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Marchenko inversion for perturbations: I

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Abstract. The Marchenko inversion method of the one-dimensional non-relativistic quantum mechanics is generalised to perturbations. The change in the potential that corresponds to a finite change in the scattering matrix is obtained by formulating a Marchenko-like integral equation. The method used here may be generalisable to higher dimensions because the inversion is formulated in terms of scattering matrices, not only in terms of reflection coefficients.

1. Introduction

Our purpose in this paper is to generalise the Marchenko inversion to perturbations, i.e., to obtain the change in the potential that corresponds to a finite change in the scattering matrix by using the method of Marchenko. In other words, our starting point is a comparison potential for which everything is assumed to be known and we try to obtain the solution to another inverse problem in terms of the known solution with the comparison potential by using the Marchenko method.

The linear integral equation of Gel'fand and Levitan and that of Marchenko have been the sources of important developments later in the field. In the Gel'fand–Levitan procedure, one sets up a linear integral equation where the integration is performed on a finite interval containing the origin (Gel'fand and Levitan 1951). The kernel of this integral equation is related to the spectral function, which is constructed from the scattering matrix. Then the potential can be recovered from the solution to the Gel'fand–Levitan integral equation. In the Marchenko procedure (Agranovich and Marchenko 1963, Marchenko 1955), one sets up a linear integral equation where the integration is performed on an interval containing $+\infty$ or $-\infty$. The kernel of this Marchenko equation is directly obtained from the scattering matrix, and the potential is recovered from the solution of the Marchenko equation. Note that many people misname the Marchenko equation and call it the Gel'fand–Levitan equation.

The generalisation of the Gel'fand–Levitan and Marchenko procedures to obtain a perturbation of the potential in terms of a perturbation of the scattering matrix has been studied by several. The generalisation of the Gel'fand–Levitan method is already completed; the interested reader is referred to the classic by Newton (1982b) for the method, development, and further references in the Gel'fand–Levitan case. The generalisation of the Marchenko method for perturbations has been less studied. In the radial case, Agranovich and Marchenko (1963) mention that one can express the solution to the Schrödinger equation with potential $V(x)$ in terms of the solution with potential $V_0(x)=l(l+1)/x^2$, i.e., in terms of Bessel functions; however, it is added that 'this

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approach leads to considerable analytical difficulties connected primarily with investigating the integral equations obtained. These difficulties are further compounded when matrix equations are considered.' So, they try another approach, but they do this only when l is an integer, and they use a Crum transformation to find the perturbation $V(x)-V_0(x)$, and they also generalise this to the matrix case where they study it explicitly with the potential matrix

$$V_0(x)=\frac{6}{x^2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

In the Marchenko formalism, the perturbations are built on the potential $V_0(x)=\frac{1}{4}(x\pm\frac{1}{2})^{-2}$ in the radial case by Dyson (1976) using the language of Fredholm determinants. Deift and Trubowitz (1979) have shown how the potential changes when a bound state is added to the scattering matrix by using a Crum transformation in one dimension. In case the reflection coefficient $R(k)$ is a rational function of k (Kay 1960), Sabatier (1983a, b) has shown how the potential changes when $R(k)$ is changed by a phase factor that is also a rational function; this is done in the one-dimensional case by using the Darboux–Bäcklund transform.

This paper is organised as follows. Section 2 is a summary of the main facts about the scattering matrix, the potential and the wavefunctions, and these results are used in later sections. The other purpose of this section is to set out the notation used in this study. In § 3, the matrix solution to the Schrödinger equation is given and this matrix will be used to obtain the matrix Riemann–Hilbert problem. In § 4, an integral expression for the Jost matrix is obtained, which will be used later. In § 5, the transformations of the scattering matrix and of the matrix solution are given when the space coordinate is shifted; hence it is shown how the matrix solution is related to the Jost matrix. In § 6, the Riemann–Hilbert problem is given in the matrix form and the matrix Marchenko equation is obtained. The matrix potential is obtained from the solution to the matrix Marchenko equation. Another matrix Marchenko equation with a different kernel is given. The solutions to these two matrix Marchenko equations will be used later to obtain the perturbation of the matrix potential from a matrix integral equation. In § 7, the matrix Riemann–Hilbert problem for perturbations is given. In § 8 the Marchenko formalism is generalised to perturbations. Since the matrix formalism is used, the method of this section may be generalisable to higher dimensions. Starting from the matrix Riemann–Hilbert problem, a matrix integral equation is obtained, and this equation is the analogue of the Marchenko equation. The perturbation of the matrix potential is obtained in a similar way one obtains the potential in the Marchenko formalism. The properties of the kernel of this Marchenko-like integral equation are given; a bound on the eigenvalues of this kernel is obtained in terms of a bound on the perturbation of the scattering matrix so that the eigenvalues can be made less than one in absolute value and hence the existence and the uniqueness of the solution can be given. In § 9 some further properties of the solution to the Marchenko-like integral equation are presented.

2. Preliminaries

The Schrödinger equation

$$\frac{d^2\psi}{dx^2}(k,x)+k^2\psi(k,x)=V(x)\psi(k,x)$$

in one dimension, if the potential $V(x) \sim 0$ in some appropriate sense, for real k, has two linearly independent solutions satisfying the boundary conditions

$$\psi_l(k, x) = T_l \exp(ikx) + o(1) \qquad \text{as } x \to \infty$$
$$\psi_l(k, x) = \exp(ikx) + L \exp(-ikx) + o(1) \qquad \text{as } x \to -\infty$$
$$\psi_r(k, x) = \exp(-ikx) + R \exp(ikx) + o(1) \qquad \text{as } x \to \infty$$
$$\psi_r(k, x) = T_r \exp(-ikx) + o(1) \qquad \text{as } x \to -\infty.$$

These are the scattering states and the subscripts l and r are used to indicate that ψ_l and ψ_r are usually called waves travelling from the left and from the right respectively.

The scattering matrix, often called the S matrix for short, is obtained from the asymptotics of ψ_l and ψ_r and it is given by

$$S(k) \equiv \begin{pmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{pmatrix}. \tag{2.1}$$

We will assume $V(x)$ is a real potential in L_2^1, where L_n^1 is the space of measurable functions $V(x)$ such that the Lebesgue integral $\int_{-\infty}^{\infty} dx(1 + |x|^n) |V(x)|$ exists. Thus we assume (Deift and Trubowitz 1979) that $S(k)$ is continuous and unitary, $S(-k) = S(k)^*$ where * denotes the complex conjugation, $T_l = T_r \equiv T$, and that $T(k)$ has a meromorphic extension to $\mathbb{C}^+$, the complex upper-half plane, with a finite number of simple poles at $\{i\beta_1, i\beta_2, \ldots, i\beta_n\}$ all located on the imaginary axis in $\mathbb{C}^+$. The characterisation problem for a potential in L_1^1 is recently given by Melin (1985); however, the analysis is simpler in class L_2^1 and hence we will work with potentials in L_2^1.

It is possible to combine the solutions ψ_l and ψ_r into a column vector

$$\psi \equiv \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix}$$

and to write the vector Schrödinger equation (Newton 1983):

$$\frac{d^2\psi}{dx^2} + k^2\psi = V\psi.$$

Let $\psi^\#(k, x) \equiv \psi(-k, x)$. We can express $\psi^\#$ in terms of ψ as (Newton 1983):

$$\psi^\# = S^{-1} q \psi \tag{2.2}$$

where the matrix q is defined as

$$q \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is known that a real potential in L_1^1 is uniquely determined by one of the reflection coefficients, bound states, and the so-called norming constants, which can be obtained from the asymptotics of the bound state solutions to the Schrödinger equation (Deift and Trubowitz 1979). Instead of determining a potential in a given class from one of the reflection coefficients, it is also desirable to have a method in one dimension that uses the whole S matrix and which can also be generalised to higher dimensions. This is because in higher dimensions the S matrix becomes an operator and there is no analogue of a reflection coefficient. The method that uses the whole S matrix is due to Newton (1980a) and its generalisation to three dimensions is also given by Newton (1980b, 1981, 1982a).

In this paper, we will use the whole S matrix to formulate the Marchenko equation for perturbations. Hence the method given here may be generalisable to higher dimensions.

3. Matrix formulation of the unperturbed problem

In order to formulate a Marchenko-like equation for perturbations and to generalise it to higher dimensions, one must use 2×2 matrix solutions of the Schrödinger equation rather than vector or scalar solutions. One way to form a matrix solution is to combine the vector solutions that correspond to the scattering matrices

$$S=\begin{pmatrix} T & R \\ L & T \end{pmatrix} \qquad \text{and} \qquad ISI=\begin{pmatrix} T & -R \\ -L & T \end{pmatrix}$$

where

$$I\equiv\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Assume that the matrices S and ISI are associated with potentials $V(x)$ and $U(x)$ respectively with the corresponding vector solutions $\psi^{(v)}$ and $\psi^{(u)}$, where the superscripts refer to the potentials. These solutions must satisfy the equations that correspond to (2.2):

$$\psi^{(v)\#}=S^{-1}q\psi^{(v)} \tag{3.1}$$

and

$$\psi^{(u)\#}=(ISI)^{-1}q\psi^{(u)}=IS^{-1}Iq\psi^{(u)}. \tag{3.2}$$

For a real potential in L_2^1, if $T(0)=0$, then $V(x)$ and $U(x)$ cannot be simultaneously in L_2^1 because the reflection coefficients for the potentials cannot both take the value of -1 at $k=0$ (Deift and Trubowitz 1979). For example, if

$$S(0)=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \text{then} \qquad ISI(0)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, if $T(0)\neq 0$, then whenever $S(k)$ satisfies the characterisation conditions for a real potential in L_2^1 given in Deift and Trubowitz (1979), so does the matrix $IS(k)I$. Hence, when $T(0)\neq 0$, the simultaneous existence of the real potentials $V(x)$ and $U(x)$ in L_2^1 is guaranteed.

In general, we can say the following. The matrices S and ISI satisfy the same conditions for the existence and uniqueness of a real potential in L_2^1, except for one condition, namely the behaviour of the corresponding reflection coefficients at $k=0$. Therefore, it is possible that when $T(0)=0$, the simultaneous existence or uniqueness of the potentials $V(x)$ and $U(x)$ in the specified class may not be known or may not be assured. One possible way to overcome this difficulty is as follows. The introduction of the second potential $U(x)$ is made for a technical reason only. The main interest lies in the potential $V(x)$ of the scattering matrix $S(k)$, not in $U(x)$. Therefore, in case $T(0)=0$, we can form a scattering matrix

$$S_\varepsilon(k)=\begin{pmatrix} T_\varepsilon(k) & R_\varepsilon(k) \\ L_\varepsilon(k) & T_\varepsilon(k) \end{pmatrix}$$

such that $\lim_{\varepsilon\to 0} S_\varepsilon(k)=S(k)$ and that $S_\varepsilon(k)$ and $S(k)$ satisfy the same conditions for the existence and uniqueness of the potential in L_2^1 except that $T_\varepsilon(0)\neq 0$. Once $V_\varepsilon(x)$ and $U_\varepsilon(x)$ are obtained as real potentials in L_2^1, we can recover $V(x)$ as $\lim_{\varepsilon\to 0} V_\varepsilon(x)$ provided that the mapping $S(k)\to V(x)$ is stable. We expect $\lim_{\varepsilon\to 0} V_\varepsilon(x)$ to be in L_2^1 if $R(0)=L(0)=-1$. In general, we cannot expect $\lim_{\varepsilon\to 0} U_\varepsilon(x)$ to be in a previously specified class. Neither the existence nor the uniqueness of $\lim_{\varepsilon\to 0} U_\varepsilon(x)$ is guaranteed. In fact, explicit examples are known where this limit is not unique (Aktosun and Newton 1985, Sabatier 1984).

We can combine the Riemann–Hilbert problems for the vector solutions $\psi^{(v)}$ and $\psi^{(u)}$ into a Riemann–Hilbert problem for a 2×2 matrix as follows. Let us define the matrix

$$\Psi\equiv\begin{pmatrix}\psi_1 & \psi_2\\ \psi_3 & \psi_4\end{pmatrix}$$

as:

$$\Psi\equiv\frac{1}{2}\begin{pmatrix}\psi_l^{(v)}+\psi_l^{(u)} & \psi_l^{(v)}-\psi_l^{(u)}\\ \psi_r^{(v)}-\psi_r^{(u)} & \psi_r^{(v)}+\psi_r^{(u)}\end{pmatrix}. \tag{3.3}$$

From (3.1) and (3.2) we obtain

$$\Psi^{\#}=S^{-1}q\Psi q. \tag{3.4}$$

Define the potential matrix $\lambda(x)$ as

$$\lambda(x)\equiv\frac{1}{2}\begin{pmatrix}V(x)+U(x) & V(x)-U(x)\\ V(x)-U(x) & V(x)+U(x)\end{pmatrix}. \tag{3.5}$$

Then the Schrödinger equation satisfied by the matrix Ψ is given by

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}+k^2\Psi=\Psi\lambda. \tag{3.6}$$

Let us call Ψ the physical solution of the matrix Schrödinger equation. From the properties of ψ_l and ψ_r we see that $\Psi^{\#}=\Psi^*$ when $k\in\mathbb{R}$.

From the Lippmann–Schwinger equations satisfied by $\psi^{(v)}$ and $\psi^{(u)}$ (Newton 1983), we obtain the Lippmann–Schwinger equation satisfied by the matrix solution $\Psi(k,x)$:

$$\Psi(k,x)=\exp(\mathrm{i}Ikx)+\frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty}\mathrm{d}y\,\exp(\mathrm{i}k|x-y|)\Psi(k,y)\lambda(y). \tag{3.7}$$

Let us define the matrix $F(k,x)$ as

$$F\equiv\exp(-\mathrm{i}Ikx)\Psi\equiv\frac{1}{2}\begin{pmatrix}f_l^{(v)}+f_l^{(u)} & f_l^{(v)}-f_l^{(u)}\\ f_r^{(v)}-f_r^{(u)} & f_r^{(v)}+f_r^{(u)}\end{pmatrix} \tag{3.8}$$

where we have defined $f_l^{(v)}\equiv\exp(-\mathrm{i}kx)\psi_l^{(v)}$, $f_r^{(v)}\equiv\exp(\mathrm{i}kx)\psi_r^{(v)}$, $f_l^{(u)}\equiv\exp(\mathrm{i}kx)\psi_l^{(u)}$ and $f_r^{(u)}\equiv\exp(\mathrm{i}kx)\psi_r^{(u)}$. It is straightforward to show that $F(k,x)$ satisfies the matrix differential equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2}+2\mathrm{i}Ik\frac{\mathrm{d}F}{\mathrm{d}x}=F\lambda. \tag{3.9}$$

Let us define another matrix $M(k,x)$ as

$$M(k,x)\equiv\frac{1}{T(k)}F(k,x).$$

Since $M(k, x)$ is a multiple of $F(k, x)$, it satisfies the same differential equation that $F(k, x)$ satisfies, but certainly with different boundary conditions.

We can also define the matrix solution $\Phi(k, x)$ by using the regular solutions $\varphi_l(k, x)$ and $\varphi_r(k, x)$ of the Schrödinger equation which satisfy the boundary conditions from Newton (1983)

$$\varphi_l(k, 0)=1 \qquad \varphi_r(k, 0)=1 \qquad \frac{d\varphi_l}{dx}(k, 0)=ik \qquad \frac{d\varphi_r}{dx}(k, 0)=-ik$$

$$\Phi(k, x)\equiv\frac{1}{2}\begin{pmatrix}\varphi_l^{(v)}+\varphi_l^{(u)} & \varphi_l^{(v)}-\varphi_l^{(u)} \\ \varphi_r^{(v)}-\varphi_r^{(u)} & \varphi_r^{(v)}+\varphi_r^{(u)}\end{pmatrix} \tag{3.10}$$

where the superscripts again refer to the potentials. Since the regular solutions satisfy the same Schrödinger equation the physical solutions satisfy, the matrix $\Phi(k, x)$ satisfies (3.6). We will call $\Phi(k, x)$ the regular matrix solution. From the boundary conditions satisfied by the regular scalar solutions, we obtain

$$\Phi(k, 0)=\mathbb{1} \qquad d\Phi(k, 0)/dx=iIk \qquad \text{where} \qquad \mathbb{1}\equiv\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.$$

For $k\in\mathbb{R}$, the matrix solutions F, M, and Φ satisfy, for real potentials,

$$F^{\#}=F^* \qquad M^{\#}=M^* \qquad \Phi^{\#}=\Phi^*$$

because for these matrices k and i appear together as ki both in the matrix equations and the boundary conditions.

4. Jost matrix

Let $J(k)$ be the Jost matrix that corresponds to the S matrix with the potential $V(x)$. We have (Newton 1983):

$$J(k)\psi^{(v)}(k, x)\equiv\varphi^{(v)}(k, x) \tag{4.1}$$

where the vector solutions $\psi^{(v)}$ and $\varphi^{(v)}$ are given by

$$\psi^{(v)}=\begin{pmatrix}\psi_l^{(v)} \\ \psi_r^{(v)}\end{pmatrix} \qquad \text{and} \qquad \varphi^{(v)}=\begin{pmatrix}\varphi_l^{(v)} \\ \varphi_r^{(v)}\end{pmatrix}.$$

The following theorem shows that the solutions Ψ and Φ of the matrix Schrödinger equation are related to each other by the same Jost matrix given in (4.1).

Theorem 4.1. $J(k)\Psi(k, x)=\Phi(k, x)$ where $J(k)$ is the Jost matrix defined in (4.1) and Ψ and Φ are the physical and regular solutions of the matrix Schrödinger equation respectively.

Proof. Let $J_u(k)$ be the Jost matrix of the scattering matrix ISI. From (4.1), we have

$$J_u\psi^{(u)}=\varphi^{(u)} \tag{4.2}$$

where $\psi^{(u)}$ and $\varphi^{(u)}$ are the physical and regular vector solutions for the potential $U(x)$. The Jost matrices satisfy (Newton 1983):

$$J^{-1\#}=S^{-1}qJ^{-1}q \tag{4.3}$$

$$J_u^{-1\#}=IS^{-1}IqJ_u^{-1}q.$$

Multiplying the last matrix equation by I both on the right and on the left, we obtain

$$IJ_u^{-1\#}I = I^2 S^{-1} I q J_u^{-1} q I = S^{-1} q I J_u^{-1} I q \tag{4.4}$$

where we have used $I^2 = \mathbb{1}$ and $Iq = -qI$, which are straightforward to verify. Subtracting (4.4) from (4.3), we obtain

$$(J^{-1} - IJ_u^{-1}I)^{\#} = S^{-1}q(J^{-1} - IJ_u^{-1}I)q.$$

Since $J^{-1} \to \mathbb{1}$ and $J_u^{-1} \to \mathbb{1}$ as $|k| \to \infty$ in $\bar{\mathbb{C}}^+$ (Newton 1983), where $\bar{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{R}$, we have $J^{-1} - IJ_u^{-1}I \to 0$ as $|k| \to \infty$ in $\mathbb{C}^+$. Then the uniqueness of the solution to (4.3) requires that $J^{-1} = IJ_u^{-1}I$ (Newton 1980a). Thus we obtain

$$J = IJ_uI.$$

From (3.3) and (3.10) it is seen that

$$\Psi \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \psi^{(v)} \qquad \Phi \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \varphi^{(v)}$$

$$I\Psi \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \psi^{(u)} \qquad I\Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \varphi^{(u)}.$$

Thus we obtain

$$J\Psi \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Phi \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.5}$$

and

$$J\Psi \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \Phi \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.6}$$

Hence the two vector equations (4.5) and (4.6) give us the matrix equation $J\Psi = \Phi$. QED

Since $\Phi(k, 0) = \mathbb{1}$, the above theorem gives us $J(k)\Psi(k, 0) = \mathbb{1}$ or equivalently

$$J^{-1}(k) = \Psi(k, 0). \tag{4.7}$$

The following theorem shows that it is possible to express $J^{-1}(k)$ in terms of the vector solution $\psi^{(v)}$ alone.

Theorem 4.2. The matrix inverse of the Jost matrix is given by

$$J^{-1}(k) = \frac{1}{2} \begin{pmatrix} \psi_l(k, 0) + \dfrac{1}{ik}\dfrac{d\psi_l}{dx}(k, 0) & \psi_l(k, 0) - \dfrac{1}{ik}\dfrac{d\psi_l}{dx}(k, 0) \\ \psi_r(k, 0) + \dfrac{1}{ik}\dfrac{d\psi_r}{dx}(k, 0) & \psi_r(k, 0) - \dfrac{1}{ik}\dfrac{d\psi_r}{dx}(k, 0) \end{pmatrix}$$

where $\binom{\psi_l}{\psi_r}$ is the vector solution of the Schrödinger equation with the potential $V(x)$.

Proof. The Wronskian

$$[\Psi; \tilde{\Phi}^*] \equiv \Psi \frac{d\tilde{\Phi}^*}{dx} - \frac{d\Psi}{dx} \tilde{\Phi}^*$$

is equal to $J^{-1}[\Phi; \tilde{\Phi}^*]$ as seen from theorem 4.1. Note that the tilde denotes the matrix transpose. Since the matrix potential $\lambda(x)$ is real and symmetric, from the matrix Schrödinger equation for $\Phi(k, x)$, we obtain $[\Phi;\tilde{\Phi}^*] = -2\mathrm{i}Ik$. In a similar way, we obtain the Wronskian

$$[\Psi; \tilde{\Phi}^*] = -\mathrm{i}kJ^{-1}I - \frac{\mathrm{d}\Psi}{\mathrm{d}x}(k, 0).$$

Thus, we have

$$J^{-1} = \frac{1}{\mathrm{i}k}\frac{\mathrm{d}\Psi}{\mathrm{d}x}(k, 0)I.$$

Combining this last result with (4.7), we obtain the expression stated in the theorem. QED

From the Lippmann–Schwinger equation for the vector solution $\psi(k, x)$ (Newton 1983), we obtain

$$\psi(k, 0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{0} \mathrm{d}y \exp(-\mathrm{i}ky)V(y)\psi(k, y) + \frac{1}{2\mathrm{i}k}\int_{0}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky)V(y)\psi(k,y)$$

and

$$\frac{\mathrm{d}\psi(k, 0)}{\mathrm{d}x} = \mathrm{i}k\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\int_{-\infty}^{0} \mathrm{d}y \exp(-\mathrm{i}ky)V(y)\psi(k, y) - \frac{1}{2}\int_{0}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky)V(y)\psi(k, y).$$

Using these two expressions in the result of theorem 4.2, we obtain

$$J^{-1} = \begin{pmatrix} 1 + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{0} \mathrm{d}y \exp(-\mathrm{i}ky)V(y)\psi_l(k, y) & \frac{1}{2\mathrm{i}k}\int_{0}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky)V(y)\psi_l(k, y) \\ \frac{1}{2\mathrm{i}k}\int_{-\infty}^{0} \mathrm{d}y \exp(-\mathrm{i}ky)V(y)\psi_r(k, y) & 1 + \frac{1}{2\mathrm{i}k}\int_{0}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky)V(y)\psi_r(k, y) \end{pmatrix}. \tag{4.8}$$

Note that when $V(x) = 0$ for $x < 0$, (4.8) reduces to equation (3.31) of Newton (1983).

5. Shifting the potential

When the space coordinate x is shifted by z, the transformed potential matrix is given by $\lambda_z(x) = \lambda(x + z)$. The transformed wavefunction $\Psi_z(k, x)$ can be obtained by using the matrix Lippmann–Schwinger equation given in (3.7) as follows:

$$\begin{aligned}\Psi(k, x) &= \exp(\mathrm{i}Ikx) + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}k|x-y|)\Psi(k, y)\lambda(y) \\ &= \exp(\mathrm{i}Ikx) + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}k|x-y-z|)\Psi(k, y + z)\lambda(y + z).\end{aligned}$$

Thus, replacing x by $x + z$, we obtain

$$\Psi(k, x + z) = \exp[\mathrm{i}Ik(x + z)] + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}k|x-y|)\Psi(k, y + z)\lambda_z(y)$$

and hence

$$\exp(-\mathrm{i}\,Ikz)\Psi(k,x+z)=\exp(\mathrm{i}Ikx)+\frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty}\mathrm{d}y\,\exp(\mathrm{i}k|x-y|)$$
$$\times\exp(-\mathrm{i}Ikz)\Psi(k,y+z)\lambda_z(y).$$

Since $\exp(-\mathrm{i}Ikz)\Psi(k,x+z)$ satisfies the matrix Lippmann–Schwinger equation with the shifted potential $\lambda_z(x)$, assuming that the Lippmann–Schwinger equation has a unique solution, we obtain

$$\Psi_z(k,x)=\exp(-\mathrm{i}Ikz)\Psi(k,x+z). \tag{5.1}$$

At $x=0$ this last expression becomes $\Psi_z(k,0)=\exp(-\mathrm{i}Ikz)\Psi(k,z)$. Since $J^{-1}(k)=\Psi(k,0)$ and $F(k,x)=\exp(-\mathrm{i}Ikx)\Psi(k,x)$, we obtain

$$J_z^{-1}(k)=F(k,z). \tag{5.2}$$

Thus $F(k,x)$ is nothing but the matrix inverse of the transformed Jost matrix when the potential is shifted.

Since $\Psi(k,x)$ satisfies (3.4), we have the canonical decomposition

$$S=q\Psi q\Psi^{\#-1}. \tag{5.3}$$

The equivalent of this expression for $S(k)$ and the expression (2.1) in terms of the asymptotics of the physical solutions is already known when $T(k)$ is holomorphic in $\mathbb{C}^+$ (Newton 1984). Since the transformation of $\Psi(k,x)$ is explicitly known, from (5.3) we obtain

$$\begin{aligned}S_z(k)&=q\Psi_z(k,x)q\Psi_z^{\#}(k,x)^{-1}\\&=q\exp(-\mathrm{i}Ikz)\Psi(k,x+z)q\Psi(k,x+z)^{\#-1}\exp(-\mathrm{i}Ikz)\\&=\exp(\mathrm{i}Ikz)q\Psi(k,x+z)q\Psi(k,x+z)^{\#-1}\exp(-\mathrm{i}Ikz)\\&=\exp(\mathrm{i}Ikz)S(k)\exp(-\mathrm{i}Ikz).\end{aligned}$$

Hence the transmission coefficient is transformed as $T_z(k)=T(k)$.

It is known that $\det J^{-1}(k)=T(k)$ (Newton 1983), and hence we have

$$\det F(k,x)=\det J_x^{-1}(k)=T_x(k)=T(k).$$

Using

$$\Psi(k,x)=\exp(\mathrm{i}Ikx)F(k,x)\qquad\text{and}\qquad M(k,x)=\frac{1}{T(k)}F(k,x)$$

we see that

$$\det\Psi(k,x)=T(k)\qquad\text{and}\qquad\det M(k,x)=\frac{1}{T(k)}.$$

Thus we can write the matrix inverse of Ψ explicitly as follows:

$$\Psi^{-1}=\begin{pmatrix}\psi_1&\psi_2\\\psi_3&\psi_4\end{pmatrix}^{-1}=\frac{1}{\det\Psi}\begin{pmatrix}\psi_1&-\psi_2\\-\psi_3&\psi_4\end{pmatrix}=\frac{1}{T}qI\tilde{\Psi}Iq. \tag{5.4}$$

Similarly we have

$$F^{-1}=\frac{1}{T}\,qI\tilde{F}Iq \qquad \text{and} \qquad M^{-1}=TqI\tilde{M}Iq.$$

6. Matrix Riemann–Hilbert problem and matrix Marchenko formalism

Using (3.8), we can write (3.4) as $F^{\#}=\exp(\mathrm{i}Ikx)S^{-1}\exp(-\mathrm{i}Ikx)qFq$ for $k\in\mathbb{R}$. Letting $\Lambda(k,x)\equiv\exp(\mathrm{i}Ikx)S^{-1}\exp(-\mathrm{i}Ikx)$, we obtain

$$F^{\#}=\Lambda qFq \qquad k\in\mathbb{R}. \tag{6.1}$$

From (3.8), we have $F(k,x)=\mathbb{1}+\mathrm{O}(k^{-1})$ as $|k|\to\infty$ in $\bar{\mathbb{C}}^{+}$. Thus (6.1) constitutes a Riemann–Hilbert problem, which we will call Newton's formulation. Here we assume that there are no bound states, in which case $F(k,x)$ has a holomorphic extension to $\mathbb{C}^{+}(k)$ for each x. The case with bound states is studied in another paper (Aktosun 1987a).

Subtracting the asymptotic value of F as $|k|\to\infty$ from both sides of (6.1), we obtain

$$F^{\#}-\mathbb{1}=(\Lambda-\mathbb{1})qFq+q(F-\mathbb{1})q. \tag{6.2}$$

Letting $\Sigma\equiv\Lambda-\mathbb{1}$ and taking the Fourier transform by $\int_{-\infty}^{\infty}\mathrm{d}k(1/2\pi)\exp(\mathrm{i}ky)$ of (6.2), we obtain

$$\eta(x,y)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\Sigma(k,x)qF(k,x)q\exp(\mathrm{i}ky)+\eta(x,-y) \tag{6.3}$$

where we have defined

$$\eta(x,y)\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,(F(k,x)-\mathbb{1})\exp(-\mathrm{i}ky). \tag{6.4}$$

$F-\mathbb{1}$ is holomorphic in $\mathbb{C}^{+}$ and as $|k|\to\infty$ in $\bar{\mathbb{C}}^{+}$, $F-\mathbb{1}=\mathrm{O}(k^{-1})$ and it belongs to the Hardy space; hence $\eta(x,y)=0$ for $y<0$. Thus (6.3) becomes

$$\eta(x,y)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\Sigma(k,x)qF(k,x)q\exp(\mathrm{i}ky) \qquad y>0. \tag{6.5}$$

In case $S(k)$ is a rational function, using (6.5), one can reduce the solution of the inverse problem to solving a system of linear equations (Aktosun and Newton 1985).

We can write (6.5) as

$$\eta(x,y)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\Sigma\exp(\mathrm{i}ky)+\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\Sigma q(F-\mathbb{1})q\exp(\mathrm{i}ky) \qquad y>0$$

and letting $g(x,y)\equiv\int_{-\infty}^{\infty}\mathrm{d}k(1/2\pi)\Sigma(k,x)\exp(\mathrm{i}ky)$, the above equation becomes

$$\eta(x,y)=g(x,y)+(g*q\eta^{\#}q)(x,y) \qquad y>0$$

where $*$ denotes the convolution as a function of y and $\eta^{\#}(x,y)=\eta(x,-y)$. Since $\eta(x,y)=0$ for $y<0$, we have

$$(g*q\eta^{\#}q)(x,y)=\int_{0}^{\infty}\mathrm{d}z\,g(x,y+z)q\eta(x,z)q \qquad y>0.$$

Hence we obtain the matrix Marchenko equation

$$\eta(x,y) = g(x,y) + \int_0^\infty \mathrm{d}z\, g(x,y+z) q \eta(x,z) q \qquad y > 0. \tag{6.6}$$

For a function $A(k,x)$ which satisfies $A^{\#} = A^*$, i.e., $A(-k,x) = A(k,x)^*$, its Fourier transform in L^2 is real. Since S, $\exp(\mathrm{i}Ikx)$, F, and their functions have this property, we see that $g(x,y)$ and $\eta(x,y)$ are both real. Furthermore, since the Fourier transform on L^2 is a unitary operator and since $S - \mathbb{1}$, $F - \mathbb{1} \in L^2(-\infty < k < \infty)$, we have $g(x,y) \in L^2(-\infty < y < \infty)$ and $\eta(x,y) \in L^2(0 < y < \infty)$ for each x.

Define

$$\eta_l^{(v)}(x,y) \equiv \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi} (f_l^{(v)}(k,x) - 1) \exp(-\mathrm{i}ky)$$

$$\eta_r^{(v)}(x,y) \equiv \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi} (f_r^{(v)}(k,x) - 1) \exp(-\mathrm{i}ky)$$

$$\eta_l^{(u)}(x,y) \equiv \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi} (f_l^{(u)}(k,x) - 1) \exp(-\mathrm{i}ky)$$

$$\eta_r^{(u)}(x,y) \equiv \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi} (f_r^{(u)}(k,x) - 1) \exp(-\mathrm{i}ky)$$

where we have used (3.8) and the superscripts refer to the potentials. Thus we can write the matrix $\eta(x,y)$ as

$$\eta = \frac{1}{2} \begin{pmatrix} \eta_l^{(v)} + \eta_l^{(u)} & \eta_l^{(v)} - \eta_l^{(u)} \\ \eta_r^{(v)} - \eta_r^{(u)} & \eta_r^{(v)} + \eta_r^{(u)} \end{pmatrix}.$$

Using the integral expressions for $\eta_l^{(v)}$, $\eta_r^{(v)}$, $\eta_l^{(u)}$, and $\eta_r^{(u)}$ (Newton 1983), we obtain the integral expression for the matrix $\eta(x,y)$:

$$\eta(x,y) = -\frac{1}{2} \int_{-\infty}^\infty \mathrm{d}z \begin{pmatrix} \theta(x - z + \frac{1}{2}y) & 0 \\ 0 & \theta(z - x + \frac{1}{2}y) \end{pmatrix} \lambda(z)$$

$$-\frac{1}{2} \int_0^y \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}z \begin{pmatrix} \theta(x - z + \frac{1}{2}y - \frac{1}{2}t) & 0 \\ 0 & \theta(z - x + \frac{1}{2}y - \frac{1}{2}t) \end{pmatrix} \eta(z,t)\lambda(z) \tag{6.7}$$

where $\lambda(x)$ is the matrix potential and $\theta(x)$ denotes the Heaviside function. From this integral expression for $\eta(x,y)$, by differentiation we can obtain the partial differential equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\eta(x,y) - 2I\frac{\partial}{\partial y}\eta(x,y)\right) = \eta(x,y)\lambda(x). \tag{6.8}$$

From (6.7) we obtain

$$\eta(x,0+) \equiv \lim_{y \downarrow 0} \eta(x,y) = -\frac{1}{2} \int_{-\infty}^\infty \begin{pmatrix} \theta(x-z) & 0 \\ 0 & \theta(z-x) \end{pmatrix} \lambda(z)$$

and hence we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta(x,0+)=-\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}z\,\delta(x-z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\lambda(z)$$

$$=-\tfrac{1}{2}I\lambda(x)$$

where $\delta(x)$ is the Dirac delta function. Thus we obtain

$$\lambda(x)=-2I\frac{\mathrm{d}}{\mathrm{d}x}\eta(x,0+). \tag{6.9}$$

Letting

$$\eta\equiv\begin{pmatrix}\eta_1 & \eta_2\\ \eta_3 & \eta_4\end{pmatrix},$$

we obtain from (6.9) the scalar potentials

$$V(x)=-2\frac{\mathrm{d}}{\mathrm{d}x}(\eta_1(x,0+)+\eta_2(x,0+))$$

$$=2\frac{\mathrm{d}}{\mathrm{d}x}(\eta_3(x,0+)+\eta_4(x,0+))$$

and

$$U(x)=2\frac{\mathrm{d}}{\mathrm{d}x}(\eta_2(x,0+)-\eta_1(x,0+))$$

$$=2\frac{\mathrm{d}}{\mathrm{d}x}(\eta_4(x,0+)-\eta_3(x,0+)).$$

We can write (6.1) in terms of $M\equiv(1/T)F$ as

$$M^{\#}=\Lambda_M qMq \qquad k\in\mathbb{R}$$

where we have defined

$$\Lambda_M\equiv\frac{T}{T^{\#}}\exp(\mathrm{i}Ikx)S^{-1}\exp(\mathrm{i}Ikx).$$

Note that the equation $M^{\#}=\Lambda_M qMq$ and (6.1) are similar. The asymptotic value of $M(k,x)$ as $|k|\to\infty$ in $\bar{\mathbb{C}}^+$ is $\mathbb{1}$ because $T(k)\to 1$ and $F(k,x)\to\mathbb{1}$. Since we assume there are no bound states, $M(k,x)$ is holomorphic in $k\in\mathbb{C}^+$ for each x as $F(k,x)$ is. Let us call this formulation of the Riemann–Hilbert problem Faddeev's formulation. Defining

$$\Sigma_M\equiv\Lambda_M-\mathbb{1}$$

$$B(x,y)\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}(M(k,x)-\mathbb{1})\exp(-\mathrm{i}ky)$$

$$g_M(x,y)\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\Sigma_M(k,x)\exp(\mathrm{i}ky)$$

we can solve the Riemann–Hilbert problem in Faddeev's formulation exactly the same way we solve it in Newton's formulation. Hence we obtain

$$B(x,y)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\Sigma_M(k,x)qM(k,x)q\exp(\mathrm{i}ky) \qquad y>0 \tag{6.10}$$

which corresponds to (6.5), and the matrix Marchenko equation in Faddeev's formulation which corresponds to (6.6):

$$B(x,y)=g_M(x,y)+\int_0^{\infty}\mathrm{d}z\,g_M(x,y+z)qB(x,z)q \qquad y>0. \tag{6.11}$$

Using exactly the same arguments as in Newton's formulation, we see that $g_M(x,y)$ and $B(x,y)$ are both real, and $g_M(x,y)\in L^2(-\infty<y<\infty)$ and $B(x,y)\in L^2(0<y<\infty)$ for each x. The integral equation for $B(x,y)$ which corresponds to (6.7) can be obtained as in Newton's formulation. Defining

$$m_l^{(v)}\equiv\frac{1}{T}\exp(-\mathrm{i}kx)\psi_l^{(v)} \qquad m_r^{(v)}\equiv\frac{1}{T}\exp(\mathrm{i}kx)\psi_r^{(v)}$$

$$m_l^{(u)}\equiv\frac{1}{T}\exp(-\mathrm{i}kx)\psi_l^{(u)} \qquad m_r^{(u)}\equiv\frac{1}{T}\exp(\mathrm{i}kx)\psi_r^{(u)}$$

we obtain

$$M(k,x)=\frac{1}{2}\begin{pmatrix} m_l^{(v)}+m_l^{(u)} & m_l^{(v)}-m_l^{(u)} \\ m_r^{(v)}-m_r^{(u)} & m_r^{(v)}+m_r^{(u)} \end{pmatrix}$$

where the superscripts again refer to the potentials. Hence we have

$$B(k,x)=\frac{1}{2}\begin{pmatrix} B_l^{(v)}+B_l^{(u)} & B_l^{(v)}-B_l^{(u)} \\ B_r^{(v)}-B_r^{(u)} & B_r^{(v)}+B_r^{(u)} \end{pmatrix}$$

where we have defined

$$B_l(x,y)\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}(m_l(k,x)-1)\exp(-\mathrm{i}ky)$$

and

$$B_r(x,y)\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}(m_r(k,x)-1)\exp(-\mathrm{i}ky)$$

with superscripts referring to the potentials. From the integral equations for $B_l^{(v)}$, $B_r^{(v)}$, $B_l^{(u)}$ and $B_r^{(u)}$ (Deift and Trubowitz 1979), we obtain

$$\begin{aligned} B(x,y)=&\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}z\begin{pmatrix}\theta(z-x-\tfrac{1}{2}y) & 0 \\ 0 & \theta(x-z-\tfrac{1}{2}y)\end{pmatrix}\lambda(z) \\ &+\frac{1}{2}\int_0^{y}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}z\begin{pmatrix}\theta(z-x-\tfrac{1}{2}y+\tfrac{1}{2}t) & 0 \\ 0 & \theta(x-z-\tfrac{1}{2}y+\tfrac{1}{2}t)\end{pmatrix}B(z,t)\lambda(z) \end{aligned} \tag{6.12}$$

which corresponds to (6.7). The partial differential equation satisfied by $B(x,y)$ can be obtained from (6.12) by differentiation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}B(x,y)-2I\frac{\partial}{\partial y}B(x,y)\right)=B(x,y)\lambda(x)$$

which corresponds to (6.8). From (6.12) we have

$$B(x,0+)\equiv\lim_{y\downarrow 0}B(x,y)=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}z\begin{pmatrix}\theta(z-x) & 0\\ 0 & \theta(x-z)\end{pmatrix}\lambda(z)$$

and thus we obtain, in a similar way we have obtained (6.9),

$$\lambda(x)=-2I\frac{\mathrm{d}B(x,0+)}{\mathrm{d}x}. \tag{6.13}$$

Each of the matrix Marchenko equations, (6.6) of Newton's formulation and (6.11) of Faddeev's formulation, is equivalent to two 2×1 vector equations, which can be obtained by multiplying these matrix Marchenko equations on the right by the column vectors $\binom{1}{1}$ and $\binom{1}{-1}$. For example, in Newton's formulation we have

$$\eta(x,y)\begin{pmatrix}1\\1\end{pmatrix}=g(x,y)\begin{pmatrix}1\\1\end{pmatrix}+\int_0^{\infty}\mathrm{d}z\,g(x,y+z)q\eta(x,z)\begin{pmatrix}1\\1\end{pmatrix}\qquad y>0$$

and

$$\eta(x,y)\begin{pmatrix}1\\-1\end{pmatrix}=g(x,y)\begin{pmatrix}1\\-1\end{pmatrix}-\int_0^{\infty}\mathrm{d}z\,g(x,y+z)q\eta(x,z)\begin{pmatrix}1\\-1\end{pmatrix}\qquad y>0$$

where we have used

$$q\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\qquad\text{and}\qquad q\begin{pmatrix}1\\-1\end{pmatrix}=-\begin{pmatrix}1\\-1\end{pmatrix}.$$

Let $G_x(y,z)\equiv g(x,y+z)q$; then $G_x(y,z)$ becomes the kernel of the operator G_x defined as

$$G_x:\xi_x(y)\to\int_0^{\infty}\mathrm{d}z\;G_x(y,z)\xi_x(z). \tag{6.14}$$

Thus, investigating the solvability of the matrix Marchenko equation in Newton's formulation is equivalent to studying the properties of the kernel $G_x(y,z)$.

According to a theorem by Titchmarsh (1937), a function $A(k)$ and its derivative $\mathrm{d}A(k)/\mathrm{d}k$ are both in $L^2(k)$ if and only if $\hat{A}(y)$ and $y\hat{A}(y)$ are both in $L^2(y)$, where the caret denotes the Fourier transform in L^2. Thus, if $S-\mathbb{1}$, $\mathrm{d}S/\mathrm{d}k\in L^2(k)$, then $g(x,y)$, $yg(x,y)\in L^2(y)$ for each x. Using this result, we have the following theorem.

Theorem 6.1. If $S-\mathbb{1}$ and $\mathrm{d}S/\mathrm{d}k$ are in $L^2(k)$, then the operator G_x defined in (6.14) is Hilbert–Schmidt.

Proof. Define the absolute value of an arbitrary matrix $A=(A_{ij})$ as

$$|A|=\max_i\sum_j|A_{ij}|. \tag{6.15}$$

Then we have (Agranovich and Marchenko 1963), provided that $A+B$ and AB are meaningful and $\int\mathrm{d}x|A(x)|$ exists, $|A+B|\leqslant|A|+|B|$, $|AB|\leqslant|A|\cdot|B|$, $|\int\mathrm{d}xA(x)|\leqslant\int\mathrm{d}x|A(x)|$.

Using (6.15) we have

$$\left| \int_0^\infty \int_0^\infty \mathrm{d}y\mathrm{d}z\, \widetilde{G_x(y,z)}^* G_x(y,z) \right| \leqslant \int_0^\infty \int_0^\infty \mathrm{d}y\mathrm{d}z |q\widetilde{g(x,y+z)}^* g(x,y+z)q|$$

$$\leqslant \int_0^\infty \int_0^\infty \mathrm{d}y\mathrm{d}z |\widetilde{g(x,y+z)}|\; |g(x,y+z)|.$$

Thus G_x is Hilbert–Schmidt if we have

$$\int_0^\infty \int_0^\infty \mathrm{d}y\mathrm{d}z |g_{ij}(x,y+z)| \cdot |g_{mn}(x,y+z)| < \infty$$

for $i,j,m,n=1,2$ where $g_{ij}(x,y)$ are the entries of the matrix $g(x,y)$. Because of the theorem of Titchmarsh mentioned above, $g_{ij} \in L^2$ in the second argument for each x. Hence the product $g_{ij}g_{mn} \in L^1$ by Hölder's inequality and the change of the order of integration below is justified:

$$\int_0^\infty \int_0^\infty \mathrm{d}y\mathrm{d}z |g_{ij}(x,y+z)g_{mn}(x,y+z)| = \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}y |g_{ij}(x,t)g_{mn}(x,t)|$$

$$= \int_0^\infty \mathrm{d}t\; t|g_{ij}(x,t)g_{mn}(x,t)|$$

$$\leqslant \left(\int_0^\infty \mathrm{d}t |tg_{ij}(x,t)|^2 \right)^{1/2} \left(\int_0^\infty \mathrm{d}t |g_{mn}(x,t)|^2 \right)^{1/2}$$

$$< \infty.$$

QED

Corollary 6.1. Being Hilbert–Schmidt, the operator G_x defined in (6.14) is compact and hence bounded.

7. Perturbations of the potential and of the S matrix

Let us interpret Newton's formulation of the Riemann–Hilbert problem as follows. We have started with $\Psi^{\#} = S^{-1}q\Psi q$ given in (3.4). Then in order to find the potential matrix $\lambda(x)$, we have used the transformation $\Psi \to \exp(-\mathrm{i}Ikx)\Psi$, $S^{-1} \to \exp(\mathrm{i}Ikx)S^{-1}\exp(-\mathrm{i}Ikx)$, and hence $S \to \exp(\mathrm{i}Ikx)S\exp(-\mathrm{i}Ikx)$, $S - \mathbb{1} \to \exp(\mathrm{i}Ikx)(S-\mathbb{1})\exp(-\mathrm{i}Ikx)$, $\lambda \to \lambda - 0$. Since $\exp(\mathrm{i}Ikx)$ is the matrix wavefunction which corresponds to the identity scattering matrix with zero matrix potential, we can rewrite the above transformation, by letting $\lambda_0 = 0$, $S_0 = \mathbb{1}$, and $\Psi_0 = \exp(\mathrm{i}Ikx)$, as follows: $\Psi \to \Psi_0^{-1}\Psi$, $S - \mathbb{1} \to (q\Psi_0 q)^{-1}(S - S_0)\Psi_0^{\#}$, $\lambda \to \lambda - \lambda_0$. With this motivation in mind, we will investigate if such a transformation may lead to a Marchenko-like method to obtain a perturbation $\lambda - \lambda_0$ of the potential matrix λ_0 which corresponds to a perturbation $S - S_0$ of the scattering matrix S_0 also in cases $\lambda_0 \neq 0$ and $S_0 \neq \mathbb{1}$. The subscript will refer to the quantities on which the perturbation is built.

Let $\Psi_0^{\#} = S_0^{-1}q\Psi_0 q$ and $\Psi^{\#} = S^{-1}q\Psi q$ for $k \in \mathbb{R}$, where Ψ_0 is the matrix wavefunction of the scattering matrix S_0 with the corresponding matrix potential λ_0, and Ψ and S are the matrix wavefunction and the S matrix for the corresponding matrix potential λ. Let us define, as in the previous sections,

$$S_0 \equiv \begin{pmatrix} T_0 & R_0 \\ L_0 & T_0 \end{pmatrix} \quad \text{and} \quad S \equiv \begin{pmatrix} T & R \\ L & T \end{pmatrix}$$

$$\lambda_0 \equiv \frac{1}{2}\begin{pmatrix} V_0+U_0 & V_0-U_0 \\ V_0-U_0 & V_0+U_0 \end{pmatrix} \quad \text{and} \quad \lambda \equiv \frac{1}{2}\begin{pmatrix} V+U & V-U \\ V-U & V+U \end{pmatrix}.$$

Since the case of bound states is studied in another paper (Aktosun 1987a), we can here assume that both S_0 and S are free of bound states.

It is already known that Ψ_0^{-1} is holomorphic in $\mathbb{C}^+$ when $T_0(0) \neq 0$ (Newton 1984) or when $T_0(k)$ vanishes linearly as $k \to 0$ (Aktosun and Newton 1985). Multiplying (3.4) on the left by $\Psi_0^{\#-1}$, we obtain

$$(\Psi_0^{-1}\Psi)^{\#} = \Psi_0^{\#-1} S^{-1} q \Psi q = (\Psi_0^{\#-1} S^{-1} q \Psi_0 q) q \Psi_0^{-1} \Psi q.$$

We can write this equation as

$$H^{\#} = \mathscr{L} q H q \qquad k \in \mathbb{R} \tag{7.1}$$

where we have defined

$$H \equiv \Psi_0^{-1} \Psi \tag{7.2}$$

and

$$\mathscr{L} \equiv \Psi_0^{\#-1} S^{-1} q \Psi_0 q. \tag{7.3}$$

Comparing (7.1) with (6.1), we see that we have a matrix Riemann–Hilbert problem for H and we may solve (7.1) by a Marchenko-like method. For this we need to know the properties of the matrices H and $\mathscr{L}$. Define the matrices $\Delta(k)$ and $\mathscr{E}(k, x)$ as

$$\Delta \equiv S - S_0 \tag{7.4}$$

$$\mathscr{E} \equiv \mathscr{L} - \mathbb{1}. \tag{7.5}$$

Since $S^{\#} = S^*$ and $\Psi^{\#} = \Psi^*$ for $k \in \mathbb{R}$, $\mathscr{L}^{\#} = \mathscr{L}^*$ and $\mathscr{E}^{\#} = \mathscr{E}^*$ for $k \in \mathbb{R}$. From (7.3) it is straightforward to show that $\mathscr{L}^{-1} = q\mathscr{L}^{\#} q$.

The equality of T_r and T_l in

$$S = \begin{pmatrix} T_l & R \\ L & T_r \end{pmatrix}$$

is known as the reciprocity and it is equivalent to the statement $\tilde{S} = qSq$ (Newton 1983). The lemma below shows that in general $\mathscr{L}$ is not reciprocal.

Lemma 7.1. In general $\tilde{\mathscr{L}} \neq q\mathscr{L}q$ and the equality holds only when $S_0 = \mathbb{1}$, and hence in general $\mathscr{L}$ is not unitary unless $S_0 = \mathbb{1}$.

Proof. Since $\mathscr{L}^* = \mathscr{L}^\#$, $\tilde{\mathscr{L}}^* = \tilde{\mathscr{L}}^\#$. Since $\mathscr{L}^{-1} = q\mathscr{L}^\# q$, $\mathscr{L}$ is unitary only when $\tilde{\mathscr{L}}^* = q\mathscr{L}^* q$, or equivalent $\tilde{\mathscr{L}} = q\mathscr{L}q$. However, from (7.3)

$$q\mathscr{L}q = q\Psi_0^{\#-1}S^{-1}q\Psi_0$$

and

$$\tilde{\mathscr{L}} = q\tilde{\Psi}_0 q\tilde{S}^{-1}\tilde{\Psi}_0^{\#-1} = q\tilde{\Psi}_0 S^{-1}q\tilde{\Psi}_0^{\#-1}.$$

Since S and Ψ_0 are independent of each other, we see that $\tilde{\mathscr{L}} = q\mathscr{L}q$ only when $\tilde{\Psi}_0 = \Psi_0^{\#-1}$, or equivalently only when $\tilde{\Psi}_0\Psi_0^\# = \mathbb{1}$. Since $\det \Psi_0 = T_0$, $\tilde{\Psi}_0\Psi_0^\# = \mathbb{1}$ implies $|T_0|^2 = 1$, which occurs only when $S_0 = \mathbb{1}$ because we assume that S_0 does not have bound states. QED

In order to apply a Marchenko-like method on (7.1), we need the Fourier transform of $\mathscr{L} - \mathbb{1}$. Hence we need $\mathscr{L}(k, x) - \mathbb{1} \in L^2(k)$ for each x. For this we need the following lemmas.

Lemma 7.2. For each x, $\mathscr{E}(k, x)$ defined in (7.5) is continuous in k when the matrix $(1/T_0)(S - S_0)$ is bounded at $k = 0$.

Proof. From (7.3) and $\Psi_0^\# = S_0^{-1}q\Psi_0 q$ we obtain $\mathscr{L} = \Psi_0^{\#-1}S^{-1}S_0\Psi_0^\#$. Replacing S^{-1} by $(S_0 + \Delta)^*$ or equivalently by $S_0^{-1} + \tilde{\Delta}^*$, we obtain $\mathscr{L} = \mathbb{1} + \Psi_0^{\#-1}\tilde{\Delta}^* S_0\Psi_0^\#$. Thus we have

$$\mathscr{E} = \Psi_0^{\#-1}\tilde{\Delta}^* S_0\Psi_0^\#. \tag{7.6}$$

From (5.4) we see that $\Psi_0^{\#-1} = (1/T_0^\#)qI\tilde{\Psi}_0^\# Iq$. Hence we obtain

$$\mathscr{E} = \frac{1}{T_0^\#}\, qI\tilde{\Psi}_0^\# Iq\tilde{\Delta}^* S_0\Psi_0^\#.$$

In terms of $F_0 \equiv \exp(-\mathrm{i}Ikx)\Psi_0$, the last expression becomes

$$\begin{aligned}\mathscr{E} &= \frac{1}{T_0^\#}\, qI\tilde{F}_0^\# \exp(-\mathrm{i}Ikx)Iq\tilde{\Delta}^* S_0 \exp(-\mathrm{i}Ikx)F_0^\# \\ &= qI\tilde{F}_0^\# Iq \exp(-\mathrm{i}Ikx)\,(\tilde{\Delta}/T_0)^* S_0 \exp(-\mathrm{i}Ikx)F_0^\#. \end{aligned} \tag{7.7}$$

By (5.2) F_0 is nothing but J_0^{-1} shifted by x; i.e., $F_0(k, x) = (J_0^{-1})_x$. Whenever $V_0(x)$ is in $L^2(x)$, it is known that J_0^{-1} is continuous (Newton 1983). Since the shift in the potential does not change the property of being in $L^2(x)$, F_0 is still continuous in k for each x. Since both S_0 and S are assumed to be continuous, it is seen from (7.7) that for each x, $\mathscr{E}(k, x)$ is continuous in k everywhere except maybe at $k = 0$ when T_0 vanishes linearly. If T_0 vanishes linearly at $k = 0$, then by requiring that Δ also vanish linearly as $k \to 0$, we can make $\mathscr{E}(k, x)$ continuous at $k = 0$. QED

Lemma 7.3. As $|k| \to \infty$, $\mathscr{E}(k, x) = \mathrm{O}(k^{-1})$, where $\mathscr{E}(k, x)$ is the matrix in (7.5).

Proof. As $|k| \to \infty$, we have $F_0 = \mathbb{1} + \mathrm{O}(k^{-1})$, $S_0 = \mathbb{1} + \mathrm{O}(k^{-1})$ and $S = \mathbb{1} + \mathrm{O}(k^{-1})$. Furthermore $1/T_0 = 1 + \mathrm{O}(k^{-1})$ (Deift and Trubowitz 1979). Hence from (7.7) we see that $\mathscr{E} = \mathrm{O}(k^{-1})$ as $|k| \to \infty$. QED

From the two previous lemmas, we have the following.

Corollary 7.1. When $(1/T_0)(S-S_0)$ is bounded at $k=0$, $\mathscr{E}(k,x)\in L^2(k)$ for each x, and hence the Fourier transform in $L^2(k)$ of $\mathscr{E}(k,x)$ exists.

The following two lemmas are about the properties of $H(k,x)$.

Lemma 7.4. The matrix $H(k,x)$ defined in (7.2) is holomorphic in $k\in\mathbb{C}^+$ for each x.

Proof. In terms of $F_0=\exp(-\mathrm{i}Ikx)\Psi_0$ and $F=\exp(-\mathrm{i}Ikx)\Psi$, we can write (7.2) as $H=F_0^{-1}F$. It is known that F_0^{-1} is holomorphic in $k\in\mathbb{C}^+$ when $T_0(0)\neq 0$ (Newton 1984) or when $T_0(k)$ vanishes linearly as $k\to 0$ (Aktosun and Newton 1985). Since F is also holomorphic in $\mathbb{C}^+$, the product $F_0^{-1}F$ is holomorphic in $k\in\mathbb{C}^+$ for each x. QED

In order to have the Fourier transform $L^2(k)$ of $H-\mathbb{1}$, we need the following lemma concerning the behaviour of $F_0(k,x)$ near $k=0$.

Lemma 7.5. Assume $T(0)=0$ and $V(x)\in L_2^1$. If the second derivative $\mathrm{d}^2T/\mathrm{d}k^2|_{k=0}$ exists, then we have $F(k,x)=F(0,x)+\mathrm{O}(k)$ as $k\to 0$.

Proof. By (5.2) $J_x^{-1}(k)=F(k,x)$, where the subscript x denotes the transformed quantity when the potential is shifted by x. Since the property $V\in L_2^1$ is independent of such a shift, it is enough to prove that $J^{-1}(k)=J^{-1}(0)+\mathrm{O}(k)$ as $k\to 0$.

We have as $k\to 0$

$$T(k)=k\frac{\mathrm{d}T}{\mathrm{d}k}(0)+\tfrac{1}{2}k^2\frac{\mathrm{d}^2T}{\mathrm{d}k^2}(0)+\mathrm{o}(k^2)$$

because we are assuming that $T(0)=0$. Letting, as in § 6, $\psi_l\equiv T\exp(\mathrm{i}kx)m_l$ and $\psi_r=T\exp(-\mathrm{i}kx)m_r$, we can write (4.8) as

$$J^{-1}=\mathbb{1}+\frac{T}{2\mathrm{i}k}\begin{pmatrix}\displaystyle\int_{-\infty}^{0}\mathrm{d}y\,V(y)m_l(k,y) & \displaystyle\int_{0}^{\infty}\mathrm{d}y\,\exp(2\mathrm{i}ky)V(y)m_l(k,y)\\ \displaystyle\int_{-\infty}^{0}\mathrm{d}y\,\exp(-2\mathrm{i}ky)V(y)m_r(k,y) & \displaystyle\int_{0}^{\infty}\mathrm{d}y\,V(y)m_r(k,y)\end{pmatrix}.$$

If $V\in L_2^1$, we have from Deift and Trubowitz (1979)

$$\left|\frac{\partial}{\partial k}m_{l,r}(k,x)\right|\leqslant\text{constant}\times(1+x^2)\qquad\text{for }k\in\bar{\mathbb{C}}^+$$

and hence

$$m_{l,r}(k,x)=m_{l,r}(0,x)+k\frac{\partial m_{l,r}}{\partial k}(0,x)+\mathrm{o}(k)$$

as $k \to 0$ and

$$J^{-1} = \mathbb{1} + \left(\frac{1}{2\mathrm{i}k}\frac{\mathrm{d}T(0)}{\mathrm{d}k} + \frac{k}{4\mathrm{i}}\frac{\mathrm{d}^2T(0)}{\mathrm{d}k^2}\right)$$

$$\times \begin{pmatrix} \int_{-\infty}^{0} \mathrm{d}y V(y) m_l(0,y) & \int_0^{\infty} \mathrm{d}y \exp(2\mathrm{i}ky) V(y) m_l(0,y) \\ \int_{-\infty}^{0} \mathrm{d}y \exp(-2\mathrm{i}ky) V(y) m_r(0,y) & \int_0^{\infty} \mathrm{d}y V(y) m_r(0,y) \end{pmatrix}$$

$$+ \frac{k}{2\mathrm{i}}\frac{\mathrm{d}T(0)}{\mathrm{d}k}$$

$$\times \begin{pmatrix} \int_{-\infty}^{0} \mathrm{d}y V(y) \frac{\partial m_l}{\partial k}(0,y) & \int_0^{\infty} \mathrm{d}y \exp(2\mathrm{i}ky) V(y) \frac{\partial m_l}{\partial k}(0,y) \\ \int_{-\infty}^{0} \mathrm{d}y \exp(-2\mathrm{i}ky) V(y) \frac{\partial m_r}{\partial k}(0,y) & \int_0^{\infty} \mathrm{d}y V(y) \frac{\partial m_r}{\partial k}(0,y) \end{pmatrix}$$

$$+ \mathrm{o}(k).$$

It is known that $|m_{l,r}(k, x)| \leqslant \text{constant} \times (1 + |x|)$ when $V \in L_2^1$ (Deift and Trubowitz 1979). Thus the integrals above in the expression for J^{-1} all converge and we obtain $J^{-1}(k) = J^{-1}(0) + \mathrm{O}(k)$ as $k \to 0$. QED

Next we have a lemma about the continuity of $H(k, x)$.

Lemma 7.6. Let $H(k, x)$ be the matrix given in (7.2). If $T_0(0) \neq 0$, $H(k, x)$ is continuous in k for each x. In case $T_0(0) = 0$, by requiring that $T(0) = 0$ and that $\mathrm{d}T_0/\mathrm{d}k|_{k=0}$ and $\mathrm{d}^2T_0/\mathrm{d}k^2|_{k=0}$ exist, we can make $H(k, x)$ to be continuous in k for each x.

Proof. Since $F_0^{-1} \equiv (1/T_0) qI\tilde{F}_0 Iq$, as given in § 5, we have

$$H = F_0^{-1}F = (1/T_0) qI\tilde{F}_0 IqF. \tag{7.8}$$

Since F_0, F and T_0 are continuous, and $T_0 \neq 0$ except maybe at $k=0$, we see from the above expression in (7.8) that H is continuous for $k \neq 0$. Furthermore, if $H(0, x)$ is bounded, from (7.8) it is seen that H becomes continuous even at $k=0$. Hence we only need to prove the second part of the lemma for the case $T_0(0) = 0$.

We can write (6.1) at $k=0$ as $S(0)F(0, x) = qF(0, x)q$. Since $T(0) = 0$, $S(0) = -q$. Thus, letting

$$F(0, x) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we obtain

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which yields

$$F(0,x)=\begin{pmatrix}-\beta & \beta\\ -\delta & \delta\end{pmatrix}.$$

Similarly, we obtain

$$F_0(0,x)=\begin{pmatrix}-\beta_0 & \beta_0\\ -\delta_0 & \delta_0\end{pmatrix}.$$

We also have $(1/T_0)=(1/k)$ (constant) $+ O(1)$ as $k\to 0$ (Deift and Trubowitz 1979). Hence from (7.8), we have

$$\begin{aligned}H&=\frac{1}{k}\,(\text{constant})\,qI\begin{pmatrix}-\beta_0 & -\delta_0\\ \beta_0 & \delta_0\end{pmatrix}Iq\begin{pmatrix}-\beta & \beta\\ -\delta & \delta\end{pmatrix}+O(1)\\ &=\frac{1}{k}\,(\text{constant})\,(\beta_0\delta-\delta_0\beta)\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}+O(1)\qquad \text{as}\qquad k\to 0.\end{aligned}$$

Hence if we choose $\beta_0/\delta_0=\beta/\delta$, we obtain $H=O(1)$ as $k\to 0$ for each x. The choice of the ratio β/δ has been shown to be equivalent to the specification of the parameter in the one-parameter family of potentials $U(x)$ (Aktosun and Newton 1985). The same remark applies to the choice of the ratio β_0/δ_0. Hence we can make $H(k,x)$ bounded at $k=0$. From now on, without further mentioning, we will assume that we choose $F_0(0,x)$ and $F(0,x)$ such that $\beta_0/\delta_0=\beta/\delta$. QED

The asymptotic value of $H(k,x)$ as $|k|\to\infty$ is given in the following.

Lemma 7.7. As $|k|\to\infty$ in $\bar{\mathbb{C}}^+$ we have $H(k,x)=\mathbb{1}+O(k^{-1})$, where $H(k,x)$ is the matrix defined in (7.2).

Proof. As $|k|\to\infty$ in $\bar{\mathbb{C}}^+$, we have $1/T_0=1+O(1/k)$ (Deift and Trubowitz 1979), $F_0=\mathbb{1}+O(1/k)$ and $F=\mathbb{1}+O(1/k)$. Hence from (7.8), we have, as $|k|\to\infty$ in $\bar{\mathbb{C}}^+$,

$$\begin{aligned}H&=(1+O(1/k))qI[\mathbb{1}+O(1/k)]Iq[\mathbb{1}+O(1/k)]\\ &=\mathbb{1}+O(1/k).\end{aligned}$$

QED

From lemma 7.6 and lemma 7.7, we have the following.

Corollary 7.2. $H(k,x)-\mathbb{1}\in L^2(k)$ for each x if we assume the hypothesis of lemma 7.6.

Having obtained the properties of $H(k,x)$ and $\mathscr{E}(k,x)$, we can solve (7.1) by a method similar to the Marchenko method given in § 6. Hence we will refer to the method to be used as the Marchenko formalism for perturbations.

8. Marchenko formalism for perturbations

Using (7.5) we can rewrite (7.1) as

$$H^{\#}=(\mathscr{E}+\mathbb{1})qHq\qquad k\in\mathbb{R}.\tag{8.1}$$

Since $H \to \mathbb{1}$ as $|k| \to \infty$, we can subtract the matrix $\mathbb{1}$ from both sides of (8.1) to obtain

$$H^{\#} - \mathbb{1} = \mathscr{E} q H q + q(H - \mathbb{1})q.$$

Note that we essentially repeat the steps following (6.1). The Fourier transform of the above equation by $\int_{-\infty}^{\infty} (\mathrm{d}k/2\pi) \exp(\mathrm{i}ky)$ gives us

$$\xi(x, y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathscr{E}(k, x) q H(k, x) q \exp(\mathrm{i}ky) + q\xi(x, -y)q \tag{8.2}$$

where we have defined

$$\xi(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} (H(k, x) - \mathbb{1}) \exp(-\mathrm{i}ky). \tag{8.3}$$

From lemma 7.4 and lemma 7.7, we obtain $\xi(x, y) = 0$ for $y < 0$. Thus (8.2) becomes

$$\xi(x, y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathscr{E}(k, x) q H(k, x) q \exp(\mathrm{i}ky) \qquad y > 0. \tag{8.4}$$

In the case where S_0 and S are rational functions of k, one can use (8.4) to reduce the solution of the inverse problems for perturbations to solving a system of linear equations, as one can use (6.5) for the same purpose. From (8.4) we obtain

$$\begin{aligned}\xi(x, y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathscr{E}(k, x) \exp(\mathrm{i}ky)& \\ + \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathscr{E}(k, x) q (H(k, x) - \mathbb{1}) q \exp(\mathrm{i}ky) &\qquad y > 0.\end{aligned} \tag{8.5}$$

Defining $w(x, y)$ as

$$w(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathscr{E}(k, x) \exp(\mathrm{i}ky) \tag{8.6}$$

we obtain from (8.5)

$$\xi(x, y) = w(x, y) + (w * q\xi^{\#} q)(x, y) \qquad y > 0$$

where $*$ denotes the convolution as a function of y and where $\xi^{\#}(x, y) = \xi(x, -y)$. Equivalently, the above equation can be written as

$$\xi(x, y) = w(x, y) + \int_{0}^{\infty} \mathrm{d}z\, w(x, y + z) q \xi(x, z) q \qquad y > 0 \tag{8.7}$$

where we have used $\xi(x, y) = 0$ for $y < 0$. Comparing (8.7) with (6.6), we see that (8.7) is a matrix Marchenko equation, and we will refer to it as the Marchenko equation for perturbations.

We have already seen in § 7 that $\mathscr{E}^{\#} = \mathscr{E}^{*}$. From (7.8) it is seen that $H^{\#} = H^{*}$ because T_0, F_0 and F satisfy the same property. Hence $w(x, y)$ and $\xi(x, y)$ are both real due to the argument given following (6.6). Furthermore, for each x, $w(x, y) \in L^2(-\infty < y < \infty)$ and $\xi(x, y) \in L^2(0 < y < \infty)$ because these are the Fourier transforms of functions in $L^2(k)$.

We can write (7.8) as

$$H = q I \tilde{M}_0 I q F \tag{8.8}$$

where $M_0 \equiv (1/T_0)F_0$, which is the matrix used in Faddeev's formulation of the Riemann–Hilbert problem given in § 6. From (8.8) we obtain

$$H - \mathbb{1} = qI(\tilde{M}_0 - \mathbb{1})Iq + (F - \mathbb{1}) + qI(\tilde{M}_0 - \mathbb{1})Iq(F - \mathbb{1}).$$

Taking the Fourier transform by $\int_{-\infty}^{\infty} (\mathrm{d}k/2\pi) \exp(-\mathrm{i}ky)$, we obtain

$$\xi(x, y) = qI\widetilde{B_0(x, y)}Iq + \eta(x, y) + \int_{-\infty}^{\infty} \mathrm{d}z\, qI\widetilde{B_0(x, y-z)}Iq\eta(x, z) \tag{8.9}$$

where $\xi(x, y)$ is defined as in (8.3) and

$$B_0(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} (M_0(k, x) - \mathbb{1}) \exp(-\mathrm{i}ky)$$

$$\eta(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} (F(k, x) - \mathbb{1}) \exp(-\mathrm{i}ky).$$

The properties of $B_0(x, y)$ and $\eta(x, y)$ are given in § 6. Hence we already know that $B_0(x, y) = 0$ when $y < 0$ and $\eta(x, y) = 0$ for $y < 0$. Thus (8.9) becomes, for $y > 0$,

$$\xi(x, y) = qI\widetilde{B_0(x, y)}Iq + \eta(x, y) + \int_0^y \mathrm{d}z\, qI\widetilde{B_0(x, y-z)}Iq\eta(x, z)$$

and hence we obtain, as $y \to 0^+$

$$\xi(x, 0+) = qI\widetilde{B_0(x, 0+)}Iq + \eta(x, 0+)$$

and taking the derivative of both sides, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} \xi(x, 0+) = qI \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{B_0(x, 0+)}Iq + \frac{\mathrm{d}}{\mathrm{d}x} \eta(x, 0+).$$

Using (6.9) and (6.13), we can write this equation as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \xi(x, 0+) &= -\tfrac{1}{2} qI(\widetilde{I\lambda_0(x)})Iq - \tfrac{1}{2} I\lambda(x) \\ &= -\tfrac{1}{2} qI\tilde{\lambda}_0(x)q - \tfrac{1}{2} I\lambda(x) \\ &= \tfrac{1}{2} I\lambda_0(x) - \tfrac{1}{2} I\lambda(x).\end{aligned}$$

Hence we have

$$\lambda(x) - \lambda_0(x) = -2I \frac{\mathrm{d}}{\mathrm{d}x} \xi(x, 0+). \tag{8.10}$$

Thus we see that the perturbation of the matrix potential is related to the solution of the matrix Marchenko equation for perturbations in exactly the same way as the potential is related to the solution of the Marchenko equation in either Newton's or Faddeev's formulation.

The matrix Marchenko equation for perturbations given in (8.7) is equivalent to two 2×1 vector equations that are obtained by multiplying (8.7) on the right by the column vectors $\binom{1}{1}$ and $\binom{1}{-1}$ respectively:

$$\xi(x, y) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = w(x, y) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^{\infty} \mathrm{d}z\, w(x, y+z)q\xi(x, z) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad y > 0$$

and

$$\xi(x,y)\begin{pmatrix}1\\-1\end{pmatrix}=w(x,y)\begin{pmatrix}1\\-1\end{pmatrix}-\int_0^\infty \mathrm{d}z\; w(x,y+z)q\xi(x,z)\begin{pmatrix}1\\-1\end{pmatrix} \qquad y>0.$$

The computations are similar to those given in § 6.

Let $\Omega_x(y,z)\equiv w(x,y+z)q$. Then $\Omega_x(y,z)$ becomes the kernel of the operator Ω_x defined as

$$\Omega_x:\ \xi_x(y)\to\int_0^\infty \mathrm{d}z\;\Omega_x(y,z)\xi_x(z). \tag{8.11}$$

Thus investigating the solvability of the Marchenko equation for perturbations becomes equivalent to the study of the properties of the kernel $\Omega_x(y,z)$. First we have the following theorem.

Theorem 8.1. The operator Ω_x defined in (8.11) becomes Hilbert–Schmidt if we assume that $(1/T_0)(S-S_0)$ is bounded at $k=0$ and that $\partial\mathscr{E}(k,x)/\partial k\in L^2(k)$ for each x, where $\mathscr{E}(k,x)$ is as defined in (7.5).

Proof. As in the proof of theorem 6.1, Ω_x is Hilbert–Schmidt if we have $\mathscr{E}(k,x)$ and $\partial\mathscr{E}(k,x)/\partial k$ in $L^2(k)$ for each x. By corollary 7.1, $\mathscr{E}(k,x)\in L^2(k)$ if we assume that $(1/T_0)(S-S_0)$ is bounded at $k=0$. Hence the boundedness of $(1/T_0)(S-S_0)$ at $k=0$ and the property $\partial\mathscr{E}(k,x)/\partial k\in L^2(k)$ are sufficient. QED

Since Hilbert–Schmidt operators are compact and hence bounded, we have the following.

Corollary 8.1. When the hypothesis of theorem 8.1 holds, the operator Ω_x defined in (8.11) is bounded and compact.

However, in general, the operator Ω_x is not self-adjoint due to the fact stated in lemma 7.1. Hence we cannot expect the eigenvalues of Ω_x to be real. Nevertheless, we will find a bound on the absolute value of the eigenvalues of Ω_x. For this we need the following two lemmas.

Lemma 8.1. For any 2×1 column vector p, we have $|\tilde{p}|\leqslant(\tilde{p}^*p)^{1/2}$ and $|p|\leqslant 2(\tilde{p}^*p)^{1/2}$, where the absolute value denotes the matrix norm defined in (6.15).

Proof. Let

$$p\equiv\begin{pmatrix}p_1\\p_2\end{pmatrix}.$$

Then by (6.15), we have

$$|\tilde{p}|=\max(|p_1|,|p_2|)\leqslant(|p_1|^2+|p_2|^2)^{1/2}$$

and

$$|p|=|p_1|+|p_2|\leqslant(|p_1|^2+|p_2|^2)^{1/2}+(|p_1|^2+|p_2|^2)^{1/2}. \qquad \text{QED}$$

Lemma 8.2. Let $\mathscr{E}(k, x)$ be the matrix defined in (7.5) and $F_0 = \exp(-\mathrm{i}Ikx)\Psi_0$, where Ψ_0 is the matrix solution of the Schrödinger equation with the matrix potential λ_0. Then we have $|\mathscr{E}|$, $|\tilde{\mathscr{E}}| \leqslant |T_0|^{-1}\,|F_0| \cdot |\tilde{F}_0| \cdot |S - S_0|$ where the absolute value denotes the matrix norm defined in (6.15).

Proof. From (6.1) we obtain $S_0 \exp(-\mathrm{i}Ikx)F_0^{\#} = \exp(-\mathrm{i}Ikx)qF_0q$, and hence we can write (7.7) as

$$\mathscr{E} = \frac{1}{T_0^{\#}}\, qI\tilde{F}_0^{\#} Iq \exp(\mathrm{i}Ikx)\tilde{\Delta}^{\#} \exp(-\mathrm{i}Ikx)qF_0q. \tag{8.12}$$

Hence we obtain

$$|\mathscr{E}| \leqslant \frac{1}{|T_0|}\, |\tilde{F}_0| \cdot |\tilde{\Delta}| \cdot |F_0|$$

where we have used $|q| = 1$, $|I| = 1$, $|\exp(\pm \mathrm{i}Ikx)| = 1$. Since

$$|\tilde{\Delta}| = |\tilde{S} - \tilde{S}_0| = |S - S_0|$$

we obtain $|\mathscr{E}| \leqslant |T_0|^{-1}\,|F_0| \cdot |\tilde{F}_0| \cdot |S - S_0|$. To find a bound on $|\tilde{\mathscr{E}}|$, we take the matrix transpose of (8.12) to obtain

$$\tilde{\mathscr{E}} = \frac{1}{T_0^{\#}}\, q\tilde{F}_0 q \exp(-\mathrm{i}Ikx)\Delta^{\#} \exp(\mathrm{i}Ikx)qIF_0^{\#} Iq.$$

Thus $|\tilde{\mathscr{E}}| \leqslant |T_0|^{-1}\,|\tilde{F}_0| \cdot |\Delta| \cdot |F_0|$. QED

The two preceding lemmas give the following.

Theorem 8.2. The eigenvalues of the operator Ω_x defined in (8.11) are bounded in absolute value by $\max_k \sqrt{2}|F_0| \cdot |\tilde{F}_0| \cdot |(S - S_0)/T_0|$, where the absolute value of a matrix is as defined in (6.15) and $F_0 = \exp(-\mathrm{i}Ikx)\Psi_0$, where Ψ_0 is the physical solution to the matrix Schrödinger equation with the matrix potential λ_0.

Proof. Let ξ_x be an eigenvector of Ω_x with the eigenvalue μ. Then we have for $y > 0$,

$$\mu\xi_x(y) = \int_0^\infty \mathrm{d}z\; \Omega_x(y, z)\xi_x(z).$$

Taking the scalar product of both sides with $\xi_x(y)$, we obtain

$$\mu \int_0^\infty \mathrm{d}y\; \widetilde{\xi_x(y)}^* \xi_x(y) = \int_{-\infty}^\infty \int_{-\infty}^\infty \mathrm{d}y\mathrm{d}z\; \widetilde{\xi_x(y)}^* \Omega_x(y, z)\xi_x(z)$$

where we have extended the domain of definition of $\xi_x(y)$ by setting $\xi_x(y) = 0$ for $y < 0$. The Fourier transform of the last equation gives us

$$\mu \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi}\, \widetilde{p(k, x)}^* p(k, x) = \int_{-\infty}^\infty \frac{\mathrm{d}k}{2\pi}\, \widetilde{p(-k, x)}^* \mathscr{E}(k, x) q p(k, x)$$

where we have defined

$$p(k, x) \equiv \int_0^\infty \mathrm{d}y\; \xi_x(y) \exp(\mathrm{i}ky).$$

The last equation can be written as $\mu\langle p, p\rangle = \langle p^{\#}, \mathscr{E}qp\rangle$ where $\langle , \rangle$ is the usual scalar product on $\mathbb{C}^2$. Using the Schwarz inequality, we obtain

$$|\mu|\,\langle p, p\rangle \leqslant \langle p^{\#}, p^{\#}\rangle^{1/2}\langle \mathscr{E}qp, \mathscr{E}qp\rangle^{1/2} = \langle p, p\rangle^{1/2}\langle p, q\tilde{\mathscr{E}}^*\mathscr{E}qp\rangle^{1/2}.$$

Thus we have

$$|\mu| \leqslant \frac{\langle p, q\tilde{\mathscr{E}}^*\mathscr{E}qp\rangle^{1/2}}{\langle p, p\rangle^{1/2}}. \tag{8.13}$$

Using the matrix norm defined in (6.15), we obtain $|\tilde{p}^*q\tilde{\mathscr{E}}^*\mathscr{E}qp| \leqslant |\tilde{p}| \cdot |\tilde{\mathscr{E}}| \cdot |\mathscr{E}| \cdot |p|$ and using lemma 8.1 and lemma 8.2, we have

$$|\tilde{p}^*q\tilde{\mathscr{E}}^*\mathscr{E}qp| \leqslant (\tilde{p}^*p)^{1/2}(|T_0|^{-1}\,|F_0|\cdot|\tilde{F}_0|\cdot|S-S_0|)^2[2(\tilde{p}^*p)^{1/2}].$$

Using this last inequality in (8.13), we obtain

$$\begin{aligned}|\mu| &\leqslant \left(\frac{\int_{-\infty}^{\infty} \mathrm{d}k[(\sqrt{2}/|T_0|)\,|F_0|\cdot|\tilde{F}_0|\cdot|S-S_0|]^2\tilde{p}^*p}{\int_{-\infty}^{\infty} \mathrm{d}k\,\tilde{p}^*p}\right)^{1/2} \\ &\leqslant \max_k \frac{\sqrt{2}}{|T_0|}\,|F_0|\cdot|\tilde{F}_0|\cdot|S-S_0|.\end{aligned}$$

QED

Note that in the Marchenko equation for perturbations given in (8.7), x appears only as a parameter. Hence we expect the eigenvalues of the operator Ω_x also contain x as a parameter. The matrix $F_0(k, x)$ is continuous in k for each x and $F_0(k, x) - \mathbb{1} = \mathrm{O}(1/k)$ as $|k| \to \infty$. Thus $|F_0|$ and $|\tilde{F}_0|$ are both bounded in k for each x. Hence for each x, if we have

$$\max_k \frac{|S-S_0|}{|T_0|} < \frac{1}{\sqrt{2}\max\limits_k |F_0|\cdot|\tilde{F}_0|}$$

then the eigenvalues of Ω_x will be less than one in absolute value, in which case the Marchenko equation for perturbations given in (8.7) has a unique solution that can be found by iteration.

Multiplying (8.10) by the row vector $(1, 1)$ on the left and by the column vector $\binom{1}{1}$ on the right, we obtain

$$V(x) - V_0(x) = -(1, 1)\frac{\mathrm{d}}{\mathrm{d}x}\xi(x, 0+)\begin{pmatrix}1\\1\end{pmatrix}. \tag{8.14}$$

Let us define the scalar $A(k, x)$ as

$$A(k, x) \equiv (1, 1)I\mathscr{E}(-k, x)qH(-k, x)q\begin{pmatrix}1\\1\end{pmatrix} \tag{8.15}$$

where $H(k, x)$ as in (7.2) and $\mathscr{E}(k, x)$ as in (8.12). We can simplify (8.15) to obtain

$$A(k, x) = \frac{1}{T_0}(\psi_{0l}, \psi_{0r})I(S-S_0)\begin{pmatrix}\psi_l^*\\\psi_r^*\end{pmatrix}. \tag{8.16}$$

Using (8.14) and (8.4), we obtain

$$V(x) - V_0(x) = \frac{\mathrm{d}}{\mathrm{d}x}\lim_{y\downarrow 0}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}A(k, x)\exp(-\mathrm{i}ky). \tag{8.17}$$

We will use (8.17) to study the stability of the Marchenko inversion in another paper (Aktosun 1987b).

9. Further properties

Using (3.6) we can find the Schrödinger equation satisfied by the matrix $H(k, x)$ defined in (7.2) as follows. Since $\Psi = \Psi_0 H$, we have

$$\frac{d^2}{dx^2}(\Psi_0 H) + k^2 \Psi_0 H = \Psi_0 H\lambda.$$

Using the matrix Schrödinger equation for Ψ_0 and multiplying the above equation by Ψ_0^{-1} on the left, we obtain the equation satisfied by $H(k, x)$:

$$\frac{d^2 H}{dx^2} + 2\Psi_0^{-1}\frac{d\Psi_0}{dx}\frac{dH}{dx} + \lambda_0 H = H\lambda. \tag{9.1}$$

The next lemma shows that $\Psi_0^{-1}\, d\Psi_0/dx$ can be given explicitly.

Lemma 9.1. Let Ψ be the physical solution to the matrix Schrödinger equation with the matrix potential

$$\lambda = \frac{1}{2}\begin{pmatrix} V+U & V-U \\ V-U & V+U \end{pmatrix}.$$

Then we have

$$\Psi^{-1}\frac{d\Psi}{dx} = -iIk + \tfrac{1}{2}q\int_{-\infty}^{x} dt(V(t) - U(t)).$$

Proof. Using (5.4) and some straightforward algebra, we obtain

$$\Psi^{-1}\frac{d\Psi}{dx} = \frac{1}{4T}\begin{pmatrix} [\psi_r^{(v)}; \psi_l^{(v)}] + [\psi_r^{(u)}; \psi_l^{(u)}] & [\psi_r^{(v)}; \psi_l^{(v)}] - [\psi_r^{(u)}; \psi_l^{(u)}] \\ -[\psi_r^{(v)}; \psi_l^{(v)}] + [\psi_r^{(u)}; \psi_l^{(u)}] & -[\psi_r^{(v)}; \psi_l^{(v)}] - [\psi_r^{(u)}; \psi_l^{(u)}] \end{pmatrix}$$
$$+\frac{1}{4T}\begin{pmatrix} \frac{d}{dx}[\psi_r^{(v)}\psi_l^{(u)} + \psi_l^{(v)}\psi_r^{(u)}] & -[\psi_r^{(v)}; \psi_l^{(u)}] - [\psi_l^{(v)}; \psi_r^{(u)}] \\ -[\psi_r^{(v)}; \psi_l^{(u)}] - [\psi_r^{(v)}; \psi_l^{(u)}] & \frac{d}{dx}[\psi_r^{(v)}\psi_l^{(u)} + \psi_l^{(v)}\psi_r^{(u)}] \end{pmatrix}.$$

The Wronskians $[\psi_r^{(v)}; \psi_l^{(v)}]$ and $[\psi_r^{(u)}; \psi_l^{(u)}]$ are both equal to $2ikT$ (Newton 1983). We have $\psi_l^{(v)}\psi_r^{(u)} + \psi_l^{(u)}\psi_r^{(v)} = 2\det\Psi = 2T$ and this is independent of x. The Wronskians $[\psi_r^{(v)}; \psi_l^{(u)}]$ and $[\psi_r^{(u)}; \psi_l^{(v)}]$ can be evaluated directly in the usual way to obtain

$$\frac{d}{dx}[\psi_l^{(u)}; \psi_r^{(v)}] = (V-U)\psi_r^{(v)}\psi_l^{(u)}$$

and

$$\frac{d}{dx}[\psi_r^{(u)}; \psi_l^{(v)}] = (V-U)\psi_r^{(u)}\psi_l^{(v)}$$

and thus using the asymptotics of $\psi_l^{(v)}$, $\psi_r^{(v)}$, $\psi_l^{(u)}$ and $\psi_r^{(u)}$, we have

$$\frac{d}{dx}([\psi_l^{(u)};\psi_r^{(v)}]+[\psi_r^{(u)};\psi_l^{(v)}])=(V-U)\,[\psi_r^{(v)}\psi_l^{(u)}+\psi_r^{(u)}\psi_l^{(v)}]$$
$$=(V-U)\,(2\det\Psi)$$
$$=2T(V-U)$$

and by integration, we obtain

$$[\psi_l^{(u)};\psi_r^{(v)}]+[\psi_r^{(u)};\psi_l^{(v)}]=2T\int_{-\infty}^{x}dt(V(t)-U(t)) \tag{9.2}$$

$$=-2T\int_{x}^{\infty}dt(V(t)-U(t)). \tag{9.3}$$

Using all these results in the expression for $\Psi^{-1}\,d\Psi/dx$, we obtain

$$\Psi^{-1}\frac{d\Psi}{dx}=\begin{pmatrix}ik & 0\\ 0 & -ik\end{pmatrix}+\tfrac{1}{2}q\int_{-\infty}^{x}dt(V(t)-U(t)).\qquad \text{QED}$$

From (9.2) and (9.3), we have the following.

Corollary 9.1. $\int_{-\infty}^{\infty}dx\,U(x)=\int_{-\infty}^{\infty}dx\,V(x)$, where $V(x)$ and $U(x)$ are the potentials for the scattering matrices

$$S=\begin{pmatrix}T & R\\ L & T\end{pmatrix}\qquad\text{and}\qquad ISI=\begin{pmatrix}T & -R\\ -L & T\end{pmatrix}$$

respectively.

Define $K(x)$ as follows:

$$K(x)\equiv\int_{-\infty}^{x}dt(V_0(t)-U_0(t)). \tag{9.4}$$

Then the Schrödinger equation satisfied by $H(k,x)$ given in (9.1) can be written as

$$\frac{d^2H}{dx^2}+2iIk\frac{dH}{dx}+Kq\frac{dH}{dx}+\lambda_0H=H\lambda. \tag{9.5}$$

The partial differential equation satisfied by $\xi(x,y)$ is given in the following.

Lemma 9.2. The matrix $\xi(x,y)$, which is defined in (8.3), satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial\xi(x,y)}{\partial x}-2I\frac{\partial\xi(x,y)}{\partial y}\right)+K(x)q\frac{\partial\xi(x,y)}{\partial x}+\lambda_0(x)\xi(x,y)=\xi(x,y)\lambda(x)$$

where $K(x)$ is the scalar defined in (9.4), and λ_0 and λ are the matrix potentials for the scattering matrices S_0 and S respectively.

Proof. From (8.3) we obtain

$$H(x,y)=\mathbb{1}+\int_{0}^{\infty}dy\,\xi(x,y)\exp(iky). \tag{9.6}$$

Hence using (9.6) in (9.5), we obtain

$$\left(\frac{d^2}{dx^2}+2iIk\frac{d}{dx}+K(x)q\frac{d}{dx}+\lambda_0(x)\right)\int_0^\infty dy\,\xi(x,y)\exp(iky)-\int_0^\infty dy\,\xi(x,y)\exp(iky)\lambda(x)$$

$$=\lambda(x)-\lambda_0(x).$$

Using integration by parts, the above equation becomes

$$\int_0^\infty dy\exp(iky)\left[\frac{\partial}{\partial x}\left(\frac{\partial\xi(x,y)}{\partial x}-2I\frac{\partial\xi(x,y)}{\partial y}\right)+K(x)q\frac{\partial\xi(x,y)}{\partial x}\right.$$

$$\left.+\lambda_0(x)\xi(x,y)-\xi(x,y)\lambda(x)\right]=\lambda(x)-\lambda_0(x)+2I\frac{d}{dx}\xi(x,0+).$$

The right side of the equation vanishes due to (8.10). Hence the left side must also vanish, and by unfolding the Fourier transform we obtain the partial differential equation stated in the lemma. QED

In order to show that the solution to the Marchenko equation for perturbations satisfies the equation given in lemma 9.2, we need to prove the following three lemmas.

Lemma 9.3. The matrix $w(x,y)$ defined in (8.6) satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}-2I\frac{\partial}{\partial y}\right)w(x,y)+K(x)q\frac{\partial w(x,y)}{\partial x}+\lambda_0(x)w(x,y)=w(x,y)\lambda_0(x)$$

where $K(x)$ is the scalar defined in (9.4) and λ_0 is the potential matrix of the scattering matrix S_0.

Proof. From (7.6) we obtain $\Psi_0\mathscr{E}^{\#}=\tilde{\Delta}S_0^{\#}\Psi_0$. The second derivative of this equation, after using the matrix Schrödinger equation for Ψ_0, gives us

$$\Psi_0\frac{d^2\mathscr{E}^{\#}}{dx^2}+2\frac{d\Psi_0}{dx}\frac{d\mathscr{E}^{\#}}{dx}+\Psi_0(\lambda_0-k^2)\mathscr{E}^{\#}=\tilde{\Delta}S_0^{\#}\Psi_0(\lambda_0-k^2).$$

Multiplying both sides by Ψ_0^{-1} on the left and using lemma 9.1, we have

$$\frac{d^2\mathscr{E}^{\#}}{dx^2}+2iIk\frac{d\mathscr{E}^{\#}}{dx}+Kq\frac{d\mathscr{E}^{\#}}{dx}+\lambda_0\mathscr{E}^{\#}=\mathscr{E}^{\#}\lambda_0. \tag{9.7}$$

Using (8.6), we can take the Fourier transform of (9.7) to obtain

$$\left(\frac{d^2}{dx^2}+2iIk\frac{d}{dx}+Kq\frac{d}{dx}+\lambda_0\right)\int_{-\infty}^\infty dy\,w(x,y)\exp(iky)=\int_{-\infty}^\infty dy\,w(x,y)\exp(iky)\lambda_0.$$

The integration by parts gives us

$$\int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky) \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - 2I\frac{\partial}{\partial y}\right) w(x,y) + K(x)q\frac{\partial w(x,y)}{\partial x} + \lambda_0(x)w(x,y)\right]$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky) w(x,y)\lambda_0(x)$$

and by unfolding the Fourier transform, we obtain the partial differential equation stated in the lemma. QED

Lemma 9.4. The matrix $w(x, y)$ which is defined in (8.6) satisfies the first order partial differential equation

$$\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y} + \tfrac{1}{2}K(x)q\right) w(x,y) = -\frac{\partial w(x,y)}{\partial y} I + \tfrac{1}{2}w(x,y)K(x)q$$

where $K(x)$ is the scalar defined in (9.4).

Proof. From (7.6) we have $\Psi_0\mathscr{E}^{\#} = \tilde{\Delta}S_0^{\#}\Psi_0$. The first derivative of this equation gives us

$$\Psi_0\frac{\mathrm{d}\mathscr{E}^{\#}}{\mathrm{d}x} + \frac{\mathrm{d}\Psi_0}{\mathrm{d}x}\mathscr{E}^{\#} = \tilde{\Delta}S_0^{\#}\frac{\mathrm{d}\Psi_0}{\mathrm{d}x}.$$

After multiplying both sides by Ψ_0^{-1} on the left and using lemma 9.1, we obtain

$$\frac{\mathrm{d}\mathscr{E}^{\#}}{\mathrm{d}x} + \mathrm{i}Ik\,\mathscr{E}^{\#} + \tfrac{1}{2}Kq\,\mathscr{E}^{\#} = \mathscr{E}^{\#}\,\mathrm{i}Ik + \tfrac{1}{2}\mathscr{E}^{\#}Kq.$$

Taking the Fourier transform of this equation and using integration by parts, we obtain

$$\int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky) \left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y} + \tfrac{1}{2}K(x)q\right) w(x,y)$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y \exp(\mathrm{i}ky) \left(-\frac{\partial w(x,y)}{\partial x} I + \tfrac{1}{2}w(x,y)K(x)q\right).$$

Unfolding the Fourier transform, we obtain the first order equation stated in the lemma. QED

Lemma 9.5. Let $\xi_x(y) \equiv \xi(x,y)$ where $\xi(x,y)$ is the matrix defined in (8.3). Then $\xi_x(y)$ satisfies

$$\mathscr{O}_x\Omega_x\xi_x q - \Omega_x\mathscr{O}_x\xi_x q = w(x,y)\,(\lambda(x) - \lambda_0(x))$$

where λ_0 and λ are the matrix potentials for S_0 and S respectively, $w(x, y)$ is the matrix given in (8.6), Ω_x is the operator defined in (8.11), and $\mathscr{O}$ is the operator defined as

$$\mathscr{O}_x\xi_x(y) \equiv \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - 2I\frac{\partial}{\partial y}\right) + K(x)q\frac{\partial}{\partial x} + \lambda_0(x)\right]\xi_x(y).$$

Proof. The straightforward computation gives us, after using integration by parts

$$
\begin{aligned}
\mathcal{O}_x\Omega_x\xi_x q-\Omega_x\mathcal{O}_x\xi_x q=&\left(2\frac{\partial\Omega_x}{\partial x}-2I\frac{\partial\Omega_x}{\partial y}+Kq\Omega_x-\Omega_x Kq-\frac{\partial\Omega_x}{\partial y}2I\right)\frac{\partial\xi_x}{\partial y}q\\
&+\left[\frac{\partial}{\partial x}\left(\frac{\partial\Omega_x}{\partial x}-2I\frac{\partial\Omega_x}{\partial y}\right)+Kq\frac{\partial\Omega_x}{\partial x}+\lambda_0\Omega_x-\Omega_x\lambda_0\right]\xi_x q\\
&+\Omega_x(y,0+)(\lambda-\lambda_0)q\\
=&\int_{-\infty}^{\infty}\mathrm{d}z\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}(x,y+z)-2I\frac{\partial w}{\partial y}(x,y+z)\right)+K(x)q\frac{\partial w}{\partial x}(x,y+z)\right.\\
&\left.+\lambda_0(x)w(x,y+z)-w(x,y+z)q\lambda_0(x)q\right]q\frac{\partial\xi(x,z)}{\partial z}q\\
&+\int_{-\infty}^{\infty}\mathrm{d}z\left(2\frac{\partial w(x,y+z)}{\partial x}-2I\frac{\partial w(x,y+z)}{\partial y}+K(x)qw(x,y+z)\right.\\
&\left.-w(x,y+z)qK(x)-\frac{\partial w(x,y+z)}{\partial y}q2Iq\right)q\xi(x,z)q\\
&+w(x,y)q(\lambda(x)-\lambda_0(x))q.
\end{aligned}
$$

The first and second integrals above vanish by lemma 9.3 and lemma 9.4 respectively. Since λ_0 and λ are symmetric matrices, the non-vanishing part of the above equation gives us the equation stated in the lemma. QED

Using the three lemmas given above, we have the following.

Theorem 9.1. If the matrix Marchenko equation for perturbations given in (8.7) has a unique solution, then this solution satisfies the partial differential equation given in lemma 9.2.

Proof. Let $\xi_x(y)\equiv\xi(x,y)$ be the unique solution to the Marchenko equation for perturbations given in (8.7). Then the corresponding homogeneous equation has only the trivial solution. Let $\mathcal{O}_x$ be the operator defined in lemma 9.5. We then only need to show that $\mathcal{O}_x\xi_x=\xi_x\lambda$. We can write (8.7) in the operator notation as $\xi_x=w_x+\Omega_x\xi_x q$, where we have defined $w_x(y)\equiv w(x,y)$. Thus we have

$$
\begin{aligned}
\mathcal{O}_x\xi_x-\xi_x\lambda&=\mathcal{O}_x(w_x+\Omega_x\xi_x q)-\xi_x\lambda\\
&=\mathcal{O}_x w_x+\mathcal{O}_x\Omega_x\xi_x q-\xi_x\lambda.
\end{aligned}\tag{9.8}
$$

From lemma 9.3 we have $\mathcal{O}_x w_x=w_x\lambda_0$. From lemma 9.5 we have

$$\mathcal{O}_x\Omega_x\xi_x q=\Omega_x\mathcal{O}_x\xi_x q+w_x\lambda-w_x\lambda_0.$$

Using these two results in (9.8), we obtain

$$\mathcal{O}_x\xi_x-\xi_x\lambda=\Omega_x\mathcal{O}_x\xi_x q+(w_x-\xi_x)\lambda.\tag{9.9}$$

From the operator form of (8.7), we have $w_x-\xi_x=-\Omega_x\xi_x q$. Thus (9.9) becomes

$$\mathcal{O}_x\xi_x-\xi_x\lambda=\Omega_x(\mathcal{O}_x\xi_x-\xi_x\lambda)q.$$

Hence $\mathcal{O}_x \xi_x - \xi_x \lambda$ satisfies the homogeneous equation corresponding to (8.7), and it must vanish. Therefore $\mathcal{O}_x \xi_x = \xi_x \lambda$. QED

10. Conclusion

In this paper we have extended the Marchenko inversion method to find the change in the potential which corresponds to a finite change in the scattering matrix. Starting from the Riemann–Hilbert problem, the Marchenko equation for perturbations is derived. The change in the potential is related to the solution of the Marchenko equation for perturbations the same way as in the regular Marchenko formulation.

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