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A unified approach to Darboux transformations

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Abstract
We analyze a certain class of integral equations related to Marchenko equations and Gel'fand–Levitan equations associated with various systems of ordinary differential operators. When the integral operator is perturbed by a finite-rank perturbation, we explicitly evaluate the change in the solution. We show how this result provides a unified approach to derive Darboux transformations associated with various systems of ordinary differential operators. We illustrate our theory by deriving the Darboux transformation for the Zakharov–Shabat system and show how the potential and wavefunction change when a discrete eigenvalue is added to the spectrum.

1. Introduction

Consider the one-parameter family of integral equations of the second kind:

$$\beta(x, y) + \zeta(x, y) + \int_x^\infty dz\, \beta(x, z)\omega(z, y) = 0, \qquad y > x, \tag{1.1}$$

where $\beta(x, y)$ is the unknown term, $\zeta(x, y)$ is the nonhomogeneous term and $\omega(z, y)$ is an integral kernel which does not depend on the parameter $x \in \mathbf{R}$ and satisfies

$$\sup_{y>x} \int_x^\infty dz(\|\omega(z, y)\| + \|\omega(y, z)\|) < +\infty, \tag{1.2}$$

with $\|\cdot\|$ denoting any $N \times N$-matrix norm. Let us write (1.1) as

$$\beta + \zeta + \beta\Omega = 0, \tag{1.3}$$

where the integral operator Ω acts from the right. From (1.2), as shown in the appendix, it follows that Ω is bounded on the complex Banach spaces $\mathcal{H}_p^{M\times N}$ of $M \times N$ matrix-valued measurable functions $F : (x, +\infty) \to \mathbf{C}^{M\times N}$ such that the matrix norm $\|F(\cdot)\|$ belongs to $L^p(x, +\infty)$ for $1 \leqslant p \leqslant +\infty$.

We assume that, for each $x \in \mathbf{R}$, $(I+\Omega)$ is an invertible operator on $\mathcal{H}_1^{N\times N}$ and on $\mathcal{H}_2^{N\times N}$, where I denotes the identity operator. Using $(I+R)$ to denote the corresponding resolvent operator, where

$$R := (I+\Omega)^{-1} - I, \qquad I+R = (I+\Omega)^{-1}, \tag{1.4}$$

the solution to (1.3) can formally be written as

$$\beta = -\zeta(I+R),$$

or equivalently as

$$\beta(x,y) = -\zeta(x,y) - \int_x^\infty \mathrm{d}z\, \zeta(x,z) r(x;z,y), \tag{1.5}$$

with $r(x;z,y)$ denoting the integral kernel of the operator R.

Let us consider (1.3) in the special case

$$\alpha + \omega + \alpha\Omega = 0, \tag{1.6}$$

where the nonhomogeneous term and the integral kernel coincide, as seen by writing (1.6) explicitly as

$$\alpha(x,y) + \omega(x,y) + \int_x^\infty \mathrm{d}z\, \alpha(x,z)\omega(z,y) = 0, \qquad y > x. \tag{1.7}$$

The solution to (1.6) can formally be written as

$$\alpha = -\omega(I+R). \tag{1.8}$$

The unique solvability of (1.7) in $\mathcal{H}_1^{N\times N}$ and the condition in (1.2) imply that

$$\sup_{y>x} \int_x^\infty \mathrm{d}z (\|\alpha(z,y)\| + \|\alpha(y,z)\|) < +\infty. \tag{1.9}$$

A fundamental question related to (1.3) is the following: Can we write $r(x;y,z)$ appearing in (1.5) explicitly in terms of $\alpha(x,y)$ appearing in (1.7)? In case the answer is affirmative, we can express the solution β to (1.3) explicitly in terms of α and ζ. In fact, such a reduction question dates back to the Armenian astrophysicist Ambarzumian whose invariance principles are used in transfer of light in planetary atmospheres [7, 11, 12, 32, 33]. Ambarzumian [6] considered (1.6) with $\omega(y,z) = (c/2)\mathrm{Ei}(|y-z|)$, where Ei is the exponential integral function and c is a constant. Similar reduction formulas were obtained [20, 21] for integral equations with convolution kernels, i.e. when $\omega(y,z)$ is a function of $(y-z)$.

One of our goals in this paper is to study the aforementioned fundamental question when the integral operator Ω in (1.3) is $N\times N$-matrix valued and J-self-adjoint in the sense that

$$\Omega = J\Omega^\dagger J, \qquad \omega(y,z) = J\omega(z,y)^\dagger J, \tag{1.10}$$

where the dagger denotes the matrix adjoint (complex conjuge and matrix transpose) and J is an $N\times N$ self-adjoint involution, i.e.

$$J = J^\dagger = J^{-1}.$$

We present one of our key results in theorem 2.2, where the resolvent kernel $r(x;y,z)$ appearing in (1.5) is explicitly expressed in terms of the solution $\alpha(x,y)$ to (1.6).

Let us note that, without loss of generality, J may be assumed to have the form

$$J := \begin{bmatrix} I_j & 0 \\ 0 & -I_{N-j} \end{bmatrix},$$

where I_j is the $j \times j$ identity matrix for some $1 \leqslant j \leqslant N$. In that case we have

$$J \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} J = \begin{bmatrix} M_1 & -M_2 \\ -M_3 & M_4 \end{bmatrix},$$

for block matrices M_1, M_2, M_3, M_4 of appropriate sizes.

Having established our first key result in theorem 2.2, we turn our attention to the integral equation

$$\tilde{\alpha} + \tilde{\omega} + \tilde{\alpha}\tilde{\Omega} = 0, \tag{1.11}$$

which is explicitly written as

$$\tilde{\alpha}(x, y) + \tilde{\omega}(x, y) + \int_x^\infty \mathrm{d}z\, \tilde{\alpha}(x, z)\tilde{\omega}(z, y) = 0, \qquad y > x, \tag{1.12}$$

obtained from (1.6) by perturbing the operator Ω to $\tilde{\Omega}$ by a finite-rank operator, i.e.

$$\tilde{\Omega} = \Omega + FG, \qquad \tilde{\omega}(x, y) = \omega(x, y) + f(x)g(y), \tag{1.13}$$

where f and g are $N \times j$ and $j \times N$ matrices with entries depending on a single independent variable and belonging to $\mathcal{H}_1^{N\times j} \cap \mathcal{H}_\infty^{N\times j}$ and $\mathcal{H}_1^{j\times N} \cap \mathcal{H}_\infty^{j\times N}$, respectively. We note that we cannot in general expect F and G to commute, and hence in general $fg \neq gf$. In our second key result, we show that (1.11) can be transformed into another integral equation in which the kernel is degenerate (i.e. separable in the independent variables) so that $\tilde{\alpha}(x, y)$ can be explicitly obtained in terms of $\alpha(x, y)$, $f(x)$ and $g(y)$, as indicated in theorem 3.4.

Our key result given in theorem 3.4 has significant implications for various linear differential equations or systems of differential equations arising in important physical applications. One important consequence of theorem 3.4 is that it provides a systematic method to derive the Darboux transformations for a wide variety of spectral problems for differential equations. Recall that the idea behind a Darboux transformation (see e.g. [9, 27, 31] and the references therein) is to determine how the (generalized) eigenvectors change when a finite number of discrete eigenvalues are added to or subtracted from the spectrum of a differential operator without changing the continuous spectrum. In the language of physics, the Darboux transformation provides the perturbed potential and wavefunction in terms of the unperturbed quantities when a finite number of bound states are added or subtracted.

In this paper we are only concerned with Darboux transformations and not with Bäcklund transformations. When a discrete eigenvalue is added to the spectrum of a differential operator, a Bäcklund transformation [18] usually consists of a first-order differential equation (or a system of first-order differential equations) involving the perturbed and unperturbed potentials. On the other hand, in a Darboux transformation the perturbed potential is explicitly expressed in terms of unperturbed quantities. Bäcklund transformations have been derived [8, 13, 19, 25, 34] for various systems of differential operators and they are useful in obtaining exact solutions to related nonlinear evolution equations such as the Korteweg–de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation and the modified Korteweg–de Vries equation.

The Darboux transformation is well understood for Sturm–Liouville problems on a finite interval [14] and the one-dimensional Schrödinger equation [15], but there are also many others for which such transformations are not yet known or only some very special cases are known. Our theorem 3.4 can be applied to the Zakharov–Shabat differential operator, matrix Zakharov–Shabat systems and other differential operators to derive in a systematic way the corresponding Darboux transformations both at the potential and wavefunction levels. Our theorem is general enough so that it applies when one eigenvalue is added or subtracted from the spectrum, several eigenvalues are added or subtracted simultaneously, and eigenvalues with nontrivial Jordan structures are added or subtracted either one at a time or simultaneously. As an example, we apply theorem 2.2 on the Zakharov–Shabat system, and in theorem 6.4 we present the Darboux transformation expressing both the change in the potential and the change in the wavefunction explicitly in terms of the wavefunction of the unperturbed problem when one bound state is added. We compare our transformation for the potential given in (6.18) and for the wavefunction given in (6.19) with the results in the literature.

Let us mention that our results remain valid if the range of the integral is over $(-\infty, x)$ so that (1.7) is replaced with

$$\alpha(x,y)+\omega(x,y)+\int_{-\infty}^{x} dz\, \alpha(x,z)\omega(z,y)=0, \qquad y<x, \tag{1.14}$$

and also valid if the integral is over $(0, x)$ so that (1.7) is replaced with

$$\alpha(x,y)+\omega(x,y)+\int_{0}^{x} dz\, \alpha(x,z)\omega(z,y)=0, \qquad 0<y<x, \tag{1.15}$$

and with the obvious appropriate replacements in (1.1), (1.2), (1.5), (1.7), (1.9) and (1.12). By using the operator notation of (1.3) and (1.6) it is straightforward to modify the proofs and to treat (1.7), (1.14) and (1.15) all at once.

Our paper is organized as follows. In section 2 we establish our first key result by expressing the resolvent kernel $r(x; y, z)$ appearing in (1.5) explicitly in terms of the solution $\alpha(x, y)$ to (1.6). In section 3 we obtain our second key result by expressing $\tilde{\alpha}(x, y)$ explicitly in terms of $\alpha(x, y)$, $f(x)$, $g(y)$ when (1.6) is perturbed to (1.11) as in (1.13). In section 4 we show how the key result in theorem 3.4 provides a unified approach to derive Darboux transformations. In section 5 we show that the key results in the previous sections are applicable to various systems such as the Zakharov–Shabat system, its matrix generalizations and the Schrödinger equations on the full and half lines; we show how an integral equation of the form (1.7), (1.14) or (1.15) arises for each system and is related to an associated Marchenko integral equation or a Gel'fand–Levitan integral equation. In section 6, we illustrate the significance of our theorem 3.4 and derive the Darboux transformation for the Zakharov–Shabat system and make a comparison with some related results in the literature.

2. Reduction of the resolvent kernel

Recall that we assume that (1.3) is uniquely solvable in $\mathcal{H}_1^{N\times N}$ and in $\mathcal{H}_2^{N\times N}$ and that the operator Ω and its integral kernel $\omega(y, z)$ satisfy (1.2) and (1.10). In this section, we analyze the resolvent kernel $r(x; y, z)$ appearing in (1.5) and present our first key result; namely, we show that $r(x; y, z)$ can be expressed explicitly in terms of the solution $\alpha(x, y)$ to (1.7).

Proposition 2.1. *Assume that (1.3) is uniquely solvable in $\mathcal{H}_2^{N\times N}$ and that Ω satisfies (1.10). Then, the operator R given in (1.4) and the corresponding kernel $r(x; y, z)$ appearing in (1.5) satisfy*

$$R = JR^{\dagger}J, \qquad r(x;y,z)=Jr(x;z,y)^{\dagger}J, \tag{2.1}$$

where J is the involution matrix appearing in (1.10).

Proof. From (1.4) we see that

$$(I+\Omega)(I+R)=I=(I+R)(I+\Omega),$$

and hence we obtain

$$R+\Omega+\Omega R=0, \tag{2.2}$$

$$R + \Omega + R\Omega = 0. \tag{2.3}$$

By taking the adjoint of the operator equation in (2.2) and applying J on both sides of the resulting equation, we get

$$JR^\dagger J + J\Omega^\dagger J + (JR^\dagger J)(J\Omega^\dagger J) = 0,$$

or equivalently, after using (1.10),

$$JR^\dagger J + \Omega + (JR^\dagger J)\Omega = 0. \tag{2.4}$$

Since (1.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, by comparing (2.3) and (2.4) we see that $R = JR^\dagger J$. In taking the adjoint, we note that the independent variables y and z are switched in the argument of the kernel and hence (2.1) is established. □

Our first key result is given in the next theorem.

Theorem 2.2. *Assume that (1.3) is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies (1.10). Then, the corresponding kernel* $r(x; y, z)$ *appearing in (1.5) can be expressed explicitly in terms of the solution* $\alpha(x, y)$ *to (1.7) as*

$$r(x; y, z) = \begin{cases} \alpha(y, z) + \displaystyle\int_x^y \mathrm{d}s\, J\alpha(s, y)^\dagger J\alpha(s, z), & x < y < z, \\ J\alpha(z, y)^\dagger J + \displaystyle\int_x^z \mathrm{d}s\, J\alpha(s, y)^\dagger J\alpha(s, z), & x < z < y, \end{cases} \tag{2.5}$$

where J *is the involution matrix appearing in (1.10).*

Proof. Since (1.3) is uniquely solvable in $\mathcal{H}_2^{N\times N}$, so is (1.6) and hence the solution R to (2.3) is unique. Thus, it suffices to prove that the quantity defined in (2.5) satisfies (2.3), i.e. the quantity in (2.5) satisfies the integral equation

$$r(x; y, z) + \omega(y, z) + \int_x^\infty \mathrm{d}s\, r(x; y, s)\omega(s, z) = 0, \qquad x < \min\{y, z\}. \tag{2.6}$$

The proof for the case $x < z < y$ is similar to the case $x < y < z$, and hence we will only give the proof in the latter case. In that case, let us use $\int_x^\infty = \int_x^y + \int_y^\infty$ in the integral appearing in (2.6). We use a direct substitution from (2.5) into (2.6), where we note that the first line of (2.5) is used in the integral $\int_y^\infty$ and the second line of (2.7) is used in the integral $\int_x^y$ in (2.6). After the substitution, the left-hand side in (2.6) becomes $v_1 + v_2 + v_3$, where we have defined

$$v_1 := \alpha(y, z) + \omega(y, z) + \int_y^\infty \mathrm{d}s\, \alpha(y, s)\omega(s, z),$$

$$v_2 := \int_x^y \mathrm{d}t\, J\alpha(t, y)^\dagger J\omega(t, z) + \int_x^y \mathrm{d}t\, J\alpha(t, y)^\dagger J\alpha(t, z),$$

$$v_3 := \int_x^y \mathrm{d}s \int_x^s \mathrm{d}t\, J\alpha(t, y)^\dagger J\alpha(t, s)\omega(s, z) + \int_y^\infty \mathrm{d}s \int_x^y \mathrm{d}t\, J\alpha(t, y)^\dagger J\alpha(t, s)\omega(s, z).$$

Note that $v_1 = 0$ from (1.7). The orders of the two iterated integrals in v_3 can be changed to $\int_x^y \mathrm{d}t \int_t^y \mathrm{d}s$ and $\int_x^y \mathrm{d}t \int_y^\infty \mathrm{d}s$, respectively. Using $\int_t^y + \int_y^\infty = \int_t^\infty$, we then get

$$v_2 + v_3 = \int_x^y \mathrm{d}t\, J\alpha(t, y)^\dagger J \left[\alpha(t, z) + \omega(t, z) + \int_t^\infty \mathrm{d}s\, \alpha(t, s)\omega(s, z)\right]. \tag{2.7}$$

We see that the quantity in the brackets in (2.7) vanishes because of (1.7). Thus, (2.6) is satisfied for $x < y < z$. A similar direct substitution for the case $x < z < y$ completes the proof. □

3. Finite-rank perturbations

Our main goal in this section is to show that the solution $\tilde{\alpha}(x, y)$ to (1.12) can be expressed explicitly in terms of $\alpha(x, y)$, $f(x)$ and $g(y)$ appearing in (1.7) and (1.13), respectively. As we will see in later sections, the key formulas given in (3.19)–(3.21) below form the basis of Darboux transformations related to a wide variety of spectral problems associated with ordinary differential operators.

Recall that we assume that (1.10) holds and that (1.7) is uniquely solvable on $\mathcal{H}_1^{N\times N}$ and on $\mathcal{H}_2^{N\times N}$. Let us now define the intermediate quantities $n(x)$ and $q(y)$ as

$$n(x) := f(x) + \int_x^\infty \mathrm{d}z\, \alpha(x, z) f(z), \qquad q(y) := g(y) + \int_y^\infty \mathrm{d}z\, g(z) J \alpha(x, z)^\dagger J, \tag{3.1}$$

where J is the involution matrix appearing in (1.10). From (1.9) it follows that $n \in \mathcal{H}_1^{N\times j} \cap \mathcal{H}_\infty^{N\times j}$ and $q \in \mathcal{H}_1^{j\times N} \cap \mathcal{H}_\infty^{j\times N}$. Note that both integration limits $\int_x^\infty$ in (3.1) can be replaced with $\int_{-\infty}^\infty$ because $\alpha(x, y) = 0$ for $x > y$.

Theorem 3.1. *We can transform (1.11) into an integral equation that has a degenerate kernel and hence obtain $\tilde{\alpha}$ explicitly by linear algebraic methods.*

Proof. Using (1.13) let us write (1.11) as

$$\tilde{\alpha}(I + \Omega + FG) = -\omega - fg. \tag{3.2}$$

Recall that all the operators act from the right. By applying on (3.2) from the right with the resolvent operator $(I + R)$ appearing in (1.4), we get

$$\tilde{\alpha}[I + FG(I + R)] = \alpha - fg(I + R), \tag{3.3}$$

where we have used (1.8) to have α on the right-hand side of (3.3). Let us define the operator $\tilde{G}$ as

$$\tilde{G} := G(I + R), \qquad \tilde{g}(x, y) := g(y) + \int_x^\infty \mathrm{d}z\, g(z) r(x; z, y), \tag{3.4}$$

where $r(x; y, z)$ is the kernel given in (2.5). We emphasize the dependence of $\tilde{g}$ both on x and y. Note that the integral equation in (3.3) has a degenerate kernel, which can be seen by writing it in the form

$$\tilde{\alpha}(I + F\tilde{G}) = \alpha - f\tilde{g}, \tag{3.5}$$

because the kernel of $F\tilde{G}$ is $f(y)\tilde{g}(x, z)$, where there is a separation of the y and z variables and x appears merely as a parameter.

Let us now solve (3.3) by using linear algebra. We look for a solution in the form

$$\tilde{\alpha}(x, y) = \alpha(x, y) + p(x)\tilde{g}(x, y), \tag{3.6}$$

where p is to be determined. Using (3.6) in (3.5), after some simplification we get

$$(\alpha F + p + p\tilde{g}F + f)\tilde{G} = 0,$$

which yields

$$p(I + \tilde{g}F) = -(f + \alpha F), \tag{3.7}$$

or written in the integral form as

$$p(x) = -n(x)\left[I + \int_x^\infty \mathrm{d}s\, \tilde{g}(x, s) f(s)\right]^{-1},$$

where we have used the definition of $n(x)$ given in (3.1). Using (3.7) in (3.6) we obtain

$$\tilde{\alpha} = \alpha - (f + \alpha F)(I + \tilde{g}F)^{-1}\tilde{g},$$

or written in the integral form as

$$\tilde{\alpha}(x, y) = \alpha(x, y) - n(x)\left[I + \int_x^\infty \mathrm{d}s\, \tilde{g}(x, s) f(s)\right]^{-1} \tilde{g}(x, y), \tag{3.8}$$

which completes the proof of our theorem. □

Note that (3.8) expresses $\tilde{\alpha}(x, y)$ in terms of $\alpha(x, y)$, $f(x)$ and $\tilde{g}(x, y)$ because as seen from (3.1) the quantity $n(x)$ is available in terms of $\alpha(x, y)$ and $f(x)$.

Next, we show that $\tilde{g}(x, y)$ can explicitly be obtained in terms of $\alpha(x, y)$ and $g(y)$, which will then imply that $\tilde{\alpha}(x, y)$ is expressed in terms of $\alpha(x, y)$, $f(x)$ and $g(y)$.

Proposition 3.2. *The quantity $\tilde{g}(x, y)$ defined in (3.4) can be expressed explicitly in terms of the solution $\alpha(x, y)$ to (1.6) and the quantities $f(x)$ and $g(y)$ appearing in (1.13) as*

$$\tilde{g}(x, y) = q(y) + \int_x^y \mathrm{d}s\, q(s)\alpha(s, y), \tag{3.9}$$

where $q(y)$ is the quantity defined in (3.1), and it is noted that $\tilde{g}(x, x) = q(x)$.

Proof. We will substitute (2.5) into (3.4). For this purpose, let us write the integral $\int_x^\infty$ in (3.4) as $\int_x^y + \int_y^\infty$. Using the first line of (2.5) in $\int_x^y$ and the second in $\int_y^\infty$, we obtain

$$\begin{aligned}\tilde{g}(x, y) = g(y) &+ \int_x^y \mathrm{d}s\, g(s)\alpha(s, y) + \int_y^\infty \mathrm{d}s\, g(s) J\alpha(y, s)^\dagger J \\ &+ \left(\int_x^y \mathrm{d}s \int_x^s \mathrm{d}t + \int_y^\infty \mathrm{d}s \int_x^y \mathrm{d}t\right) g(s) J\alpha(t, s)^\dagger J\alpha(t, y).\end{aligned}$$

The sum of the two iterated integrals above can be written first as a double integral and then as an iterated integral by changing the order of integration to get

$$\int_x^y \mathrm{d}s \int_x^s \mathrm{d}t + \int_y^\infty \mathrm{d}s \int_x^y \mathrm{d}t = \int_x^y \mathrm{d}t \int_t^\infty \mathrm{d}s. \tag{3.10}$$

Using (3.10) and combining terms as in (3.1), we then obtain (3.9). □

We note that the integral $\int_x^y$ in (3.9) can also be written as $\int_x^\infty$ because $\alpha(s, y) = 0$ for $s > y$.

Let us define the matrix $\Gamma(x)$ as the quantity whose inverse appearing in (3.8), namely as

$$\Gamma(x) := I + \int_x^\infty \mathrm{d}s\, \tilde{g}(x, s) f(s). \tag{3.11}$$

Proposition 3.3. *The quantity $\Gamma(x)$ defined in (3.11) can be expressed explicitly in terms of the solution $\alpha(x, y)$ to (1.6) and the quantities $f(x)$ and $g(y)$ appearing in (1.13) as*

$$\Gamma(x) = I + \int_x^\infty \mathrm{d}s\, q(s)n(s), \tag{3.12}$$

where $n(x)$ and $q(x)$ are the quantities defined in (3.1).

Proof. Using (3.9) in (3.11) we get

$$\Gamma(x) = I + \int_x^\infty \mathrm{d}s\, q(s) f(s) + \int_x^\infty \mathrm{d}s \int_x^s \mathrm{d}t\, q(t)\alpha(t, s) f(s). \tag{3.13}$$

Changing the order of integration in the last integral in (3.13) and using (3.11), we get (3.12). Since $n(s)$ and $q(s)$ are expressed in terms of $\alpha(x, y)$, $f(x)$ and $g(y)$, we see from (3.12) that $\Gamma(x)$ is explicitly expressed in terms of $\alpha(x, y)$, $f(x)$ and $g(y)$, as well. □

The Fourier transform of the $N \times N$ matrix-valued quantity $\alpha(x, y)$ in (1.7), usually called a wavefunction, can be written as

$$\Psi(\lambda, x) := \mathrm{e}^{-\mathrm{i}\lambda J x} + \int_x^\infty \mathrm{d}y\, \alpha(x, y)\, \mathrm{e}^{-\mathrm{i}\lambda J y}, \tag{3.14}$$

where J is the involution matrix appearing in (1.10). Using the inverse Fourier transform on (3.14) we get

$$\alpha(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}\lambda [\Psi(\lambda, x) - \mathrm{e}^{-\mathrm{i}\lambda J x}]\, \mathrm{e}^{\mathrm{i}\lambda J y}. \tag{3.15}$$

Similarly, for $\tilde{\alpha}(x, y)$, we have the associated $N \times N$ matrix-valued wavefunction $\tilde{\Psi}(\lambda, x)$, where

$$\begin{aligned} \tilde{\Psi}(\lambda, x) &:= \mathrm{e}^{-\mathrm{i}\lambda J x} + \int_x^\infty \mathrm{d}y\, \tilde{\alpha}(x, y)\, \mathrm{e}^{-\mathrm{i}\lambda J y}, \\ \tilde{\alpha}(x, y) &= \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}\lambda [\tilde{\Psi}(\lambda, x) - \mathrm{e}^{-\mathrm{i}\lambda J x}]\, \mathrm{e}^{\mathrm{i}\lambda J y}. \end{aligned} \tag{3.16}$$

Let us introduce

$$\gamma(\lambda, x) := \int_x^\infty \mathrm{d}y\, \tilde{g}(x, y)\, \mathrm{e}^{-\mathrm{i}\lambda J y}. \tag{3.17}$$

Using (3.9) and (3.14) in (3.17) and the fact that $\alpha(x, y) = 0$ for $y < x$, we get

$$\gamma(\lambda, x) = \int_x^\infty \mathrm{d}s\, q(s) \Psi(\lambda, s). \tag{3.18}$$

The following theorem and in particular (3.20) describes the effect of a finite-rank perturbation on the wavefunction.

Theorem 3.4. *Let α and $\tilde{\alpha}$ be the solutions to the integral equations (1.6) and (1.11), respectively, and let $n(x)$, $\Gamma(x)$, $\tilde{g}(x, y)$ and $\gamma(\lambda, x)$ be the quantities given in (3.1), (3.12), (3.9) and (3.18), respectively. Then, $\tilde{\alpha}(x, y) - \alpha(x, y)$ and $\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x)$ can explicitly be written in terms of $\alpha(x, y)$, $f(x)$, $g(y)$ as*

$$\tilde{\alpha}(x, y) - \alpha(x, y) = -n(x)\Gamma(x)^{-1}\tilde{g}(x, y), \tag{3.19}$$

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = -n(x)\Gamma(x)^{-1}\gamma(\lambda, x). \tag{3.20}$$

Furthermore, we have

$$\tilde{\alpha}(x, x) - \alpha(x, x) = -n(x)\Gamma(x)^{-1}q(x). \tag{3.21}$$

Proof. Note that using (3.1), (3.11) and (3.12) in (3.8) we obtain (3.19). Using $\tilde{g}(x, x) = q(x)$ from proposition 3.2 in (3.19) we get (3.21). Finally, we obtain (3.20) with the help of (3.14), (3.16), (3.17) and (3.19). □

We conclude this section with a result on the trace of the left-hand side of (3.21).

Proposition 3.5. *Let α and $\tilde{\alpha}$ be the solutions of the integral equations (1.6) and (1.11), respectively, and let Γ be the matrix given in (3.12). The trace of the difference $\tilde{\alpha}(x,x)-\alpha(x,x)$ can be expressed as the logarithmic derivative of the determinant of $\Gamma(x)$ as*

$$\mathrm{tr}[\tilde{\alpha}(x,x)-\alpha(x,x)] = \mathrm{tr}\left[\frac{\mathrm{d}\Gamma(x)}{\mathrm{d}x}\Gamma(x)^{-1}\right] = \frac{1}{\det\Gamma(x)}\frac{\mathrm{d}\det\Gamma(x)}{\mathrm{d}x}. \tag{3.22}$$

Proof. From (3.12) we get

$$\frac{\mathrm{d}\Gamma(x)}{\mathrm{d}x} = -q(x)n(x). \tag{3.23}$$

Using the well-known matrix properties

$$\mathrm{tr}[M_1M_2] = \mathrm{tr}[M_2M_1], \qquad \mathrm{tr}\left[\frac{\mathrm{d}M(x)}{\mathrm{d}x}M(x)^{-1}\right] = \frac{1}{\det M(x)}\frac{\mathrm{d}\det M(x)}{\mathrm{d}x},$$

from (3.21) and (3.23) we get (3.22). □

4. Darboux transformations

The integral equations (1.7), (1.14) and (1.15) arise in the study of various scattering and spectral problems, some of which are described in section 5. In this section we will elaborate on (3.19)–(3.21) and show how they provide a unified approach to Darboux transformations for a variety of scattering and spectral problems. Recall that a Darboux transformation describes how the wavefunction and the potential change when a finite number of discrete eigenvalues are added (or subtracted) from the spectrum of a differential operator without changing its continuous spectrum.

Suppose we add a discrete eigenvalue λ_j with multiplicity n_j to the existing spectrum. Then, associated with the eigenvalue λ_j, there are n_j parameters $c_{j0},\ldots,c_{j(n_j-1)}$, usually known as norming constants. Formulas (3.19)–(3.21) tell us how the wavefunction changes from $\Psi(\lambda,x)$ to $\tilde{\Psi}(\lambda,x)$, how the potential changes from $u(x)$ to $\tilde{u}(x)$, and how the quantity $\alpha(x,y)$ related to the Fourier transform of the wavefunction changes to $\tilde{\alpha}(x,y)$. Consequently, for each discrete eigenvalue λ_j added to the spectrum, there will be an n_j-parameter family of potentials $\tilde{u}(x)$, where the norming constants act as the parameters. In case several discrete eigenvalues $\lambda_1,\ldots,\lambda_N$ are added all at once, it is convenient to use [4] a square matrix A whose eigenvalues are related to λ_j for $j=1,\ldots,N$ in a simple manner; it is also convenient to use [4] a matrix C whose entries are related to the norming constants c_{js} for $j=1,\ldots,N$ and $s=0,1,\ldots,n_j-1$.

The quantities $f(x)$ and $g(x)$ appearing in (1.13) can usually be represented in the form

$$f(x) = \begin{bmatrix} 0 & B^\dagger \mathrm{e}^{-A^\dagger x} \\ C\,\mathrm{e}^{-Ax} & 0 \end{bmatrix}, \qquad g(y) = \begin{bmatrix} \mathrm{e}^{-Ay}B & 0 \\ 0 & -\mathrm{e}^{-A^\dagger y}C^\dagger \end{bmatrix}, \tag{4.1}$$

where A is a constant square matrix with all eigenvalues having positive real parts (the bound-state λ-values are usually obtained [4] by multiplying the eigenvalues of A by the imaginary unit i), and B and C are constant matrices of appropriate sizes so that the matrix product $f(x)g(y)$ is well defined and given by

$$f(x)g(y) = \begin{bmatrix} 0 & -B^\dagger \mathrm{e}^{-A^\dagger(x+y)}C^\dagger \\ C\,\mathrm{e}^{-A(x+y)}B & 0 \end{bmatrix}.$$

For $f(x)$ and $g(y)$ given in (4.1), let us evaluate $n(x)$, $q(x)$ and $\tilde{g}(x,y)$ given in (3.1) and (3.9), respectively, explicitly in terms of the wavefunction $\Psi(\lambda,x)$ evaluated at the eigenvalues of A. First, by taking the matrix adjoint, from (3.15) we get

$$J\alpha(x,y)^\dagger J = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}\lambda Jy}[J\Psi(\lambda,x)^\dagger J - \mathrm{e}^{\mathrm{i}\lambda Jx}]. \tag{4.2}$$

Using (3.15) in (3.1) and the fact that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}s\, \mathrm{e}^{\pm \mathrm{i}as} = \delta(a), \tag{4.3}$$

where δ is the Dirac delta distribution, we evaluate $n(x)$ as

$$n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \Psi(\lambda, x)\int_{x}^{\infty} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}\lambda J y} f(y). \tag{4.4}$$

Using (4.1) in (4.4), we obtain

$$n(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \Psi(\lambda, x)\, \mathrm{e}^{\mathrm{i}\lambda J x}\mathcal{N}(\lambda, x), \tag{4.5}$$

where we have defined

$$\mathcal{N}(\lambda, x) := \begin{bmatrix} 0 & -B^\dagger(\lambda I + \mathrm{i}A^\dagger)^{-1}\, \mathrm{e}^{-A^\dagger x} \\ C(\lambda I - \mathrm{i}A)^{-1}\, \mathrm{e}^{-Ax} & 0 \end{bmatrix}.$$

Similarly, using (4.2) and (4.3) in (3.1) we obtain

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{x}^{\infty} \mathrm{d}y\, g(y)\, \mathrm{e}^{-\mathrm{i}\lambda J y} J\Psi(\lambda, x)^\dagger J, \tag{4.6}$$

and using (4.1) in (4.6) we conclude that

$$q(x) = \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathcal{Q}(\lambda, x)\, \mathrm{e}^{-\mathrm{i}\lambda J x} J\Psi(\lambda, x)^\dagger J, \tag{4.7}$$

where we have defined

$$\mathcal{Q}(\lambda, x) := \begin{bmatrix} \mathrm{e}^{-Ax}(\lambda I - \mathrm{i}A)^{-1}B & 0 \\ 0 & \mathrm{e}^{-A^\dagger x}(\lambda I + \mathrm{i}A^\dagger)^{-1}C^\dagger \end{bmatrix}.$$

Proceeding in a similar manner, with the help of (3.9), (3.14), (4.3) and (4.7) we first get

$$\tilde{g}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\mu \int_{x}^{\infty} \mathrm{d}z\, q(z)\Psi(\mu, z)\, \mathrm{e}^{\mathrm{i}\mu J y},$$

and then obtain

$$\tilde{g}(x, y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\mu \int_{x}^{\infty} \mathrm{d}z\, E(\lambda, \mu, z)\, \mathrm{e}^{\mathrm{i}\mu J y}, \tag{4.8}$$

where we have defined

$$E(\lambda, \mu, x) := \frac{1}{4\pi^2 \mathrm{i}}\mathcal{Q}(\lambda, x)\, \mathrm{e}^{-\mathrm{i}\lambda J x} J\Psi(\lambda, x)^\dagger J\Psi(\mu, x). \tag{4.9}$$

Then, with the help of (4.5) and (4.7) the matrix $\Gamma(x)$ given in (3.12) can be explicitly written in terms of the wavefunction Ψ as

$$\Gamma(x) = I - \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\mu \int_{x}^{\infty} \mathrm{d}y\, E(\lambda, \mu, y)\, \mathrm{e}^{\mathrm{i}\mu J y}\mathcal{N}(\mu, y). \tag{4.10}$$

Finally, using (4.7) and (4.9) in (3.18), we obtain

$$\gamma(\lambda, x) = 2\pi \int_{x}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}\lambda\, E(\lambda, \lambda, s). \tag{4.11}$$

Note that the integrals in (4.5), (4.7), (4.10) and (4.11) can be performed as residue integrals in the complex λ-plane with the poles at the eigenvalues of $\mathrm{i}A$ and $-\mathrm{i}A^\dagger$. Evaluating those integrals and using the result in (4.8) and (4.11) we can obtain $\tilde{g}(x, y)$ and $\gamma(\lambda, x)$ explicitly in terms of $\Psi(\lambda, x)$ evaluated at the eigenvalues of $\mathrm{i}A$ and $-\mathrm{i}A^\dagger$. If some bound states have multiplicities, i.e. if some of the eigenvalues of A have nontrivial Jordan structures, then the explicit expressions for $\tilde{g}(x, y)$ and $\gamma(\lambda, x)$ also contain some λ-derivatives of $\Psi(\lambda, x)$ evaluated at the eigenvalues of $\mathrm{i}A$ and $-\mathrm{i}A^\dagger$.

In section 6 we will use the procedure described here to obtain the Darboux transformation for the Zakharov–Shabat system given in (5.1).

5. Applications to specific systems

In this section we present some specific systems on which the theory presented in the previous sections is applicable. In the first four examples we analyze the Zakharov–Shabat system and its matrix generalizations. In the remaining three examples we analyze the Schrödinger equation on the full and half lines. For each system, we identify the quantities $\omega(x, y)$ and $\alpha(x, y)$ appearing in (1.7) or in one of its variants (1.14) and (1.15), and we identify $f(x)$ and $g(y)$ appearing in (1.13). We identify the involution J for which (1.10) is satisfied by $\omega(y, z)$ for each system. Then, for each system we indicate how the potential $u(x)$ appearing in the system is related to $\alpha(x, x)$ so that one sees clearly how the perturbation $\tilde{u}(x) - u(x)$ in the potential can be recovered from (3.21). For each system we indicate how the wavefunction $\Psi(\lambda, x)$ is related to certain specific solutions to the corresponding system. We also relate the integral equations (1.7), (1.14) and (1.15) to the Marchenko equations or the Gel'fand–Levitan equation corresponding to each system. In some cases we observe that (1.7), (1.14) or (1.15) is exactly the same as a Marchenko equation or a Gel'fand–Levitan equation, and in some other cases one needs to rearrange the Marchenko equations or the Gel'fand–Levitan equation in order to get the integral equation (1.7), (1.14) or (1.15).

We first discuss four examples involving the Zakharov–Shabat system and its matrix generalization, two with the range of the integral over $(x, +\infty)$ and two with the range of the integral over $(-\infty, x)$. We shall discuss the details of example 5.1 in the next section, where we derive the Darboux transformation for the Zakharov–Shabat system and compare our results with the existing results in the literature [10, 19, 23, 24, 26, 29].

Example 5.1. Consider the Zakharov–Shabat system

$$\frac{d\varphi(\lambda, x)}{dx} = \begin{bmatrix} -i\lambda & u(x) \\ -u(x)^* & i\lambda \end{bmatrix} \varphi(\lambda, x), \tag{5.1}$$

where the asterisk denotes complex conjugation, u is a (scalar) complex-valued integrable potential and φ is a column vector with two components. The corresponding (left) Marchenko equations are given by [30]

$$\begin{cases} \bar{K}(x, y) + \begin{bmatrix} 0 \\ \Omega_l(x+y) \end{bmatrix} + \displaystyle\int_x^\infty dz\, K(x, z)\Omega_l(z+y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & y > x, \\ K(x, y) - \begin{bmatrix} \Omega_l(x+y)^\dagger \\ 0 \end{bmatrix} - \displaystyle\int_x^\infty dz\, \bar{K}(x, z)\Omega_l(z+y)^\dagger = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & y > x, \end{cases} \tag{5.2}$$

where $K(x, y)$ and $\bar{K}(x, y)$ are 2×1 matrix valued and $\Omega_l(x + y)$ is a scalar function. Let us stress that an overline does not indicate complex conjugation. The potential is recovered as [30]

$$u(x) = -2[1 \quad 0]K(x, x) = 2\bar{K}(x, x)^\dagger \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.3}$$

By letting

$$\alpha(x, y) = [\bar{K}(x, y) \quad K(x, y)], \qquad \omega(x, y) = \begin{bmatrix} 0 & -\Omega_l(x+y)^\dagger \\ \Omega_l(x+y) & 0 \end{bmatrix}, \tag{5.4}$$

we can write (5.2) as (1.7), which is now a 2×2 system of integral equations. We note that $\omega(x, y)$ given in (5.4) satisfies (1.10) with $J = \mathrm{diag}[1, -1]$. As seen from (5.3), u is then recovered from the solution to (1.7) as

$$u(x) = -2[1 \quad 0]\alpha(x, x) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.5}$$

In this case, the degenerate perturbation on $\Omega_l(x+y)$ is given [4] by $C_l\,\mathrm{e}^{-A_l(x+y)}B_l$, where A_l is a constant $p\times p$ matrix with all eigenvalues having positive real parts, B_l is a constant $p\times 1$ matrix and C_l is a constant $1\times p$ matrix. The functions f and g appearing in (1.13) are then $2\times 2p$ and $2p\times 2$ matrices, respectively, given by

$$f(x)=\begin{bmatrix}0 & B_l^\dagger\,\mathrm{e}^{-A_l^\dagger x}\\ C_l\,\mathrm{e}^{-A_lx} & 0\end{bmatrix},\qquad g(y)=\begin{bmatrix}\mathrm{e}^{-A_ly}B_l & 0\\ 0 & -\mathrm{e}^{-A_l^\dagger y}C_l^\dagger\end{bmatrix}. \tag{5.6}$$

For the Zakharov–Shabat system the wavefunction appearing in (3.14) is the 2×2 matrix given by

$$\Psi(\lambda,x)=\begin{bmatrix}\bar\psi_1(\lambda,x) & \psi_1(\lambda,x)\\ \bar\psi_2(\lambda,x) & \psi_2(\lambda,x)\end{bmatrix}=\begin{bmatrix}\psi_2(\lambda^*,x)^* & \psi_1(\lambda,x)\\ -\psi_1(\lambda^*,x)^* & \psi_2(\lambda,x)\end{bmatrix}, \tag{5.7}$$

where $\left[\begin{smallmatrix}\psi_1(\lambda,x)\\ \psi_2(\lambda,x)\end{smallmatrix}\right]$ is the Jost solution to (5.1) with the asymptotics $\left[\begin{smallmatrix}0\\ \mathrm{e}^{\mathrm{i}\lambda x}\end{smallmatrix}\right]$ as $x\to+\infty$ and $\left[\begin{smallmatrix}\bar\psi_1(\lambda,x)\\ \bar\psi_2(\lambda,x)\end{smallmatrix}\right]$ is the solution behaving as $\left[\begin{smallmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}\\ 0\end{smallmatrix}\right]+o(1)$ as $x\to+\infty$.

Example 5.2. Consider the matrix generalization of example 5.1 with the Zakharov–Shabat system

$$\frac{\mathrm{d}\varphi(\lambda,x)}{\mathrm{d}x}=\begin{bmatrix}-\mathrm{i}\lambda I_m & u(x)\\ -u(x)^\dagger & \mathrm{i}\lambda I_n\end{bmatrix}\varphi(\lambda,x), \tag{5.8}$$

where u is an $m\times n$ matrix with integrable entries. The corresponding (left) Marchenko equations are given by

$$\begin{cases}\bar K(x,y)+\begin{bmatrix}0_{mn}\\ \Omega_l(x+y)\end{bmatrix}+\displaystyle\int_x^\infty \mathrm{d}z\,K(x,z)\Omega_l(z+y)=\begin{bmatrix}0_{mn}\\ 0_{nn}\end{bmatrix}, & y>x,\\ K(x,y)-\begin{bmatrix}\Omega_l(x+y)^\dagger\\ 0_{nm}\end{bmatrix}-\displaystyle\int_x^\infty \mathrm{d}z\,\bar K(x,z)\Omega_l(z+y)^\dagger=\begin{bmatrix}0_{mm}\\ 0_{nm}\end{bmatrix}, & y>x,\end{cases} \tag{5.9}$$

where 0_{jk} is the zero matrix of size $j\times k$, and K, $\bar K$, and Ω_l have sizes $(m+n)\times n$, $(m+n)\times m$ and $n\times m$, respectively. The $m\times n$ potential matrix u is recovered from the solution to (5.9) as

$$u(x)=-2\begin{bmatrix}I_m & 0_{mn}\end{bmatrix}K(x,x)=2\bar K(x,x)^\dagger\begin{bmatrix}0_{mn}\\ I_n\end{bmatrix}. \tag{5.10}$$

By letting

$$\alpha(x,y)=[\bar K(x,y)\quad K(x,y)],\qquad \omega(x,y)=\begin{bmatrix}0_{mm} & -\Omega_l(x+y)^\dagger\\ \Omega_l(x+y) & 0_{nn}\end{bmatrix}, \tag{5.11}$$

we can write (5.9) as (1.7), which is now an $(m+n)\times(m+n)$ system of integral equations. Note that $\omega(x,y)$ given in (5.11) satisfies (1.10) with $J=I_m\oplus(-I_n)$. As seen from (5.10), the potential u is recovered from the solution to (1.7) as

$$u(x)=-2[I_m\quad 0_{mn}]\alpha(x,x)\begin{bmatrix}0_{mn}\\ I_n\end{bmatrix}.$$

In this case, the degenerate perturbation on the $n\times m$ matrix quantity $\Omega_l(x+y)$ is given [4, 16, 17] by $C_l\,\mathrm{e}^{-A_l(x+y)}B_l$, where A_l is a constant $p\times p$ matrix with all eigenvalues having positive real parts, B_l is a constant $p\times m$ matrix and C_l is a constant $n\times p$ matrix. The functions f and g appearing in (1.13) are then $(m+n)\times 2p$ and $2p\times(m+n)$ matrices, respectively, given by

$$f(x)=\begin{bmatrix}0_{mp} & B_l^\dagger\,\mathrm{e}^{-A_l^\dagger x}\\ C_l\,\mathrm{e}^{-A_lx} & 0_{np}\end{bmatrix},\qquad g(y)=\begin{bmatrix}\mathrm{e}^{-A_ly}B_l & 0_{pn}\\ 0_{pm} & -\mathrm{e}^{-A_l^\dagger y}C_l^\dagger\end{bmatrix}.$$

The wavefunction appearing in (3.14) has the form

$$\Psi(\lambda,x)=\begin{bmatrix}\bar\psi_1(\lambda,x) & \psi_1(\lambda,x)\\ \bar\psi_2(\lambda,x) & \psi_2(\lambda,x)\end{bmatrix},$$

where $\left[\begin{smallmatrix}\psi_1(\lambda,x)\\ \psi_2(\lambda,x)\end{smallmatrix}\right]$ is the $(m+n)\times n$ Jost solution to (5.8) with the asymptotics $\left[\begin{smallmatrix}0\\ \mathrm{e}^{\mathrm{i}\lambda x}I_n\end{smallmatrix}\right]$ as $x\to+\infty$ and $\left[\begin{smallmatrix}\bar\psi_1(\lambda,x)\\ \bar\psi_2(\lambda,x)\end{smallmatrix}\right]$ is the $(m+n)\times m$ Jost solution to (5.8) with the asymptotics $\left[\begin{smallmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}I_m\\ 0_{nm}\end{smallmatrix}\right]$ as $x\to+\infty$.

Example 5.3. For the Zakharov–Shabat system in (5.1), the (right) Marchenko integral equations are given by

$$\begin{cases}\bar M(x,y)+\begin{bmatrix}\Omega_{\mathrm r}(x+y)\\ 0\end{bmatrix}+\displaystyle\int_{-\infty}^{x}\mathrm{d}y\,M(x,z)\Omega_{\mathrm r}(z+y)=\begin{bmatrix}0\\0\end{bmatrix}, & y<x,\\ M(x,y)-\begin{bmatrix}0\\ \Omega_{\mathrm r}(x+y)^{\dagger}\end{bmatrix}-\displaystyle\int_{-\infty}^{x}\mathrm{d}y\,\bar M(x,z)\Omega_{\mathrm r}(z+y)^{\dagger}=\begin{bmatrix}0\\0\end{bmatrix}, & y<x.\end{cases}\tag{5.12}$$

The scalar potential u is recovered from the solution to (5.12) as

$$u(x)=2[1\quad 0]\bar M(x,x)=-2M(x,x)^{\dagger}\begin{bmatrix}0\\1\end{bmatrix}.\tag{5.13}$$

By letting

$$\alpha(x,y)=[M(x,y)\quad \bar M(x,y)],\qquad \omega(x,y)=\begin{bmatrix}0 & \Omega_{\mathrm r}(x+y)\\ -\Omega_{\mathrm r}(x+y)^{\dagger} & 0\end{bmatrix},\tag{5.14}$$

we can transform (5.12) into (1.14), which is a 2×2 system of integral equations. Note that $\omega(x,y)$ given in (5.14) satisfies (1.10) with $J=\mathrm{diag}[1,-1]$. As seen from (5.13), u is recovered from the solution to (1.14) as

$$u(x)=2[1\quad 0]\alpha(x,x)\begin{bmatrix}0\\1\end{bmatrix}.$$

In this case, $f(x)$ and $g(y)$ appearing in (1.13) are given by

$$f(x)=\begin{bmatrix}0 & C_{\mathrm r}\mathrm{e}^{A_{\mathrm r}x}\\ B_{\mathrm r}^{\dagger}\,\mathrm{e}^{A_{\mathrm r}^{\dagger}x} & 0\end{bmatrix},\qquad g(y)=\begin{bmatrix}-\mathrm{e}^{A_{\mathrm r}^{\dagger}y}C_{\mathrm r}^{\dagger} & 0\\ 0 & \mathrm{e}^{A_{\mathrm r}y}B_{\mathrm r}\end{bmatrix},$$

where $A_{\mathrm r}$ is a constant $p\times p$ matrix with all eigenvalues having positive real parts, $B_{\mathrm r}$ is a constant $p\times 1$ matrix and $C_{\mathrm r}$ is a constant $1\times p$ matrix. In this case the wavefunction appearing in (3.14) is given by

$$\Psi(\lambda,x)=\begin{bmatrix}\phi_1(\lambda,x) & \bar\phi_1(\lambda,x)\\ \phi_2(\lambda,x) & \bar\phi_2(\lambda,x)\end{bmatrix}=\begin{bmatrix}\phi_1(\lambda,x) & -\phi_2(\lambda^*,x)^*\\ \phi_2(\lambda,x) & \phi_1(\lambda^*,x)^*\end{bmatrix},$$

where $\left[\begin{smallmatrix}\phi_1(\lambda,x)\\ \phi_2(\lambda,x)\end{smallmatrix}\right]$ is the Jost solution to (5.1) with the asymptotics $\left[\begin{smallmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}\\ 0\end{smallmatrix}\right]$ as $x\to-\infty$ and $\left[\begin{smallmatrix}\bar\phi_1(\lambda,x)\\ \bar\phi_2(\lambda,x)\end{smallmatrix}\right]$ is the solution behaving as $\left[\begin{smallmatrix}0\\ \mathrm{e}^{\mathrm{i}\lambda x}\end{smallmatrix}\right]+o(1)$ as $x\to-\infty$.

Example 5.4. Example 5.3 is also valid for the Zakharov–Shabat system given in (5.8), where $M,\bar M,\Omega_{\mathrm r},A_{\mathrm r},B_{\mathrm r},C_{\mathrm r},f,g$ have now sizes $(m+n)\times m$, $(m+n)\times n$, $m\times n$, $p\times p$, $p\times n$, $m\times p$, $(m+n)\times 2p$, $2p\times(m+n)$, respectively, and the eigenvalues of the constant matrix $A_{\mathrm r}$ have all positive real parts, and $B_{\mathrm r}$ and $C_{\mathrm r}$ are constant matrices. The matrix system in (5.12) can be written as in (1.14), which is now an $(m+n)\times(m+n)$ system of integral equations. The $m\times n$ potential matrix u is recovered from the solution to (1.14) as

$$u(x)=2[I_m\quad 0_{mn}]\alpha(x,x)\begin{bmatrix}0_{mn}\\ I_n\end{bmatrix}.$$

In this case the wavefunction appearing in (3.14) is given by

$$\Psi(\lambda, x) = \begin{bmatrix} \phi_1(\lambda, x) & \bar{\phi}_1(\lambda, x) \\ \phi_2(\lambda, x) & \bar{\phi}_2(\lambda, x) \end{bmatrix},$$

where $\left[\begin{smallmatrix} \phi_1(\lambda,x) \\ \phi_2(\lambda,x) \end{smallmatrix}\right]$ is the $(m+n) \times m$ Jost solution to (5.8) with the asymptotics $\left[\begin{smallmatrix} \mathrm{e}^{-\mathrm{i}\lambda x} I_m \\ 0_{nm} \end{smallmatrix}\right]$ as $x \to -\infty$ and $\left[\begin{smallmatrix} \bar{\phi}_1(\lambda,x) \\ \bar{\phi}_2(\lambda,x) \end{smallmatrix}\right]$ is the $(m+n) \times n$ Jost solution to (5.8) with the asymptotics $\left[\begin{smallmatrix} 0_{mn} \\ \mathrm{e}^{\mathrm{i}\lambda x} I_n \end{smallmatrix}\right]$ as $x \to -\infty$.

In the next three examples we discuss the Schrödinger equation on the full and half lines. In each example we have n bound states at $k = i\kappa_j$ with the corresponding norming constant c_j. Here $\kappa_1, \ldots, \kappa_n$ are distinct positive numbers and $c_1, \ldots, c_n$ are positive. We begin with the familiar example of the Schrödinger equation on the full line [15] and then discuss the Schrödinger equation on the half line with various boundary conditions at $x = 0$ [5].

Example 5.5. Consider the Schrödinger equation on the full line

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + u(x)\varphi = k^2\varphi, \qquad x \in (-\infty, +\infty),$$

where u is a real-valued integrable potential with a finite first moment. The corresponding (left) Marchenko equation is a scalar integral equation and is given by (1.7), from whose solution the potential u is recovered as

$$u(x) = -2\frac{\mathrm{d}\alpha(x, x)}{\mathrm{d}x}.$$

In this case, $f(x)$ and $g(y)$ appearing in (1.13) are given [5] by

$$f(x) = \begin{bmatrix} c_1\,\mathrm{e}^{-\kappa_1 x} & \ldots & c_n\,\mathrm{e}^{-\kappa_n x} \end{bmatrix}, \qquad g(y) = \begin{bmatrix} c_1\,\mathrm{e}^{-\kappa_1 y} \\ \vdots \\ c_n\,\mathrm{e}^{-\kappa_n y} \end{bmatrix},$$

where c_j is the norming constant for the bound state at $k = \mathrm{i}\kappa_j$. The wavefunction appearing in (3.14) corresponds to the Jost solution from the left satisfying $\Psi(k, x) = \mathrm{e}^{\mathrm{i}kx}[1 + o(1)]$ as $x \to +\infty$. In this case (3.14) holds with the involution J being the scalar quantity equal to -1.

Example 5.6. Consider the Schrödinger equation on the half line with the Dirichlet boundary condition at the origin, i.e.

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + u(x)\varphi = k^2\varphi, \qquad \varphi(k, 0) = 0. \tag{5.15}$$

The Gel'fand–Levitan integral equation arising in the related inverse scattering theory is given by

$$\alpha(x, y) + \omega(x, y) + \int_0^x \mathrm{d}y\, \alpha(x, z)\omega(z, y) = 0, \qquad 0 < y < x, \tag{5.16}$$

where the kernel ω is real valued and symmetric, i.e. $\omega(x, y) = \omega(y, x)$, and thus it satisfies (1.10). Note that (5.16) is already in the form of (1.15). The potential is recovered as

$$u(x) = 2\frac{\mathrm{d}\alpha(x, x)}{\mathrm{d}x}.$$

In this case, $f(x)$ and $g(y)$ appearing in (1.13) are given by given [5] by

$$f(x) = \begin{bmatrix} \dfrac{c_1}{\kappa_1}\sinh(\kappa_1 x) & \dots & \dfrac{c_n}{\kappa_n}\sinh(\kappa_n x) \end{bmatrix}, \qquad g(y) = \begin{bmatrix} \dfrac{c_1}{\kappa_1}\sinh(\kappa_1 y) \\ \vdots \\ \dfrac{c_n}{\kappa_n}\sinh(\kappa_n y) \end{bmatrix},$$

where c_j is the norming constant corresponding to the bound state at $k = \mathrm{i}\kappa_j$. In this case the wavefunction $\Psi(k, x)$ is the regular solution to (5.15) satisfying the initial conditions $\Psi(k, 0) = 0$ and $\Psi'(k, 0) = 1$, where the prime denotes the x-derivative. The relationship between the wavefunction $\Psi(k, x)$ and $\alpha(x, y)$, instead of (3.14), is given by [5]

$$\Psi(k, x) = \frac{\sin(kx)}{k} + \int_0^x \mathrm{d}y\, \alpha(x, y)\frac{\sin(ky)}{k}, \qquad 0 < y < x,$$

with the inverse transform given by

$$\alpha(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, k\Psi(k, x)\sin(ky), \qquad 0 < y < x.$$

Example 5.7. Consider the Schrödinger equation on the half line with a self-adjoint boundary condition at the origin, i.e.

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + u(x)\varphi = k^2\varphi, \qquad \varphi'(k, 0) + \cot\theta\,\varphi(k, 0) = 0, \tag{5.17}$$

where θ is a constant in the interval $(0, \pi)$. The Gel'fand–Levitan integral equation arising in the related inverse scattering theory is given by

$$\alpha(x, y) + \omega(x, y) + \int_0^x \mathrm{d}z\, \alpha(x, z)\omega(z, y) = 0, \qquad 0 < y < x, \tag{5.18}$$

where the kernel ω is real valued and symmetric, i.e. $\omega(x, y) = \omega(y, x)$, and thus it satisfies (1.10). Note that (5.18) is already in the form of (1.15). The potential is recovered as

$$u(x) = 2\frac{\mathrm{d}\alpha(x, x)}{\mathrm{d}x}.$$

In this case, $f(x)$ and $g(y)$ appearing in (1.13) are given [5] by

$$f(x) = [c_1\cosh(\kappa_1 x) \quad \dots \quad c_n\cosh(\kappa_n x)], \qquad g(y) = \begin{bmatrix} c_1\cosh(\kappa_1 y) \\ \vdots \\ c_n\cosh(\kappa_n y) \end{bmatrix},$$

where c_j is the norming constant for the bound state at $k = \mathrm{i}\kappa_j$ as in example 5.6. In this case the wavefunction $\Psi(k, x)$ is the regular solution to (5.17) satisfying the initial conditions $\Psi(k, 0) = 1$ and $\Psi'(k, 0) = -\cot\theta$. The relationship between $\Psi(k, x)$ and $\alpha(x, y)$, instead of (3.14), is given by

$$\Psi(k, x) = \cos(kx) + \int_0^x \mathrm{d}y\, \alpha(x, y)\cos(ky), \qquad 0 < y < x,$$

with the inverse transform given by

$$\alpha(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \Psi(k, x)\cos(ky), \qquad 0 < y < x.$$

6. Darboux transform for the Zakharov–Shabat system

In order to illustrate the significance of the results presented in this paper, we will derive the Darboux transformation for the Zakharov–Shabat system given in (5.1) when one bound state is added to the spectrum. In particular, to the spectrum of (5.1) we will add one bound state at $\lambda = \lambda_1 \in \mathbf{C}^+$ with the norming constant c_1, where c_1 is a complex constant and we use $\mathbf{C}^+$ to denote the upper-half complex plane. The potential appearing in (5.1) will then change from $u(x)$ to $\tilde{u}(x)$ and the wavefunction appearing in (5.7) will change from $\Psi(\lambda, x)$ to $\tilde{\Psi}(\lambda, x)$. With the help of (3.20), (3.21) and (5.5), we will explicitly evaluate $\tilde{u}(x) - u(x)$ and $\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x)$ in terms of $u(x)$, λ_1, c_1 and $\Psi(\lambda, x)$. Note that $\tilde{u}(x)$ and $\tilde{\Psi}(\lambda, x)$ each consist of a one-parameter family of potentials and wavefunctions, respectively, with c_1 being the parameter. From (5.7) we see that, in evaluating $\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x)$, it is sufficient to obtain $\begin{bmatrix} \tilde{\psi}_1(\lambda,x) \\ \tilde{\psi}_2(\lambda,x) \end{bmatrix} - \begin{bmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{bmatrix}$, where $\begin{bmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{bmatrix}$ is the Jost solution to (5.1) with the asymptotics $\begin{bmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\lambda x} \end{bmatrix}$ as $x \to +\infty$. At the end of this section we will compare our result with those in the literature.

Before we derive our Darboux transformation, we need some identities related to the solutions to (5.1). Let us use an overdot to indicate the derivative with respect to λ. The following result is already known and its proof is omitted. Its proof can easily be obtained by using the integral representation of the Jost solution [30].

Proposition 6.1. *Let $\begin{bmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{bmatrix}$ be the Jost solution to (5.1) with the asymptotics $\begin{bmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\lambda x} \end{bmatrix}$ as $x \to +\infty$, where u is an integrable potential. If $\lambda_1 \in \mathbf{C}^+$, then $\psi_1(\lambda_1, x)$, $\psi_2(\lambda_1, x)$, $\dot{\psi}_1(\lambda_1, x)$ and $\dot{\psi}_2(\lambda_1, x)$ all vanish as $x \to +\infty$.*

Even though the identities in the following proposition are known [1–3, 30, 35], we provide a brief proof for the convenience of the reader.

Proposition 6.2. *Let $\begin{bmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{bmatrix}$ be the Jost solution to (5.1) with the asymptotics $\begin{bmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\lambda x} \end{bmatrix}$ as $x \to +\infty$, where u is an integrable potential. We then have the following identities:*

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{vmatrix} \psi_1(\lambda, x) & -\psi_2(\lambda, x)^* \\ \psi_2(\lambda, x) & \psi_1(\lambda, x)^* \end{vmatrix} = 2\,\mathrm{Im}[\lambda][|\psi_1(\lambda, x)|^2 - |\psi_2(\lambda, x)|^2], \tag{6.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{vmatrix} \psi_1(\lambda, x) & \dot{\psi}_1(\lambda, x) \\ \psi_2(\lambda, x) & \dot{\psi}_2(\lambda, x) \end{vmatrix} = 2\mathrm{i}\psi_1(\lambda, x)\psi_2(\lambda, x), \tag{6.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{vmatrix} \psi_1(\lambda, x) & \psi_1(\lambda_1, x) \\ \psi_2(\lambda, x) & \psi_2(\lambda_1, x) \end{vmatrix} = -\mathrm{i}(\lambda - \lambda_1)[\psi_1(\lambda, x)\psi_2(\lambda_1, x) + \psi_2(\lambda, x)\psi_1(\lambda_1, x)], \tag{6.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{vmatrix} \psi_1(\lambda, x) & -\psi_2(\lambda_1, x)^* \\ \psi_2(\lambda, x) & \psi_1(\lambda_1, x)^* \end{vmatrix} = -\mathrm{i}(\lambda - \lambda_1^*)[\psi_1(\lambda, x)\psi_1(\lambda_1, x)^* - \psi_2(\lambda, x)\psi_2(\lambda_1, x)^*], \tag{6.4}$$

where $\mathrm{Im}[\lambda]$ *is used to denote the imaginary part of* λ.

Proof. From (5.1) we obtain

$$\begin{cases} \psi_1'(\lambda, x)^* = \mathrm{i}\lambda^*\psi_1(\lambda, x)^* + u(x)^*\psi_2(\lambda, x)^*, \\ \psi_2'(\lambda, x)^* = -\mathrm{i}\lambda^*\psi_2(\lambda, x)^* - u(x)\psi_1(\lambda, x)^*, \end{cases} \tag{6.5}$$

$$\begin{cases} \dot{\psi}_1'(\lambda, x) = -\mathrm{i}\psi_1(\lambda, x) - \mathrm{i}\lambda\dot{\psi}_1(\lambda, x) + u(x)\dot{\psi}_2(\lambda, x), \\ \dot{\psi}_2'(\lambda, x) = \mathrm{i}\psi_2(\lambda, x) + \mathrm{i}\lambda\dot{\psi}_2(\lambda, x) - u(x)^*\dot{\psi}_1(\lambda, x). \end{cases} \tag{6.6}$$

The identities given in (6.1)–(6.4) are derived directly from (5.1), (6.5) and (6.6). □

Using propositions 6.1 and 6.2, we have the following result.

Corollary 6.3. *Let* $\begin{bmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{bmatrix}$ *be the Jost solution to (5.1) with the asymptotics* $\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}$ *as* $x \to +\infty$, *where* u *is an integrable potential. We then have the following identities:*

$$\int_x^\infty \mathrm{d}s\, \psi_1(\lambda_1, s)\psi_2(\lambda_1, s) = \frac{\mathrm{i}}{2}[\psi_1(\lambda_1, x)\dot{\psi}_2(\lambda_1, x) - \psi_2(\lambda_1, x)\dot{\psi}_1(\lambda_1, x)], \tag{6.7}$$

$$\int_x^\infty \mathrm{d}s(|\psi_1(\lambda_1, s)|^2 - |\psi_2(\lambda_1, s)|^2) = -\frac{|\psi_1(\lambda_1, x)|^2 + |\psi_2(\lambda_1, x)|^2}{2\,\mathrm{Im}[\lambda_1]}, \tag{6.8}$$

$$\int_x^\infty \mathrm{d}s[\psi_2(\lambda_1, s)\psi_1(\lambda, s) + \psi_1(\lambda_1, s)\psi_2(\lambda, s)]$$
$$= \frac{\mathrm{i}}{\lambda - \lambda_1}[\psi_1(\lambda_1, x)\psi_2(\lambda, x) - \psi_2(\lambda_1, x)\psi_1(\lambda, x)], \tag{6.9}$$

$$\int_x^\infty \mathrm{d}s[\psi_1(\lambda_1, s)^*\psi_1(\lambda, s) - \psi_2(\lambda_1, s)^*\psi_2(\lambda, s)]$$
$$= \frac{-\mathrm{i}}{\lambda - \lambda_1^*}[\psi_1(\lambda_1, x)^*\psi_1(\lambda, x) + \psi_2(\lambda_1, x)^*\psi_2(\lambda, x)]. \tag{6.10}$$

Let us now use our systematic approach to derive the one-parameter family of the Darboux transformations for (5.1) when one bound state at $\lambda = \lambda_1 \in \mathbf{C}^+$ with the norming constant c_1 is added to the spectrum. Let us choose in (5.6)

$$A_1 = -\mathrm{i}\lambda_1, \qquad B_1 = 1, \qquad C_1 = c_1.$$

Then, using (5.6) and (5.7) in (4.5), (4.7) and (4.10) we obtain

$$n(x) = \begin{bmatrix} c_1\psi_1(\lambda_1, x) & \psi_2(\lambda_1, x)^* \\ c_1\psi_2(\lambda_1, x) & -\psi_1(\lambda_1, x)^* \end{bmatrix}, \tag{6.11}$$

$$q(x) = \begin{bmatrix} \psi_2(\lambda_1, x) & \psi_1(\lambda_1, x) \\ c_1^*\psi_1(\lambda_1, x)^* & -c_1^*\psi_2(\lambda_1, x)^* \end{bmatrix}, \tag{6.12}$$

$$\Gamma(x) = \begin{bmatrix} \Gamma_1(x) & \Gamma_2(x) \\ \Gamma_3(x) & \Gamma_4(x) \end{bmatrix}, \tag{6.13}$$

where we have defined

$$\Gamma_1(x) := \Gamma_4(x)^* := 1 + 2c_1 \int_x^\infty \mathrm{d}s\, \psi_1(\lambda_1, s)\psi_2(\lambda_1, s), \tag{6.14}$$

$$\Gamma_2(x) := -\frac{\Gamma_3(x)}{|c_1|^2} := -\int_x^\infty \mathrm{d}s(|\psi_1(\lambda_1, s)|^2 - |\psi_2(\lambda_1, s)|^2). \tag{6.15}$$

Using (6.7) in (6.14) and using (6.8) in (6.15) we get

$$\Gamma_4(x)^* = \Gamma_1(x) = 1 + \mathrm{i}c_1[\psi_1(\lambda_1, x)\dot{\psi}_2(\lambda_1, x) - \psi_2(\lambda_1, x)\dot{\psi}_1(\lambda_1, x)], \tag{6.16}$$

$$-\frac{\Gamma_3(x)}{|c_1|^2} = \Gamma_2(x) = \frac{|\psi_1(\lambda_1, x)|^2 + |\psi_2(\lambda_1, x)|^2}{2\,\mathrm{Im}[\lambda_1]}. \tag{6.17}$$

Next, we present the main result in this section, namely, the Darboux transformation for the Zakharov–Shabat system given in (5.1).

Theorem 6.4. *When one bound state at $\lambda = \lambda_1 \in \mathbf{C}^+$ with the norming constant c_1 is added to the spectrum of (5.1), the potential $u(x)$ changes to $\tilde{u}(x)$ and the Jost solution $\begin{bmatrix}\psi_1(\lambda,x)\\ \psi_2(\lambda,x)\end{bmatrix}$ changes to $\begin{bmatrix}\tilde\psi_1(\lambda,x)\\ \tilde\psi_2(\lambda,x)\end{bmatrix}$ according to the following Darboux transformation:*

$$\tilde{u}(x) - u(x) = \frac{P_0(x)}{|\Gamma_1(x)|^2 + |c_1|^2\Gamma_2(x)^2}, \tag{6.18}$$

$$\begin{bmatrix}\tilde\psi_1(\lambda,x)\\ \tilde\psi_2(\lambda,x)\end{bmatrix} - \begin{bmatrix}\psi_1(\lambda,x)\\ \psi_2(\lambda,x)\end{bmatrix} = \frac{1}{|\Gamma_1(x)|^2 + |c_1|^2\Gamma_2(x)^2}\begin{bmatrix}P_1 & P_2\\ P_3 & P_4\end{bmatrix}\begin{bmatrix}P_5\\ P_6\end{bmatrix}, \tag{6.19}$$

where $\Gamma_1(x)$ and $\Gamma_2(x)$ are as in (6.16) and (6.17), respectively, and

$$P_0 := 2c_1\psi_1(\lambda_1,x)^2\Gamma_1(x)^* - 2c_1^*[\psi_2(\lambda_1,x)^*]^2\Gamma_1(x) + 4|c_1|^2\psi_1(\lambda_1,x)\psi_2(\lambda_1,x)^*\Gamma_2(x),$$
$$P_1 := -|c_1|^2\psi_2(\lambda_1,x)^*\Gamma_2(x) - c_1\psi_1(\lambda_1,x)\Gamma_1(x)^*,$$
$$P_2 := |c_1|^2\psi_1(\lambda_1,x)\Gamma_2(x) - c_1^*\psi_2(\lambda_1,x)^*\Gamma_1(x),$$
$$P_3 := |c_1|^2\psi_1(\lambda_1,x)^*\Gamma_2(x) - c_1\psi_2(\lambda_1,x)\Gamma_1(x)^*,$$
$$P_4 := |c_1|^2\psi_2(\lambda_1,x)\Gamma_2(x) + c_1^*\psi_1(\lambda_1,x)^*\Gamma_1(x),$$
$$P_5 := \frac{\mathrm{i}}{\lambda-\lambda_1}[\psi_1(\lambda_1,x)\psi_2(\lambda,x) - \psi_2(\lambda_1,x)\psi_1(\lambda,x)],$$
$$P_6 := \frac{-\mathrm{i}}{\lambda-\lambda_1^*}[\psi_1(\lambda_1,x)^*\psi_1(\lambda,x) + \psi_2(\lambda_1,x)^*\psi_2(\lambda,x)].$$

Proof. From (3.21) and (5.5) we have

$$\tilde{u}(x) - u(x) = 2[1 \quad 0]n(x)\Gamma(x)^{-1}q(x)\begin{bmatrix}0\\1\end{bmatrix}. \tag{6.20}$$

Thus, using (6.11)–(6.13), (6.16) and (6.17) in (6.20) we obtain (6.18). From (3.18), (3.20), (5.7) and (6.12) it follows that

$$\begin{bmatrix}\tilde\psi_1(\lambda,x)\\ \tilde\psi_2(\lambda,x)\end{bmatrix} - \begin{bmatrix}\psi_1(\lambda,x)\\ \psi_2(\lambda,x)\end{bmatrix} = -n(x)\Gamma(x)^{-1}\int_x^\infty \mathrm{d}s\, q(s)\begin{bmatrix}\psi_1(\lambda,s)\\ \psi_2(\lambda,s)\end{bmatrix}. \tag{6.21}$$

Using (6.11)–(6.13), (6.16) and (6.17) on the right-hand side of (6.21), with the help of (6.9) and (6.10) we obtain (6.19). □

Let us now compare our systematic approach to the Darboux transformation for the Zakharov–Shabat system given in theorem 6.4 with some of the results in the literature. Some special cases of the Darboux transformations were obtained explicitly [10, 23, 24] for the Zakharov–Shabat system when a bound state is added at $\lambda = \lambda_1$, whereas our own method provides all the Darboux transformations in this case. One special Darboux transformation, named an elementary Darboux transform [10, 23, 24], corresponds to (cf (1.9) of [10])

$$\tilde{u}(x) = \frac{\psi_2(\lambda_1,x)^*}{\psi_1(\lambda_1,x)^*} = u'(x) + 2\mathrm{i}\lambda_1 u(x) - \frac{\psi_2(\lambda_1,x)}{\psi_1(\lambda_1,x)}u(x)^2. \tag{6.22}$$

A second elementary Darboux transform [10, 23, 24] is given by (cf (1.11) of [10])

$$\tilde{u}(x) = \frac{\psi_1(\lambda_1,x)}{\psi_2(\lambda_1,x)} = -u'(x) - 2\mathrm{i}\lambda_1^* u(x) - \frac{\psi_1(\lambda_1,x)^*}{\psi_2(\lambda_1,x)^*}u(x)^2. \tag{6.23}$$

A combination of (6.22) and (6.23) yields the following Darboux transformation (cf (1.14) of [10]) when a bound state at $\lambda = \lambda_1$:

$$\tilde{u}(x) - u(x) = \frac{4\,\mathrm{Im}[\lambda_1]\psi_1(\lambda_1, x)\psi_2(\lambda_1, x)^*}{|\psi_1(\lambda_1, x)|^2 + |\psi_2(\lambda_1, x)|^2}. \tag{6.24}$$

The Jost solution is then transformed according to (cf (2.41) of [10])

$$\begin{bmatrix} \tilde{\psi}_1(\lambda, x) \\ \tilde{\psi}_2(\lambda, x) \end{bmatrix} = \left(-\frac{\mathrm{i}}{2}(\lambda - \lambda_1) + \frac{\mathrm{Im}[\lambda_1] S(\lambda_1, x)}{|\psi_1(\lambda_1, x)|^2 + |\psi_2(\lambda_1, x)|^2} \right) \begin{bmatrix} \psi_1(\lambda, x) \\ \psi_2(\lambda, x) \end{bmatrix}, \tag{6.25}$$

where the matrix $S(\lambda_1, x)$ is defined as

$$S(\lambda_1, x) := \begin{bmatrix} |\psi_2(\lambda_1, x)|^2 & -\psi_1(\lambda_1, x)\psi_2(\lambda_1, x)^* \\ -\psi_1(\lambda_1, x)^*\psi_2(\lambda_1, x) & |\psi_1(\lambda_1, x)|^2 \end{bmatrix}.$$

Note that (6.25) implies that

$$\tilde{\psi}_1(\lambda_1, x) = \tilde{\psi}_2(\lambda_1, x) = 0.$$

The norming constant c_1 appearing in (6.18) is given by

$$c_1 = \frac{-1}{2\int_{-\infty}^{\infty} \mathrm{d}s\, \tilde{\psi}_1(\lambda_1, s)\tilde{\psi}_2(\lambda_1, s)},$$

and hence the Darboux transformation given in (6.24) and (6.25) corresponds to a very particular choice of the norming constant c_1, namely $1/c_1 = 0$.

We can also see that each of the three Darboux transforms given in (6.22)–(6.24) is very special by considering the easiest case where $u(x) = 0$. In that case (6.22) is understood as $\tilde{u}(x) = 0$ and (6.23) and (6.24) each impose $\tilde{u}(x) = 0$. When $u(x) = 0$, the Jost solution $\left[\begin{smallmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{smallmatrix}\right]$ to (5.1) with the asymptotics $\left[\begin{smallmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\lambda x} \end{smallmatrix}\right]$ as $x \to +\infty$ is given by

$$\psi_1(\lambda, x) = 0, \qquad \psi_2(\lambda, x) = \mathrm{e}^{\mathrm{i}\lambda x}. \tag{6.26}$$

Then, we see that (6.24) also imposes $\tilde{u}(x) = 0$. On the other hand, using (6.26) in (6.11), (6.12), (6.16), (6.17) and theorem 6.4, our own procedure yields the one-parameter family of potentials and wavefunctions

$$\tilde{u}(x) = \frac{-8c_1^*(\mathrm{Im}[\lambda_1])^2\,\mathrm{e}^{-2\mathrm{i}\lambda_1^* x}}{4(\mathrm{Im}[\lambda_1])^2 + |c_1|^2\,\mathrm{e}^{-4\mathrm{Im}[\lambda_1]x}}, \tag{6.27}$$

$$\tilde{\psi}_1(\lambda, x) = \frac{4\mathrm{i}c_1^*(\mathrm{Im}[\lambda_1])^2\,\mathrm{e}^{-2\mathrm{i}\lambda_1^* x + \mathrm{i}\lambda x}}{(\lambda - \lambda_1^*)[4(\mathrm{Im}[\lambda_1])^2 + |c_1|^2\,\mathrm{e}^{-4\mathrm{Im}[\lambda_1]x}]},$$

$$\tilde{\psi}_2(\lambda, x) = \mathrm{e}^{\mathrm{i}\lambda x} - \frac{2\mathrm{i}|c_1|^2(\mathrm{Im}[\lambda_1])\,\mathrm{e}^{-4\mathrm{Im}[\lambda_1]x + \mathrm{i}\lambda x}}{(\lambda - \lambda_1^*)[4(\mathrm{Im}[\lambda_1])^2 + |c_1|^2\,\mathrm{e}^{-4\mathrm{Im}[\lambda_1]x}]}.$$

It is not surprising that we have $\tilde{u}(x) = 0$ in this particular case because the special transformation in (6.24) corresponds to choosing $1/c_1 = 0$ in (6.27). Using [1–3, 29]

$$\tilde{\psi}_2(\lambda, x) = \mathrm{e}^{\mathrm{i}\lambda x}\left[\frac{1}{\tilde{T}(\lambda)} + o(1)\right], \qquad x \to -\infty,$$

where $\tilde{T}(\lambda)$ is the transmission coefficient corresponding to $\tilde{u}(x)$, we get $\tilde{T}(\lambda) = \frac{\lambda - \lambda_1^*}{\lambda - \lambda_1}$. We note that the quantity given in (6.27) is the one-soliton potential and the corresponding reflection coefficients are zero.

In [29] a Darboux transformation was studied for the Zakharov–Shabat system in two spatial variables, and by eliminating one of the variables, the corresponding Bäcklund transformation was provided in the form of an ordinary differential equation (cf (4.9a) of [29]) involving $\tilde{u}'(x)$, $u'(x)$, $\tilde{u}(x)$ and $u(x)$. The same differential equation was derived earlier (cf (4.7) of [26]) by Levi, Ragnisco and Sym, who studied the equivalence of the dressing method [35] and the Darboux transformation for the Schrödinger equation and indicated that the result also holds for the Zakharov–Shabat system. These authors obtained a formula, which is given as (4.6) in their paper [26], connecting $\tilde{u}(x)$ and $u(x)$ through an intermediate function $\bar{\xi}$ expressed in terms of the four entries of a matrix-valued solution to (5.1). A first-order differential equation similar to (4.9a) of [29] was earlier derived by Gerdzhikov and Kulish (cf (14) of [19]). In [28], when N bound states are added to the spectrum, the change in the potential is expressed (cf (20) of [28]) in terms of ratios of the determinants of two $2N \times 2N$ matrices differing only in their last columns; such matrices are constructed by determining the zeros of certain polynomial equations in λ and by solving certain linear algebraic equations. Various other authors (see e.g. [22]) presented similar formulas for the change in the potential when bound states are added to the spectrum. In most of these papers a 'Darboux matrix' is constructed connecting the wavefunctions of the original and perturbed problems.

One criticism of the result of [26] is that a matrix solution to (5.1) was evaluated (cf (3.6) of [26]) at a λ-value on the upper-half complex plane and also evaluated (cf (3.8) of [26]) at a λ-value on the lower-half complex plane. The same concern also applies to other works (see e.g. (19) of [28] and (3.8) of [22]). In general, we cannot expect the entries of a matrix solution to (5.1) to have extensions in λ to both upper-half and lower-half complex planes, unless the class of potentials $u(x)$ is very restrictive.

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Appendix. Some norm estimates

In this appendix we derive some boundedness properties of the integral operator Ω defined in section 1.

Suppose $\omega(x, y)$ is an $N \times N$ matrix function satisfying (1.2). Let

$$N_1(x) := \sup_{y>x} \int_x^\infty dz \|\omega(y,z)\|, \qquad N_\infty(x) := \sup_{y>x} \int_x^\infty dz \|\omega(z,y)\|,$$

so that (1.2) amounts to $N_1(x) + N_\infty(x) < +\infty$.

Proposition A.1. *If the assumption in (1.2) is satisfied, then the integral operator* Ω *appearing in (1.3) is bounded on* $\mathcal{H}_1^{M\times N}$ *with norm bound* $N_1(x)$ *and on* $\mathcal{H}_\infty^{M\times N}$ *with norm bound* $N_\infty(x)$.

Proof. We directly verify that

$$\int_x^\infty dy \|(\beta\Omega)(x,y)\| \leqslant \int_x^\infty dy \int_x^\infty dz \|\beta(x,z)\| \|\omega(z,y)\| \leqslant N_1(x) \int_x^\infty dz \|\beta(x,z)\|,$$

and

$$\|(\beta\Omega)(x,y)\| \leqslant N_\infty(x) \sup_{y>x} \|\beta(x,y)\|,$$

which proves the proposition. □

It is now clear that the (scalar) integral operator L_ω defined by

$$(L_\omega h)(y) := \int_x^\infty \mathrm{d}z\, h(z)\|\omega(z, y)\|,$$

is bounded on $L^1(x, +\infty)$ with norm bound $N_1(x)$ and on $L^\infty(x, +\infty)$ with norm bound $N_\infty(x)$. By the Riesz–Thorin interpolation theorem (cf [36], vol II, section XII.1), L_ω is bounded on $L^p(x, +\infty)$ for $p \in (1, +\infty)$ with norm bounded above by

$$N_1(x)^{1/p} N_\infty(x)^{1-(1/p)}.$$

Since Ω maps a dense linear subspace of $\mathcal{H}_p^{M\times N}$ (namely, its intersection with $\mathcal{H}_1^{M\times N} \cap \mathcal{H}_\infty^{M\times N}$) into $\mathcal{H}_1^{M\times N} \cap \mathcal{H}_\infty^{M\times N}$, with $h(z) = \|\beta(x, z)\|$ we obtain the estimate

$$\left[\int_x^\infty \mathrm{d}z \|(\beta\Omega)(x, z)\|^p\right]^{1/p} \leqslant \|L_\omega h\|_p \leqslant N_1(x)^{1/p} N_\infty(x)^{1-(1/p)} \|h\|_p$$
$$= N_1(x)^{1/p} N_\infty(x)^{1-(1/p)} \left[\int_x^\infty \mathrm{d}z \|\beta(x, z)\|^p\right]^{1/p},$$

where $\|\cdot\|_p$ is the L^p-norm. Hence, Ω is bounded on $\mathcal{H}_p^{M\times N}$ for $p \in (1, +\infty)$ as well.

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