

Inverse scattering problem for the third-order equation on the line*

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Abstract

We consider the third-order linear differential equation

$$\frac{d^3\psi}{dx^3} + Q(x) \frac{d\psi}{dx} + P(x) \psi = k^3 \psi, \quad x \in \mathbb{R},$$

where the complex-valued potentials Q and P are assumed to belong to the Schwartz class. We describe the basic solutions, the scattering coefficients, and the bound-state information, and we introduce the dependency constants and the normalization constants at the bound states. When the secondary reflection coefficients are zero, we provide a method to solve the corresponding inverse scattering problem, where the goal is to recover the two potentials Q and P from the scattering data set consisting of the transmission and primary reflection coefficients and the bound-state information. We formulate the corresponding inverse scattering problem as a Riemann–Hilbert problem on the complex k -plane and describe how the potentials are recovered from the solution to the Riemann–Hilbert problem. In the absence of bound states, we introduce a linear integral equation, which is the analog of the Marchenko integral equation used in the inverse scattering theory for the full-line Schrödinger equation. We

* Dedicated to the memory of Pierre C. Sabatier, a pioneer in inverse problems.

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describe the recovery of the two potentials from the solution to the aforementioned linear integral equation.

Keywords: inverse scattering for the third-order equation,
bound-state dependency constants, Riemann–Hilbert problem,
Marchenko integral equation

1. Introduction

In this paper, we analyze the direct and inverse scattering problems for the third-order equation

$$\psi''' + Q(x) \psi' + P(x) \psi = k^3 \psi, \quad x \in \mathbb{R}, \quad (1.1)$$

where x is the independent variable taking values on the real axis $\mathbb{R}$, the prime denotes the x -derivative, k^3 is the spectral parameter, ψ is the wavefunction, and Q and P are the complex-valued potentials satisfying certain restrictions. For simplicity, we assume that Q and P belong to the Schwartz class $S(\mathbb{R})$ even though our results hold under weaker assumptions on those two potentials. We recall that the Schwartz class consists of infinitely differentiable functions $\theta(x)$ in such a way that, for any pair of nonnegative integers j and l , the quantity $x^j d^l \theta(x)/dx^l$ vanishes as $x \rightarrow \pm\infty$.

The direct scattering problem for (1.1) consists of the determination of the scattering coefficients and the bound-state information when the potentials Q and P are known. On the other hand, the inverse scattering problem for (1.1) deals with the determination of the potentials Q and P when the scattering coefficients and the bound-state information are known. The bound-state information usually consists of the bound-states poles of a transmission coefficient and the specification of a dependency constant or a normalization constant at each bound state.

The physical importance of the analysis of the inverse scattering problem for (1.1) comes from the fact that the solution to that inverse problem is used to determine solutions to various integrable nonlinear partial differential equations arising in fluid dynamics, nonlinear optics, and plasma physics. The association between (1.1) and those integrable nonlinear equations is established via the method of IST (inverse scattering transform) [17] and by determining the time evolution of the potentials Q and P by the use of the corresponding Lax pairs [25]. For example, the fourth-order nonlinear partial differential equation [11] introduced by the French mathematician and physicist Boussinesq is used to describe the propagation of surface water waves when those waves have relatively long wavelength and propagate in a weakly nonlinear medium. We refer the reader to [14] for a comprehensive analysis of the Boussinesq equation, to [10] for the analysis of soliton solutions to that equation, and to [15] for the analysis of a modified version of that equation.

The solution to the inverse problem for (1.1) is also relevant in solving the fifth-order nonlinear partial differential equations known as the Sawada–Kotera equation [31] and Kaup–Kupershmidt equation [24]. Like the third-order nonlinear partial differential equation known as the Korteweg–de Vries equation [23], the Sawada–Kotera equation and the Kaup–Kupershmidt equation are used to describe the propagation of surface water waves in long, shallow, narrow canals when those waves have steeper wavefronts and shorter wavelengths.

A mathematical significance of our paper is that it introduces a linear integral equation, to which we refer as the Marchenko integral equation, associated with the inverse scattering problem for third-order equation (1.1) on the full line. A linear integral equation to solve an

inverse scattering problem was first introduced [26] by Vladimir Marchenko, and it was used to recover the potential in the half-line Schrödinger equation from the corresponding scattering matrix. Our integral equation is the counterpart of the Marchenko integral equation introduced by Faddeev [16] to solve the inverse scattering problem for the full-line Schrödinger equation. Such linear integral equations used to solve inverse scattering problems are usually referred to as Marchenko integral equations, and the method to recover the potential from the solution to a Marchenko integral equation is usually called a Marchenko method. We refer the reader to the research monograph [13] by Chadan and Sabatier or the monograph [27] by Marchenko for the history and use of the Marchenko integral equations and the Marchenko methods to solve the inverse scattering problems associated with the Schrödinger equation on the half line and the Schrödinger equation on the full line. The Marchenko integral equation was also derived [1] to solve the inverse scattering problem associated with the first-order AKNS system and later generalized to solve the inverse scattering problems associated with other first-order systems of linear differential and difference equations. Although the Marchenko integral equation we introduce is the counterpart of the Marchenko equations available for various second-order differential and difference equations and also a counterpart of the Marchenko equations available for various first-order systems of differential and difference equations, it is not a straightforward extension of those Marchenko equations. The common feature of Marchenko integral equations is that both the nonhomogeneous term and the integral kernel are constructed by using the scattering data as input. A Marchenko integral equation is usually derived by formulating the associated inverse scattering problem as a Riemann–Hilbert problem in the complex plane determined by the spectral parameter and then by applying a Fourier transformation to derive the corresponding Marchenko integral equation. For the third-order equation (1.1), the formulation of the corresponding Riemann–Hilbert problem and the establishment of the Fourier transformation are not straightforward. The challenges in those two steps are the primary reasons for the difficulty of establishing a Marchenko integral equation associated with the third-order equation (1.1).

Another mathematical significance of our paper is that it also establishes a properly posed Riemann–Hilbert problem to solve the inverse scattering problem for (1.1). Thus, our paper enables the use of the IST method to solve the integrable evolution equations associated with the third-order linear operator for (1.1). The bilinear method of Hirota has been the only viable method to obtain soliton solutions to such integrable evolution equation. We refer the reader to [19] for Hirota’s method to obtain soliton solutions to the Boussinesq equation, to Hirota’s paper [20] to obtain soliton solutions to the Sawada–Kotera equation, to Parker’s paper [28] to obtain soliton solutions to the Kaup–Kupershmidt equation, and to Parker’s paper [29] to obtain soliton solutions for the Sawada–Kotera equation. However, Hirota’s bilinear method is an ad hoc algebraic procedure without any physical motivation. Parker stated in his 2000 paper [28] that ‘it seems to us that obtaining the multisoliton solutions to the Kaup–Kupershmidt equation via IST, or one of its variants, remains a worthwhile goal.’ The same observation is true also for other integrable evolution equations such as the Sawada–Kotera equation and the Boussinesq equation. We refer the reader to [29] for the difficulties encountered in using the dressing method [33] of Zakharov and Shabat to derive a linear integral equation to obtain soliton solutions to the Sawada–Kotera equation.

Contrary to the second-order linear differential equations such as the Schrödinger equation, the theory of inverse scattering problem for the third-order equation (1.1) is not well developed. Kaup [21] started the study of the direct and inverse scattering problems for (1.1) with the aim of finding solutions to the Sawada–Kotera equation and also to the Kaup–Kupershmidt equation. As stated in [20] by Hirota, Kaup ‘was unsuccessful’ in his efforts to solve the inverse

scattering problem for (1.1). In fact, Kaup was not even able to formulate [21] a proper scattering matrix to analyze the direct scattering problem for (1.1). On p. 190 of his 1980 pioneering paper, Kaup mentions that ‘Due to its complexity, at the present time we leave many questions unanswered.’ To deal with many other difficulties, on p. 203 of his paper Kaup states that ‘With all of this, we may now obtain integral ‘dispersion relation’ for all of our eigenfunctions in terms of the others and the scattering data. To do this, we shall assume compact support, so that all functions are entire.’ For various other difficulties encountered by Kaup, we refer the reader especially to section III and section V of his original paper [21].

As indicated in [22] by Kaup himself, he always desired to find the analog of the Marchenko integral equation for (1.1) so that the solution to that linear integral equation would yield solutions to the Sawada–Kotera equation and the Kaup–Kupershmidt equation. In the last section entitled ‘Unsolved Problems’ of Kaup’s paper [22], his relentless pursuit of a Marchenko integral equation for (1.1) is clearly explained where he states that ‘Now we will discuss what the hard problems are. They are hard because they have not been solved. This is not a complete list, but is the start of a list of problems in need of solutions. Although we have a solution for the IST of the cubic eigenvalue problem, there still is no equivalent of the GLM equations for this system. What we do have is the solution of the Riemann–Hilbert problem for the eigenfunctions. What is next needed is a representation of the eigenfunctions in terms of some transformation kernels (like the $K(x, y; t)$ for the KdV). Once these are known, then the equivalent GLM equations will follow.’ We remark that, by a GLM (Gel’fand–Levitan–Marchenko) equation, Kaup actually means a Marchenko integral equation. Even though the phrase ‘Gel’fand–Levitan–Marchenko equation’ is commonly used by researchers in the field of integrable systems, this is a misnomer. There are two separate linear integral equations used in inverse scattering problems. The Gel’fand–Levitan equation [18] uses as input the spectral function which is related to the relevant Jost solution at the spacial location $x = 0$. The integral in the Gel’fand–Levitan equation is over a finite interval such as $(0, x)$ containing the point $x = 0$, and the Gel’fand–Levitan integral equation usually arises in inverse scattering problems on the half line and not on the full line. On the other hand, the Marchenko integral equation uses as input the scattering data set which is related to the relevant Jost solution’s spacial asymptotics as $x \rightarrow +\infty$ or as $x \rightarrow -\infty$. The integral in the Marchenko integral equation is over a half line containing either $x = +\infty$ or $x = -\infty$.

Even though Kaup initiated [21] the study of the direct and inverse scattering problems for (1.1), he was not able to formulate a proper Riemann–Hilbert problem related to the inverse scattering theory for (1.1). We refer to a Riemann–Hilbert problem as properly posed when that Riemann–Hilbert problem is formulated along a full line separating the complex plane into two half planes, as in the formulation used in the inverse scattering theory for the Schrödinger equation. A proper formulation of the Riemann–Hilbert problem allows the use of a Fourier transformation along the aforementioned full line separating the two complex half planes. With the help of the Fourier transformation, a properly posed Riemann–Hilbert problem allows the formulation of a linear integral equation, which is the analog of the Marchenko integral equation for the Schrödinger equation. Deift, Tomei, and Trubowitz studied [15] the direct and inverse scattering problems for a special case of (1.1), when the corresponding third-order linear differential operator is selfadjoint. They introduced [15] a proper Riemann–Hilbert problem in the complex plane to solve the corresponding inverse scattering problem. Their physical motivation was to analyze the solution to a modified version [15] of the Boussinesq equation. However, in order to formulate their Riemann–Hilbert problem properly, Deift *et al* made [15] the severe assumption that the two transmission coefficients had to be identically equal to 1 for all k -values in the complex plane. They also made the assumption that the two secondary reflection coefficients are identically zero for all k -values. Their restriction on the secondary

reflection coefficients is not as severe as their restriction on the transmission coefficients, and in fact the former restriction helps avoid solutions to the modified Boussinesq equation that blow up at a finite time. The severe restriction on the transmission coefficients prevented Deift *et al* from obtaining explicit solutions to the modified Boussinesq equation. The reason for this is that explicit solutions to integrable nonlinear partial differential equations are usually obtained when all the reflection coefficients are zero and by using the bound-state poles of the transmission coefficients. Inspired by the important paper [15] of Deift *et al*, the second author of the present paper in his recent Ph.D. thesis [32] formulated a proper Riemann–Hilbert problem for the inverse scattering theory for (1.1) by using only the assumption that the two secondary reflection coefficients are zero but without using the assumption that the transmission coefficients are identically equal to 1 for all k -values.

Another mathematical significance of our paper is the innovative way we overcome the difficulty encountered by Deift, Tomei, and Trubowitz in their paper [15] that forced them to use the severe restriction on the transmission coefficients. In the standard formulation of a Riemann–Hilbert problem, a sectionally analytic function is recovered from the jump specified on a line separating the complex plane into two regions. In the standard formulation of the corresponding Riemann–Hilbert problem in [15], the only way to preserve the analyticity in both regions is to impose the severe restriction by choosing the transmission coefficients identically equal to 1. In our own formulation of the Riemann–Hilbert problem, a sectionally meromorphic function is recovered from the jump specified on a line separating the complex plane into two regions. With this innovative approach, the jump is determined solely by the two primary reflection coefficients and the poles of the transmission coefficients do not affect the jump because we are dealing with a sectionally meromorphic function rather than a sectionally analytic function. This approach we use readily allows us to construct soliton solutions to all integrable evolution equations associated with (1.1). In the reflectionless case, the jump in our formulation of the Riemann–Hilbert problem is zero, and the corresponding Marchenko integral equation is readily set up by using the input consisting of the bound-state poles of the transmission coefficients and the corresponding bound-state dependency constants. In a future publication, we will present the Marchenko equations whose solutions yield the N -soliton solution for each of the Sawada–Kotera equation, the Kaup–Kupershmidt equation, and the modified version of the Boussinesq equation.

The innovative way we overcome the difficulty encountered by Deift, Tomei, and Trubowitz in their 1982 seminal paper is not just a significant achievement to include the consideration of bound states in the analysis of inverse problems for higher-order differential equations. It brings a solution to the following important problem arising in the formulation of a Marchenko integral equation by using the Fourier transformation on a corresponding Riemann–Hilbert problem. We elaborate on this critical issue as follows. In the formulation of a Marchenko integral equation, it is necessary that the transmission and reflection coefficients are separated from each other before applying the Fourier transform. In the spacial asymptotics of Jost solutions, the transmission and reflection coefficients do not appear separately but appear as ratios of a reflection coefficient to a transmission coefficient. Consequently, in the first formulation of a Riemann–Hilbert problem a reflection coefficient usually appears not by itself but as a ratio in combination with a transmission coefficient. The severe assumption of setting the transmission coefficients identically equal to 1 has enabled Deift, Tomei, and Trubowitz to use the Fourier transforms on those reflection coefficients because that severe restriction has resulted in the automatic separation of reflection coefficients from the transmission coefficients. In our formulation of the Marchenko integral equation, we do not need to use that severe restriction on the transmission coefficients. This is because we separate the reflection and transmission

coefficients from each other in our own analysis. That also clarifies why the reflection coefficients in our analysis do not just appear as ratios in combination with transmission coefficients, but they appear explicitly by themselves.

The most relevant papers in the literature for our work is the 1980 paper [21] by Kaup, the 1982 paper [15] by Deift, Tomei, and Trubowitz, the 2024 Ph.D. thesis [32] by Toledo, and the recent paper [2] where soliton solutions to some integrable nonlinear evolution equations are obtained by solving our properly formulated Riemann–Hilbert problem when all the reflection coefficients are zero. We also mention the relevant work [7] by Beals, the series of two papers [8, 9] by Beals and Coifman, the paper [12] by Caudrey, and the review paper [22] by Kaup. In these references, a standard Riemann–Hilbert problem is formulated on a set of two intersecting full lines on the complex plane. However, such a formulation does not allow the use of scattering coefficients for (1.1) as input to solve the corresponding inverse scattering problem and it does not allow the use of a Fourier transformation to derive a corresponding linear integral equation.

Our paper is organized as follows. In section 2, we introduce the four basic solutions to (1.1) and the six scattering coefficients associated with (1.1), and we provide the relevant properties of those solutions and scattering coefficients. In section 3, we present a brief description of the bound states for (1.1), and we introduce the dependency constants and the normalization constants for the bound states. In section 4, we consider the important special case where the secondary reflection coefficients are zero but without the severe restriction on the transmission coefficients used [15] by Deift *et al.* We formulate the relevant proper Riemann–Hilbert problem, and we describe the recovery of the potentials Q and P from the solution to that Riemann–Hilbert problem. In section 5, we derive the linear integral equation associated with (1.1), which is the counterpart of the Marchenko equation used in the inverse scattering theory for the Schrödinger equation. For simplicity, we derive our linear integral equation in the absence of bound states. Finally, in section 6 we present our conclusion by summarizing the novelty and significance of our work and our plans for relevant work in the future.

2. The basic solutions and scattering coefficients

We recall that we assume that the potentials Q and P appearing in (1.1) belong to the Schwartz class. In this section we present the four basic solutions to (1.1), denoted by $f(k, x)$, $g(k, x)$, $m(k, x)$, $n(k, x)$, respectively, and the six scattering coefficients denoted by $T_1(k)$, $L(k)$, $M(k)$, $T_r(k)$, $R(k)$, $N(k)$, respectively. We provide the large spacial and spectral asymptotics of the four basic solutions, and we present some relevant properties of the scattering coefficients. For further details and proofs, we refer the reader to [2].

For each fixed $k \in \mathbb{C}$, we are interested in identifying a set of three fundamental solutions to (1.1). Since (1.1) is a linear homogeneous differential equation, we can use three fundamental solutions to (1.1) at a k -value to determine the general solution. We remark that if $\psi(k, x)$ is a solution to (1.1), then $\psi(zk, x)$ and $\psi(z^2k, x)$ are also solutions, where z is the special complex constant $e^{2\pi i/3}$, or equivalently

$$z := -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \quad (2.1)$$

This is because the parameter k appears as k^3 in (1.1) and we have $(zk)^3 = k^3$ and $(z^2k)^3 = k^3$. The k -domain of $\psi(zk, x)$ is obtained by rotating the k -domain of $\psi(k, x)$ around the origin of the complex k -plane by $2\pi/3$ radians clockwise. Similarly, the k -domain of $\psi(z^2k, x)$ is

obtained by rotating the k -domain of $\psi(k, x)$ by $4\pi/3$ radians clockwise. This observation is useful to construct a fundamental set of three solutions to (1.1) at each k -value.

We divide the complex k -plane into six open sectors Ω_1^{up} , Ω_1^{down} , Ω_2 , Ω_3^{down} , Ω_3^{up} , and Ω_4 as shown in figure 1. These six open sectors are parametrized, respectively, as

$$\begin{aligned}\Omega_1^{\text{up}} &:= \left\{ k \in \mathbb{C} : \frac{2\pi}{3} < \arg[k] < \pi \right\}, \\ \Omega_1^{\text{down}} &:= \left\{ k \in \mathbb{C} : \pi < \arg[k] < \frac{4\pi}{3} \right\}, \\ \Omega_2 &:= \left\{ k \in \mathbb{C} : -\frac{2\pi}{3} < \arg[k] < -\frac{\pi}{3} \right\}, \\ \Omega_3^{\text{down}} &:= \left\{ k \in \mathbb{C} : -\frac{\pi}{3} < \arg[k] < 0 \right\}, \\ \Omega_3^{\text{up}} &:= \left\{ k \in \mathbb{C} : 0 < \arg[k] < \frac{\pi}{3} \right\}, \\ \Omega_4 &:= \left\{ k \in \mathbb{C} : \frac{\pi}{3} < \arg[k] < \frac{2\pi}{3} \right\},\end{aligned}\tag{2.2}$$

$$\tag{2.3}$$

where we use $\arg[k]$ to denote the argument function taking values in the interval $(-2\pi/3, 4\pi/3)$. We use $\overline{\Omega_1^{\text{up}}}$, $\overline{\Omega_1^{\text{down}}}$, $\overline{\Omega_2}$, $\overline{\Omega_3^{\text{down}}}$, $\overline{\Omega_3^{\text{up}}}$, $\overline{\Omega_4}$ to denote the closures of the respective open sets, where we recall that the closure of an open set is obtained by including the boundary of that open set. We also define Ω_1 and Ω_3 as the open sectors parametrized, respectively, as

$$\Omega_1 := \left\{ k \in \mathbb{C} : \frac{2\pi}{3} < \arg[k] < \frac{4\pi}{3} \right\},\tag{2.4}$$

$$\Omega_3 := \left\{ k \in \mathbb{C} : -\frac{\pi}{3} < \arg[k] < \frac{\pi}{3} \right\},\tag{2.5}$$

and we use $\overline{\Omega_1}$ and $\overline{\Omega_3}$ to denote their closures, respectively.

The left Jost solution $f(k, x)$ is the solution to (1.1) satisfying the asymptotics as $x \rightarrow +\infty$ given by

$$\begin{cases} f(k, x) = e^{kx} [1 + o(1)], \\ f'(k, x) = k e^{kx} [1 + o(1)], \\ f''(k, x) = k^2 e^{kx} [1 + o(1)], \end{cases}\tag{2.6}$$

and its k -domain is given by $\overline{\Omega_1}$. The left scattering coefficients $T_1(k)$, $L(k)$, $M(k)$ are defined via the spacial asymptotics of $f(k, x)$ as $x \rightarrow -\infty$ given by

$$f(k, x) = \begin{cases} e^{kx} T_1(k)^{-1} [1 + o(1)] + e^{2kx} L(k) T_1(k)^{-1} [1 + o(1)], & k \in \mathcal{L}_1, \\ e^{kx} T_1(k)^{-1} [1 + o(1)], & k \in \Omega_1, \\ e^{kx} T_1(k)^{-1} [1 + o(1)] + e^{2kx} M(k) T_1(k)^{-1} [1 + o(1)], & k \in \mathcal{L}_2, \end{cases}\tag{2.7}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the directed half lines shown in figure 1 corresponding to the upper and lower boundaries of Ω_1 , respectively. We note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are parametrized, respectively, as

$$\mathcal{L}_1 := \{k \in \mathbb{C} : k = zs \text{ for } s \in [0, +\infty)\},$$

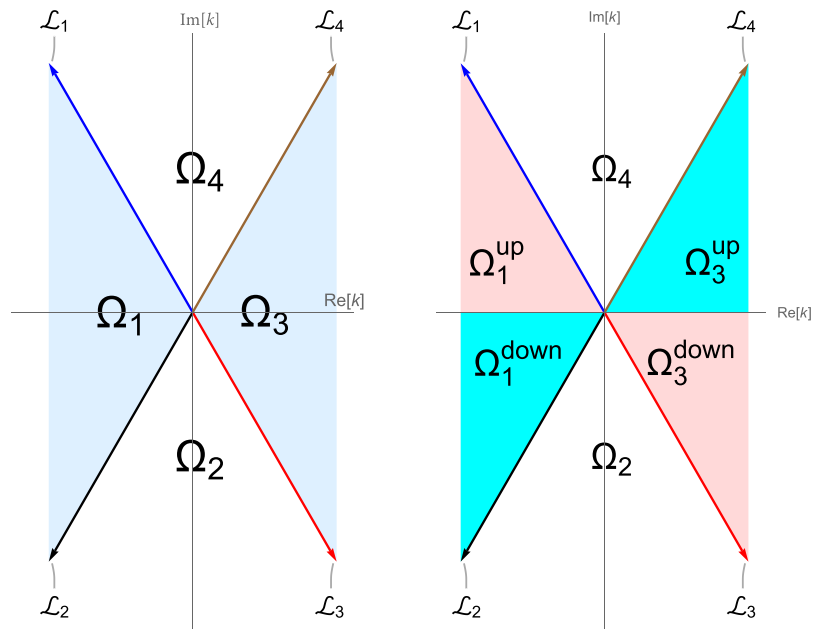


Figure 1. The complex k -plane is divided into the four sectors Ω_1 , Ω_2 , Ω_3 , and Ω_4 as shown on the left plot, with the directed half lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ acting as the boundaries. The sectors Ω_1 and Ω_3 are each divided into two subsectors by the directed half lines $-\mathbb{R}^-$ and $\mathbb{R}^+$, respectively, as shown on the right plot.

$$\mathcal{L}_2 := \{k \in \mathbb{C} : k = z^2 s \text{ for } s \in [0, +\infty)\}.$$

The left transmission coefficient $T_l(k)$ is defined on $\overline{\Omega_1}$, the left primary reflection coefficient $L(k)$ is defined on $\mathcal{L}_1$, and the left secondary reflection coefficient $M(k)$ is defined on $\mathcal{L}_2$.

The right Jost solution $g(k, x)$ is the solution to (1.1) satisfying the spacial asymptotics as $x \rightarrow -\infty$ given by

$$\begin{cases} g(k, x) = e^{kx} [1 + o(1)], \\ g'(k, x) = k e^{kx} [1 + o(1)], \\ g''(k, x) = k^2 e^{kx} [1 + o(1)], \end{cases} \quad (2.8)$$

and its k -domain is given by $\overline{\Omega_3}$. The right scattering coefficients $T_r(k)$, $R(k)$, $N(k)$ are defined via the spacial asymptotics of $g(k, x)$ as $x \rightarrow +\infty$ given by

$$g(k, x) = \begin{cases} e^{kx} T_r(k)^{-1} [1 + o(1)] + e^{zx} R(k) T_r(k)^{-1} [1 + o(1)], & k \in \mathcal{L}_3, \\ e^{kx} T_r(k)^{-1} [1 + o(1)], & k \in \Omega_3, \\ e^{kx} T_r(k)^{-1} [1 + o(1)] + e^{z^2 kx} N(k) T_r(k)^{-1} [1 + o(1)], & k \in \mathcal{L}_4, \end{cases} \quad (2.9)$$

where $\mathcal{L}_3$ and $\mathcal{L}_4$ are the directed half lines shown in figure 1 corresponding to the lower and upper boundaries of Ω_3 . We remark that $\mathcal{L}_3$ and $\mathcal{L}_4$ have the respective parametric representations given by

$$\mathcal{L}_3 := \{k \in \mathbb{C} : k = -zs \text{ for } s \in [0, +\infty)\},$$

$$\mathcal{L}_4 := \{k \in \mathbb{C} : k = -z^2 s \text{ for } s \in [0, +\infty)\}.$$

The right transmission coefficient $T_r(k)$ is defined in $\overline{\Omega_3}$, the right primary reflection coefficient $R(k)$ is defined on $\mathcal{L}_3$, and the right secondary reflection coefficient $N(k)$ is defined on $\mathcal{L}_4$.

Our use of the terms ‘left and right Jost solutions’ and ‘left and right scattering coefficients’ is motivated by the corresponding terminology [6] for the full-line Schrödinger equation. In the inverse scattering problem for the Schrödinger equation, we construct a single potential by using a single reflection coefficient. We remark that (1.1) contains the two potentials Q and P , and it is associated with the four reflection coefficients $L(k)$, $R(k)$, $M(k)$, $N(k)$. Our use of the term ‘secondary reflection coefficients’ for $M(k)$ and $N(k)$ is motivated by the following fact related to the initial-value problem for the modified version of the bad Boussinesq equation studied by Deift *et al* in [15]. As indicated on p. 568 of [15], by setting $M(k) \equiv 0$ and $N(k) \equiv 0$ and by using the corresponding time-evolved versions for $L(k)$ and $R(k)$, one obtains a global solution to the aforementioned initial-value problem. We recall that a global solution is valid for all $t > 0$, and hence it is a solution that does not blow up at a finite time. Motivated by this fact, we refer to $L(k)$ and $R(k)$ as the primary reflection coefficients and refer to $M(k)$ and $N(k)$ as the secondary reflection coefficients.

The basic solution $m(k, x)$ to (1.1) has the k -domain $\overline{\Omega_2}$, it satisfies the spacial asymptotics as $x \rightarrow +\infty$ given by

$$m(k, x) = \begin{cases} e^{kx} T_1(z^2 k)^{-1} T_r(zk) [1 + o(1)] \\ \quad - e^{z^2 kx} M(k) T_1(k)^{-1} T_r(zk) [1 + o(1)], & k \in \mathcal{L}_2, \\ e^{kx} T_1(z^2 k)^{-1} T_r(zk) [1 + o(1)], & k \in \Omega_2, \\ e^{kx} T_1(z^2 k)^{-1} T_r(zk) [1 + o(1)], & k \in \mathcal{L}_3, \end{cases} \quad (2.10)$$

and it satisfies the spacial asymptotics as $x \rightarrow -\infty$ given by

$$m(k, x) = \begin{cases} e^{kx} [1 + o(1)], & k \in \mathcal{L}_2, \\ e^{kx} [1 + o(1)], & k \in \Omega_2, \\ e^{kx} [1 + o(1)] - e^{z kx} R(k) T_r(k)^{-1} T_r(zk) [1 + o(1)], & k \in \mathcal{L}_3. \end{cases} \quad (2.11)$$

Similarly, the basic solution $n(k, x)$ to (1.1) has the k -domain $\overline{\Omega_4}$, it satisfies the spacial asymptotics as $x \rightarrow +\infty$ given by

$$n(k, x) = \begin{cases} e^{kx} T_1(zk)^{-1} T_r(z^2 k) [1 + o(1)] \\ \quad - e^{z kx} L(k) T_1(k)^{-1} T_r(z^2 k) [1 + o(1)], & k \in \mathcal{L}_1, \\ e^{kx} T_1(zk)^{-1} T_r(z^2 k) [1 + o(1)], & k \in \Omega_4, \\ e^{kx} T_1(zk)^{-1} T_r(z^2 k) [1 + o(1)], & k \in \mathcal{L}_4, \end{cases} \quad (2.12)$$

and it has the spacial asymptotics as $x \rightarrow -\infty$ given by

$$n(k, x) = \begin{cases} e^{kx} [1 + o(1)], & k \in \mathcal{L}_1, \\ e^{kx} [1 + o(1)], & k \in \Omega_4, \\ e^{kx} [1 + o(1)] - e^{z^2 kx} N(k) T_r(k)^{-1} T_r(z^2 k) [1 + o(1)], & k \in \mathcal{L}_4. \end{cases} \quad (2.13)$$

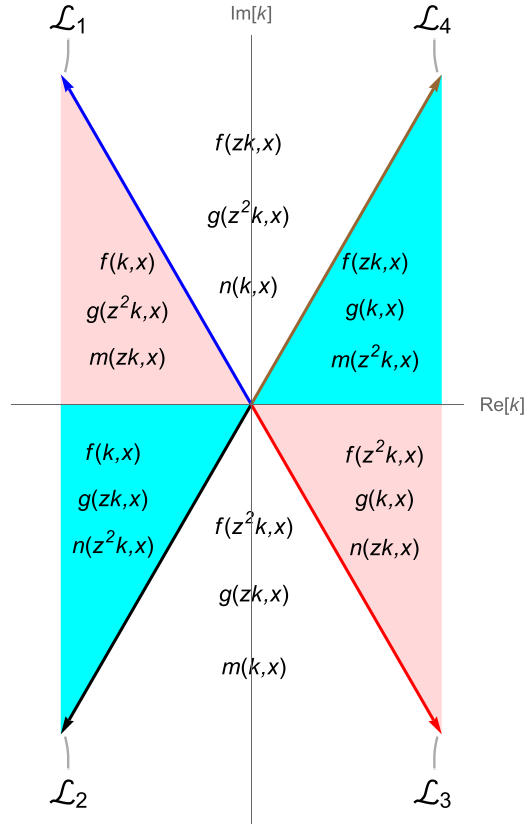


Figure 2. The k -domains of the basic solutions to (1.1) in each of the six regions, respectively.

The large k -asymptotics of the basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, and $n(k, x)$, respectively, are given by

$$f(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_1}, \quad (2.14)$$

$$g(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_3}, \quad (2.15)$$

$$m(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_2}, \quad (2.16)$$

$$n(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_4}. \quad (2.17)$$

Using the four basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, and $n(k, x)$, we can form a fundamental set of three solutions at each k -value in the complex k -plane, as shown in figure 2.

Let us briefly explain how the solutions $m(k, x)$ and $n(k, x)$ to (1.1) arise. Associated with (1.1), we have the adjoint equation

$$\overline{\psi}'''(k, x) + \overline{Q}(x) \overline{\psi}'(k, x) + \overline{P}(x) \overline{\psi}(k, x) = k^3 \overline{\psi}(k, x), \quad x \in \mathbb{R}, \quad (2.18)$$

where the adjoint potentials $\overline{Q}$ and $\overline{P}$ are defined in terms of the potentials Q and P as

$$\overline{Q}(x) := Q(x)^*, \quad x \in \mathbb{R}, \quad (2.19)$$

$$\overline{P}(x) := Q'(x)^* - P(x)^*, \quad x \in \mathbb{R}. \quad (2.20)$$

We remark that the overbar used in (2.18)–(2.20) does not denote complex conjugation, and it only refers to the quantities associated with the adjoint equation (2.18). We recall that an asterisk is used to denote complex conjugation and the prime in (2.20) denotes the x -derivative. From (2.19) and (2.20) we observe that the adjoint potentials $\overline{Q}$ and $\overline{P}$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ because Q and P belong to $\mathcal{S}(\mathbb{R})$. The left and right Jost solutions to the adjoint equation (2.18), denoted by $\overline{f}(k, x)$ and $\overline{g}(k, x)$, respectively, are the solutions to (2.18) satisfying the spacial asymptotics given in (2.6) and (2.8), respectively. Since k appears as k^3 in (2.18), we see that $\overline{f}(zk, x)$, $\overline{f}(z^2k, x)$, $\overline{g}(zk, x)$, $\overline{g}(z^2k, x)$ are also solutions to (2.18). The k -domains of $\overline{f}(k, x)$ and $\overline{g}(k, x)$ coincide with the k -domains of $f(k, x)$ and $g(k, x)$, respectively. It is known [2] that $m(k, x)$ and $n(k, x)$ are defined in terms of the adjoint Jost solutions, respectively, as

$$m(k, x) := \frac{T_r(zk) [\overline{f}(-z^2k^*, x)^* ; \overline{g}(-zk^*, x)^*]}{z(1-z)k}, \quad k \in \overline{\Omega_2}, \quad x \in \mathbb{R}, \quad (2.21)$$

$$n(k, x) := \frac{T_r(z^2k) [\overline{f}(-zk^*, x)^* ; \overline{g}(-z^2k^*, x)^*]}{-z(1-z)k}, \quad k \in \overline{\Omega_4}, \quad x \in \mathbb{R}, \quad (2.22)$$

where we recall that $T_r(k)$ is the right transmission coefficient appearing in (2.9). We remark that in (2.21) and (2.22) we use the 2-Wronskian of the adjoint Jost solutions, where we have defined

$$[F(x); G(x)] := F(x) G'(x) - F'(x) G(x).$$

In comparing $m(k, x)$ and $[\overline{f}(-z^2k^*, x)^* ; \overline{g}(-zk^*, x)^*]$, we remark that the former solution to (1.1) has simpler spacial asymptotics and the latter quantity is analytic in $k \in \Omega_2$ and continuous $k \in \overline{\Omega_2}$. Similarly, in comparing the solutions $n(k, x)$ and $[\overline{f}(-zk^*, x)^* ; \overline{g}(-z^2k^*, x)^*]$ to (1.1), we remark that the former has simpler spacial asymptotics and the latter solution has the advantage that it is analytic in $k \in \Omega_4$ and continuous $k \in \overline{\Omega_4}$.

Let us define the 3-Wronskian of three functions $F(x)$, $G(x)$, and $H(x)$ as

$$[F(x); G(x); H(x)] := \begin{vmatrix} F(x) & G(x) & H(x) \\ F'(x) & G'(x) & H'(x) \\ F''(x) & G''(x) & H''(x) \end{vmatrix},$$

where the right-hand side consists of the determinant of the relevant 3×3 matrix. We recall that the 3-Wronskian of any three solutions to (1.1) at a k -value is zero if and only if those three solutions are linearly dependent. Furthermore, since the term ψ'' is absent in (1.1), the 3-Wronskian of any three solutions to (1.1) is independent of x and hence its value can be evaluated at any x -value. With the help of the spacial asymptotics in (2.6)–(2.13), we obtain the 3-Wronskians

$$[f(k, x); g(zk, x); n(z^2k, x)] = -3z(1-z)k^3 T_1(k)^{-1}, \quad k \in \overline{\Omega_1^{\text{down}}}, \quad (2.23)$$

$$[f(k, x); g(z^2k, x); m(zk, x)] = 3(1-z^2)k^3 T_1(k)^{-1}, \quad k \in \overline{\Omega_1^{\text{up}}}. \quad (2.24)$$

Thus, from (2.23) we observe that the solutions $f(k, x)$, $g(zk, x)$, $n(z^2k, x)$ to (1.1) are linearly independent in $k \in \Omega_1^{\text{down}} \setminus \{0\}$ except at the k -values corresponding to the poles of $T_1(k)$ there.

Similarly, from (2.24) we see that the solutions $f(k, x)$, $g(z^2k, x)$, $m(zk, x)$ to (1.1) are linearly independent in $k \in \overline{\Omega_1^{\text{up}}} \setminus \{0\}$ except at the k -values corresponding to the poles of $T_l(k)$ there. We refer the reader to [2] for further properties of the basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, $n(k, x)$ to (1.1) and to [32] for the basic properties of the six scattering coefficients $T_l(k)$, $L(k)$, $M(k)$, $T_r(k)$, $R(k)$, $N(k)$.

The scattering coefficients for (1.1) satisfy various properties. For example, the left reflection coefficients $L(k)$ and $M(k)$ are related to the transmission coefficients $T_l(k)$ and $T_r(k)$ as

$$T_r(z^2k)^{-1} = T_l(k)^{-1} T_l(zk)^{-1} [1 - L(k) M(zk)], \quad k \in \mathcal{L}_1. \quad (2.25)$$

Similarly, the right reflection coefficients $R(k)$ and $N(k)$ are related to the transmission coefficients $T_l(k)$ and $T_r(k)$ as

$$T_l(z^2k)^{-1} = T_r(k)^{-1} T_r(zk)^{-1} [1 - R(k) N(zk)], \quad k \in \mathcal{L}_3. \quad (2.26)$$

For the proof of (2.25) and (2.26), we refer the reader to theorem 3.3.3 of [32]. We remark that when $M(k) \equiv 0$, from (2.25) we get

$$T_r(z^2k)^{-1} = T_l(k)^{-1} T_l(zk)^{-1}, \quad k \in \mathcal{L}_1. \quad (2.27)$$

Similarly, when $N(k) \equiv 0$, from (2.26) we obtain

$$T_l(z^2k)^{-1} = T_r(k)^{-1} T_r(zk)^{-1}, \quad k \in \mathcal{L}_3.$$

3. The bound states

A bound-state solution to (1.1) corresponds to a nontrivial solution which is square integrable in $x \in \mathbb{R}$. Let us assume that (1.1) has a bound state at the k -value k_j in the complex k -plane. Then, (1.1) at $k = k_j$, i.e. the equation

$$\psi''' + Q(x) \psi' + P(x) \psi = k_j^3 \psi, \quad x \in \mathbb{R}, \quad (3.1)$$

has a solution $\psi(k_j, x)$ satisfying

$$\int_{-\infty}^{\infty} dx |\psi(k_j, x)|^2 < +\infty.$$

The solution $\psi(k_j, x)$ is a bound-state solution to (1.1) at $k = k_j$. Since (3.1) is a linear homogeneous equation, any constant multiple of a bound-state solution is also a bound-state solution. The number of linearly independent bound-state solutions to (3.1) determines the multiplicity of the bound state at $k = k_j$. In this paper, we only consider simple bound states, i.e. bound states of multiplicity one. We refer the reader to section 2 of [3] for the analysis of nonsimple bound states for a first-order linear system of differential equations, to section 3 of [4] for the analysis of nonsimple bound states for a first-order system of difference equations, and to section 3 of [5] for the analysis of nonsimple bound states for a first-order linear system of differential equations. The analysis of nonsimple bound states for other linear differential or difference equations can be carried out in a similar way. We remark that by a nonsimple bound state we mean a bound state with multiplicity greater than one. We refer the reader to the analysis in section 3 of [2] for a detailed description of the bound states for (1.1). In the present paper, we provide a brief summary of the bound-state analysis relevant to the formulation of the inverse scattering problem investigated in sections 4 and 5.

Suppose that the bound state at $k = k_j$ with $k_j \neq 0$ occurs when the argument of k_j lies in the interval $[7\pi/6, 4\pi/3)$. In that case we have $T_1(k_j)^{-1} = 0$, where we recall that $T_1(k)$ is the left reflection coefficient appearing in (2.7). We then observe that the right-hand side of (2.23) becomes zero, indicating that the three solutions $f(k_j, x)$, $g(zk_j, x)$, and $n(z^2k_j, x)$ to (3.1) are linearly dependent. The analysis in section 3 of [2] shows that, when $T_1(k_j)^{-1} = 0$ with $\arg[k_j] \in [7\pi/6, 4\pi/3)$, the solution $n(z^2k_j, x)$ to (3.1) does not vanish as $x \rightarrow +\infty$ but $f(k_j, x)$ and $g(zk_j, x)$ both vanish exponentially as $x \rightarrow \pm\infty$. Thus, there exists a nonzero constant $D(k_j)$ such that

$$f(k_j, x) = D(k_j) g(zk_j, x), \quad \arg[k_j] \in \left[\frac{7\pi}{6}, \frac{4\pi}{3} \right), \quad x \in \mathbb{R}. \quad (3.2)$$

We refer to the nonzero complex constant $D(k_j)$ as the dependency constant at the bound state at $k = k_j$. The constant $D(k_j)$ is the analog of the bound-state dependency constant defined in (1.3) of [6] for the full-line Schrödinger equation. As in (1.3) of [6], the bound-state dependency constant corresponds to the ratio of the left Jost solution to the right Jost solution at the bound state.

From (3.2) we observe that each of $f(k_j, x)$ and $g(zk_j, x)$ are square-integrable solutions to (3.1). By introducing the positive constant $c_1(k_j)$ via

$$c_1(k_j) := \frac{1}{\sqrt{\int_{-\infty}^{\infty} dx |f(k_j, x)|^2}}, \quad \arg[k_j] \in \left[\frac{7\pi}{6}, \frac{4\pi}{3} \right),$$

we observe that the quantity $c_1(k_j)f(k_j, x)$ is a normalized square-integrable solution to (3.1), i.e. we have

$$\int_{-\infty}^{\infty} dx |c_1(k_j)f(k_j, x)|^2 = 1.$$

Since $f(k_j, x)$ is the left Jost solution to (3.1), it is appropriate to refer to $c_1(k_j)$ as the left bound-state normalization constant, which is the analog of the left bound-state normalization constant appearing in (1.7) of [6] for the full-line Schrödinger equation. Similarly, we can introduce the right bound-state normalization constant $c_r(k_j)$ via

$$c_r(k_j) := \frac{1}{\sqrt{\int_{-\infty}^{\infty} dx |g(zk_j, x)|^2}}, \quad \arg[k_j] \in \left[\frac{7\pi}{6}, \frac{4\pi}{3} \right).$$

Then, the quantity $c_r(k_j)g(zk_j, x)$ corresponds to a normalized bound-state solution to (1.1) at $k = k_j$.

In case we have a bound state occurring at a nonzero k -value with $\arg[k] \in (2\pi/3, 5\pi/6]$, then we can assume that this particular k -value occurs at $k = k_j^*$ with $\arg[k_j] \in (7\pi/6, 4\pi/3]$, where we recall that an asterisk denotes complex conjugation. Since the points k_j and k_j^* are located symmetrically with respect to the negative real axis in the complex k -plane, the bound-state analysis at $k = k_j$ can be used to describe the bound-state analysis at $k = k_j^*$ with $\arg[k_j^*] \in (2\pi/3, 5\pi/6]$. Thus, at the bound state occurring at $k = k_j^*$, we have $T_1(k_j^*)^{-1} = 0$. Furthermore, the solutions $f(k_j^*, x)$ and $g(z^2k_j^*, x)$ to (1.1) at $k = k_j^*$ become linearly dependent, and they are related to each other as

$$f(k_j^*, x) = D(k_j^*) g(z^2k_j^*, x), \quad \arg[k_j^*] \in \left(\frac{2\pi}{3}, \frac{5\pi}{6} \right],$$

where $D(k_j^*)$ is a nonzero complex constant, which we call the dependency constant at the bound state at $k = k_j^*$.

For the analysis of the bound states occurring at k -values in the sector $\arg[k] \in (5\pi/6, \pi)$ or in the sector $\arg[k] \in (\pi, 7\pi/6)$, we refer the reader to section 3 of [2], where the dependency constants at those k -values are described.

4. The case of zero secondary reflection coefficients

In this section we consider the important special case for (1.1) when the secondary reflection coefficients $M(k)$ and $N(k)$ are both zero. This special case is relevant in introducing a properly posed Riemann–Hilbert problem in the complex k -plane and obtaining the associated linear integral equation, which is the analog of the Marchenko integral equation [16] available arising in the analysis of the inverse scattering theory for the full-line Schrödinger equation.

In the special case when the left secondary reflection coefficient $M(k)$ is zero, we have [2]

$$T_1(k)f(k, x) = m(k, x), \quad x \in \mathbb{R}, \quad k \in \mathcal{L}_2. \quad (4.1)$$

The following argument shows why we have (4.1) when $M(k) \equiv 0$. In order to show that (4.1) holds when $M(k) \equiv 0$, it is enough to show that both $m(k, x)$ and $T_1(k)f(k, x)$ satisfy (1.1) for $k \in \mathcal{L}_2$ and they both have the same spacial asymptotics as $x \rightarrow -\infty$ for $k \in \mathcal{L}_2$. Since (1.1) is a linear homogeneous equation and both $m(k, x)$ and $f(k, x)$ are solutions to (1.1), it is clear that $m(k, x)$ and $T_1(k)f(k, x)$ are both solutions to (1.1). Furthermore, comparing the third line of (2.7) with $M(k) \equiv 0$ and the first line of (2.11), we see that each of $m(k, x)$ and $T_1(k)f(k, x)$ behaves as $e^{kx}[1 + o(1)]$ as $x \rightarrow -\infty$ for $k \in \mathcal{L}_2$. Thus, (4.1) holds when $M(k) \equiv 0$.

Similarly, when the right secondary reflection coefficient $N(k)$ is zero, we have [2]

$$g(k, x) = n(k, x), \quad x \in \mathbb{R}, \quad k \in \mathcal{L}_4. \quad (4.2)$$

The following argument shows why (4.2) holds when $N(k) \equiv 0$. For (4.2) to hold, it is sufficient to show that each side of (4.2) is a solution to (1.1) and has the same spacial asymptotics as $x \rightarrow +\infty$ for $k \in \mathcal{L}_4$. We already know that both $f(k, x)$ and $n(k, x)$ are solution to (1.1) when $k \in \mathcal{L}_4$. When $N(k) \equiv 0$, from the first line of (2.8) and from the third line of (2.13), we see that each of $g(k, x)$ and $n(k, x)$ behaves as $e^{kx}[1 + o(1)]$ as $x \rightarrow -\infty$ for $k \in \mathcal{L}_4$. Thus, (4.2) holds when $N(k) \equiv 0$.

When both secondary reflection coefficients are zero, with the help of (4.1) and (4.2) we introduce the solution $\Phi_+(k, x)$ to (1.1) with the k -domain $\overline{\Omega_1} \cup \overline{\Omega_2}$ and the solution $\Phi_-(k, x)$ to (1.1) with the k -domain $\overline{\Omega_3} \cup \overline{\Omega_4}$, where we have defined

$$\Phi_+(k, x) := \begin{cases} T_1(k)f(k, x), & k \in \overline{\Omega_1}, \\ m(k, x), & k \in \overline{\Omega_2}, \end{cases} \quad (4.3)$$

$$\Phi_-(k, x) := \begin{cases} g(k, x), & k \in \overline{\Omega_3}, \\ n(k, x), & k \in \overline{\Omega_4}. \end{cases} \quad (4.4)$$

From (4.1) and (4.3), we conclude that the solution $m(k, x)$ to (1.1) has a meromorphic continuation from Ω_2 to Ω_1 , or equivalently the meromorphic function $T_1(k)f(k, x)$ defined in $k \in \Omega_1$ has a meromorphic continuation to $\overline{\Omega_2}$. Similarly, from (4.2) and (4.4), we see that $g(k, x)$ has a meromorphic extension from Ω_3 to Ω_4 . or equivalently $n(k, x)$ has an analytic continuation from Ω_4 to Ω_3 .

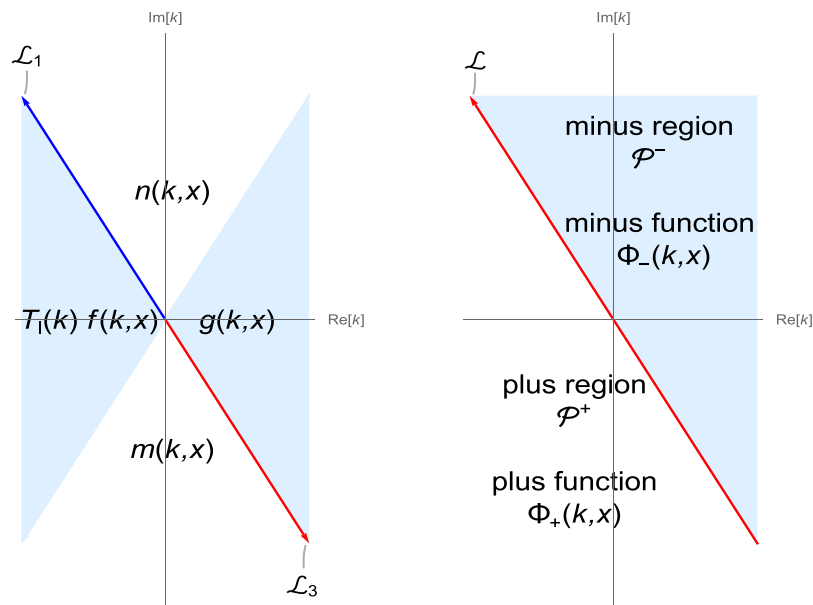


Figure 3. The k -domains of $T_1(k)f(k,x)$, $m(k,x)$, $g(k,x)$, and $n(k,x)$, respectively, are shown on the left plot. The right plot shows the plus and minus regions in the complex k -plane separated by the directed full line $\mathcal{L}$, as well as the plus and minus functions in their respective k -domains.

Let us define the full directed line $\mathcal{L}$ as the union of the half lines $\mathcal{L}_1$ and $-\mathcal{L}_3$, where $-\mathcal{L}_3$ denotes $\mathcal{L}_3$ but with its direction reversed. Thus, $\mathcal{L}$ has the parametrization given by

$$\mathcal{L} := \{k \in \mathbb{C} : k = zs \text{ for } s \in (-\infty, +\infty)\}.$$

The line $\mathcal{L}$ separates the complex k -plane into two open half planes $\mathcal{P}^+$ and $\mathcal{P}^-$, respectively, as shown on the right plot of figure 3. We note that the open half plane $\mathcal{P}^+$ corresponds to the interior of $\overline{\Omega}_1 \cup \overline{\Omega}_2$ and the open half plane $\mathcal{P}^-$ corresponds to the interior of $\overline{\Omega}_3 \cup \overline{\Omega}_4$. We recall that $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 are the open sectors defined in (2.4), (2.2), (2.5), and (2.3), respectively. From (4.3), we observe that, for each fixed $x \in \mathbb{R}$, the quantity $\Phi_+(k,x)$ is meromorphic in $k \in \mathcal{P}^+$. Similarly, from (4.4), we observe that, for each fixed $x \in \mathbb{R}$, the quantity $\Phi_-(k,x)$ is meromorphic in $k \in \mathcal{P}^-$. We refer to $\mathcal{P}^+$ as the plus region and $\mathcal{P}^-$ as the minus region. We use $\overline{\mathcal{P}^+}$ to denote the closure of $\mathcal{P}^+$, i.e. we let $\overline{\mathcal{P}^+} := \mathcal{P}^+ \cup \mathcal{L}$. Similarly, we use $\overline{\mathcal{P}^-}$ to denote the closure of $\mathcal{P}^-$, i.e. we let $\overline{\mathcal{P}^-} := \mathcal{P}^- \cup \mathcal{L}$.

Using $\Phi_+(k,x)$ and $\Phi_-(k,x)$ we are interested in the formulation of a Riemann–Hilbert problem for (1.1) when the secondary reflection coefficients $M(k)$ and $N(k)$ are both zero. This Riemann–Hilbert problem is posed on the complex k -plane separated by the full directed line $\mathcal{L}$ as shown on the right plot of figure 3. We refer to $\Phi_+(k,x)$ as the plus function with the k -domain $\overline{\mathcal{P}^+}$. Similarly, we refer to $\Phi_-(k,x)$ as the minus function with the k -domain $\overline{\mathcal{P}^-}$. In the formulation of our Riemann–Hilbert problem, the complex k -plane is divided into two regions by one single full line. We refer to such a formulation as a proper formulation of the Riemann–Hilbert problem. In case the boundary separating the plus and minus regions were not a single full line, then we would refer to the formulation of that Riemann–Hilbert problem as an improper formulation.

Our Riemann–Hilbert problem relevant to (1.1) arises as follows. The complex k -plane is separated by the directed full line $\mathcal{L}$ into the two regions $\mathcal{P}^+$ and $\mathcal{P}^-$. According to the commonly used convention, the region to the left of the directed line $\mathcal{L}$ is the plus region $\mathcal{P}^+$ and the region to the right of $\mathcal{L}$ is the minus region $\mathcal{P}^-$. In a standard Riemann–Hilbert problem, we would have a sectionally analytic function in $k \in \mathbb{C}$ with a jump on $\mathcal{L}$. In other words, for each fixed $x \in \mathbb{R}$, we would have the sectionally analytic function $\Phi(k, x)$, where $\Phi_+(k, x)$ and $\Phi_-(k, x)$ make up the two pieces of $\Phi(k, x)$ in such a way that $\Phi_+(k, x)$ is analytic in $k \in \mathcal{P}^+$ and $\Phi_-(k, x)$ is analytic in $k \in \mathcal{P}^-$. In our own formulation of the Riemann–Hilbert problem, we deal with a sectionally meromorphic function $\Phi(k, x)$, where the piece $\Phi_+(k, x)$ is meromorphic in $k \in \mathcal{P}^+$ and the piece $\Phi_-(k, x)$ is meromorphic in $k \in \mathcal{P}^-$. As in the standard Riemann–Hilbert problem we choose $\Phi_+(k, x)$ and $\Phi_-(k, x)$ so that $\Phi_+(k, x)$ vanishes as $k \rightarrow \infty$ in $\overline{\mathcal{P}^+}$ and $\Phi_-(k, x)$ vanishes as $k \rightarrow \infty$ in $\overline{\mathcal{P}^-}$. That normalization is chosen so that the Riemann–Hilbert problem has a unique solution when an appropriate input data set is provided to solve the Riemann–Hilbert problem. The solution to the Riemann–Hilbert problem is obtained by constructing $\Phi_+(k, x)$ and $\Phi_-(k, x)$ when their difference $\Phi_+(k, x) - \Phi_-(k, x)$ is known for $k \in \mathcal{L}$. In other words, knowing the jump on $\mathcal{L}$ for the sectionally meromorphic function $\Phi(k, x)$, we would like to construct $\Phi(k, x)$ when $k \in \mathbb{C}$.

When the secondary reflection coefficients $M(k)$ and $N(k)$ are both zero, the functions $\Phi_+(k, x)$ and $\Phi_-(k, x)$, defined in (4.3) and (4.4), respectively satisfy [32] the jump condition on $\mathcal{L}$ given by

$$\Phi_+(k, x) - \Phi_-(k, x) = J(k, x), \quad k \in \mathcal{L}. \quad (4.5)$$

The jump $J(k, x)$ appearing on the right-hand side of (4.5) is expressed in terms of the Jost solutions $f(k, x)$ and $g(k, x)$ and the remaining scattering coefficients $T_l(k)$, $T_r(k)$, $L(k)$, and $R(k)$ as

$$J(k, x) = \begin{cases} L(k) T_l(zk) f(zk, x), & k \in \mathcal{L}_1, \\ -\frac{R(k) T_r(zk) g(zk, x)}{T_r(k)}, & k \in -\mathcal{L}_3. \end{cases} \quad (4.6)$$

We recall that $-\mathcal{L}_3$ appearing in (4.6) corresponds to the half line $\mathcal{L}_3$ with the direction reversed.

In the following we briefly describe how (4.6) is obtained when $M(k) \equiv 0$ and $N(k) \equiv 0$. For the details of the derivation, we refer the reader to the proof of theorem 4.0.4 of [32]. In order to establish the first line of (4.6), as seen from (4.3) and (4.4), we need to show that

$$T_l(k) f(k, x) - n(k, x) = L(k) T_l(zk) f(zk, x), \quad k \in \mathcal{L}_1. \quad (4.7)$$

For this, it is sufficient to prove that each side of (4.7) is a solution to (1.1) satisfying the same spacial asymptotics as $x \rightarrow +\infty$ for $k \in \mathcal{L}_1$. We note that (1.1) is a linear homogeneous differential equation, and furthermore $f(k, x)$, $n(k, x)$, and $f(zk, x)$ are solutions to (1.1). Thus, it follows that each side of (4.7) is a solution to (1.1). From the first line of (2.6) we see that the right-hand side of (4.7) has the spacial asymptotics as $x \rightarrow +\infty$ given by

$$L(k) T_l(zk) f(zk, x) = L(k) T_l(zk) e^{z k x} [1 + o(1)], \quad k \in \mathcal{L}_1. \quad (4.8)$$

On the other hand, from the first line of (2.6) and the first line of (2.12) we see that the left-hand side of (4.7) has the spacial asymptotics as $x \rightarrow +\infty$ given by

$$T_1(k)f(k,x) - n(k,x) = e^{kx} \left[T_1(k)^{-1} - T_1(zk)^{-1} T_r(z^2k) \right] [1 + o(1)] \\ + e^{zkx} L(k) T_1(k)^{-1} T_r(z^2k) [1 + o(1)], \quad k \in \mathcal{L}_1. \quad (4.9)$$

From (2.27), we know that when $M(k) \equiv 0$, the coefficient of e^{kx} on the right-hand side of (4.9) vanishes and (4.9) reduces to the spacial asymptotics as $x \rightarrow +\infty$ given by

$$T_1(k)f(k,x) - n(k,x) = e^{zkx} L(k) T_1(k)^{-1} T_r(z^2k) [1 + o(1)], \quad k \in \mathcal{L}_1. \quad (4.10)$$

We observe that the right-hand sides of (4.8) and (4.10) coincide as a consequence of (2.27). Thus, we have proved that (4.7) holds. To complete the proof of (4.6), as seen from (4.3)–(4.6), we also need to show that

$$m(k,x) - g(k,x) = -\frac{R(k) T_r(zk) g(zk,x)}{T_r(k)}, \quad k \in \mathcal{L}_3. \quad (4.11)$$

For the proof of (4.11), we proceed as follows. We note that each side of (4.11) is a solution to (1.1) due to the fact that (1.1) is a linear homogeneous differential equation and that $m(k,x)$, $g(k,x)$, and $g(zk,x)$ are all solutions to (1.1). Thus, to establish (4.11) it is sufficient to prove that each side of (4.11) has the same spacial asymptotics as $x \rightarrow -\infty$. From the first line of (2.8), we see that the right-hand side of (4.11) has the spacial asymptotics as $x \rightarrow -\infty$ given by

$$-\frac{R(k) T_r(zk) g(zk,x)}{T_r(k)} = -\frac{R(k) T_r(zk) e^{zkx} [1 + o(1)]}{T_r(k)}, \quad k \in \mathcal{L}_3. \quad (4.12)$$

On the other hand, from the first line to (2.8) and the third line of (2.11) we observe that the left-hand side of (4.11) has the spacial asymptotics as $x \rightarrow -\infty$ given by

$$m(k,x) - g(k,x) = -e^{zkx} R(k) T_r(k)^{-1} T_r(zk), \quad k \in \mathcal{L}_3. \quad (4.13)$$

Hence, the right-hand sides of (4.12) and (4.13) agree and consequently (4.11) holds.

With the help of (4.5) and (4.6), we formulate our Riemann–Hilbert problem as follows, by recalling that we assume that the secondary reflection coefficients $M(k)$ and $N(k)$ for (1.1) are both zero. We are given the left scattering coefficients $T_1(k)$ and $L(k)$ for $k \in \mathcal{L}_1$. Similarly, we are given the right scattering coefficients $T_r(k)$ and $R(k)$ for $k \in \mathcal{L}_3$. Furthermore, we know the poles of the extension of $T_1(k)$ from $k \in \mathcal{L}_1$ to $k \in \Omega_1$. For each fixed $x \in \mathbb{R}$, we would like to determine the function $\Phi_+(k,x)$, which is meromorphic in $k \in \mathcal{P}^+$ with the poles coinciding with the poles of $T_1(k)$ in Ω_1 in such a way that $\Phi_+(k,x)$ has the asymptotics e^{kx} as $k \rightarrow \infty$ in $\mathcal{P}^+$. For each fixed $x \in \mathbb{R}$, we also would like to determine the function $\Phi_-(k,x)$, which is meromorphic in $k \in \mathcal{P}^-$ and has the asymptotics e^{kx} as $k \rightarrow \infty$ in $\overline{\mathcal{P}^-}$.

The solution to the aforementioned Riemann–Hilbert problem yields the solution to the inverse scattering problem for (1.1). This can be seen as follows. Once we recover $\Phi_+(k,x)$ from the input scattering data set, as seen from (4.3), we have the solutions $f(k,x)$ and $n(k,x)$ to (1.1) with their respective k -domains $\overline{\Omega_1}$ and $\overline{\Omega_2}$. For example, using $f(k,x)$ in (1.1) we can recover the potentials Q and P appearing in (1.1). The recovery of these two potentials can be accomplished by using $f(k,x)$, $f'(k,x)$, and $f'''(k,x)$ in (1.1). Alternatively, if we have the solution $\Phi_-(k,x)$ to our Riemann–Hilbert problem, as seen from (4.4), we get the solutions

$g(k, x)$ and $n(k, x)$ to (1.1) in their respective k -domains $\overline{\Omega_3}$ and $\overline{\Omega_4}$. We can then recover the potentials Q and P by using either of $g(k, x)$ and $n(k, x)$ and their x -derivatives in (1.1).

The recovery of the potentials Q and P from the left Jost solution $f(k, x)$ can also be achieved by exploiting the large k -asymptotics of $f(k, x)$ as follows. It is known [2] that we have

$$f(k, x) = e^{kx} \left[1 + \frac{u_1(x)}{k} + \frac{u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_1}, \quad x \in \mathbb{R}, \quad (4.14)$$

where we have let

$$u_1(x) := \frac{1}{3} \int_x^\infty dy Q(y), \quad x \in \mathbb{R}, \quad (4.15)$$

$$u_2(x) := -\frac{1}{3} \int_x^\infty dy [Q'(y) - P(y)] + \frac{1}{18} \left[\int_x^\infty dy Q(y) \right]^2, \quad x \in \mathbb{R}. \quad (4.16)$$

From (4.15) and (4.16), we recover Q and P as

$$Q(x) = -3 \frac{du_1(x)}{dx}, \quad x \in \mathbb{R}, \quad (4.17)$$

$$P(x) = 3 \left[u_1(x) \frac{du_1(x)}{dx} - \frac{d^2 u_1(x)}{dx^2} - \frac{du_2(x)}{dx} \right], \quad x \in \mathbb{R}. \quad (4.18)$$

Alternatively, the potentials Q and P can be recovered from the large k -asymptotics of the right Jost solution $g(k, x)$ as follows. We have [2] the asymptotics

$$g(k, x) = e^{kx} \left[1 + \frac{v_1(x)}{k} + \frac{v_2(x)}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_3}, \quad (4.19)$$

where we have let

$$v_1(x) := -\frac{1}{3} \int_{-\infty}^x dy Q(y), \quad x \in \mathbb{R}, \quad (4.20)$$

$$v_2(x) := \frac{1}{3} \int_{-\infty}^x dy [Q'(y) - P(y)] + \frac{1}{18} \left[\int_{-\infty}^x dy Q(y) \right]^2, \quad x \in \mathbb{R}. \quad (4.21)$$

Using (4.20) and (4.21), we recover Q and P as

$$Q(x) = -3 \frac{dv_1(x)}{dx}, \quad x \in \mathbb{R}, \quad (4.22)$$

$$P(x) = 3 \left[v_1(x) \frac{dv_1(x)}{dx} - \frac{d^2 v_1(x)}{dx^2} - \frac{dv_2(x)}{dx} \right], \quad x \in \mathbb{R}. \quad (4.23)$$

Having described how we recover the potentials Q and P from the Jost solutions $f(k, x)$ and $g(k, x)$, let us explain how to obtain those potentials from the solutions $\Phi_+(k, x)$ and $\Phi_-(k, x)$ to the Riemann–Hilbert problem posed in (4.5). Since we already know $T_1(k)$ in $\overline{\Omega_1}$ as a part of the input data set for our Riemann–Hilbert problem, we also know its large k -asymptotics given by

$$T_1(k) = 1 + \frac{t_{11}}{k} + \frac{t_{12}}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ in } \overline{\Omega_1}, \quad (4.24)$$

where t_{11} and t_{12} are some complex constants. From the first line of (4.3), with the help of (4.14) and (4.24) we obtain

$$e^{-kx} \Phi_+(k, x) = 1 + \frac{t_{11} + u_1(x)}{k} + \frac{t_{12} + t_{11} u_1(x) + u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ in } \overline{\Omega_1}. \quad (4.25)$$

Thus, we can obtain $u_1(x)$ and $u_2(x)$ from the asymptotics of $\Phi_+(k, x)$ as $k \rightarrow +\infty$ in $\overline{\Omega_1}$, and by using those quantities in (4.17) and (4.18) we recover Q and P .

In a similar manner, we can recover Q and P directly from the solution $\Phi_-(k, x)$ to the Riemann–Hilbert problem by proceeding as follows. From the first line of (4.4) we see that $\Phi_-(k, x)$ and $g(k, x)$ coincide when $k \in \overline{\Omega_3}$. Hence, with the help of (4.19) we have

$$e^{-kx} \Phi_-(k, x) = 1 + \frac{v_1(x)}{k} + \frac{v_2(x)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ in } \overline{\Omega_3}. \quad (4.26)$$

Since we already know how to recover Q and P from $g(k, x)$ via (4.22) and (4.23), the same procedure also describes the recovery of Q and P from the solution $\Phi_-(k, x)$ to the Riemann–Hilbert problem.

In the reflectionless case, i.e. when all the four reflection coefficients are zero, the Riemann–Hilbert problem posed in (4.2) reduces to

$$\Phi_+(k, x) = \Phi_-(k, x), \quad k \in \mathcal{L}, \quad (4.27)$$

as a result of the fact that the jump $J(k, x)$ given in (4.6) vanishes. This special case is important because the solution to the Riemann–Hilbert problem posed in (4.27) yields explicit closed-form solutions to various integrable evolution equations. We refer the reader to [2], where soliton solutions to certain integrable evolution equations, including the Sawada–Kotera equation [31] and a modified form of the bad Boussinesq [11] equation, are constructed by solving the Riemann–Hilbert problem (4.27) with an appropriate input data set consisting of the left transmission coefficient, the bound-state poles of the left transmission coefficient, and the bound-state dependency constants with the appropriate time evolution.

5. The Marchenko integral equation

In this section, when the secondary reflection coefficients $M(k)$ and $N(k)$ are both zero, we show how the Riemann–Hilbert problem formulation of the inverse scattering problem for (1.1) can be converted to a linear integral equation, which is the analog of the Marchenko integral equation [16] arising in the inverse scattering theory for the full-line Schrödinger equation. We derive the Marchenko equation for (1.1) in the absence of bound states. We postpone to a future paper the derivation of the Marchenko equation in the presence of bound states.

By multiplying both sides of (4.5) with e^{-kx} and subtracting 1 from each side, we obtain

$$e^{-kx} \Phi_+(k, x) - 1 = e^{-kx} \Phi_-(k, x) - 1 + e^{-kx} J(k, x), \quad k \in \mathcal{L}, \quad x \in \mathbb{R}. \quad (5.1)$$

Let us write (5.1) as

$$F(x, s) = G(x, s) + H(x, s), \quad s \in \mathbb{R}, \quad (5.2)$$

where we have defined

$$F(x, s) := e^{-zsx} \Phi_+(zs, x) - 1, \quad s \in \mathbb{R}, \quad (5.3)$$

$$G(x, s) := e^{-zsx} \Phi_-(zs, x) - 1, \quad s \in \mathbb{R}, \quad (5.4)$$

$$H(x, s) := e^{-zsx} J(zs, x), \quad s \in \mathbb{R}, \quad (5.5)$$

by recalling that we have $k = zs$ for $s \in \mathbb{R}$ when $k \in \mathcal{L}$. In the absence of bound states, we already know [2] that for each fixed $x \in \mathbb{R}$, the quantity $f(k, x)$ is analytic in $k \in \Omega_1$ and continuous in $k \in \overline{\Omega}_1$, and it behaves as $e^{kx}[1 + O(1/k)]$ as $k \rightarrow \infty$ in $\overline{\Omega}_1$. Similarly, the quantity $g(k, x)$ is analytic in $k \in \Omega_3$ and continuous in $k \in \overline{\Omega}_3$, and it behaves as $e^{kx}[1 + O(1/k)]$ as $k \rightarrow \infty$ in $\overline{\Omega}_3$. In the absence of bound states, the quantity $m(k, x)$ is analytic $k \in \Omega_2$ and continuous in $k \in \overline{\Omega}_2$, and it has the spectral asymptotics $e^{kx}[1 + O(1/k)]$ as $k \rightarrow \infty$ in $\overline{\Omega}_2$. Similarly, the quantity $n(k, x)$ is analytic in $k \in \Omega_4$ and continuous in $k \in \overline{\Omega}_4$, and it has the spectral asymptotics $e^{kx}[1 + O(1/k)]$ as $k \rightarrow \infty$ in $\overline{\Omega}_4$. In fact, those large k -asymptotics are already indicated in (2.14)–(2.17). Thus, for each fixed $x \in \mathbb{R}$ it follows that the quantity $F(x, s)$ is analytic in $s \in \mathbb{C}^+$ and continuous in $s \in \overline{\mathbb{C}^+}$, and we have $F(x, s) = O(1/s)$ as $s \rightarrow \infty$ in $s \in \overline{\mathbb{C}^+}$. Similarly, for each fixed $x \in \mathbb{R}$ it follows that the quantity $G(x, s)$ is analytic in $s \in \mathbb{C}^-$ and continuous in $s \in \overline{\mathbb{C}^-}$, and we have $G(x, s) = O(1/s)$ as $s \rightarrow \infty$ in $s \in \overline{\mathbb{C}^-}$. Consequently, the Fourier transforms in $L^2(\mathbb{R})$ with $s \in \mathbb{R}$ for $F(x, s)$ and $G(x, s)$ exist. From (5.2), it follows that the Fourier transform of $H(x, s)$ also exists. We define the Fourier transforms $\hat{F}(x, y)$ and $\hat{G}(x, y)$ by letting

$$\hat{F}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-isy} F(x, s), \quad y \in \mathbb{R}, \quad (5.6)$$

$$\hat{G}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-isy} G(x, s), \quad y \in \mathbb{R}. \quad (5.7)$$

From (5.6) and (5.7), respectively, we get the inverse Fourier transforms given by

$$F(x, s) = \int_{-\infty}^{\infty} dy e^{isy} \hat{F}(x, y), \quad s \in \mathbb{R}, \quad (5.8)$$

$$G(x, s) = \int_{-\infty}^{\infty} dy e^{isy} \hat{G}(x, y), \quad s \in \mathbb{R}. \quad (5.9)$$

Using the analyticity of $F(x, s)$ in $s \in \mathbb{C}^+$, we get

$$\hat{F}(x, y) = 0, \quad y < 0. \quad (5.10)$$

Similarly, using the analyticity of $G(x, s)$ in $s \in \mathbb{C}^-$, we have

$$\hat{G}(x, y) = 0, \quad y > 0. \quad (5.11)$$

Thus, we can write (5.8) and (5.9) in their respective equivalent forms as

$$F(x, s) = \int_0^{\infty} dy e^{isy} \hat{F}(x, y), \quad s \in \mathbb{R}, \quad (5.12)$$

$$G(x, s) = \int_{-\infty}^0 dy e^{isy} \hat{G}(x, y), \quad s \in \mathbb{R}. \quad (5.13)$$

Next, we are interested in the analysis of $H(x, s)$ appearing in (5.5) and its Fourier transform. With the help of (4.5), (4.6), (5.1) and (5.2), we see that

$$H(x, s) = \begin{cases} e^{-zsx} L(zs) T_l(z^2 s) f(z^2 s, x), & s > 0, \\ -\frac{R(zs) T_r(z^2 s) g(z^2 s, x)}{T_r(zs)}, & s < 0. \end{cases} \quad (5.14)$$

Using (5.3) and (5.4) in (5.14), we obtain

$$H(x, s) = \begin{cases} e^{(z^2 - z)sx} L(zs) [F(x, zs) + 1], & s > 0, \\ -\frac{e^{(z^2 - z)sx} R(zs) T_r(z^2 s) [G(x, zs) + 1]}{T_r(zs)}, & s < 0. \end{cases} \quad (5.15)$$

We introduce the scalar quantity $\rho(s)$ as

$$\rho(s) := \begin{cases} \rho_+(s) := L(zs), & s > 0, \\ \rho_-(s) := -\frac{R(zs) T_r(z^2 s)}{T_r(zs)}, & s < 0. \end{cases} \quad (5.16)$$

From (5.15) and (5.16), we have

$$H(x, s) = \begin{cases} e^{-\sqrt{3}isx} \rho_+(s) [F(x, zs) + 1], & s > 0, \\ e^{-\sqrt{3}isx} \rho_-(s) [G(x, zs) + 1], & s < 0. \end{cases} \quad (5.17)$$

In the absence of bound states, let us show that

$$F(x, zs) = 0, \quad s < 0, \quad (5.18)$$

$$G(x, zs) = 0, \quad s > 0. \quad (5.19)$$

For this, we proceed as follows. Since $F(x, \eta)$ is analytic in $\eta \in \mathbb{C}^+$, is continuous in $\eta \in \overline{\mathbb{C}^+}$, and behaves as $O(1/\eta)$ as $\eta \rightarrow \infty$ in $\eta \in \mathbb{C}^+$, we have

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\eta \frac{F(x, \eta)}{\eta - \alpha} = \begin{cases} F(x, \alpha), & \alpha \in \mathbb{C}^+, \\ 0, & \alpha \in \mathbb{C}^-. \end{cases} \quad (5.20)$$

We remark that (5.20) can be established by writing the integral over $\eta \in \mathbb{R}$ as an integral over the boundary of the upper-half complex η -plane and by using Jordan's lemma from the theory of complex variables [30]. Similarly, since $G(x, \eta)$ is analytic in $\eta \in \mathbb{C}^-$, is continuous in $\eta \in \overline{\mathbb{C}^-}$, and behaves as $O(1/\eta)$ as $\eta \rightarrow \infty$ in $\eta \in \mathbb{C}^-$, it follows that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\eta \frac{G(x, \eta)}{\eta - \alpha} = \begin{cases} 0, & \alpha \in \mathbb{C}^+, \\ -G(x, \alpha), & \alpha \in \mathbb{C}^-. \end{cases} \quad (5.21)$$

One can establish (5.21) by writing the integral over $\eta \in \mathbb{R}$ as an integral over the boundary of the lower-half of the complex η -plane and by using Jordan's lemma. From (2.1) it follows that $zs \in \mathbb{C}^+$ when $s > 0$ and that $zs \in \mathbb{C}^-$ when $s < 0$. Thus, (5.20) yields (5.18) by choosing $\alpha = zs$ in (5.20) with $s > 0$. Similarly, (5.21) yields (5.19) by choosing $\alpha = zs$ in (5.21) with $s < 0$.

Having established (5.18) and (5.19), we proceed as follows. Using (5.16), (5.18), and (5.19) in (5.17), we get

$$H(x, s) = e^{-\sqrt{3}isx} \rho(s) + [F(x, zs) + G(x, zs)] e^{-\sqrt{3}isx} \rho(s), \quad s \in \mathbb{R}. \quad (5.22)$$

We use $\hat{\rho}(y)$ to denote the Fourier transform of $\rho(s)$ defined in (5.16), i.e. we let

$$\hat{\rho}(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-isy} \rho(s), \quad y \in \mathbb{R}. \quad (5.23)$$

From (5.23), by using the inverse Fourier transform we obtain

$$\rho(s) = \int_{-\infty}^{\infty} dy e^{isy} \hat{\rho}(y), \quad s \in \mathbb{R}.$$

From (5.16) and (5.23), we see that

$$\hat{\rho}(y) = \hat{\rho}_+(y) + \hat{\rho}_-(y), \quad y \in \mathbb{R},$$

where we have let

$$\hat{\rho}_+(y) := \frac{1}{2\pi} \int_0^{\infty} ds e^{-isy} \rho_+(s), \quad y \in \mathbb{R}, \quad (5.24)$$

$$\hat{\rho}_-(y) := \frac{1}{2\pi} \int_{-\infty}^0 ds e^{-isy} \rho_-(s), \quad y \in \mathbb{R}. \quad (5.25)$$

With the help of (5.6), (5.7), (5.22), (5.23), we obtain

$$\hat{H}(x, y) = \hat{\rho}(\sqrt{3}x + y) + \int_{-\infty}^{\infty} d\zeta [\hat{F}(x, \zeta) + \hat{G}(x, \zeta)] \hat{\rho}(\sqrt{3}x - z\zeta + y), \quad y \in \mathbb{R}, \quad (5.26)$$

where we have let

$$\hat{H}(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-isy} H(x, s), \quad y \in \mathbb{R}.$$

With the help of (5.6), (5.7), and (5.26), we take the Fourier transform of both sides of (5.2), and we obtain

$$\begin{aligned} \hat{F}(x, y) &= \hat{G}(x, y) + \hat{\rho}(\sqrt{3}x + y) \\ &\quad + \int_{-\infty}^{\infty} d\zeta [\hat{F}(x, \zeta) + \hat{G}(x, \zeta)] \hat{\rho}(\sqrt{3}x - z\zeta + y), \quad y \in \mathbb{R}. \end{aligned} \quad (5.27)$$

Using (5.10) and (5.11) in (5.27), we get

$$\begin{cases} \hat{F}(x, y) = \hat{\rho}(\sqrt{3}x + y) + \int_0^{\infty} d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) \\ \quad + \int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y), & y > 0, \\ 0 = \hat{G}(x, y) + \hat{\rho}(\sqrt{3}x + y) + \int_0^{\infty} d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) \\ \quad + \int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y), & y < 0. \end{cases} \quad (5.28)$$

We would like to simplify the integral terms in (5.28) by replacing $\hat{\rho}(\eta)$ with $\hat{\rho}_+(\eta)$ and $\hat{\rho}_-(\eta)$ when $\eta > 0$ and $\eta < 0$, respectively, where we recall that $\hat{\rho}_+(y)$ and $\hat{\rho}_-(y)$ are the quantities defined in (5.24) and (5.25), respectively. In other words, we would like to show that

$$\int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}_+(\sqrt{3}x - z\zeta + y), \quad y \in \mathbb{R}, \quad (5.29)$$

$$\int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \hat{\rho}_-(\sqrt{3}x - z\zeta + y), \quad y \in \mathbb{R}. \quad (5.30)$$

For the proof of (5.29) we proceed as follows. With the help of (5.23), we write the left-hand side of (5.29) as

$$\begin{aligned} & \int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) \\ &= \int_0^\infty d\zeta \hat{F}(x, \zeta) \int_{-\infty}^\infty \frac{ds}{2\pi} e^{-is(\sqrt{3}x - z\zeta + y)} \rho(s), \quad y \in \mathbb{R}. \end{aligned} \quad (5.31)$$

By changing the order of integration on the right-hand side of (5.31), we get

$$\begin{aligned} & \int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) \\ &= \int_{-\infty}^\infty \frac{ds}{2\pi} e^{-is(\sqrt{3}x + y)} \rho(s) \int_0^\infty d\zeta \hat{F}(x, \zeta) e^{isz\zeta}, \quad y \in \mathbb{R}. \end{aligned} \quad (5.32)$$

Using (5.12) on the right-hand side of (5.32), we obtain

$$\int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_{-\infty}^\infty ds e^{-is(\sqrt{3}x + y)} \rho(s) F(x, zs), \quad y \in \mathbb{R}. \quad (5.33)$$

From (5.18), we know that $F(x, zs)$ vanishes when $s < 0$. Hence, we can write (5.33) as

$$\int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_0^\infty ds e^{-is(\sqrt{3}x + y)} \rho(s) F(x, zs), \quad y \in \mathbb{R}. \quad (5.34)$$

Using (5.16) and (5.24) on the right-hand side of (5.34), we get

$$\int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_0^\infty ds e^{-is(\sqrt{3}x + y)} \rho_+(s) F(x, zs), \quad y \in \mathbb{R}. \quad (5.35)$$

Since $F(x, zs) = 0$ for $s < 0$, we can write the right-hand side of (5.35) over the integral on $s \in \mathbb{R}$, and hence we get

$$\int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_{-\infty}^\infty ds e^{-is(\sqrt{3}x + y)} \rho_+(s) F(x, zs), \quad y \in \mathbb{R}. \quad (5.36)$$

Finally, comparing (5.36) with (5.33), we conclude that

$$\int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}(\sqrt{3}x - z\zeta + y) = \int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}_+(\sqrt{3}x - z\zeta + y), \quad y \in \mathbb{R},$$

which coincides with (5.29). The proof of (5.30) is obtained in a similar manner by exploiting the fact that $G(x, zs) = 0$ when $s > 0$, which is stated in (5.19). Thus, using (5.29) and (5.30) in (5.28), we obtain the Marchenko integral equation for (1.1) as

$$\begin{cases} \hat{F}(x, y) = \hat{\rho}_+ \left(\sqrt{3}x + y \right) + \int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}_+ \left(\sqrt{3}x - z\zeta + y \right) \\ \quad + \int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \hat{\rho}_- \left(\sqrt{3}x - z\zeta + y \right), & y > 0, \\ 0 = \hat{G}(x, y) + \hat{\rho}_- \left(\sqrt{3}x + y \right) + \int_0^\infty d\zeta \hat{F}(x, \zeta) \hat{\rho}_+ \left(\sqrt{3}x - z\zeta + y \right) \\ \quad + \int_{-\infty}^0 d\zeta \hat{G}(x, \zeta) \hat{\rho}_- \left(\sqrt{3}x - z\zeta + y \right), & y < 0. \end{cases} \quad (5.37)$$

Having obtained the Marchenko integral equation for (1.1) given in (5.37), we now show how to recover the potentials Q and P from the solution $\hat{F}(x, y)$ to the Marchenko equation. For this, we proceed as follows. From (5.12) we get

$$F(x, s) = \int_0^\infty dy \hat{F}(x, y) \frac{d}{dy} \left(\frac{e^{isy}}{is} \right). \quad (5.38)$$

Using integration by parts on the right-hand side of (5.38), we obtain

$$F(x, s) = \hat{F}(x, y) \frac{e^{isy}}{is} \Big|_{y=0^+}^{+\infty} - \frac{1}{is} \int_0^\infty dy \hat{F}_y(x, y) e^{isy}, \quad (5.39)$$

where the subscript y denotes the partial derivative with respect to y . From (5.39) we get

$$F(x, s) = -\frac{1}{is} \hat{F}(x, 0^+) - \frac{1}{is} \int_0^\infty dy \hat{F}_y(x, y) e^{isy}, \quad s \in \mathbb{R}. \quad (5.40)$$

A second integration by parts on the right-hand side of (5.40) yields

$$F(x, s) = -\frac{1}{is} \hat{F}(x, 0^+) - \frac{1}{s^2} \hat{F}_y(x, 0^+) + O\left(\frac{1}{s^3}\right), \quad s \rightarrow \pm\infty. \quad (5.41)$$

We recall that we have $k = zs$ for $s \in \mathbb{R}$ when $k \in \mathcal{L}$, and hence we can write (5.41) as

$$e^{-kx} \Phi_+(k, x) - 1 = -\frac{1}{iz^2k} \hat{F}(x, 0^+) - \frac{1}{zk^2} \hat{F}_y(x, 0^+) + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ on } \mathcal{L}_1, \quad (5.42)$$

where we have used (5.3) on the left-hand side and used $z^4 = z$ on the right-hand side. In fact, the asymptotics given in (5.42) holds when $k \in \overline{\Omega}_1$. Comparing (5.42) with (4.25), we obtain

$$\begin{cases} t_{11} + u_1(x) = -\frac{1}{iz^2} \hat{F}(x, 0^+), \\ t_{12} + t_{11} u_1(x) + u_2(x) = -\frac{1}{z} \hat{F}_y(x, 0^+), \end{cases} \quad (5.43)$$

where we recall that t_{11} and t_{12} are the constants appearing in (4.24). By solving the linear algebraic system (5.43) for $u_1(x)$ and $u_2(x)$, we obtain

$$u_1(x) = -t_{11} + iz\hat{F}(x, 0^+), \quad x \in \mathbb{R}, \quad (5.44)$$

$$u_2(x) = -t_{12} + t_{11}^2 - izt_{11}\hat{F}(x, 0^+) - z^2\hat{F}_y(x, 0^+), \quad x \in \mathbb{R}. \quad (5.45)$$

Finally, using (5.44) and (5.45) in (4.24) and (4.25) we recover the potentials Q and P from the solution $\hat{F}(x, y)$ to the Marchenko equation as

$$Q(x) = -3iz \frac{d\hat{F}(x, 0^+)}{dx}, \quad x \in \mathbb{R},$$

$$P(x) = -3z^2\hat{F}(x, 0^+) \frac{d\hat{F}(x, 0^+)}{dx} - 3iz \frac{d^2\hat{F}(x, 0^+)}{dx^2} + 3z^2 \frac{d\hat{F}_y(x, 0^+)}{dx}, \quad x \in \mathbb{R}.$$

Alternatively, we can recover the potentials Q and P from the solution $\hat{G}(x, y)$ to the Marchenko equation. For this, we proceed as follows. From (5.13) we get

$$G(x, s) = \int_{-\infty}^0 dy \hat{G}(x, y) \frac{d}{dy} \left(\frac{e^{isy}}{is} \right). \quad (5.46)$$

Using integration by parts on the right-hand side of (5.46), we obtain

$$G(x, s) = \hat{G}(x, y) \frac{e^{isy}}{is} \Big|_{y=-\infty}^{0^-} - \frac{1}{is} \int_{-\infty}^0 dy \hat{G}_y(x, y) e^{isy}, \quad s \in \mathbb{R},$$

or equivalently

$$G(x, s) = \frac{1}{is} \hat{G}(x, 0^-) - \frac{1}{is} \int_{-\infty}^0 dy \hat{G}_y(x, y) e^{isy}, \quad s \in \mathbb{R}. \quad (5.47)$$

Another integration by parts on the right-hand side of (5.47) yields

$$G(x, s) = \frac{1}{is} \hat{G}(x, 0^-) + \frac{1}{s^2} \hat{G}_y(x, 0^-) + O\left(\frac{1}{s^3}\right), \quad s \rightarrow \pm\infty. \quad (5.48)$$

Since the equality $k = zs$ for $s \in \mathbb{R}$ is equivalent to having $k \in \mathcal{L}$, from (5.48) we obtain

$$e^{-kx} \Phi_-(k, x) - 1 = \frac{1}{iz^2k} \hat{G}(x, 0^-) + \frac{1}{zk^2} \hat{G}_y(x, 0^-) + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ on } \mathcal{L}_3, \quad (5.49)$$

where we have used $z^4 = z$ on the right-hand side. In fact, the asymptotic expression given in (5.49) holds when $k \in \overline{\Omega}_3$. Comparing (5.49) with (4.26), we obtain

$$v_1(x) = -iz\hat{G}(x, 0^-), \quad x \in \mathbb{R}, \quad (5.50)$$

$$v_2(x) = z^2\hat{G}_y(x, 0^-), \quad x \in \mathbb{R}. \quad (5.51)$$

Finally, using (5.50) and (5.51) in (4.22) and (4.23) we recover the potentials Q and P as

$$Q(x) = 3iz \frac{d\hat{G}(x, 0^-)}{dx}, \quad x \in \mathbb{R},$$

$$P(x) = -3z^2\hat{G}(x, 0^-) \frac{d\hat{G}(x, 0^-)}{dx} + 3iz \frac{d^2\hat{G}(x, 0^-)}{dx^2} - 3z^2 \frac{d\hat{G}_y(x, 0^-)}{dx}, \quad x \in \mathbb{R}.$$

6. Conclusion

In this paper we present a solution method for an inverse scattering problem associated with the third-order equation (1.1). This method is the counterpart of the Marchenko method used to solve the inverse scattering problem for the full-line Schrödinger equation. We establish the Marchenko equation for (1.1), which is a linear integral equation using the appropriate scattering data as input to the integral kernel and the nonhomogeneous term. The two potentials Q and P in (1.1) are recovered from the solution to the established Marchenko integral equation.

The study of direct and inverse scattering problems for (1.1) was pioneered by Kaup [21]. Kaup described the establishment of a Marchenko integral equation for (1.1) as a ‘hard’ and ‘unsolved’ problem [22]. We bring a solution to that open problem by establishing a Riemann–Hilbert problem differing from a standard Riemann–Hilbert problem in several ways. In our method, the reflection and transmission coefficients are separated from each other so that the jump factor in the Riemann–Hilbert problem is determined by the primary reflection coefficients. Our Riemann–Hilbert problem involves the determination of a sectionally meromorphic function rather than a sectionally analytic function. By choosing the secondary reflection coefficients equal to zero, our Riemann–Hilbert problem is formulated on a single full line separating the complex plane into two regions. This allows the use of a Fourier transformation along the aforementioned full line, and it yields the establishment of the Marchenko integral equation for (1.1). Thus, our method of establishing a Marchenko integral equation is not a straightforward extension of existing work.

The innovative approach used in our paper allows the construction of soliton solutions to the integrable nonlinear systems of partial differential equations associated with (1.1), such as the Sawada–Kotera equation, the Kaup–Kupershmidt equation, and a modified version of the Boussinesq equation. Based on the Marchenko method developed for (1.1) in this paper, in a different publication in the near future we plan to present the corresponding Marchenko integral equations for the N -soliton solutions to the three aforementioned nonlinear evolution equations using as input the bound-state poles of a transmission coefficient and the corresponding time-evolved bound-state dependency constants.

Data availability statement

No new data were created or analysed in this study.

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