathbf{B}_N^{-1} D_N [I + o(1)], \quad x \rightarrow +\infty. \quad (6.103)$$

From (6.3) we have

$$W_N(x) = o(1), \quad x \rightarrow +\infty, \quad (6.104)$$

and hence (6.103) and (6.104) imply

$$W_N(x) = \frac{1}{2\kappa_N} e^{-2\kappa_N x} D_N^\dagger \mathbf{B}_N^{-1} D_N [I + o(1)], \quad x \rightarrow +\infty. \quad (6.105)$$

From (6.3) with the help of the second equality in (3.28), we get

$$W_N(x) = Q_N W_N(x) Q_N.$$

Using (4.14) and after premultiplying and postmultiplying both sides of (6.105) by Q_N , we obtain

$$W_N(x) = \frac{1}{2\kappa_N} e^{-2\kappa_N x} D_N^\dagger \mathbf{B}_N^{-1} D_N [I + Q_N o(1) Q_N], \quad x \rightarrow +\infty. \quad (6.106)$$

From (6.106) we conclude that $[W_N(x)]_1$ defined in (6.11) has the asymptotics

$$[W_N(x)]_1 = \frac{1}{2\kappa_N} e^{-2\kappa_N x} D_N^\dagger \mathbf{B}_N^{-1} D_N [I + Q_N o(1) Q_N], \quad x \rightarrow +\infty,$$

and with the help of (6.17) we obtain

$$W_N(x)^+ = 2\kappa_N e^{2\kappa_N x} \left(D_N^\dagger \mathbf{B}_N^{-1} D_N \right)^+ [I + Q_N o(1) Q_N] \oplus 0, \quad x \rightarrow +\infty. \quad (6.107)$$

From (6.107) we get

$$W_N(x)^+ = 2\kappa_N e^{2\kappa_N x} \left(D_N^\dagger \mathbf{B}_N^{-1} D_N \right)^+ [I + o(1)], \quad x \rightarrow +\infty, \quad (6.108)$$

where we recall that the superscript $+$ denotes the Moore–Penrose inverse. With the help of (2.8), (6.100), and (6.101), as $x \rightarrow +\infty$ we obtain

$$\Phi'_N(x)^\dagger f(k, x) - \phi_N(x)^\dagger f'(k, x) = -(\kappa_N + ik) e^{ikx - \kappa_N x} D_N^\dagger P_N \mathbf{B}_N^{-1/2} [I + o(1)]. \quad (6.109)$$

Finally, using (2.8) for $f(k, x)$, the analog of (2.8) for $\tilde{f}(k, x)$, (6.98), (6.99), (6.108), and (6.109) on the right-hand side of (6.92), we establish (6.93) with $q_8(k)$ defined as in (6.94), provided we can show that

$$\mathbf{B}_N^{-1/2} P_N D_N \left(D_N^\dagger \mathbf{B}_N^{-1} D_N \right)^+ D_N^\dagger P_N \mathbf{B}_N^{-1/2} = P_N,$$

or equivalently if we can show that

$$P_N \mathbf{B}_N^{-1/2} D_N \left(D_N^\dagger \mathbf{B}_N^{-1} D_N \right)^+ D_N^\dagger \mathbf{B}_N^{-1/2} P_N = P_N, \quad (6.110)$$

because P_N commutes with $\mathbf{B}_N^{-1/2}$. We establish (6.110) as follows. We know from Sec. III that the matrix $\mathbf{B}_N^{-1/2}$ is selfadjoint, and hence we have

$$D_N^\dagger \mathbf{B}_N^{-1} D_N = \left(\mathbf{B}_N^{-1/2} D_N \right)^\dagger \left(\mathbf{B}_N^{-1/2} D_N \right). \quad (6.111)$$

For the Moore–Penrose inverse of the matrix product $\mathbf{B}_N^{-1/2} D_N$, we have

$$\left(\mathbf{B}_N^{-1/2} D_N \right)^+ = D_N^+ \mathbf{B}_N^{1/2}, \quad (6.112)$$

which can be verified by showing that the four equalities in (3.2) hold if we let $\mathbf{M} = \mathbf{B}_N^{-1/2} D_N$ and $\mathbf{M}^+ = D_N^+ \mathbf{B}_N^{1/2}$ there. The verification of the first equality in (3.2) is obtained with the help of (4.17) and the first equality of (4.19). The second equality of (3.2) is verified with the help of (4.17) and the second equality of (4.19). The verification of the third equality of (3.2) is obtained by using (4.16), (4.17), the fact that P_N commutes with $\mathbf{B}_N^{1/2}$ and $\mathbf{B}_N^{-1/2}$, and that the matrix product $\mathbf{M}\mathbf{M}^+$ becomes equal to the selfadjoint matrix P_N . The fourth equality of (3.2) is verified by using (4.15), (4.17), and the fact that $\mathbf{M}^+\mathbf{M}$ becomes equal to the selfadjoint matrix Q_N . By expressing the right-hand side of (6.111) and then the left-hand side of (6.110) in terms of $\mathbf{M}$ and $\mathbf{M}^+$, we see that (6.110) is equivalent to

$$P_N \mathbf{M} \left(\mathbf{M}^+ \mathbf{M} \right)^+ \mathbf{M}^+ P_N = P_N. \quad (6.113)$$

Using (3.4), we observe that (6.113) is equivalent to

$$P_N \mathbf{M} \mathbf{M}^+ \left(\mathbf{M}^+ \right)^+ \mathbf{M}^+ P_N = P_N,$$

which, after using (3.3), can also be written as

$$P_N \mathbf{M} \mathbf{M}^+ \left(\mathbf{M} \mathbf{M}^+ \right)^\dagger P_N = P_N. \quad (6.114)$$

Using $\mathbf{M} \mathbf{M}^+ = P_N$ and the fact that P_N is an orthogonal projection, we see that (6.114) indeed holds. Thus, the proof of (a) is complete. For the proof of (b) we proceed as follows. By taking the matrix adjoint of (6.81), we obtain

$$\tilde{J}(k)^\dagger = J(k)^\dagger \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right]^\dagger. \quad (6.115)$$

Since κ_N is positive and P_N is an orthogonal projection, we can directly verify that

$$\left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right]^\dagger = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right]^{-1} = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_N \right]. \quad (6.116)$$

Using (6.81) and (6.115) on the left-hand side of (6.82), with the help of (6.116) we establish (6.82). Hence, the proof of (b) is complete. We now turn to the proof of (c). Since the orthogonal projection matrix P_N has m_N eigenvalues equal to +1 and the remaining eigenvalues all equal to zero, we have

$$\det \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{m_N}. \quad (6.117)$$

Using (6.81) and (6.117) we obtain (6.83). Thus, the proof of (c) is complete. From Theorems 3.11.1 and 3.11.16 of Ref. 7 we know that the bound states for the perturbed and unperturbed Schrödinger operators correspond to the zeros of $\det[\tilde{J}(k)]$ and $\det[J(k)]$, respectively, on the positive imaginary axis and the multiplicities of the bound states are determined by the order of the zeros of $\det[\tilde{J}(k)]$ and $\det[J(k)]$, respectively. Hence, (d) follows from (6.83). We establish (e) by using (6.81) in (6.80) and by simplifying the right-hand side of the resulting equation with the help of (2.11) and the second equality of (6.116). We obtain the proof of (f) by taking the determinant of both sides of (6.84) and using the analog of (6.117) when κ_N there is replaced with $-\kappa_N$. Finally, we prove (g) as follows. Using (2.1) for $f(k, x)$ and the adjoint of (6.31), we obtain

$$\Phi_N''(x)^\dagger f(k, x) - \Phi_N(x)^\dagger f''(k, x) = (k^2 + \kappa_N^2) \Phi_N(x)^\dagger f(k, x),$$

which is equivalent to

$$q_7'(k, x) = (k^2 + \kappa_N^2) \Phi_N(x)^\dagger f(k, x), \quad (6.118)$$

where $q_7'(k, x)$ is the x -derivative of the quantity $q_7(k, x)$ defined in (6.87). Integrating (6.118) and using the asymptotics listed in (2.8), (6.43), and (6.44), we get

$$\frac{1}{k^2 + \kappa_N^2} q_7(k, x) = - \int_x^\infty dy \Phi_N(y)^\dagger f(k, y). \quad (6.119)$$

We introduce the quantity $\tilde{h}(k, x)$ as

$$\tilde{h}(k, x) := f(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_N(x) W_N(x)^+ q_7(k, x), \quad (6.120)$$

which differs from the right-hand side of (6.86) by the second term in the brackets there. In other words, the quantity $\tilde{h}(k, x)$ is related to the quantity $\tilde{f}(k, x)$ appearing in (6.86) as

$$\tilde{f}(k, x) = \tilde{h}(k, x) \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right]. \quad (6.121)$$

From (6.119) and (6.120) we observe that $\tilde{h}(k, x)$ does not have a singularity at $k = i\kappa_N$. We would like to show that $\tilde{h}(k, x)$ satisfies the Schrödinger equation (2.1) with the potential $\tilde{V}(x)$ given in (6.21). For this we proceed as follows. The x -derivative of (6.120), after using (6.22) and (6.118), is evaluated as

$$\tilde{h}'(k, x) = f'(k, x) + \frac{\Phi_N'(x) W_N(x)^+ q_7(k, x)}{k^2 + \kappa_N^2} + \Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger \tilde{h}(k, x). \quad (6.122)$$

By taking the x -derivative of (6.122), we obtain the expression for $\tilde{h}''(k, x)$. Using that expression for $\tilde{h}''(k, x)$, the expression for $f''(k, x)$ obtained by using $f(k, x)$ in (2.1), the expression for $\Phi_N''(x)$ obtained from (6.31), the expression for $\tilde{h}'(k, x)$ from (6.122), the expression for $q_7'(k, x)$ from (6.118), and the expression for $\tilde{V}(x)$ from (6.21), we prove that

$$-\tilde{h}''(x) + \tilde{V}(x) \tilde{h}(k, x) - k^2 \tilde{h}(k, x) = 0, \quad (6.123)$$

which confirms that $\tilde{h}(k, x)$ indeed satisfies the Schrödinger equation (2.1) when $V(x)$ there is replaced with $\tilde{V}(x)$ in (6.21). Next, using (2.8) and (6.98)–(6.101), we obtain the asymptotics

$$\tilde{h}(k, x) = e^{ikx} \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_N \right] (I + o(1)), \quad x \rightarrow +\infty. \quad (6.124)$$

From (6.121), (6.123), and (6.124), we see that the quantity $\tilde{f}(k, x)$ appearing in (6.86) is indeed the Jost solution to the Schrödinger equation (2.1) when the potential $V(x)$ there is replaced with the perturbed potential $\tilde{V}(x)$. Using (6.117), (6.119), and (6.120), we see that the perturbed Jost solution $\tilde{f}(k, x)$ has the alternate expression given in (6.88). Thus, the proof of (g) is complete. $\square$

In the following remark, we emphasize that the perturbed Jost solution $\tilde{f}(k, x)$ appearing in (6.86) and (6.88) does not have a singularity at $k = i\kappa_N$ despite the factor $k - i\kappa_N$ appearing in the denominators in (6.86) and (6.88).

Remark 6.5. As stated in the proof of Theorem 6.4, the quantity $\tilde{h}(k, x)$ introduced in (6.120) does not have a singularity at $k = i\kappa_N$. On the other hand, (6.121) suggests that the perturbed Jost solution $\tilde{f}(k, x)$ might have a pole at $k = i\kappa_N$ due to the factor $k - i\kappa_N$ in the denominator on the right-hand side of (6.121). To prove that the factor $k - i\kappa_N$ in the denominator in (6.121) yields a removable singularity, we proceed as follows. From (6.124) we have

$$\tilde{h}(i\kappa_N, x) = e^{-\kappa_N x} [I - P_N] (I + o(1)), \quad x \rightarrow +\infty,$$

which yields

$$\tilde{h}(i\kappa_N, x) P_N = e^{-\kappa_N x} o(1), \quad x \rightarrow +\infty. \quad (6.125)$$

From Proposition 3.2.2 of Ref. 7 it follows that that (2.1) with the potential $V(x)$ there replaced with $\tilde{V}(x)$, for each fixed $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, has an $n \times n$ matrix solution $\tilde{g}(k, x)$ satisfying

$$\tilde{g}(k, x) = e^{-ikx} [I + o(1)], \quad \tilde{g}'(k, x) = -e^{-ikx} [ik I + o(1)], \quad x \rightarrow +\infty. \quad (6.126)$$

From (6.126) we see that $\tilde{g}(i\kappa_N, x)$ increases exponentially as $x \rightarrow +\infty$ because it has the asymptotics

$$\tilde{g}(i\kappa_N, x) = e^{\kappa_N x} (I + o(1)), \quad x \rightarrow +\infty. \quad (6.127)$$

In contrast to $\tilde{g}(i\kappa_N, x)$, the perturbed Jost solution $\tilde{f}(i\kappa_N, x)$ decreases exponentially as $x \rightarrow +\infty$ because it has the asymptotics

$$\tilde{f}(i\kappa_N, x) = e^{-\kappa_N x} (I + o(1)), \quad x \rightarrow +\infty. \quad (6.128)$$

From Proposition 3.2.2 of Ref. 7 we know that $\tilde{f}(i\kappa_N, x)$ and $\tilde{g}(i\kappa_N, x)$ form a fundamental set of solutions to (2.1) at $k = i\kappa_N$ with the potential $\tilde{V}(x)$. By expressing $\tilde{h}(i\kappa_N, x) P_N$ as a linear combination of $\tilde{f}(i\kappa_N, x)$ and $\tilde{g}(i\kappa_N, x)$, from (6.127) and (6.128) we conclude that there exists a constant $n \times n$ matrix M such that

$$\tilde{h}(i\kappa_N, x) P_N = \tilde{f}(i\kappa_N, x) M. \quad (6.129)$$

From (6.125), (6.128), and (6.129), we see that we must have $M = 0$ in (6.129). Hence, (6.129) yields

$$\tilde{h}(i\kappa_N, x) P_N = 0. \quad (6.130)$$

Using (6.130) we can write (6.121) in the equivalent form as

$$\tilde{f}(k, x) = \tilde{h}(k, x) + \frac{2i\kappa_N}{k - i\kappa_N} (\tilde{h}(k, x) - \tilde{h}(i\kappa_N, x)) P_N. \quad (6.131)$$

Since we have

$$\tilde{h}(k, x) - \tilde{h}(i\kappa_N, x) = O(k - i\kappa_N), \quad k \rightarrow i\kappa_N,$$

it follows that the right-hand side of (6.131) remains bounded as $k \rightarrow i\kappa_N$. Consequently, the singularity of (6.121) at $k = i\kappa_N$ is a removable singularity.

In the next theorem, when we remove a bound state from the Schrödinger operator, we show that the projection matrices and the Gel'fand–Levitan normalization matrices associated with the remaining bound states stay unaffected. We also show how the spectral measure changes when a bound state is removed. We recall that the bound-state energies are not required to be ordered in an increasing or decreasing order, and hence without any loss of generality we assume that we remove the bound state at $k = i\kappa_N$ with the Gel'fand–Levitan normalization matrix C_N and that the remaining bound states occur at $k = i\kappa_j$ with the Gel'fand–Levitan normalization matrices C_j for $1 \leq j \leq N - 1$. As in Theorem 6.4, we impose the stronger assumption that the unperturbed potential V belongs to $L_2^1(\mathbb{R}^+)$ rather than $L_1^1(\mathbb{R}^+)$ so that the perturbed potential $\tilde{V}$ belongs to $L_1^1(\mathbb{R}^+)$, which is ensured by Theorem 6.3(b). We recall that this is a sufficiency assumption because the asymptotic estimates used in Theorem 6.3(b) on the potentials are not necessarily sharp.

Theorem 6.6. Consider the unperturbed Schrödinger operator associated with (2.1) and (2.5), where the potential V satisfies (2.2) and belongs to $L_2^1(\mathbb{R}^+)$ and the boundary matrices A and B appearing in (2.5) satisfy (2.6) and (2.7). Let $\varphi(k, x)$ be the corresponding regular solution satisfying the initial conditions (2.9), $J(k)$ be the Jost matrix defined in (2.10), $d\rho$ be the spectral measure given in (4.1), and each of $k = i\kappa_j$ for $1 \leq j \leq N$ correspond to the bound state with the energy $-\kappa_j^2$, the Gel'fand–Levitan normalization matrix C_j , the orthogonal projection Q_j onto $\text{Ker}[J(i\kappa_j)]$, and the Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$. Let us use a tilde to identify the quantities associated with the perturbed Schrödinger operator with the potential $\tilde{V}(x)$ expressed as in (6.21) and the boundary condition (2.5) where the boundary matrices A and B are replaced with $\tilde{A}$ and $\tilde{B}$, respectively, given in (6.40). We use $\tilde{\varphi}(k, x)$ for the perturbed regular solution satisfying the initial conditions (5.14), the quantity $\tilde{J}(k)$ for the perturbed Jost matrix defined as in (6.79), the matrix $\tilde{Q}_j$ for the orthogonal projection onto $\text{Ker}[\tilde{J}(i\kappa_j)]$, the quantity $\tilde{C}_j$ for the Gel'fand–Levitan normalization matrix defined as in (3.24), (3.25), and (3.27) but by using $\tilde{Q}_j$ instead of Q_j and by using $\tilde{\varphi}(i\kappa_j, x)$ instead of $\varphi(i\kappa_j, x)$ there for $1 \leq j \leq N - 1$, and we define the perturbed spectral measure $d\tilde{\rho}$ as

$$d\tilde{\rho} := \begin{cases} \frac{\sqrt{\lambda}}{\pi} \left(\tilde{J}(\sqrt{\lambda})^\dagger \tilde{J}(\sqrt{\lambda}) \right)^{-1} d\lambda, & \lambda \geq 0, \\ \sum_{j=1}^{N-1} \tilde{C}_j^2 \delta(\lambda - \lambda_j) d\lambda, & \lambda < 0. \end{cases} \quad (6.132)$$

We then have the following:

(a) Under the perturbation, the rest of the projection matrices Q_j are unchanged, i.e. we have

$$\tilde{Q}_j = Q_j, \quad 1 \leq j \leq N - 1. \quad (6.133)$$

(b) Under the perturbation, the rest of the Gel'fand–Levitan normalization matrices are unchanged, i.e. we have

$$\tilde{C}_j = C_j, \quad 1 \leq j \leq N - 1. \quad (6.134)$$

(c) The perturbed spectral measure $d\tilde{\rho}$ is related to the unperturbed spectral measure $d\rho$ as

$$d\tilde{\rho} = d\rho - C_N^2 \delta(\lambda - \lambda_N) d\lambda, \quad \lambda \in \mathbb{R}. \quad (6.135)$$

Proof. The proof of (a) directly follows from (6.81) and the fact that for $1 \leq j \leq N-1$ the matrices Q_j and $\tilde{Q}_j$ are the orthogonal projections onto $\text{Ker}[J(i\kappa_j)]$ and $\text{Ker}[\tilde{J}(i\kappa_j)]$, respectively, which indicate that any nonzero vector in the kernel of $J(i\kappa_j)$ is also in the kernel of $\tilde{J}(i\kappa_j)$ and vice versa. For the proof of (b), we proceed as follows. From (3.27) and (6.133) we see that (6.134) holds if and only if $\mathbf{H}_j$ is unchanged. With the help of (3.25) we see that $\mathbf{H}_j$ is unchanged if and only if $\mathbf{G}_j$ is unchanged. Thus, the proof of (b) can be obtained by showing that the integral on the right-hand side of (3.24) is unchanged for $1 \leq j \leq N-1$. Let us define the $n \times n$ matrices $\mathbf{S}_j(x)$ and $\tilde{\mathbf{S}}_j(x)$ as

$$\mathbf{S}_j(x) := \varphi(i\kappa_j, x) Q_j, \quad \tilde{\mathbf{S}}_j(x) := \tilde{\varphi}(i\kappa_j, x) Q_j, \quad 1 \leq j \leq N-1. \quad (6.136)$$

From the second equality of (3.28), by using the fact that $\mathbf{H}_j^{-1/2}$ and Q_j commute, we obtain

$$\Phi_j(x) := \varphi(i\kappa_j, x) Q_j \mathbf{H}_j^{-1/2}, \quad 1 \leq j \leq N-1. \quad (6.137)$$

From (6.136) and (6.137) it follows that

$$\mathbf{S}_j(x) = \Phi_j(x) \mathbf{H}_j^{1/2}, \quad 1 \leq j \leq N-1.$$

Evaluating (6.39) at $k = i\kappa_j$ and postmultiplying the resulting equation with Q_j , we obtain

$$\tilde{\mathbf{S}}_j(x) = \mathbf{S}_j(x) + \frac{1}{\kappa_N^2 - \kappa_j^2} \Phi_N(x) W_N(x)^+ q_9(x) \mathbf{H}_j^{1/2}, \quad 1 \leq j \leq N-1, \quad (6.138)$$

where we have defined

$$q_9(x) := \Phi_N'(x)^\dagger \Phi_j(x) - \Phi_N(x)^\dagger \Phi_j'(x).$$

By taking the matrix adjoint of (6.138), we get

$$\tilde{\mathbf{S}}_j(x)^\dagger = \mathbf{S}_j(x)^\dagger + \frac{1}{\kappa_N^2 - \kappa_j^2} \mathbf{H}_j^{1/2} q_9(x)^\dagger W_N(x)^+ \Phi_N(x)^\dagger, \quad 1 \leq j \leq N-1, \quad (6.139)$$

where we have used the fact that $\mathbf{H}_j^{1/2}$ and $W_N(x)^+$ are both selfadjoint. The selfadjointness of $W_N(x)$ is seen from (6.3). From (3.3) we know that the operations for the adjoint and the Moore–Penrose inverse for any square matrix commute. Consequently, from (6.3) we conclude that $W_N(x)^+$ is also selfadjoint. From (6.138) and (6.139) we get

$$\mathbf{H}_j^{-1/2} [\tilde{\mathbf{S}}_j(x)^\dagger \tilde{\mathbf{S}}_j(x) - \mathbf{S}_j(x)^\dagger \mathbf{S}_j(x)] \mathbf{H}_j^{-1/2} = \frac{1}{\kappa_N^2 - \kappa_j^2} [q_{10}(x) + q_{11}(x) + q_{12}(x)], \quad (6.140)$$

where we have defined

$$q_{10}(x) := \Phi_j(x)^\dagger \Phi_N(x) W_N(x)^+ q_9(x), \quad (6.141)$$

$$q_{11}(x) := q_9(x)^\dagger W_N(x)^+ \Phi_N(x)^\dagger \Phi_j(x), \quad (6.142)$$

$$q_{12}(x) := q_9(x)^\dagger \frac{W_N(x)^+ \Phi_N(x)^\dagger \Phi_N(x) W_N(x)^+}{\kappa_N^2 - \kappa_j^2} q_9(x). \quad (6.143)$$

We have

$$\Phi_j(x)^\dagger \Phi_N(x) = \frac{1}{\kappa_N^2 - \kappa_j^2} \frac{d}{dx} [q_9(x)^\dagger], \quad (6.144)$$

$$W_N(x)^+ \Phi_N(x)^\dagger \Phi_N(x) W_N(x)^+ = \frac{dW_N(x)^+}{dx}, \quad (6.145)$$

$$\Phi_N(x)^\dagger \Phi_j(x) = \frac{1}{\kappa_N^2 - \kappa_j^2} \frac{d}{dx} [q_9(x)]. \quad (6.146)$$

Note that (6.145) follows from (6.22). We can verify (6.144) and (6.146) with the help of (6.31) and the corresponding Schrödinger equation

$$-\Phi_j''(x) + V(x) = -\kappa_j^2 \Phi_j(x), \quad 1 \leq j \leq N-1.$$

Using (6.144)–(6.146) in (6.141)–(6.143), respectively, we observe that the right-hand side of (6.140) is proportional to the derivative of the product of three matrix-valued functions. Thus, we can write (6.140) as

$$\mathbf{H}_j^{-1/2} [\tilde{\mathbf{S}}_j(x)^\dagger \tilde{\mathbf{S}}_j(x) - \mathbf{S}_j(x)^\dagger \mathbf{S}_j(x)] \mathbf{H}_j^{1/2} = \frac{1}{(\kappa_N^2 - \kappa_j^2)^2} \frac{dY(x)}{dx}, \quad (6.147)$$

where we have defined

$$Y(x) := q_9(x)^\dagger W_N(x)^+ q_9(x). \quad (6.148)$$

Because $\mathbf{H}_j^{-1/2}$ is an invertible matrix and $\kappa_j \neq \kappa_N$ for $1 \leq j \leq N-1$, from (6.147) we conclude that

$$\int_0^\infty dx [\tilde{\mathbf{S}}_j(x)^\dagger \tilde{\mathbf{S}}_j(x) - \mathbf{S}_j(x)^\dagger \mathbf{S}_j(x)] = 0, \quad (6.149)$$

if and only if we have

$$Y(+\infty) = Y(0). \quad (6.150)$$

With the help of (6.43), (6.44), their adjoints, and their analogs for $\Phi_j(x)$, we get

$$q_9(x) = O(e^{-(\kappa_N + \kappa_j)x}), \quad x \rightarrow +\infty, \quad (6.151)$$

$$q_9(x)^\dagger = O(e^{-(\kappa_j + \kappa_N)x}), \quad x \rightarrow +\infty. \quad (6.152)$$

On the other hand, from (6.52) we have

$$W_N(x)^+ = O(e^{2\kappa_N x}), \quad x \rightarrow +\infty. \quad (6.153)$$

Using (6.151)–(6.153), we obtain

$$Y(x) = O(e^{-2\kappa_j x}), \quad x \rightarrow +\infty,$$

which yields

$$Y(+\infty) = 0. \quad (6.154)$$

Note that we have the analogs of (6.32) given by

$$\Phi_j(0) = AC_j, \quad \Phi_j'(0) = BC_j, \quad 1 \leq j \leq N-1. \quad (6.155)$$

Using (6.32), (6.69), and (6.155) on the right-hand side of (6.148), we obtain

$$Y(0) = C_j (A^\dagger B - B^\dagger A) C_N Q_N C_N (B^\dagger A - A^\dagger B) C_j, \quad 1 \leq j \leq N-1, \quad (6.156)$$

where we have used the fact that C_j and C_N are selfadjoint matrices. Using (2.6) on the right-hand side of (6.156), we see that

$$Y(0) = 0, \quad (6.157)$$

and hence (6.154) and (6.157) yield (6.150), which in turn implies that (6.149) holds. Comparing (6.149) with (3.24), we see that the matrix $\mathbf{G}_j$ is unchanged and we have

$$\tilde{\mathbf{G}}_j = \mathbf{G}_j, \quad 1 \leq j \leq N-1.$$

Thus, (6.134) holds and the proof of (b) is complete. Finally, we establish (6.135) by using (5.1), (6.82), (d) of Theorem 6.4, (6.132), and (6.134). Hence, the proof of (c) is complete. $\square$

VII. THE TRANSFORMATION TO DECREASE THE MULTIPLICITY OF A BOUND STATE

In Sec. VI we have presented a method to completely remove an eigenvalue from the discrete spectrum of the matrix-valued Schrödinger operator with the general selfadjoint boundary condition, without changing the continuous spectrum. We have provided the transformations for all relevant quantities resulting from the removal. Without loss of generality, we have removed the bound state at $k = i\kappa_N$ with the multiplicity m_N and the Gel'fand–Levitan normalization matrix C_N so that after the removal we have left with the remaining bound states at $k = i\kappa_j$ for $1 \leq j \leq N-1$. We have seen that the remaining bound-state energies $-\kappa_j^2$, multiplicities m_j , and Gel'fand–Levitan normalization matrices C_j are not affected by the removal. In the scalar case, i.e. when $n = 1$, the multiplicity m_N is equal to 1, and hence the reduction of the multiplicity m_N is always equivalent to the complete removal of the bound state at $k = i\kappa_N$. On the other hand, when $n \geq 2$ the multiplicity

m_N is a fixed positive integer satisfying $1 \leq m_N \leq n$ and hence the reduction in the multiplicity m_N is not necessarily the same as the complete removal of the bound state.

This section is complementary to Sec. VI. Instead of completely removing the bound state at $k = i\kappa_N$, we reduce its multiplicity from m_N to $\tilde{m}_N$ so that we have $1 \leq \tilde{m}_N < m_N$. Thus, we assume that the number N of bound states is at least 1 and that the multiplicity m_N is at least 2. As in the previous sections, we use a tilde to identify the perturbed quantities obtained after the multiplicity of the bound state is reduced. We assume that our unperturbed potential $V(x)$ satisfies (2.2) and belongs to $L^1_1(\mathbb{R}^+)$. Our unperturbed regular solution is $\varphi(k, x)$, unperturbed Jost solution is $f(k, x)$, unperturbed Jost matrix is $J(k)$, unperturbed scattering matrix is $S(k)$, unperturbed matrix Q_N corresponds to the orthogonal projection onto the kernel of $J(i\kappa_N)$, and unperturbed boundary matrices are given by A and B . Our perturbed potential is $\tilde{V}(x)$, perturbed regular solution is $\tilde{\varphi}(k, x)$, perturbed Jost solution is $\tilde{f}(k, x)$, perturbed Jost matrix is $\tilde{J}(k)$, perturbed scattering matrix is $\tilde{S}(k)$, and perturbed boundary matrices are given by $\tilde{A}$ and $\tilde{B}$.

We assume that the multiplicity of the bound state at $k = i\kappa_N$ is reduced by m_{Nr} , and hence we define the positive integer representing the reduction in the multiplicity of the bound state at $k = i\kappa_N$ as

$$m_{Nr} := m_N - \tilde{m}_N. \quad (7.1)$$

The subscript Nr indicates that we refer to the N th bound state at $k = i\kappa_N$ and that we reduce its multiplicity. From (7.1) we see that m_{Nr} satisfies the inequality $1 \leq m_{Nr} \leq n - 1$. The reduction of the multiplicity of the bound state at $k = i\kappa_N$ by m_{Nr} is achieved by introducing the matrix Q_{Nr} in such a way that Q_{Nr} is an orthogonal projection onto a proper subspace of $Q_N \mathbb{C}^n$, it has its rank equal to m_{Nr} , and it satisfies $Q_{Nr} \leq Q_N$. This last matrix inequality is equivalent to $Q_N - Q_{Nr} \geq 0$, which indicates that the matrix $Q_N - Q_{Nr}$ is nonnegative. In other words, the eigenvalues of the matrix $Q_N - Q_{Nr}$ are all real and nonnegative.

The transformations for the relevant quantities are obtained as follows. We omit the proofs in this section because they are similar to those presented in Secs. III and VI. We note that we have

$$Q_{Nr} Q_N = Q_N Q_{Nr} = Q_{Nr}.$$

Consequently, the columns of the matrix $\varphi(i\kappa_N, x) Q_{Nr}$ are square integrable because the columns of the matrix $\varphi(i\kappa_N, x) Q_N$ are square integrable. Analogous to (3.24) and (3.25), we let

$$\mathbf{G}_{Nr} := \int_0^\infty dx Q_{Nr} \varphi(i\kappa_N, x)^\dagger \varphi(i\kappa_N, x) Q_{Nr}, \quad (7.2)$$

$$\mathbf{H}_{Nr} := I - Q_{Nr} + \mathbf{G}_{Nr}. \quad (7.3)$$

As in Sec. III, we prove that the matrix $\mathbf{H}_{Nr}$ is positive and hence invertible. Thus, the positive matrix $\mathbf{H}_{Nr}^{1/2}$ and its inverse $\mathbf{H}_{Nr}^{-1/2}$ are uniquely defined. Furthermore, we have

$$\mathbf{H}_{Nr} Q_{Nr} = Q_{Nr} \mathbf{H}_{Nr}.$$

Analogous to (3.27), with the help of (7.2) and (7.3) we introduce the $n \times n$ matrix C_{Nr} as

$$C_{Nr} := \mathbf{H}_{Nr}^{-1/2} Q_{Nr}.$$

In a manner similar to (3.20) we introduce the $n \times n$ matrix solution $\Phi_{Nr}(x)$ to the Schrödinger equation (3.11) with $\kappa_j = \kappa_N$ as

$$\Phi_{Nr}(x) := \varphi(i\kappa_N, x) C_{Nr}. \quad (7.4)$$

Analogous to (3.21) we obtain the normalization property

$$\int_0^\infty dx \Phi_{Nr}(x)^\dagger \Phi_{Nr}(x) = Q_{Nr}.$$

Let $\{w_N^{(l)}\}_{l=1}^{m_{Nr}}$ be an orthonormal basis for $Q_{Nr} \mathbb{C}^n$. Analogous to the second equality in (3.8), we have

$$Q_{Nr} = \sum_{l=1}^{m_{Nr}} w_N^{(l)} (w_N^{(l)})^\dagger. \quad (7.5)$$

Each column of $\varphi(i\kappa_N, x) Q_{Nr}$ is square integrable and satisfies the boundary condition (2.5). Consequently, from Theorem 3.11.1 of Ref. 7 it follows that for each column vector $w_N^{(l)}$ there exists a unique column vector $\beta_N^{(l)}$ in $\text{Ker}[J(i\kappa_N)^\dagger]$ such that

$$\varphi(i\kappa_N, x) w_N^{(l)} = f(i\kappa_N, x) \beta_N^{(l)}, \quad 1 \leq l \leq m_{Nr}. \quad (7.6)$$

From (7.5) we have

$$Q_{N_r} w_N^{(l)} = w_N^{(l)},$$

and hence (7.6) is equivalent to

$$\varphi(i\kappa_N, x) Q_{N_r} w_N^{(l)} = f(i\kappa_N, x) \beta_N^{(l)}, \quad 1 \leq l \leq m_{N_r}. \quad (7.7)$$

Let us use P_{N_r} to denote the orthogonal projection onto the subspace of $\text{Ker}[J(i\kappa_N)^\dagger]$ generated by the set $\{\beta_N^{(l)}\}_{l=1}^{m_{N_r}}$. From Theorem 3.11.1(f) of Ref. 7 it follows that the map $w_N^{(l)} \mapsto \beta_N^{(l)}$ is one-to-one. Consequently, the m_{N_r} vectors in the set $\{\beta_N^{(l)}\}_{l=1}^{m_{N_r}}$ are linearly independent. Hence, the rank of P_{N_r} is equal to m_{N_r} . Consider any column vector w in $Q_{N_r} \mathbb{C}^n$, which can be expressed as

$$w = \sum_{l=1}^{m_{N_r}} a_l w_N^{(l)}, \quad (7.8)$$

for some appropriate coefficients a_l . By letting

$$v := \sum_{l=1}^{m_{N_r}} a_l \beta_N^{(l)}, \quad (7.9)$$

we see that $v \in P_{N_r} \mathbb{C}^n$ and we also have $P_{N_r} v = v$. Consequently, using (7.6)–(7.9), we get

$$\varphi(i\kappa_N, x) Q_{N_r} w = f(i\kappa_N, x) P_{N_r} v. \quad (7.10)$$

As already indicated, each column of the matrix $\varphi(i\kappa_N, x) Q_{N_r}$ is a square-integrable solution to (3.11) and satisfies the boundary condition (2.5). From (7.10) it follows that each column of the matrix $f(i\kappa_N, x) P_{N_r}$ is a square-integrable solution to (3.11) and satisfies the boundary condition (2.5). Hence, we can use the matrix $f(i\kappa_N, x) P_{N_r}$ to construct an $n \times n$ matrix M_{N_r} as in (3.19). For this, we use the analogs of (3.16) and (3.17) by letting

$$\begin{aligned} \mathbf{A}_{N_r} &:= \int_0^\infty dx P_{N_r} f(i\kappa_N, x)^\dagger f(i\kappa_N, x) P_{N_r}, \\ \mathbf{B}_{N_r} &:= I - P_{N_r} + \mathbf{A}_{N_r}. \end{aligned}$$

The matrix $\mathbf{B}_{N_r}$ is positive and hence invertible, and it commutes with P_{N_r} . With the help of the analogs of (3.18) and (3.19), we define the $n \times n$ matrix M_{N_r} as

$$M_{N_r} := \mathbf{B}_{N_r}^{-1/2} P_{N_r},$$

where $\mathbf{B}_{N_r}^{-1/2}$ is the positive matrix corresponding to the inverse of the positive matrix $\mathbf{B}_{N_r}^{1/2}$. Analogous to (3.13), we define the $n \times n$ matrix solution Ψ_{N_r} to (3.11) as

$$\Psi_{N_r} := f(i\kappa_N, x) M_{N_r}. \quad (7.11)$$

In a manner similar to (4.1) and (4.2), using (7.4) and (7.11) we introduce the dependency matrix D_{N_r} so that

$$\Phi_{N_r} = \Psi_{N_r} D_{N_r},$$

$$D_{N_r} = P_{N_r} D_{N_r}.$$

Analogous to (4.13), we prove that the dependency matrix D_{N_r} satisfies

$$D_{N_r} = P_{N_r} \mathbf{B}_{N_r}^{1/2} \sum_{l=1}^{m_{N_r}} \beta_N^{(l)} [w_N^{(l)}]^\dagger \mathbf{H}_{N_r}^{-1/2} Q_{N_r}.$$

We recall that we assume that the unperturbed potential V satisfies (2.2) and belongs to $L_1^1(\mathbb{R}^+)$. As in Sec. VI we show the following:

- (1) We establish the appropriate analogs of (a)–(h) of Theorem 4.1 for the quantities D_{N_r} , Q_{N_r} , P_{N_r} , Ψ_{N_r} , and Φ_{N_r} , where m_{N_r} corresponds to the common rank of the orthogonal projections Q_{N_r} and P_{N_r} .
- (2) We show that the analogs of (4.36) and (4.44) hold. This is done by proving that

$$D_{N_r} = M_{N_r}^+ f(i\kappa_N, x)^{-1} \Phi_{N_r}(x),$$

where the right-hand side is evaluated at any x -value at which the matrix $f(i\kappa_N, x)$ is invertible. We also show that

$$D_{N_r} = M_{N_r}^+ f'(i\kappa_N, x)^{-1} \Phi'_{N_r}(x),$$

where the right-hand side is evaluated at any x -value at which the matrix $f'(i\kappa_N, x)$ is invertible.

- (3) We show that the analog of Theorem 4.3 holds. In other words, we establish that
 - (a) The matrix D_{Nr} is a partial isometry with the initial subspace $Q_{Nr} \mathbb{C}^n$ and the final subspace $P_{Nr} \mathbb{C}^n$.
 - (b) The adjoint matrix $D_{Nr}^\dagger$ is a partial isometry with the initial subspace $P_{Nr} \mathbb{C}^n$ and the final subspace $Q_{Nr} \mathbb{C}^n$.
- (4) We solve the Gel'fand–Levitan system (5.38) of integral equations with the kernel $G(x, y)$ defined as

$$G(x, y) := -\varphi(i\kappa_N, x) C_{Nr}^2 \varphi(i\kappa_N, y)^\dagger,$$

which is the analog of (6.1). Then, by proceeding as in the proof of Theorem 6.1, we show that the solution $\mathcal{A}(x, y)$ to (5.38) is given by

$$\mathcal{A}(x, y) = \Phi_{Nr}(x) W_{Nr}(x)^+ \Phi_{Nr}(y)^\dagger, \quad 0 \leq y < x, \quad (7.12)$$

which is the analog of (6.2). Here, we have defined

$$W_{Nr}(x) := \int_x^\infty dy \Phi_{Nr}(y)^\dagger \Phi_{Nr}(y),$$

which is the analog of (6.3). We recall that $W_{Nr}(x)^+$ denotes the Moore–Penrose inverse of $W_{Nr}(x)$.

- (5) We define the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) := V(x) + 2 \frac{d}{dx} \left[\Phi_{Nr}(x) W_{Nr}(x)^+ \Phi_{Nr}(x)^\dagger \right], \quad (7.13)$$

which is the analog of (6.21). We then show that $\mathcal{A}(x, y)$ and $\tilde{V}(x)$ satisfy (a)–(d) of Theorem 6.2, and this is done by proceeding as in the proof of Theorem 6.2.

- (6) By proceeding as in the proof of Theorem 6.3, we establish the following:
 - (a) We construct the quantity $\tilde{\varphi}(k, x)$ as in (5.15) but by using (7.12), and we obtain

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \int_0^x dy \mathcal{A}(x, y) \varphi(k, y). \quad (7.14)$$

We show that $\tilde{\varphi}(k, x)$ given in (7.14) is a solution to the perturbed Schrödinger equation (2.1) with the potential $\tilde{V}(x)$ in (7.13). Furthermore, we show that $\tilde{\varphi}(k, x)$ given in (7.14) satisfies the analog of (6.34), and hence we have

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \Phi_{Nr}(x) W_{Nr}(x)^+ \int_0^x dy \Phi_{Nr}(y)^\dagger \varphi(k, y).$$

- (b) The perturbed potential $\tilde{V}(x)$ appearing in (7.13) satisfies (2.2), and the asymptotics for the potential increment $\tilde{V}(x) - V(x)$ is given by

$$\tilde{V}(x) - V(x) = O(q_6(x)), \quad x \rightarrow +\infty,$$

which is the analog of (6.35) and we recall that $q_6(x)$ is the quantity defined in (6.36). We further show that, if the unperturbed potential V belongs to $L_{1+\epsilon}^1(\mathbb{R}^+)$ for some fixed $\epsilon > 0$, then the perturbed potential $\tilde{V}$ belongs to $L_e^1(\mathbb{R}^+)$.

- (c) If the unperturbed potential $V(x)$ is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (7.15)$$

for some positive constants α and x_0 , then the potential increment $\tilde{V}(x) - V(x)$ satisfies

$$|\tilde{V}(x) - V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0. \quad (7.16)$$

Here, c denotes a generic constant not necessarily taking the same value in different appearances. We remark that (7.15) and (7.16) are the analogs of (6.37) and (6.38), respectively.

- (d) If the support of V is contained in the interval $[0, x_0]$, then the support of $\tilde{V}$ is also contained in $[0, x_0]$.
- (e) For $k \neq i\kappa_N$, the perturbed quantity $\tilde{\varphi}(k, x)$ can be expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_{Nr}(x) W_{Nr}(x)^+ \left[\Phi'_{Nr}(x)^\dagger \varphi(k, x) - \Phi_{Nr}(x)^\dagger \varphi'(k, x) \right],$$

which is the analog of (6.39).

- (f) The perturbed quantity $\tilde{\varphi}(k, x)$ satisfies the initial conditions (5.14) with the matrices $\tilde{A}$ and $\tilde{B}$ expressed in terms of the unperturbed boundary matrices A and B and the matrix C_{N_r} as

$$\tilde{A} = A, \quad \tilde{B} = B + A C_{N_r}^2 A^\dagger, \quad (7.17)$$

which is the analog of (6.40).

- (g) The matrices $\tilde{A}$ and $\tilde{B}$ appearing in (7.17) satisfy (2.6) and (2.7). Hence, as a consequence of (a) and (f), it follows that the quantity $\tilde{\varphi}(k, x)$ is the regular solution to the Schrödinger equation with the potential $\tilde{V}(x)$ in (7.13) and with the selfadjoint boundary condition (2.5) with A and B there replaced with $\tilde{A}$ and $\tilde{B}$, respectively. In other words, $\tilde{\varphi}(k, x)$ satisfies (5.13) and (5.14).
- (7) If we further assume that V belongs to $L_2^1(\mathbb{R}^+)$, then we proceed as in the proof of Theorem 6.4 and establish the following:
- (a) The unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_{N_r} \right] J(k), \quad k \in \overline{\mathbb{C}^+},$$

which is the analog of (6.81).

- (b) The matrix product $J(k)^\dagger J(k)$ for $k \in \mathbb{R}$ does not change under the perturbation, i.e. we have

$$\tilde{J}(k)^\dagger \tilde{J}(k) = J(k)^\dagger J(k), \quad k \in \mathbb{R},$$

which is the analog of (6.82). Consequently, the continuous part of the spectral measure $d\rho$ does not change under the perturbation.

- (c) Under the perturbation, the determinant of the Jost matrix is transformed as

$$\det[\tilde{J}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{m_{N_r}} \det[J(k)], \quad k \in \overline{\mathbb{C}^+}, \quad (7.18)$$

which is the analog of (6.83). We recall that m_{N_r} corresponds to the rank of the orthogonal projection Q_{N_r} .

- (d) Under the perturbation, the bound state with the energy $-\kappa_N^2$ remains but its multiplicity is reduced to $m_N - m_{N_r}$. No new bound states are added, and the bound states with the energies $-\kappa_j^2$ and multiplicities m_j for $1 \leq j \leq N - 1$ are unchanged. We confirm that the multiplicity of the bound state with the energy $-\kappa_N^2$ is reduced to $m_N - m_{N_r}$ by the following observation. The unperturbed Schrödinger equation has a bound state with the energy $-\kappa_N^2$ and multiplicity m_N . From Theorem 3.11.6 of Ref. 7 we know that $\det[J(k)]$ has a zero of order m_N at $k = i\kappa_N$. Moreover, from (7.18) we see that $\det[\tilde{J}(k)]$ has a zero of order $m_N - m_{N_r}$ at $k = i\kappa_N$.
- (e) Under the perturbation, the scattering matrix $S(k)$ undergoes the transformation

$$\tilde{S}(k) = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_{N_r} \right] S(k) \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_{N_r} \right], \quad k \in \mathbb{R},$$

which is the analog of (6.84).

- (f) Under the perturbation, the determinant of the scattering matrix is transformed as

$$\det[\tilde{S}(k)] = \left(\frac{k - i\kappa_N}{k + i\kappa_N} \right)^{2m_{N_r}} \det[S(k)], \quad k \in \mathbb{R},$$

which is the analog of (6.85).

- (g) Under the perturbation, for $k \in \overline{\mathbb{C}^+}$ the unperturbed Jost solution $f(k, x)$ is transformed into the perturbed Jost solution $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_{N_r}(x) W_{N_r}(x)^+ q_{13}(x) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_{N_r} \right], \quad (7.19)$$

where we have defined

$$q_{13}(x) := \Phi'_{N_r}(x)^\dagger f(k, x) - \Phi_{N_r}(x)^\dagger f'(k, x),$$

which are the analogs of (6.86) and (6.87), respectively. We can express (7.19) equivalently as

$$\tilde{f}(k, x) = \left[f(k, x) - \Phi_{N_r}(x) W_{N_r}(x)^+ \int_x^\infty dy \Phi_{N_r}(y)^\dagger f(k, y) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_{N_r} \right],$$

which is the analog of (6.88).

(8) Further assuming that the unperturbed potential V belongs to $L_2^1(\mathbb{R}^+)$, we proceed as in the proof of Theorem 6.6, and we establish the following:

(a) Under the perturbation, the projection matrices Q_j for $1 \leq j \leq N-1$ remain unchanged, i.e. we have

$$\tilde{Q}_j = Q_j, \quad 1 \leq j \leq N-1.$$

(b) Under the perturbation, the Gel'fand–Levitan normalization matrices C_j remain unchanged for $1 \leq j \leq N-1$, i.e. we have

$$\tilde{C}_j = C_j, \quad 1 \leq j \leq N-1.$$

VIII. THE TRANSFORMATION TO ADD A BOUND STATE

In Sec. VI the spectrum of the matrix Schrödinger operator has been changed by completely removing a bound state from the discrete spectrum without changing the continuous spectrum. In this section, we add a new bound state to the spectrum of the matrix Schrödinger operator without changing the existing bound states and without changing the continuous spectrum. We determine the transformations of all relevant quantities when the new bound state is added. Since several bound states can be added in succession, the technique presented in this section can be used to add any number of new bound states to the discrete spectrum.

We start with the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_1^1(\mathbb{R}^+)$, the boundary matrices A and B describing the boundary condition as in (2.5)–(2.7), the regular solution $\varphi(k, x)$ satisfying the initial conditions (2.9), the Jost solution $f(k, x)$ satisfying (2.8), the spectral measure $d\rho$ described in (5.1), the Jost matrix $J(k)$ defined in (2.10), the scattering matrix $S(k)$ in (2.11), and N bound states with the energies $-\kappa_j^2$, the Gel'fand–Levitan normalization matrices C_j , the orthogonal projections Q_j onto $\text{Ker}[J(i\kappa_j)]$, the orthogonal projections P_j onto $\text{Ker}[J(i\kappa_j)^\dagger]$, the Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$, the Marchenko normalized bound-state solutions $\Psi_j(x)$, the dependency matrices D_j , and the bound-state multiplicities m_j for $1 \leq j \leq N$. We then add a new bound state with the energy $-\tilde{\kappa}_{N+1}^2$ with the Gel'fand–Levitan normalization matrix $\tilde{C}_{N+1}$, where the $n \times n$ matrix $\tilde{C}_{N+1}$ is hermitian and nonnegative and has rank $\tilde{m}_{N+1}$. We use a tilde to identify the quantities obtained after the addition of the bound state at $k = i\tilde{\kappa}_{N+1}$ with the Gel'fand–Levitan normalization matrix $\tilde{C}_{N+1}$. Thus, the perturbed Schrödinger operator involves the potential $\tilde{V}(x)$, the boundary matrices $\tilde{A}$ and $\tilde{B}$, the regular solution $\tilde{\varphi}(k, x)$, the Jost solution $\tilde{f}(k, x)$, the spectral measure $d\tilde{\rho}$, the Jost matrix $\tilde{J}(k)$, the scattering matrix $\tilde{S}(k)$, and $N+1$ bound states with the energies $-\tilde{\kappa}_j^2$, the Gel'fand–Levitan normalization matrices $\tilde{C}_j$, the orthogonal projections $\tilde{Q}_j$ onto $\text{Ker}[\tilde{J}(i\tilde{\kappa}_j)]$, the orthogonal projections $\tilde{P}_j$ onto $\text{Ker}[\tilde{J}(i\tilde{\kappa}_j)^\dagger]$, the Gel'fand–Levitan normalized bound-state solutions $\tilde{\Phi}_j(x)$, the Marchenko normalized bound-state solutions $\tilde{\Psi}_j(x)$, the dependency matrices $\tilde{D}_j$, and the bound-state multiplicities $\tilde{m}_j$ for $1 \leq j \leq N+1$. Since the first N bound states are unchanged, we know that $\tilde{\kappa}_j = \kappa_j$ and $\tilde{m}_j = m_j$ for $1 \leq j \leq N$. Initially, it may be unclear if we have $\tilde{Q}_j = Q_j$ and $\tilde{C}_j = C_j$ for $1 \leq j \leq N$. In Theorem 8.8 we confirm that we indeed have $\tilde{Q}_j = Q_j$ and $\tilde{C}_j = C_j$ for $1 \leq j \leq N$.

To determine the transformations of all relevant quantities resulting from the addition of the new bound state, we use the Gel'fand–Levitan method described in Sec. V with the input consisting the unperturbed quantities and the perturbation specified by $\tilde{\kappa}_{N+1}$ and $\tilde{C}_{N+1}$. The Gel'fand–Levitan method enables us to explicitly determine (5.15), (5.18), and (5.20) when the spectral measures $d\tilde{\rho}$ and $d\rho$ differ from each other only in the bound states. By specifying $\tilde{C}_{N+1}$ we also specify the positive integer $\tilde{m}_{N+1}$ satisfying $1 \leq \tilde{m}_{N+1} \leq n$. The specification of $\tilde{C}_{N+1}$ also yields an orthogonal projection matrix $\tilde{Q}_{N+1}$ with rank $\tilde{m}_{N+1}$ in such a way that

$$\tilde{C}_{N+1} \tilde{Q}_{N+1} = \tilde{Q}_{N+1} \tilde{C}_{N+1} = \tilde{C}_{N+1}. \quad (8.1)$$

The matrix $\tilde{C}_{N+1}$ must be related to the orthogonal projection $\tilde{Q}_{N+1}$ and a hermitian positive matrix $\tilde{\mathbf{H}}_{N+1}^{-1/2}$ as

$$\tilde{C}_{N+1} = \tilde{\mathbf{H}}_{N+1}^{-1/2} \tilde{Q}_{N+1}, \quad (8.2)$$

which is the analog of (3.27). For the construction of the $n \times n$ matrix $\tilde{\mathbf{H}}_{N+1}^{-1/2}$, we proceed as follows. We use an $n \times n$ hermitian nonnegative matrix $\tilde{\mathbf{G}}_{N+1}$ satisfying the equalities

$$\tilde{\mathbf{G}}_{N+1} \tilde{Q}_{N+1} = \tilde{Q}_{N+1} \tilde{\mathbf{G}}_{N+1} = \tilde{\mathbf{G}}_{N+1}, \quad (8.3)$$

in such a way that the restriction of $\tilde{\mathbf{G}}_{N+1}$ to $\tilde{Q}_{N+1} \mathcal{C}^n$ is invertible. We then define the matrix $\tilde{\mathbf{H}}_{N+1}$ as

$$\tilde{\mathbf{H}}_{N+1} := I - \tilde{Q}_{N+1} + \tilde{\mathbf{G}}_{N+1}. \quad (8.4)$$

From (8.4) it follows that $\tilde{\mathbf{H}}_{N+1}$ is indeed hermitian and nonnegative. In fact, $\tilde{\mathbf{H}}_{N+1}$ is positive and hence invertible. The positivity can be established by showing that the kernel of $\tilde{\mathbf{H}}_{N+1}$ contains only the zero vector in $\mathcal{C}^n$. For the proof we proceed as follows. Since $\tilde{\mathbf{G}}_{N+1}$ is nonnegative, there exists a nonnegative matrix $\tilde{\mathbf{G}}_{N+1}^{1/2}$ so that $\tilde{\mathbf{G}}_{N+1}^{1/2} \tilde{\mathbf{G}}_{N+1}^{1/2} = \tilde{\mathbf{G}}_{N+1}$. Thus, we can write (8.4) in the equivalent form

$$\tilde{\mathbf{H}}_{N+1} = I - \tilde{Q}_{N+1} + \tilde{\mathbf{G}}_{N+1}^{1/2} \tilde{\mathbf{G}}_{N+1}^{1/2}. \quad (8.5)$$

For any column vector v in $\mathbb{C}^N$, from (8.5) we get

$$v^\dagger \tilde{\mathbf{H}}_{N+1} v = |(I - \tilde{Q}_{N+1})v|^2 + |\tilde{\mathbf{G}}_{N+1}^{1/2} v|^2. \quad (8.6)$$

If v belongs to the kernel of $\tilde{\mathbf{H}}_{N+1}$, then the left-hand side in (8.6) is zero, and hence the right-hand side yields

$$v = \tilde{Q}_{N+1} v, \quad \tilde{\mathbf{G}}_{N+1}^{1/2} v = 0. \quad (8.7)$$

After premultiplying the second equality in (8.7) by $\tilde{\mathbf{G}}_{N+1}^{1/2}$, we conclude that $\tilde{\mathbf{G}}_{N+1} v = 0$ when v belongs to $\tilde{Q}_{N+1} \mathbb{C}^n$. On the other hand, we know that the restriction of $\tilde{\mathbf{G}}_{N+1}$ to $\tilde{Q}_{N+1} \mathbb{C}^n$ is invertible. Thus, the column vector v must be the zero vector. This establishes the fact that $\tilde{\mathbf{H}}_{N+1}$ is invertible. We remark that, with the help of (8.4), we prove that $\tilde{\mathbf{H}}_{N+1}$ satisfies

$$\tilde{\mathbf{H}}_{N+1} \tilde{Q}_{N+1} = \tilde{Q}_{N+1} \tilde{\mathbf{H}}_{N+1}. \quad (8.8)$$

We remark that we cannot choose $\tilde{C}_{N+1}$ arbitrarily as any nonnegative hermitian matrix. The equalities in (8.1)–(8.4) impose a restriction on the eligible choices for $\tilde{C}_{N+1}$. Different choices for $\tilde{C}_{N+1}$ *a priori* yield different $\tilde{V}(x)$, $\tilde{\varphi}(k, x)$, $\tilde{f}(k, x)$, $\tilde{f}(k)$, $\tilde{P}_{N+1}$, and $\tilde{B}$. The orthogonal projection matrix $\tilde{P}_{N+1}$ is defined as in (8.54). The boundary matrix $\tilde{A}$, however, is the same for any of those different choices for $\tilde{C}_{N+1}$ and it is equal to A .

In the next proposition we show that the orthogonal projection $\tilde{Q}_{N+1}$, the matrix $\tilde{\mathbf{G}}_{N+1}$, and the matrix $\tilde{\mathbf{H}}_{N+1}$ are uniquely determined by $\tilde{C}_{N+1}$.

Proposition 8.1. Assume that a nonnegative matrix $\tilde{C}_{N+1}$ is related to a positive invertible matrix $\tilde{\mathbf{H}}_{N+1}$ and an orthogonal projection $\tilde{Q}_{N+1}$ as in (8.2) in such a way that (8.1), (8.3), and (8.4) hold, with $\tilde{\mathbf{G}}_{N+1}$ being the matrix appearing in (8.3) and (8.4). Then, $\tilde{C}_{N+1}$ uniquely determines the matrices $\tilde{Q}_{N+1}$, $\tilde{\mathbf{G}}_{N+1}$, and $\tilde{\mathbf{H}}_{N+1}$.

Proof. Let us decompose $\mathbb{C}^n$ as a direct sum given by

$$\mathbb{C}^n = [\tilde{Q}_{N+1} \mathbb{C}^n] \oplus [(I - \tilde{Q}_{N+1}) \mathbb{C}^n]. \quad (8.9)$$

From (8.3) we get

$$\tilde{\mathbf{G}}_{N+1} = [\tilde{\mathbf{G}}_{N+1}]_1 \oplus 0, \quad (8.10)$$

where we use $[\tilde{\mathbf{G}}_{N+1}]_1$ to denote the restriction of $\tilde{\mathbf{G}}_{N+1}$ to $\tilde{Q}_{N+1} \mathbb{C}^n$ and use 0 to denote the zero operator on $[(I - \tilde{Q}_{N+1}) \mathbb{C}^n]$. From (8.4) and (8.10) we see that we have

$$\tilde{\mathbf{H}}_{N+1} = [\tilde{\mathbf{G}}_{N+1}]_1 \oplus I, \quad (8.11)$$

where I on the right-hand side denotes the identity operator on $[(I - \tilde{Q}_{N+1}) \mathbb{C}^n]$. Since $\tilde{\mathbf{H}}_{N+1}$ is positive and invertible in $\mathbb{C}^n$, it follows that the positive matrix $\tilde{\mathbf{H}}_{N+1}^{1/2}$ is well defined by using the analog of (3.26). Hence, the positive matrix $\tilde{\mathbf{H}}_{N+1}^{-1/2}$ is also well defined as the inverse of $\tilde{\mathbf{H}}_{N+1}^{1/2}$. From (8.11) we obtain

$$\tilde{\mathbf{H}}_{N+1}^{-1/2} = ([\tilde{\mathbf{G}}_{N+1}]_1)^{-1/2} \oplus I, \quad (8.12)$$

where we used the fact that $[\tilde{\mathbf{G}}_{N+1}]_1$ is invertible on $\tilde{Q}_{N+1} \mathbb{C}^n$. From (8.2) and (8.12) we observe the decomposition

$$\tilde{C}_{N+1} = ([\tilde{\mathbf{G}}_{N+1}]_1)^{-1/2} \oplus 0. \quad (8.13)$$

Since $([\tilde{\mathbf{G}}_{N+1}]_1)^{-1/2}$ is bijective on $\tilde{Q}_{N+1} \mathbb{C}^n$, it follows from (8.13) that $\tilde{C}_{N+1}$ is onto $\tilde{Q}_{N+1} \mathbb{C}^n$. Consequently, we conclude that the range of $\tilde{C}_{N+1}$ is $\tilde{Q}_{N+1} \mathbb{C}^n$ and that $\tilde{Q}_{N+1}$ is the orthogonal projection onto the range of $\tilde{C}_{N+1}$. Thus, $\tilde{Q}_{N+1}$ is uniquely determined by $\tilde{C}_{N+1}$. From (8.13) it follows that $([\tilde{\mathbf{G}}_{N+1}]_1)^{-1/2}$ is uniquely determined by $\tilde{C}_{N+1}$, and hence $[\tilde{\mathbf{G}}_{N+1}]_1$ is also uniquely determined by $\tilde{C}_{N+1}$. Moreover, from (8.10) we see that $\tilde{\mathbf{G}}_{N+1}$ is also uniquely determined by $\tilde{C}_{N+1}$. Finally, with the help of (8.4) we see that $\tilde{\mathbf{H}}_{N+1}$ is also uniquely determined by $\tilde{C}_{N+1}$. $\square$

Having related $\tilde{C}_{N+1}$ to $\tilde{\mathbf{H}}_{N+1}$ and $\tilde{Q}_{N+1}$ as in (8.2), we form the Gel'fand–Levitan kernel $G(x, y)$ as

$$G(x, y) = \varphi(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1}^2 \varphi(i\tilde{\kappa}_{N+1}, y)^\dagger, \quad (8.14)$$

where we recall that $\varphi(k, x)$ is the unperturbed regular solution. The next theorem shows how we obtain the solution to the Gel'fand–Levitan system (5.38) of integral equations with the kernel $G(x, y)$ given in (8.14).

Theorem 8.2. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_1^1(\mathbb{R}^+)$, with the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), with the regular solution $\varphi(k, x)$

satisfying the initial conditions (2.9), and containing N bound states with the energies $-\kappa_j^2$ and the Gel'fand–Levitan normalization matrices C_j for $1 \leq j \leq N$. Choose the kernel $G(x, y)$ of the Gel'fand–Levitan system (5.38) of integral equations as in (8.14), where $\tilde{\kappa}_{N+1}$ is a positive constant distinct from κ_j for $1 \leq j \leq N$ and $\tilde{C}_{N+1}$ is the $n \times n$ selfadjoint nonnegative matrix in (8.2). Then, the corresponding solution $\mathcal{A}(x, y)$ to (5.38) is given by

$$\mathcal{A}(x, y) = -\xi_{N+1}(x) \Omega_{N+1}(x)^+ \xi_{N+1}(y)^\dagger, \quad 0 \leq y < x, \quad (8.15)$$

with the $n \times n$ matrices $\xi_{N+1}(x)$ and $\Omega_{N+1}(x)$ defined as

$$\xi_{N+1}(x) := \varphi(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1}, \quad (8.16)$$

$$\Omega_{N+1}(x) := \tilde{Q}_{N+1} + \int_0^x dy \xi_{N+1}(y)^\dagger \xi_{N+1}(y), \quad (8.17)$$

where we recall that $\Omega_{N+1}(x)^+$ denotes the Moore–Penrose inverse of $\Omega_{N+1}(x)$.

Proof. The proof is similar to the proof of Theorem 6.1. We decompose $\mathbb{C}^n$ into the direct sum as in (8.9). From (8.17) we observe that $\Omega_{N+1}(x)$ maps $\tilde{Q}_{N+1} \mathbb{C}^n$ into $\tilde{Q}_{N+1} \mathbb{C}^n$ and that it acts as the zero mapping on $(I - \tilde{Q}_{N+1}) \mathbb{C}^n$. Hence, we can decompose $\Omega_{N+1}(x)$ as

$$\Omega_{N+1}(x) = [\Omega_{N+1}(x)]_1 \oplus 0, \quad (8.18)$$

where we use $[\Omega_{N+1}(x)]_1$ to denote the restriction of $\Omega_{N+1}(x)$ to $\tilde{Q}_{N+1} \mathbb{C}^n$. Next, we prove that $[\Omega_{N+1}(x)]_1$ is invertible on $\tilde{Q}_{N+1} \mathbb{C}^n$ by proceeding as follows. If v is a column vector in $\tilde{Q}_{N+1} \mathbb{C}^n$, then as seen from (8.18) we have $\Omega_{N+1}(x)v = [\Omega_{N+1}(x)]_1 v$. Hence, using v in (8.17) we obtain

$$v^\dagger [\Omega_{N+1}(x)]_1 v = |v|^2 + \int_0^x dy |\xi_{N+1}(y)v|^2. \quad (8.19)$$

If v belongs to the kernel of $[\Omega_{N+1}(x)]_1$, then the left-hand side in (8.19) must be zero. Each of the two terms on the right-hand side is nonnegative and hence we must also have $|v|^2 = 0$, which implies that v is the zero vector in $\tilde{Q}_{N+1} \mathbb{C}^n$. This establishes the fact that $[\Omega_{N+1}(x)]_1$ is invertible on $\tilde{Q}_{N+1} \mathbb{C}^n$. We then use the analog of (3.5) and conclude that

$$\Omega_{N+1}(x)^+ = [\Omega_{N+1}(x)]_1^{-1} \oplus 0. \quad (8.20)$$

From (8.18) and (8.20) we observe that

$$\Omega_{N+1}(x) \Omega_{N+1}(x)^+ = \Omega_{N+1}(x)^+ \Omega_{N+1}(x) = I \oplus 0 = \tilde{Q}_{N+1}, \quad (8.21)$$

which yields the analog of (6.18) and (6.19). The proof of the theorem is completed by exploiting the analogy between $\Omega_{N+1}(x)$ defined in (8.17) and $W_N(x)$ defined in (6.9) and by using the steps given in the proof of Theorem 6.1. $\square$

In the next theorem we present some relevant properties of the quantity $\mathcal{A}(x, y)$ appearing in (8.15). This theorem is the counterpart of Theorem 6.2, and hence their proofs are similar.

Theorem 8.3. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_1^1(\mathbb{R}^+)$, with the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), and containing N bound states with the energies $-\kappa_j^2$ and the Gel'fand–Levitan normalization matrices C_j for $1 \leq j \leq N$. Let $\mathcal{A}(x, y)$ be the matrix-valued quantity given in (8.15), and define the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) := V(x) - 2 \frac{d}{dx} \left[\xi_{N+1}(x) \Omega_{N+1}(x)^+ \xi_{N+1}(x)^\dagger \right], \quad (8.22)$$

where $\xi_{N+1}(x)$ and $\Omega_{N+1}(x)$ are the $n \times n$ matrices defined in (8.16) and (8.17), respectively. Then, we have the following:

- The quantity $\mathcal{A}(x, y)$ is continuously differentiable in the set $\mathbb{D}$, where $\mathbb{D}$ is the subset of $\mathbb{R}^2$ defined in (5.16).
- The second partial derivatives $\mathcal{A}_{xx}(x, y)$ and $\mathcal{A}_{yy}(x, y)$ are locally integrable in $\mathbb{D}$.
- The quantity $\mathcal{A}(x, y)$ satisfies the second-order matrix-valued partial differential equation (5.17).
- The perturbed potential $\tilde{V}(x)$ defined in (8.22) is related to the unperturbed potential $V(x)$ and the quantity $\mathcal{A}(x, y)$ as in (5.18).

Proof. By arguing as in the proof of (6.22), we prove that the x -derivative of $\Omega_{N+1}(x)^+$ is given by

$$[\Omega_{N+1}(x)^+]' = -\Omega_{N+1}(x)^+ \xi_{N+1}(x)^\dagger \xi_{N+1}(x) \Omega_{N+1}(x)^+. \quad (8.23)$$

We then use the steps analogous to the steps in the proof of Theorem 6.2 and establish the results stated in (a)–(d). $\square$

The results presented in the next proposition are needed later on. Those results are related to the large spacial asymptotics of the unperturbed regular solution $\varphi(k, x)$ and an exponentially growing matrix solution to (2.1) when $k = i\tilde{\kappa}_{N+1}$ with the unperturbed potential $V(x)$.

Proposition 8.4. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_1^1(\mathbb{R}^+)$, with the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), with the regular solution $\varphi(k, x)$ satisfying the initial conditions (2.9), and containing N bound states with the eigenvalues $-\kappa_j^2$ and the Gel'fand–Levitan normalization matrices C_j for $1 \leq j \leq N$. Let $\tilde{\kappa}_{N+1}$ be a positive constant distinct from κ_j for $1 \leq j \leq N$. We have the following:

- (a) For each nonnegative constant a and each fixed $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, the Schrödinger equation (2.1) has an $n \times n$ matrix-valued solution $g(k, x)$ satisfying

$$g(k, x) = e^{-ikx}[I + o(1)], \quad g'(k, x) = -e^{-ikx}[ikI + o(1)], \quad x \rightarrow +\infty.$$

Furthermore, for each fixed $k \in \overline{\mathbb{C}^+}$, the Jost solution $f(k, x)$ and the solution $g(k, x)$ form a fundamental set of solutions to (2.1).

- (b) For each fixed parameter ε with $0 < \varepsilon < 1$, the quantity $g(i\tilde{\kappa}_{N+1}, x)$ has the asymptotics

$$g(i\tilde{\kappa}_{N+1}, x) = e^{\tilde{\kappa}_{N+1}x}[I + O(q_6(x) + q_{14}(a, \varepsilon, x) + q_{15}(\varepsilon, x))], \quad x \rightarrow +\infty, \quad (8.24)$$

where $q_6(x)$ is the quantity in (6.36) and we have defined

$$q_{14}(a, \varepsilon, x) := e^{-2\varepsilon\tilde{\kappa}_{N+1}x} \int_a^{(1-\varepsilon)x} dy |V(y)|, \quad (8.25)$$

$$q_{15}(\varepsilon, x) := \int_{(1-\varepsilon)x}^x dy e^{-2\tilde{\kappa}_{N+1}(x-y)} |V(y)|. \quad (8.26)$$

- (c) The x -derivative $g'(i\tilde{\kappa}_{N+1}, x)$ has the asymptotics

$$g'(i\tilde{\kappa}_{N+1}, x) = \tilde{\kappa}_{N+1} e^{\tilde{\kappa}_{N+1}x}[I + O(q_6(x) + q_{14}(a, \varepsilon, x) + q_{15}(\varepsilon, x))], \quad x \rightarrow +\infty. \quad (8.27)$$

- (d) The quantity $\varphi(i\tilde{\kappa}_{N+1}, x)$ has the representation

$$\varphi(i\tilde{\kappa}_{N+1}, x) = f(i\tilde{\kappa}_{N+1}, x) K_{N+1} + g(i\tilde{\kappa}_{N+1}, x) L_{N+1}, \quad (8.28)$$

for some $n \times n$ matrices K_{N+1} and L_{N+1} , where the matrix L_{N+1} is invertible.

Proof. We remark that (a) follows from Propositions 3.2.2 and 3.2.3 of Ref. 7. For any $a \geq 0$, from (3.2.39) and (3.2.63) of Ref. 7 we have

$$e^{-\tilde{\kappa}_{N+1}x} g(i\tilde{\kappa}_{N+1}, x) = I + O(q_6(x) + q_{16}(a, x)), \quad x \rightarrow +\infty, \quad (8.29)$$

where we have defined $q_{16}(a, x)$ as

$$q_{16}(a, x) := \int_a^x dy e^{-2\tilde{\kappa}_{N+1}(x-y)} |V(y)|.$$

We remark that, even though the result of (3.2.63) in Ref. 7 is derived for a large positive a , it also holds for any smaller a -value, and hence (8.29) holds for any nonnegative a . We also remark that we have suppressed the a -dependence in $g(k, x)$ in our notation. Using any ε satisfying $0 < \varepsilon < 1$, we can write (8.29) as

$$e^{-\tilde{\kappa}_{N+1}x} g(i\tilde{\kappa}_{N+1}, x) = I + O(q_6(x) + q_{17}(a, \varepsilon, x) + q_{15}(\varepsilon, x)), \quad x \rightarrow +\infty, \quad (8.30)$$

where we have defined $q_{17}(a, \varepsilon, x)$ as

$$q_{17}(a, \varepsilon, x) := \int_a^{(1-\varepsilon)x} dy e^{-2\tilde{\kappa}_{N+1}(x-y)} |V(y)|.$$

We note that (8.24) follows from (8.30) after using $y \leq (1 - \varepsilon)x$, and hence the proof of (b) is complete. For the proof of (c), we use

$$\frac{d}{dx} [e^{-\tilde{\kappa}_{N+1}x} g(i\tilde{\kappa}_{N+1}, x)] = O(q_{16}(a, x)), \quad x \rightarrow +\infty, \quad (8.31)$$

which is implied by (3.2.71) of Ref. 7. By proceeding as in the proof of (8.24), with the help of (8.30) and (8.31) we obtain (8.27), which completes the proof of (c). For the proof of (d) we proceed as follows. From (a) we know that $f(i\tilde{\kappa}_{N+1}, x)$ and $g(i\tilde{\kappa}_{N+1}, x)$ form a fundamental

set of solutions to (2.1) when $k = i\tilde{\kappa}_{N+1}$. Hence, $\varphi(i\tilde{\kappa}_{N+1}, x)$ can be expressed as in (8.28). The invertibility of L_{N+1} is proved by showing that the kernel of L_{N+1} contains only the zero vector in $\mathbb{C}^n$, and for the proof we proceed as follows. For any column vector v satisfying $L_{N+1}v = 0$, from (8.28) we get

$$\varphi(i\tilde{\kappa}_{N+1}, x)v = f(i\tilde{\kappa}_{N+1}, x) K_{N+1}v. \quad (8.32)$$

From the first equality of (2.8) we observe that the right-hand side of (8.32) behaves as $O(e^{-\tilde{\kappa}_{N+1}x})$ as $x \rightarrow +\infty$, and hence that right-hand side is a square-integrable solution to (2.1) at $k = i\tilde{\kappa}_{N+1}$. Further, the vector $K_{N+1}v$ cannot be zero unless we have $v = 0$. Otherwise, we would have $\varphi(i\tilde{\kappa}_{N+1}, x)v = 0$ for all $x \in [0, +\infty)$, which would contradict (2.7) and (2.9). On the other hand, the left-hand side of (8.32) cannot be a square-integrable solution to (2.1) because otherwise we would have a bound-state at $k = i\tilde{\kappa}_{N+1}$ for the unperturbed problem unless v is the zero vector. Thus, the proof of (d) is complete. $\square$

The next theorem is the counterpart of Theorem 6.3. It shows that the quantity $\tilde{\varphi}(k, x)$ in (5.15) with $\mathcal{A}(x, y)$ in (8.15) is the perturbed regular solution to (2.1) with the perturbed potential $\tilde{V}(x)$ in (8.22) and with the appropriately determined boundary matrices $\tilde{A}$ and $\tilde{B}$.

Theorem 8.5. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L^1_1(\mathbb{R}^+)$, with the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), with the regular solution $\varphi(k, x)$ satisfying the initial conditions (2.9), and with N bound states with the energies $-\kappa_j^2$ and the Gel'fand–Levitan normalization matrices C_j for $1 \leq j \leq N$. Let $\tilde{\varphi}(k, x)$ be the quantity defined in (5.15) with $\mathcal{A}(x, y)$ expressed as in (8.15). Then, we have the following:

- (a) The quantity $\tilde{\varphi}(k, x)$ is a solution to the perturbed Schrödinger equation (2.1) with the potential $\tilde{V}(x)$ given in (8.22). We can express $\tilde{\varphi}(k, x)$ also as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \xi_{N+1}(x) \Omega_{N+1}(x)^+ \int_0^x dy \xi_{N+1}(y)^\dagger \varphi(k, y). \quad (8.33)$$

- (b) The perturbed potential $\tilde{V}(x)$ appearing in (8.22) satisfies (2.2). Moreover, for every nonnegative constant a and the parameter ε satisfying $0 < \varepsilon < 1$, the potential increment $\tilde{V}(x) - V(x)$ has the asymptotic behavior

$$\tilde{V}(x) - V(x) = O(q_{18}(a, \varepsilon, x)), \quad x \rightarrow +\infty,$$

with $q_{18}(a, \varepsilon, x)$ defined as

$$q_{18}(a, \varepsilon, x) := x e^{-2\tilde{\kappa}_{N+1}x} + q_6(x) + q_{14}(a, \varepsilon, x) + q_{15}(\varepsilon, x),$$

and where we recall that $q_6(x)$, $q_{14}(a, \varepsilon, x)$, and $q_{15}(\varepsilon, x)$ are the quantities in (6.36), (8.25), and (8.26), respectively. If the unperturbed potential V is further restricted to $L^1_{1+\varepsilon}(\mathbb{R}^+)$ for some fixed $\varepsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L^1_\varepsilon(\mathbb{R}^+)$.

- (c) If the unperturbed potential $V(x)$ is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0,$$

for some positive constants α and x_0 and with c denoting a generic constant, then for every $0 < \varepsilon < 1$ the potential increment $\tilde{V}(x) - V(x)$ has the asymptotic behavior as $x \rightarrow +\infty$ given by

$$\tilde{V}(x) - V(x) = \begin{cases} O(e^{-\alpha x} + e^{-2\varepsilon\tilde{\kappa}_{N+1}x}), & \alpha \leq 2\tilde{\kappa}_{N+1}, \\ O(e^{-2\varepsilon\tilde{\kappa}_{N+1}x}), & \alpha > 2\tilde{\kappa}_{N+1}. \end{cases} \quad (8.34)$$

- (d) If the unperturbed potential $V(x)$ has compact support, then the perturbed potential $\tilde{V}(x)$ has the asymptotic behavior

$$\tilde{V}(x) = O(x e^{-2\tilde{\kappa}_{N+1}x}), \quad x \rightarrow +\infty.$$

- (e) For $k \neq i\tilde{\kappa}_{N+1}$, the perturbed quantity $\tilde{\varphi}(k, x)$ can be expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \frac{1}{k^2 + \tilde{\kappa}_{N+1}^2} \xi_{N+1}(x) \Omega_{N+1}(x)^+ \left[\xi'_{N+1}(x)^\dagger \varphi(k, x) - \xi_{N+1}(x)^\dagger \varphi'(k, x) \right]. \quad (8.35)$$

- (f) The perturbed quantity $\tilde{\varphi}(k, x)$ satisfies the initial conditions (5.14) where the matrices $\tilde{A}$ and $\tilde{B}$ are expressed in terms of the unperturbed boundary matrices A and B and the Gel'fand–Levitan normalization matrix $\tilde{C}_{N+1}$ in (8.2) as

$$\tilde{A} = A, \quad \tilde{B} = B - A \tilde{C}_{N+1}^2 A^\dagger. \quad (8.36)$$

- (g) The matrices $\tilde{A}$ and $\tilde{B}$ appearing in (8.36) satisfy (2.6) and (2.7). Hence, as a consequence of (a) and (f), the quantity $\tilde{\varphi}(k, x)$ corresponds to the regular solution to the matrix Schrödinger equation (2.1) with the potential $\tilde{V}(x)$ in (8.22) and with the selfadjoint boundary condition (2.5) with A and B there replaced with $\tilde{A}$ and $\tilde{B}$, respectively.

Proof. The proof of (a) is obtained by using Theorems 5.2 and 8.3. The proof of (d) is obtained by using (b). The proofs of (e)–(g) are similar to the proofs of (e)–(g) of Theorem 6.3. In order to prove (b) and (c), we first obtain some estimates by proceeding as follows. With the help of (6.41), (6.42), (8.16), (8.24), (8.27), and (8.28), we get the asymptotics for $\xi_{N+1}(x)$ defined in (8.16) as

$$\xi_{N+1}(x) = e^{\tilde{\kappa}_{N+1}x} [L_{N+1} + O(q_{19}(a, \varepsilon, x))] \tilde{C}_{N+1}, \quad x \rightarrow +\infty, \quad (8.37)$$

$$\xi'_{N+1}(x) = \tilde{\kappa}_{N+1} e^{\tilde{\kappa}_{N+1}x} [L_{N+1} + O(q_{19}(a, \varepsilon, x))] \tilde{C}_{N+1}, \quad x \rightarrow +\infty, \quad (8.38)$$

where we have defined

$$q_{19}(a, \varepsilon, x) := e^{-2\tilde{\kappa}_{N+1}x} + q_6(x) + q_{14}(a, \varepsilon, x) + q_{15}(\varepsilon, x),$$

by recalling that L_{N+1} is the invertible matrix appearing in (8.28). In terms of the parameters a and ε , let us choose a positive constant b satisfying $b \geq a/(1 - \varepsilon)$ and

$$q_{19}(a, \varepsilon, x) < 1, \quad x \geq b. \quad (8.39)$$

Using (8.37)–(8.39), for the quantity $\Omega_{N+1}(x)$ given in (8.17) we obtain the asymptotics as $x \rightarrow +\infty$ given by

$$\Omega_{N+1}(x) - \Omega_{N+1}(b) = \frac{1}{2\tilde{\kappa}_{N+1}} \left(e^{2\tilde{\kappa}_{N+1}x} - e^{2\tilde{\kappa}_{N+1}b} \right) \tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} + O(q_{20}(b, \varepsilon, x)), \quad (8.40)$$

where $q_{20}(b, \varepsilon, x)$ is some quantity with the asymptotic behavior

$$O(q_{20}(b, \varepsilon, x)) = \tilde{C}_{N+1} \int_b^x dy e^{2\tilde{\kappa}_{N+1}y} [O(q_{19}(a, \varepsilon, y))] \tilde{C}_{N+1}, \quad x \rightarrow +\infty. \quad (8.41)$$

Using integration by parts, we obtain

$$\int_b^x dy e^{2\tilde{\kappa}_{N+1}y} q_6(y) = \frac{1}{2\tilde{\kappa}_{N+1}} \left[e^{2\tilde{\kappa}_{N+1}x} q_6(x) - e^{2\tilde{\kappa}_{N+1}b} q_6(b) + q_{21}(b, x) \right], \quad (8.42)$$

$$\int_b^x dy e^{2\tilde{\kappa}_{N+1}y} q_{14}(a, \varepsilon, y) = -\frac{1}{2\tilde{\kappa}_{N+1}} q_{22}(b, \varepsilon, x) + \frac{1}{2(1 - \varepsilon)\tilde{\kappa}_{N+1}} \left[e^{2\tilde{\kappa}_{N+1}x} q_{14}(a, \varepsilon, x) - e^{2\tilde{\kappa}_{N+1}b} q_{14}(a, \varepsilon, b) \right], \quad (8.43)$$

$$\int_b^x dy e^{2\tilde{\kappa}_{N+1}y} q_{15}(\varepsilon, y) = \frac{1}{2\tilde{\kappa}_{N+1}} \left[e^{2\tilde{\kappa}_{N+1}x} q_{15}(\varepsilon, x) - e^{2\tilde{\kappa}_{N+1}b} q_{15}(\varepsilon, b) - q_{21}(b, x) + (1 - \varepsilon) q_{22}(b, \varepsilon, x) \right], \quad (8.44)$$

where we have defined

$$q_{21}(b, x) := \int_b^x dy e^{2\tilde{\kappa}_{N+1}y} |V(y)|, \quad q_{22}(b, \varepsilon, x) := \int_b^x dy e^{2(1-\varepsilon)\tilde{\kappa}_{N+1}y} |V((1 - \varepsilon)y)|.$$

Using (8.40)–(8.44), we get the asymptotics for $\Omega_{N+1}(x)$ as

$$\Omega_{N+1}(x) = \frac{1}{2\tilde{\kappa}_{N+1}} e^{2\tilde{\kappa}_{N+1}x} \tilde{C}_{N+1} \left[L_{N+1}^\dagger L_{N+1} + O(q_{18}(a, \varepsilon, x)) \right] \tilde{C}_{N+1}, \quad x \rightarrow +\infty. \quad (8.45)$$

From (8.2) we see that we can decompose $\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}$ as in the direct sum given in (8.9), where we use $[\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]_1$ to denote the restriction of $\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}$ to $\tilde{Q}_{N+1} \mathbb{C}^n$. Thus, we have

$$\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} = [\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]_1 \oplus 0. \quad (8.46)$$

We remark that $[\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]_1$ is invertible, and this is proved by showing that the only vector v in $\tilde{Q}_{N+1} \mathbb{C}^n$ satisfying

$$[\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]_1 v = 0, \quad (8.47)$$

is the zero vector. For the proof we proceed as follows. From (8.1) and (8.47), we see that we equivalently need to show that

$$\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} v = 0, \quad v \in \tilde{Q}_{N+1} \mathbb{C}^n, \quad (8.48)$$

is possible only when v is the zero vector. From (8.48), with the help of (8.1) and (8.2) we obtain

$$v^\dagger \tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} v = \left| L_{N+1} \tilde{\mathbf{H}}_{N+1}^{-1/2} v \right|^2. \quad (8.49)$$

As a result of (8.48), the left-hand side of (8.49) is zero. Hence, the right-hand side of (8.49) implies $L_{N+1} \tilde{H}_{N+1}^{-1/2} v = 0$. Since the matrices L_{N+1} and $\tilde{H}_{N+1}^{-1/2}$ are both invertible, we conclude that v must be the zero vector. With the help of (3.5) and (8.46), we get

$$\left(\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} \right)^+ = \left([\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]_1 \right)^{-1} \oplus 0. \quad (8.50)$$

Using (3.5), (8.45), and (8.50), we obtain the asymptotics for $\Omega_{N+1}(x)^+$ as $x \rightarrow +\infty$ as

$$\Omega_{N+1}(x)^+ = 2\tilde{\kappa}_{N+1} e^{-2\tilde{\kappa}_{N+1}x} \tilde{Q}_{N+1} \left[\left(\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} \right)^+ + O(q_{18}(a, \varepsilon, x)) \right] \tilde{Q}_{N+1} \oplus 0,$$

which is equivalent to the asymptotics as $x \rightarrow +\infty$ given by

$$\Omega_{N+1}(x)^+ = 2\tilde{\kappa}_{N+1} e^{-2\tilde{\kappa}_{N+1}x} \tilde{Q}_{N+1} \left[\left(\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} \right)^+ + O(q_{18}(a, \varepsilon, x)) \right] \tilde{Q}_{N+1}. \quad (8.51)$$

Next, by using (8.23), (8.37), (8.51), and the second equality of (3.2), we obtain the asymptotics as $x \rightarrow +\infty$ for the x -derivative of $\Omega_{N+1}(x)^+$ as

$$[\Omega(x)_{N+1}^+] = -4\tilde{\kappa}_{N+1}^2 e^{-2\tilde{\kappa}_{N+1}x} \tilde{Q}_{N+1} \left[\left(\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} \right)^+ + O(q_{18}(a, \varepsilon, x)) \right] \tilde{Q}_{N+1}. \quad (8.52)$$

Finally, we use (8.37), (8.38), (8.45), (8.51), (8.52), and we proceed as in the proof of Theorem 6.3 and complete the proofs of (b) and (c). $\square$

The next theorem is the counterpart of Theorem 6.4. When a new bound state is added to the spectrum, it describes how the Jost matrix, the scattering matrix, and the Jost solution transform and it also shows that the existing bound-state energies and their multiplicities remain unchanged. As in Theorem 6.4 we use the stronger assumption that the unperturbed potential V belongs to $L_2^1(\mathbb{R}^+)$ rather than $L_1^1(\mathbb{R}^+)$ so that the perturbed potential $\tilde{V}$ belongs to $L_1^1(\mathbb{R}^+)$, which is ensured by Theorem 8.5(b). We remark that this is a sufficiency assumption because the asymptotic estimates used in Theorem 8.5(b) on the potentials are not necessarily sharp.

Theorem 8.6. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_2^1(\mathbb{R}^+)$, with the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), the Jost solution $f(k, x)$ satisfying (2.8), the Jost matrix $J(k)$ defined as in (2.10), the scattering matrix $S(k)$ defined in (2.11), and with N bound states with the energies $-\kappa_j^2$ and the Gel'fand–Levitan normalization matrices C_j for $1 \leq j \leq N$. Let us use a tilde to identify the quantities associated with the perturbed Schrödinger operator so that we have the perturbed potential $\tilde{V}(x)$ expressed as in (8.22), the boundary condition (2.5) where the boundary matrices A and B are replaced with $\tilde{A}$ and $\tilde{B}$, respectively, given in (8.36), the perturbed Jost solution $\tilde{f}(k, x)$ satisfying the asymptotics (2.8), the perturbed regular solution $\tilde{\phi}(k, x)$ satisfying the initial conditions (5.14), the perturbed Jost matrix $\tilde{J}(k)$ defined as in (6.79), and the perturbed scattering matrix $\tilde{S}(k)$ defined in (6.80). We then have the following:

(a) The unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right] J(k), \quad k \in \overline{\mathbb{C}^+}, \quad (8.53)$$

where we let

$$\tilde{P}_{N+1} := L_{N+1} \tilde{C}_{N+1} [\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]^+ \tilde{C}_{N+1} L_{N+1}^\dagger, \quad (8.54)$$

with L_{N+1} being the invertible matrix defined in (8.28). Moreover, $\tilde{P}_{N+1}$ is the orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})^\dagger$ and it has rank $\tilde{m}_{N+1}$. We recall that $\tilde{m}_{N+1}$ is the rank of $\tilde{C}_{N+1}$.

(b) The matrix product $J(k)^\dagger J(k)$ for $k \in \mathbb{R}$ does not change under the perturbation, i.e. we have

$$\tilde{J}(k)^\dagger \tilde{J}(k) = J(k)^\dagger J(k), \quad k \in \mathbb{R}. \quad (8.55)$$

Hence, the continuous part of the spectral measure dp does not change under the perturbation.

(c) Under the perturbation, the determinant of the Jost matrix $J(k)$ is transformed as

$$\det [\tilde{J}(k)] = \left(\frac{k - i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \right)^{\tilde{m}_{N+1}} \det [J(k)], \quad k \in \overline{\mathbb{C}^+}. \quad (8.56)$$

(d) The perturbation adds a new bound state with the energy $-\tilde{\kappa}_{N+1}^2$ and multiplicity $\tilde{m}_{N+1}$ in such a way that the remaining bound states with the energies $-\kappa_j^2$ and multiplicities m_j for $1 \leq j \leq N$ are unchanged.

(e) Under the perturbation, the scattering matrix $S(k)$ undergoes the transformation

$$\tilde{S}(k) = \left[I + \frac{2i\tilde{\kappa}_{N+1}}{k - i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right] S(k) \left[I + \frac{2i\tilde{\kappa}_N}{k - i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right], \quad k \in \mathbb{R},$$

where $\tilde{P}_{N+1}$ is the orthogonal projection in (8.54).

(f) The perturbation changes the determinant of the scattering matrix as

$$\det[\tilde{S}(k)] = \left(\frac{k + i\tilde{\kappa}_{N+1}}{k - i\tilde{\kappa}_{N+1}} \right)^{2\tilde{m}_{N+1}} \det[S(k)], \quad k \in \mathbb{R}.$$

(g) Under the perturbation the Jost solution $f(k, x)$ is transformed into the perturbed Jost solution $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) - \frac{1}{k^2 + \tilde{\kappa}_{N+1}^2} \xi_{N+1}(x) \Omega_{N+1}(x)^+ q_{23}(k, x) \right] \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right], \quad (8.57)$$

where we have defined $q_{23}(k, x)$ as

$$q_{23}(k, x) := \left[\xi'_{N+1}(x)^\dagger f(k, x) - \xi_{N+1}(x)^\dagger f'(k, x) \right], \quad (8.58)$$

which is the analog of $q_7(k, x)$ defined in (6.87).

Proof. We establish (8.53) by proceeding as in the proof of Theorem 6.4(a), and for this we use (2.8) for $\tilde{f}(k, x)$, the analog of (2.14) for $\tilde{\varphi}(k, x)$, the analog of (6.91), as well as (8.35), (8.37), (8.38), and (8.51). To prove that $\tilde{P}_{N+1}$ is an orthogonal projection, we need to show that $\tilde{P}_{N+1}$ is selfadjoint and it satisfies

$$\tilde{P}_{N+1}^2 = \tilde{P}_{N+1}. \quad (8.59)$$

Using (3.2) and the fact that $\tilde{C}_{N+1}$ is selfadjoint, we prove that $\tilde{P}_{N+1}$ given in (8.54) is selfadjoint. We establish (8.59) by proceeding as follows. Using (8.50) we obtain

$$\left[\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} \right]^\dagger \left[\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1} \right] = I \oplus 0 = \tilde{Q}_{N+1}. \quad (8.60)$$

Then, using (8.1), (8.54), and (8.60), we confirm that (8.59) holds. Since L_{N+1} and $\tilde{H}_{N+1}^{-1/2}$ are invertible matrices, using (8.50) and (8.54) we prove that the rank of $\tilde{P}_{N+1}$ is $\tilde{m}_{N+1}$. To show that $\tilde{P}_{N+1}$ is the orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})^\dagger$, we proceed as follows. Using (8.53) and the fact that $\tilde{J}(i\tilde{\kappa}_{N+1})$ is invertible, we obtain

$$\text{Ker}[\tilde{J}(i\tilde{\kappa}_{N+1})^\dagger] = \text{Ker}[I - \tilde{P}_{N+1}],$$

which implies that the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})^\dagger$ is precisely the range of $\tilde{P}_{N+1}$. Hence, $\tilde{P}_{N+1}$ is the orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})^\dagger$. Thus, the proof of (a) is complete. By using (8.53) and by proceeding as in the proof of Theorem 6.4, we establish the stated results in (b), (c), (e), and (f). We note that (d) follows from (8.56) and Theorems 3.11.1 and 3.11.6 of Ref. 7. Finally, we prove (g) as in the proof of Theorem 6.4(g). $\square$

The following remark is the counterpart of Remark 6.5. It stresses the fact that the perturbed Jost solution $\tilde{f}(k, x)$ appearing in (8.57) does not have a singularity at $k = i\tilde{\kappa}_{N+1}$ despite the factor $k^2 + \tilde{\kappa}_{N+1}^2$ in a denominator on the right-hand side of (8.57).

Remark 8.7. In this remark we show that the right-hand side of (8.57) has a removable singularity at $k = i\tilde{\kappa}_{N+1}$, and hence the factor $k^2 + \tilde{\kappa}_{N+1}^2$ in a denominator there does not cause a singularity for the perturbed Jost solution $\tilde{f}(k, x)$. For the proof we proceed as follows. From (8.58) we get

$$q_{23}(i\tilde{\kappa}_{N+1}, x) = \xi'_{N+1}(x)^\dagger f(i\tilde{\kappa}_{N+1}, x) - \xi_{N+1}(x)^\dagger f'(i\tilde{\kappa}_{N+1}, x). \quad (8.61)$$

Since $\xi_{N+1}(x)$ and $f(i\tilde{\kappa}_{N+1}, x)$ are matrix-valued solutions to (2.1) with $k = i\tilde{\kappa}_{N+1}$, the Wronskian appearing on the right-hand side in (8.61) is independent of x . Thus, the constant value of $q_{23}(i\tilde{\kappa}_{N+1}, x)$ can be evaluated by letting $x \rightarrow +\infty$ in (8.61). Using (2.8), (8.37), and (8.38) on the right-hand side of (8.61), we obtain

$$q_{23}(i\tilde{\kappa}_{N+1}, x) = 2 \tilde{\kappa}_{N+1} \tilde{C}_{N+1} L_{N+1}^\dagger. \quad (8.62)$$

We can isolate the part of the term containing $k^2 + \tilde{\kappa}_{N+1}^2$ on the right-hand side of (8.57) by defining $q_{24}(k, x)$ as

$$q_{24}(k, x) := \frac{q_{23}(k, x)}{k^2 + \tilde{\kappa}_{N+1}^2} \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right]. \quad (8.63)$$

Comparing (8.57) and (8.63) we see that $\tilde{f}(k, x)$ does not have a singularity at $k = i\tilde{\kappa}_{N+1}$ provided the singularity of $q_{24}(k, x)$ at $k = i\tilde{\kappa}_{N+1}$ is removable. We can write the right-hand side of (8.63) as a sum of two terms by letting

$$q_{24}(k, x) = q_{25}(k, x) + q_{26}(k, x),$$

where we have defined

$$q_{25}(k, x) := \frac{q_{23}(k, x) - q_{23}(i\tilde{\kappa}_{N+1}, x)}{k^2 + \tilde{\kappa}_{N+1}^2} \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right], \quad (8.64)$$

$$q_{26}(k, x) := \frac{q_{23}(i\tilde{\kappa}_{N+1}, x)}{k^2 + \tilde{\kappa}_{N+1}^2} \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right]. \quad (8.65)$$

Since $f(k, x)$ and $f'(k, x)$ are analytic in k at $k = i\tilde{\kappa}_{N+1}$, the quantity $q_{23}(k, x) - q_{23}(i\tilde{\kappa}_{N+1}, x)$ has the behavior $O(k - i\tilde{\kappa}_{N+1})$ as $k \rightarrow i\tilde{\kappa}_{N+1}$, and hence the right-hand side of (8.64) has a removable singularity at $k = i\tilde{\kappa}_{N+1}$. On the other hand, using (8.62) in (8.65) we get

$$q_{26}(k, x) = \frac{2\tilde{\kappa}_{N+1} \tilde{C}_{N+1} L_{N+1}^\dagger}{k^2 + \tilde{\kappa}_{N+1}^2} \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right]. \quad (8.66)$$

From (8.1), (8.54), and (8.60), we conclude that

$$\tilde{C}_{N+1} L_{N+1}^\dagger \tilde{P}_{N+1} = \tilde{C}_{N+1} L_{N+1}^\dagger. \quad (8.67)$$

Using (8.67) on the right-hand side of (8.65), we obtain

$$q_{26}(k, x) = \frac{2\tilde{\kappa}_{N+1} \tilde{C}_{N+1} L_{N+1}^\dagger}{(k + i\tilde{\kappa}_{N+1})^2},$$

which indicates that $q_{26}(k, x)$ is analytic at $k = i\tilde{\kappa}_{N+1}$. Thus, $q_{24}(k, x)$ has only a removable singularity, confirming that $\tilde{f}(k, x)$ does not have a singularity at $k = i\tilde{\kappa}_{N+1}$.

The next theorem is the counterpart of Theorem 6.6. When we add a new bound state to the Schrödinger operator, it shows that the existing orthogonal projection matrices and Gel'fand–Levitan normalization matrices remain unchanged. It also shows how the spectral measure changes when the bound state is added. As in Theorem 8.6 we use the stronger assumption that the unperturbed potential V belongs to $L_2^1(\mathbb{R}^+)$ rather than $L_1^1(\mathbb{R}^+)$ so that the perturbed potential $\tilde{V}$ belongs to $L_1^1(\mathbb{R}^+)$, which is ensured by Theorem 8.5(b). We recall that this is a sufficiency assumption because the asymptotic estimates in Theorem 8.5(b) on the potentials are not necessarily sharp.

Theorem 8.8. Consider the unperturbed Schrödinger operator with the potential V satisfying (2.2) and belonging to $L_2^1(\mathbb{R}^+)$, with the selfadjoint boundary condition (2.5) described by the boundary matrices A and B satisfying (2.6) and (2.7), the regular solution $\varphi(k, x)$ satisfying the initial conditions (2.9), the Jost matrix $J(k)$ defined as in (2.10), the spectral measure $d\rho$ as in (5.1), and containing N bound states with the energies $-\kappa_j^2$, the Gel'fand–Levitan normalization matrices C_j , and the orthogonal projections Q_j onto $\text{Ker}[J(i\kappa_j)]$ for $1 \leq j \leq N$. Let us use a tilde to identify the quantities associated with the perturbed Schrödinger operator so that $\tilde{\varphi}(k, x)$ is the perturbed regular solution satisfying the initial conditions (5.14), $\tilde{J}(k)$ is the perturbed Jost matrix defined as in (6.79), $\tilde{Q}_j$ is the orthogonal projection onto $\text{Ker}[\tilde{J}(i\tilde{\kappa}_j)]$, $\tilde{C}_j$ is the Gel'fand–Levitan normalization matrix defined as in (3.24), (3.25), and (3.27) but by using $\tilde{Q}_j$ instead of Q_j and by using $\tilde{\varphi}(i\tilde{\kappa}_j, x)$ instead of $\varphi(i\kappa_j, x)$ for $1 \leq j \leq N + 1$, and $d\tilde{\rho}$ is the perturbed spectral measure defined as

$$d\tilde{\rho} := \begin{cases} \frac{\sqrt{\lambda}}{\pi} \left(\tilde{J}(\sqrt{\lambda})^\dagger \tilde{J}(\sqrt{\lambda}) \right)^{-1} d\lambda, & \lambda \geq 0, \\ \sum_{j=1}^{N+1} \tilde{C}_j^2 \delta(\lambda - \tilde{\lambda}_j) d\lambda, & \lambda < 0, \end{cases} \quad (8.68)$$

where $\tilde{\lambda}_j := -\tilde{\kappa}_j^2$ with $\tilde{\lambda}_j$ for $1 \leq j \leq N + 1$ denoting the eigenvalues of the perturbed operator. It is understood that $\tilde{\kappa}_j = \kappa_j$ for $1 \leq j \leq N$. We then have the following:

(a) Under the perturbation, the projection matrices Q_j for $1 \leq j \leq N$ remain unchanged, i.e. we have

$$\tilde{Q}_j = Q_j, \quad 1 \leq j \leq N.$$

(b) Under the perturbation, the Gel'fand–Levitan normalization matrices for $1 \leq j \leq N$ remain unchanged, i.e. we have

$$\tilde{C}_j = C_j, \quad 1 \leq j \leq N.$$

- (c) The orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})$ is the matrix $\tilde{Q}_{N+1}$ appearing in (8.1). The matrix $\tilde{C}_{N+1}$ appearing in (8.1) is indeed the Gel'fand–Levitan normalization matrix for the bound-state with the energy $-\tilde{\kappa}_{N+1}^2$ for the perturbed Schrödinger operator.
- (d) The perturbed spectral measure $d\tilde{\rho}$ is related to the unperturbed spectral measure $d\rho$ as

$$d\tilde{\rho} = d\rho + \tilde{C}_{N+1}^2 \delta(\lambda - \tilde{\lambda}_{N+1}) d\lambda, \quad \lambda \in \mathbb{R}.$$

Proof. The proof of (a) is similar to the proof of Theorem 6.6(a). The proof of (b) is obtained by proceeding as in the proof of Theorem 6.6(b) and by using (6.43) and (6.44), the counterparts for (6.43) and (6.44) for the adjoint quantities, the analogs of those four asymptotics but by replacing N in them with j for $1 \leq j \leq N+1$, and by also using (8.37), (8.38), and (8.51). For the proof of (c) we proceed as follows. From Theorem 8.6(d) we already know that the multiplicity of the added bound state at $k = i\tilde{\kappa}_{N+1}$ is equal to $\tilde{m}_{N+1}$. From (8.1), (8.16), (8.17), (8.21), and (8.33), we obtain

$$\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1} = \xi_{N+1}(x) \Omega(x)_{N+1}^+. \quad (8.69)$$

Using (8.17), (8.23), and (8.69), and the hermitian property of $\Omega_{N+1}(x)^+$, we get

$$\int_0^x dy [\tilde{\varphi}(i\tilde{\kappa}_{N+1}, y) \tilde{C}_{N+1}]^\dagger \tilde{\varphi}(i\tilde{\kappa}_{N+1}, y) \tilde{C}_{N+1} = \tilde{Q}_{N+1} - \Omega_{N+1}(x)^+. \quad (8.70)$$

Using (8.51) in (8.70) we have

$$\int_0^\infty dx [\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1}]^\dagger \tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1} = \tilde{Q}_{N+1}. \quad (8.71)$$

On the other hand, using (8.2), (8.8), and (8.71), we obtain

$$\int_0^\infty dx [\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{Q}_{N+1}]^\dagger \tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{Q}_{N+1} = \tilde{\mathbf{H}}_{N+1} \tilde{Q}_{N+1}. \quad (8.72)$$

From (8.72) we conclude that each column of $\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{Q}_{N+1}$ is square integrable, and hence $\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{Q}_{N+1}$ is a matrix-valued bound-state solution to the perturbed Schrödinger equation with the energy $-\tilde{\kappa}_{N+1}^2$. Since the rank of $\tilde{Q}_{N+1}$ is $\tilde{m}_{N+1}$, any basis for $\tilde{Q}_{N+1} \mathbb{C}^n$ contains exactly $\tilde{m}_{N+1}$ linearly independent vectors. Let $\{\tilde{w}_{N+1}^{(l)}\}_{l=1}^{\tilde{m}_{N+1}}$ be an orthonormal basis for $\tilde{Q}_{N+1} \mathbb{C}^n$. Using the analog of the second equality of (3.8), we get

$$\tilde{Q}_{N+1} = \sum_{l=1}^{\tilde{m}_{N+1}} \tilde{w}_{N+1}^{(l)} (\tilde{w}_{N+1}^{(l)})^\dagger,$$

which yields

$$\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{Q}_{N+1} = \sum_{l=1}^{\tilde{m}_{N+1}} [\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{w}_{N+1}^{(l)}] (\tilde{w}_{N+1}^{(l)})^\dagger. \quad (8.73)$$

From (8.73) it follows that each column vector $\tilde{\varphi}(i\tilde{\kappa}_{N+1}, x) \tilde{w}_{N+1}^{(l)}$ for $1 \leq l \leq \tilde{m}_{N+1}$ is a bound state for the perturbed Schrödinger operator with the eigenvalue $-\tilde{\kappa}_{N+1}^2$. Then, from Theorem 3.11.1(e) of Ref. 7 we conclude that each column vector $\tilde{w}_{N+1}^{(l)}$ for $1 \leq l \leq \tilde{m}_{N+1}$ belongs to the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})$. Recall that from Theorem 8.6(d) we already know that the multiplicity of the added bound state at $k = i\tilde{\kappa}_{N+1}$ is equal to $\tilde{m}_{N+1}$. Furthermore, by Theorem 3.11.1(d) of Ref. 7 we know that the dimension of the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})$ is precisely $\tilde{m}_{N+1}$. This shows that the map $\tilde{Q}_{N+1}$ is an orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})$. Moreover, from (3.24), (3.25), (3.27), and (8.72) it follows that $\tilde{C}_{N+1}$ is the Gel'fand–Levitan normalization matrix for the bound state with the energy $-\tilde{\kappa}_{N+1}^2$ for the perturbed Schrödinger operator. This completes the proof of (c). We obtain the proof of (d) by using (b) and (c) with the help of (8.55), Theorem 8.6(d), and (8.68). $\square$

In (IV.1.30) of Ref. 11 it is claimed that, when adding a bound state of energy $-\kappa^2$ to the spectrum of the scalar-valued Schrödinger operator with the Dirichlet boundary condition, the potential increment $\tilde{V}(x) - V(x)$ decays as $Fe^{-2\kappa x}$ as $x \rightarrow +\infty$, where F is a nonzero constant. This is incorrect, as we demonstrate with an explicitly solved example below.

Example 8.9. Consider the scalar case, i.e. when $n = 1$, with the unperturbed potential

$$V(x) = -\frac{8e^{2x}}{(1+e^{2x})^2}, \quad x \in \mathbb{R}^+, \quad (8.74)$$

and the boundary matrices A and B , which are scalars in this case, chosen as

$$A = 0, \quad B = -1. \quad (8.75)$$

Using (8.75) in (2.5), we observe that the corresponding boundary condition is the Dirichlet boundary condition $\psi(0) = 0$. Using (8.74) and (8.75) in (2.1), (2.8), (2.10), and (2.11), we obtain the corresponding Jost solution $f(k, x)$, regular solution $\varphi(k, x)$, Jost matrix $J(k)$, and the scattering matrix $S(k)$ as

$$\begin{aligned} f(k, x) &= e^{ikx} \left[1 - \frac{2i}{(k+i)(1+e^{2x})} \right], \\ \varphi(k, x) &= -\frac{k \sin(kx) + \cos(kx) \tanh x}{k^2 + 1}, \\ J(k) &= -\frac{k}{k+i}, \\ S(k) &= -\frac{k+i}{k-i}, \end{aligned} \quad (8.76)$$

which are all scalar valued. Since $J(k)$ in (8.76) does not have any zeros on the positive imaginary axis in the complex k -plane, we see that the unperturbed Schrödinger operator does not have any bound states. Using the method of Sec. VIII, we add a bound state to the unperturbed Schrödinger operator at $k = i\kappa_1$ with the Gel'fand–Levitan normalization constant C_1 , where we choose

$$\kappa_1 = 1, \quad C_1 = 4.$$

The corresponding perturbed Jost matrix $\tilde{J}(k)$, perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$, and perturbed potential $\tilde{V}(x)$ are obtained as

$$\begin{aligned} \tilde{J}(k) &= -\frac{k(k-i)}{(k+i)^2}, \\ \tilde{A} &= 0, \quad \tilde{B} = -1, \\ \tilde{V}(x) &= \frac{q_{27}(x) + q_{28}(x)}{[(1-2x) \cosh x + (1+4x^2 + \cosh(2x)) \sinh x]^2}, \end{aligned} \quad (8.77)$$

where we have defined

$$\begin{aligned} q_{27}(x) &:= 7 - 24x + 32x^4 + 64x^2 \cosh(2x) - (16 + 32x) \sinh(2x), \\ q_{28}(x) &:= -(9 + 8x^2) \cosh(4x) + (-2 + 20x) \sinh(4x). \end{aligned}$$

We observe that the unperturbed potential $V(x)$ in (8.74) behaves as $O(e^{-2x})$ and the perturbed potential $\tilde{V}(x)$ in (8.77) behaves as $O(x^2 e^{-2x})$ as $x \rightarrow +\infty$, and hence the potential difference $\tilde{V}(x) - V(x)$ also behaves as $O(x^2 e^{-2x})$ as $x \rightarrow +\infty$. This is in contradiction to the result stated in (IV.1.30) of Ref. 11, which incorrectly predicts $\tilde{V}(x) - V(x) = O(e^{-2x})$ as $x \rightarrow +\infty$. We remark that the decay rate for $\tilde{V}(x) - V(x)$ as $x \rightarrow +\infty$ in this example is consistent with our general estimate (8.34).

IX. THE TRANSFORMATION TO INCREASE THE MULTIPLICITY OF A BOUND STATE

This section is complementary to Sec. VIII. In Sec. VIII we have started with the unperturbed Schrödinger operator and added a new bound state to the discrete spectrum without changing the continuous spectrum. We have developed the transformation formulas for all relevant quantities under the aforementioned perturbation. In this complementary section, we still do not change the continuous spectrum, we do not add any new bound states or remove any existing bound states, but we only increase the multiplicity of one of the existing bound states. Under such a perturbation, we show how all relevant quantities transform. We omit the proofs as those proofs are similar to the proofs of the results in Sec. VIII.

The changing of the multiplicity of a bound state does not apply in the scalar case, i.e. when $n = 1$, because in the scalar case all bound states have multiplicity one. Hence, we assume that we are in the matrix case with $n \geq 2$ and that there exists at least one bound state, i.e. we have $N \geq 1$. As in the previous sections, we use a tilde to identify the perturbed quantities obtained after the multiplicity of the bound state is increased. We assume that our unperturbed potential V satisfies (2.2) and belongs to $L_1^1(\mathbb{R}^+)$. In our notation, our unperturbed regular solution is $\varphi(k, x)$, unperturbed Jost solution is $f(k, x)$, unperturbed Jost matrix is $J(k)$, unperturbed scattering matrix is $S(k)$, unperturbed matrix Q_j corresponds to the orthogonal projection onto the kernel of $J(ik_j)$ for $1 \leq j \leq N$, and unperturbed boundary matrices are given by A and B . Our perturbed potential is $\tilde{V}(x)$, perturbed regular solution is $\tilde{\varphi}(k, x)$, perturbed Jost solution is $\tilde{f}(k, x)$, perturbed Jost matrix is $\tilde{J}(k)$, perturbed scattering matrix is $\tilde{S}(k)$, and perturbed boundary matrices are given by $\tilde{A}$ and $\tilde{B}$.

Without loss of generality, we increase the multiplicity of the bound state at $k = i\kappa_N$ from m_N to $\tilde{m}_N$ by the positive integer m_{Ni} , which is defined as

$$m_{Ni} := \tilde{m}_N - m_N. \quad (9.1)$$

Thus, m_{N_i} satisfies the restriction $1 \leq m_{N_i} \leq n - m_N$. By using the subscript N_i in (9.1), we indicate that we refer to the N th bound state at $k = i\kappa_N$ and that we increase its multiplicity. We recall that Q_N denotes the orthogonal projection onto the kernel of $J(i\kappa_N)$, the positive integer m_N corresponds to the dimension of $\text{Ker}[J(i\kappa_N)]$, and we use C_N to denote the Gel'fand–Levitan normalization matrix associated with the bound state at $k = i\kappa_N$ in the unperturbed case. We introduce the $n \times n$ matrix $\tilde{Q}_{N_i}$ with the rank m_{N_i} in order to denote the orthogonal projection onto a subspace of the orthogonal complement in $\mathbb{C}^n$ of $Q_N \mathbb{C}^n$. Thus, we have

$$\tilde{Q}_{N_i} \mathbb{C}^n \subset (Q_N \mathbb{C}^n)^\perp. \quad (9.2)$$

We use an $n \times n$ nonnegative hermitian matrix $\tilde{G}_{N_i}$ so that we have the equalities

$$\tilde{G}_{N_i} \tilde{Q}_{N_i} = \tilde{Q}_{N_i} \tilde{G}_{N_i} = \tilde{G}_{N_i},$$

which is the analog of (8.3). We further assume that the restriction of $\tilde{G}_{N_i}$ to the subspace $\tilde{Q}_{N_i} \mathbb{C}^n$ is invertible. We define the $n \times n$ matrix $\tilde{H}_{N_i}$ as

$$\tilde{H}_{N_i} := I - \tilde{Q}_{N_i} + \tilde{G}_{N_i},$$

which is analogous to (8.4). By proceeding as in Sec. VIII, we prove that $\tilde{H}_{N_i}$ is a nonnegative hermitian matrix and that it satisfies

$$\tilde{H}_{N_i} \tilde{Q}_{N_i} = \tilde{Q}_{N_i} \tilde{H}_{N_i},$$

which is analogous to (8.8). By proceeding as in (8.6), we prove that the matrix $\tilde{H}_{N_i}$ is invertible, and hence it is a positive matrix. Thus, the positive matrix $\tilde{H}_{N_i}^{1/2}$ and its inverse $\tilde{H}_{N_i}^{-1/2}$ are well defined. Analogous to (8.2), we define the $n \times n$ matrix $\tilde{C}_{N_i}$ as

$$\tilde{C}_{N_i} := \tilde{H}_{N_i}^{-1/2} \tilde{Q}_{N_i}. \quad (9.3)$$

It follows that $\tilde{C}_{N_i}$ is hermitian and nonnegative, its rank is equal to m_{N_i} , and it satisfies the equalities

$$\tilde{C}_{N_i} \tilde{Q}_{N_i} = \tilde{Q}_{N_i} \tilde{C}_{N_i} = \tilde{C}_{N_i}.$$

We remark that we cannot choose $\tilde{C}_{N_i}$ arbitrarily as any nonnegative hermitian matrix. The definition in (9.3) imposes a restriction on the eligible choices for $\tilde{C}_{N_i}$. Different choices for $\tilde{C}_{N_i}$ *a priori* yield different $\tilde{V}(x)$, $\tilde{\varphi}(k, x)$, $\tilde{f}(k, x)$, $\tilde{J}(k)$, $\tilde{P}_{N_i}$, and $\tilde{B}$. We define the orthogonal projection matrix $\tilde{P}_{N_i}$ later in this section. The boundary matrix $\tilde{A}$, however, is the same for any of those different choices for $\tilde{C}_{N_i}$ and it is equal to A .

Proceeding as in the proof of Proposition 8.4(d) and using the analog of (8.28), we can express the matrix solution $\varphi(i\kappa_N, x) \tilde{Q}_{N_i}$ to the Schrödinger equation (2.1) with $k = i\kappa_N$ as

$$\varphi(i\kappa_N, x) \tilde{Q}_{N_i} = f(i\kappa_N, x) K_{N_i} \tilde{Q}_{N_i} + g(i\kappa_N, x) L_{N_i} \tilde{Q}_{N_i}, \quad (9.4)$$

for some $n \times n$ matrices K_{N_i} and L_{N_i} . We claim that the restriction of $L_{N_i} \tilde{Q}_{N_i}$ to the subspace $\tilde{Q}_{N_i} \mathbb{C}^n$ is invertible. Otherwise, we would have $L_{N_i} \tilde{Q}_{N_i} v = 0$ for some column vector $v \in \tilde{Q}_{N_i} \mathbb{C}^n$. Then, (9.4) would imply that

$$\varphi(i\kappa_N, x) \tilde{Q}_{N_i} v = f(i\kappa_N, x) K_{N_i} v. \quad (9.5)$$

Further, the vector $K_{N_i} v$ cannot be zero unless we have $v = 0$. Otherwise, we would have $\varphi(i\kappa_{N+1}, x) v = 0$ for all $x \in [0, +\infty)$, which would contradict (2.7) and (2.9). Using (6.41) on the right-hand side of (9.5), we would get

$$\varphi(i\kappa_N, x) \tilde{Q}_{N_i} v = e^{-\kappa_N x} \left[I + O\left(\int_x^\infty dy |V(y)|\right) \right] Q_{N_i} v, \quad x \rightarrow +\infty. \quad (9.6)$$

Thus, from (9.6) we would conclude that the column vector $\varphi(i\kappa_N, x) \tilde{Q}_{N_i} v$ would be a square-integrable solution to (2.1) with $k = i\kappa_N$ and satisfying the boundary condition (2.5), and that would imply that $\varphi(i\kappa_N, x) \tilde{Q}_{N_i} v$ would be a bound-state solution to (2.1) at $k = i\kappa_N$. However, (9.2) would imply that $v \notin Q_N \mathbb{C}^n$, which would contradict Theorem 3.11.1(e) of Ref. 7. This completes the proof that the restriction of $L_{N_i} \tilde{Q}_{N_i}$ to $\tilde{Q}_{N_i} \mathbb{C}^n$ is indeed invertible.

We recall that the unperturbed potential V is assumed to satisfy (2.2) and belong to $L_1^1(\mathbb{R}^+)$. By proceeding as in Sec. VIII, we establish the following:

- (1) We solve the Gel'fand–Levitan system (5.38) of integral equations with the kernel $G(x, y)$ chosen as

$$G(x, y) = \varphi(i\kappa_N, x) \tilde{C}_{N_i}^2 \varphi(i\kappa_N, y)^\dagger, \quad (9.7)$$

where $\tilde{C}_{Ni}$ is the $n \times n$ nonnegative Hermitian matrix defined in (9.3). We remark that (9.7) is analogous to (8.14). By proceeding as in the proof of Theorem 8.2, we show that the solution $\mathcal{A}(x, y)$ to (5.38) is given by

$$\mathcal{A}(x, y) = -\xi_{Ni}(x) \Omega_{Ni}(x)^+ \xi_{Ni}(y)^\dagger, \quad 0 \leq y < x, \quad (9.8)$$

which is the analog of (8.15). We note that the $n \times n$ matrix $\xi_{Ni}(x)$ appearing in (9.8) is defined as

$$\xi_{Ni}(x) := \varphi(i\kappa_N, x) \tilde{C}_{Ni}, \quad (9.9)$$

which is the analog of (8.16), and the $n \times n$ matrix $\Omega_{Ni}(x)$ there is defined as

$$\Omega_{Ni}(x) := \tilde{Q}_{Ni} + \int_0^x dy \xi_{Ni}(y)^\dagger \xi_{Ni}(y), \quad (9.10)$$

which is the analog of (8.17). We recall that $\Omega_{Ni}(x)^+$ denotes the Moore–Penrose inverse of $\Omega_{Ni}(x)$.

- (2) We define the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) := V(x) - 2 \frac{d}{dx} \left[\xi_{Ni}(x) \Omega_{Ni}(x)^+ \xi_{Ni}(x)^\dagger \right]. \quad (9.11)$$

By proceeding as in the proof of Theorem 8.3, we show that $\mathcal{A}(x, y)$ and $\tilde{V}(x)$ satisfy (a)–(d) of Theorem 8.3.

- (3) Similarly, by proceeding as in the proof of Theorem 8.5, we establish the following:

- (a) The quantity $\tilde{\varphi}(k, x)$ expressed in terms of the unperturbed regular solution $\varphi(k, x)$ and the quantity $\mathcal{A}(x, y)$ in (9.8) as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \int_0^x dy \mathcal{A}(x, y) \varphi(k, y),$$

is a solution to the perturbed Schrödinger equation (2.1) with the potential $\tilde{V}(x)$ given in (9.11). Furthermore, with the help of (9.9) and (9.10), we prove that the quantity $\tilde{\varphi}(k, x)$ can also be expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \xi_{Ni}(x) \Omega_{Ni}(x)^+ \int_0^x dy \xi_{Ni}(y)^\dagger \varphi(k, y). \quad (9.12)$$

- (b) The perturbed potential $\tilde{V}(x)$ appearing in (9.11) satisfies (2.2). Moreover, for any a and ε with $a \geq 0$ and $0 < \varepsilon < 1$, the corresponding potential increment $\tilde{V}(x) - V(x)$ has the asymptotic behavior

$$\tilde{V}(x) - V(x) = O(q_{29}(x, a, \varepsilon)), \quad x \rightarrow +\infty,$$

where we have defined

$$q_{29}(x, a, \varepsilon) := x e^{-2\kappa_N x} + \int_x^\infty dy |V(y)| + e^{-2\varepsilon\kappa_N x} \int_a^{(1-\varepsilon)x} dy |V(y)| + \int_{(1-\varepsilon)x}^x dy e^{-2\kappa_N(x-y)} |V(y)|.$$

Furthermore, if the unperturbed potential V belongs to $L^1_{1+\varepsilon}(\mathbb{R}^+)$ for some fixed $\varepsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L^1_e(\mathbb{R}^+)$.

- (c) Assume that the unperturbed potential V is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0,$$

for some positive constants α and x_0 , where c denotes a generic constant. Then, for every $0 < \varepsilon < 1$, the potential increment $\tilde{V}(x) - V(x)$ has the asymptotic behavior as $x \rightarrow +\infty$ given by

$$\tilde{V}(x) - V(x) = \begin{cases} O(e^{-\alpha x} + e^{-2\varepsilon\kappa_N x}), & \alpha \leq 2\kappa_N, \\ O(e^{-2\varepsilon\kappa_N x}), & \alpha > 2\kappa_N. \end{cases}$$

- (d) If the unperturbed potential V has compact support, then the perturbed potential $\tilde{V}$ has the asymptotic behavior

$$\tilde{V}(x) = O(x e^{-2\kappa_N x}), \quad x \rightarrow +\infty.$$

- (e) For $k \neq i\kappa_N$, the perturbed quantity $\tilde{\varphi}(k, x)$ given in (9.12) can be expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \frac{1}{k^2 + \kappa_N^2} \xi_{Ni}(x) \Omega_{Ni}(x)^+ \left[\xi'_{Ni}(x)^\dagger \varphi(k, x) - \xi_{Ni}(x)^\dagger \varphi'(k, x) \right].$$

- (f) The perturbed quantity $\tilde{\varphi}(k, x)$ satisfies the initial conditions (5.14), where the matrices $\tilde{A}$ and $\tilde{B}$ are expressed in terms of the unperturbed boundary matrices A and B and the nonnegative matrix $\tilde{C}_{N_1}$ defined in (9.3), and we have

$$\tilde{A} = A, \quad \tilde{B} = B - A \tilde{C}_{N_1}^2 A^\dagger. \quad (9.13)$$

- (g) The matrices $\tilde{A}$ and $\tilde{B}$ appearing in (9.13) satisfy (2.6) and (2.7). Hence, as a consequence of (a) and (f), the quantity $\tilde{\varphi}(k, x)$ is the regular solution to the matrix Schrödinger equation with the potential $\tilde{V}(x)$ in (9.11) and with the selfadjoint boundary condition (2.5) with A and B there replaced with $\tilde{A}$ and $\tilde{B}$, respectively.
- (4) Under the additional assumption $V \in L_2^1(\mathbb{R}^+)$, by proceeding as in the proof of Theorem 8.6 we establish the following:
- (a) The unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix as

$$\tilde{J}(k) = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} \tilde{P}_{N_1} \right] J(k), \quad k \in \overline{\mathbb{C}^+},$$

where we have defined

$$\tilde{P}_{N_1} := L_{N_1} \tilde{C}_{N_1} \left(\tilde{C}_{N_1} L_{N_1}^\dagger L_{N_1} \tilde{C}_{N_1} \right)^+ \tilde{C}_{N_1} L_{N_1}^\dagger,$$

with L_{N_1} being the $n \times n$ matrix appearing in (9.4). Moreover, $\tilde{P}_{N_1}$ is an orthogonal projection with the rank m_{N_1} , where we recall that m_{N_1} is also equal to the rank of the orthogonal projection $\tilde{Q}_{N_1}$ appearing in (9.2). We also prove that $\tilde{P}_{N_1}$ projects onto a subspace of the kernel of $\tilde{J}(i\kappa_N)^\dagger$, where that subspace has dimension m_{N_1} .

- (b) The matrix product $J(k)^\dagger J(k)$ for $k \in \mathbb{R}$ does not change under the perturbation, i.e. we have

$$\tilde{J}(k)^\dagger \tilde{J}(k) = J(k)^\dagger J(k), \quad k \in \mathbb{R},$$

and hence the continuous part of the spectral measure $d\rho$ does not change under the perturbation.

- (c) The perturbation changes the determinant of the Jost matrix as

$$\det[\tilde{J}(k)] = \left(\frac{k - i\kappa_N}{k + i\kappa_N} \right)^{m_{N_1}} \det[J(k)], \quad k \in \overline{\mathbb{C}^+}. \quad (9.14)$$

- (d) Under the perturbation, the bound state with the energy $-\kappa_N^2$ remains, but its multiplicity is increased from m_N to $m_N + m_{N_1}$. No new bound states are added, and the bound states with the energies $-\kappa_j^2$ and multiplicities m_j for $1 \leq j \leq N-1$ are unchanged. We confirm that the multiplicity of the bound state with the energy $-\kappa_N^2$ is increased by m_{N_1} by using the following argument. The multiplicity of the bound state at $k = i\kappa_N$ for the unperturbed problem is m_N . It follows from Theorem 3.11.6 of Ref. 7 that $\det[J(k)]$ has a zero of order m_N at $k = i\kappa_N$. From (9.14) we see that $\det[\tilde{J}(k)]$ has a zero of order $m_N + m_{N_1}$ at $k = i\kappa_N$. Thus, the multiplicity of the bound state with the energy $-\kappa_N^2$ for the perturbed problem is $m_N + m_{N_1}$.
- (e) Under the perturbation, the scattering matrix $S(k)$ undergoes the transformation

$$\tilde{S}(k) = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} \tilde{P}_{N_1} \right] S(k) \left[I + \frac{2i\kappa_N}{k - i\kappa_N} \tilde{P}_{N_1} \right], \quad k \in \mathbb{R}.$$

- (f) The perturbation changes the determinant of the scattering matrix as

$$\det[\tilde{S}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{2m_{N_1}} \det[S(k)], \quad k \in \mathbb{R}.$$

- (g) Under the perturbation, the Jost solution $f(k, x)$ is transformed into the perturbed Jost solution $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) - \frac{1}{k^2 + \kappa_N^2} \xi_{N_1}(x) \Omega_{N_1}(x)^+ q_{30}(x) \right] \left[I - \frac{2i\kappa_N}{k + i\kappa_N} \tilde{P}_{N_1} \right], \quad (9.15)$$

where we have defined

$$q_{30}(x) := \xi'_{N_1}(x)^\dagger f(k, x) - \xi_{N_1}(x)^\dagger f'(k, x).$$

We prove that the singularity at $k = i\kappa_N$ appearing on the right-hand side of (9.15) is actually a removable singularity, as discussed in Remark 8.7.

- (5) Finally, assuming further that $V \in L_2^1(\mathbb{R}^+)$, by proceeding as in the proof of Theorem 8.8 we establish the following:
- (a) Under the perturbation, the projection matrices Q_j for $1 \leq j \leq N-1$ remain unchanged, i.e. we have

$$\tilde{Q}_j = Q_j, \quad 1 \leq j \leq N-1.$$

- (b) Under the perturbation, the Gel'fand–Levitan normalization matrices C_j remain unchanged for $1 \leq j \leq N-1$, i.e. we have

$$\tilde{C}_j = C_j, \quad 1 \leq j \leq N-1.$$

- (c) The orthogonal projection $\tilde{Q}_{N1}$ projects into a subspace of dimension m_{N1} of the kernel of $\tilde{J}(i\kappa_N)$.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Tuncay Aktosun: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Ricardo Weder:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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