

Direct and inverse scattering problems for a first-order system with energy-dependent potentials

T Aktosun¹  and R Ercan

Department of Mathematics, University of Texas at Arlington, Arlington,
TX 76019-0408, United States of America

E-mail: aktosun@uta.edu

Received 13 December 2018, revised 1 March 2019

Accepted for publication 9 May 2019

Published 30 July 2019



CrossMark

Abstract

The direct and inverse scattering problems on the full line are analyzed for a first-order system of ordinary linear differential equations associated with the derivative nonlinear Schrödinger equation and related equations. The system contains a spectral parameter and two potentials, where the potentials are proportional to the spectral parameter and hence are called energy-dependent potentials. Using the two potentials as input, the direct problem is solved by determining the scattering coefficients and the bound-state information consisting of bound-state energies, their multiplicities, and the corresponding norming constants. Using two different methods, the corresponding inverse problem is solved by determining the two potentials when the scattering data set is used as input. The first method involves the transformation of the energy-dependent system into two distinct energy-independent systems. The second method involves the establishment of the so-called alternate Marchenko system of linear integral equations and the recovery of the energy-dependent potentials from the solution to the alternate Marchenko system.

Keywords: alternate Marchenko system, inverse scattering for first-order systems, energy-dependent potentials

1. Introduction

In this paper we are interested in analyzing the direct and inverse scattering problems for the first-order system

$$\frac{d}{dx} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\zeta^2 & \zeta q(x) \\ \zeta r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad x \in \mathbb{R}, \quad (1.1)$$

¹ Author to whom any correspondence should be addressed.

where ζ is the spectral parameter, x is the spatial coordinate, \mathbb{R} denotes the real axis $(-\infty, +\infty)$, the scalar-valued potentials $q(x)$ and $r(x)$ belong to the Schwartz class, and $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is the wavefunction depending on ζ and x . Because of the appearance of the spectral parameter ζ in front of the potentials $q(x)$ and $r(x)$ in (1.1), the system (1.1) is referred to as an energy-dependent system. We remark that $q(x)$ and $r(x)$ are not necessarily real valued.

As indicated in section 3, the system (1.1) is associated with the first-order AKNS system [1–3]

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad x \in \mathbb{R}, \quad (1.2)$$

where the prime denotes the x -derivative, the spectral parameter λ is related to ζ as

$$\lambda = \zeta^2, \quad (1.3)$$

and the potentials $u(x)$ and $v(x)$ are related to $q(x)$ and $r(x)$ appearing in (1.1) as

$$u(x) := q(x) E^{-2}, \quad (1.4)$$

$$v(x) := \left(-\frac{i}{2} r'(x) + \frac{1}{4} q(x) r(x)^2 \right) E^2, \quad (1.5)$$

with E being the function of x given by

$$E := \exp \left(\frac{i}{2} \int_{-\infty}^x dy q(y) r(y) \right). \quad (1.6)$$

For notational simplicity, we suppress the dependence of E on x .

As also indicated in section 3, the system (1.1) is associated with another energy-independent AKNS system, namely

$$\begin{bmatrix} \gamma \\ \epsilon \end{bmatrix}' = \begin{bmatrix} -i\lambda & p(x) \\ s(x) & i\lambda \end{bmatrix} \begin{bmatrix} \gamma \\ \epsilon \end{bmatrix}, \quad x \in \mathbb{R}, \quad (1.7)$$

where the potentials $p(x)$ and $s(x)$ are related to $q(x)$ and $r(x)$ as

$$p(x) := \left(\frac{i}{2} q'(x) + \frac{1}{4} q(x)^2 r(x) \right) E^{-2}, \quad (1.8)$$

$$s(x) := r(x) E^2. \quad (1.9)$$

In contrast with (1.1), we refer to (1.2) and (1.7) as the energy-independent systems.

The goal in the direct scattering problem for (1.1) is, given the potentials $q(x)$ and $r(x)$, to determine the scattering coefficients associated with (1.1) and also to determine the bound-state information related to (1.1). The inverse problem for (1.1) is to determine the potentials $q(x)$ and $r(x)$ in terms of the input data set containing the scattering coefficients and the bound-state information for (1.1). The direct and inverse scattering problems for the energy-independent systems (1.2) and (1.7) are well understood [1–3, 16] when the potentials $u(x)$, $v(x)$, $p(x)$, and $s(x)$ belong to the Schwartz class and satisfy the additional restriction that each bound state is simple (i.e. the assumption that there is only one linearly independent square-integrable solution for each λ -value corresponding to a bound state of (1.2) and of (1.7)). By using the method of [4, 5, 9], the bound states with any multiplicities can easily and in an

elegant way be handled, and hence in our paper we do not make the additional assumption that the bound states are simple.

Our goal in this paper is to solve the direct and inverse scattering problems for (1.1) with the help of the solutions to the direct and inverse scattering problems for (1.2) and (1.7). We accomplish our goal by establishing the appropriate transformations among certain particular solutions known as the Jost solutions, the scattering coefficients, and the bound-state information for (1.1), (1.2), and (1.7). The bound states for these three systems are not necessarily simple, and the multiplicities of the bound states and the corresponding norming constants are treated in our paper without assuming simplicity of bound states.

The research presented is based on the second author's doctoral thesis [10] under the supervision of the first author. Our paper is organized as follows. In section 2 we establish our notation and provide the preliminaries needed for our analysis in later sections. We describe the scattering data sets for the energy-independent systems (1.2) and (1.7), and this is done by paying particular attention to the bound states and to their multiplicities and by expressing the corresponding bound-state information in terms of certain triplets of constant matrices, whereas in the main references [1–3, 16] the bound states are assumed to be simple. We present a brief summary of the solution to the corresponding inverse problems via the so-called Marchenko method. In section 3 the energy-dependent system (1.1) is transformed into the two energy-independent systems (1.2) and (1.7). The transformations are presented relating the Jost solutions for (1.1) to the Jost solutions for (1.2) and for (1.7). In section 4 a brief description is presented for the determination of the scattering data set for (1.1) when the potentials $q(x)$ and $r(x)$ appearing in (1.1) are used as input. In section 5 the inverse problem for (1.1) is solved by exploiting the relations among the scattering data sets for (1.1), (1.2) and (1.7). In section 6 the same inverse problem is solved by using a method different from the one used in section 5. This is done by deriving a system of uncoupled linear integral equations, which we refer to as the alternate Marchenko system. The scattering data set for (1.1) is used as input to the alternate Marchenko system, and the potentials $q(x)$ and $r(x)$ are recovered from the solution to the alternate Marchenko system.

When the potentials $q(x)$ and $r(x)$ appearing in (1.1) depend on an additional parameter t , which can be interpreted as the time parameter, the system (1.1) is replaced with

$$\frac{d}{dx} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\zeta^2 & \zeta q(x,t) \\ \zeta r(x,t) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad x \in \mathbb{R}, \quad t > 0. \quad (1.10)$$

Via the inverse scattering transform, the system (1.10) is related to the system of integrable evolution equations [1, 3, 15]

$$\begin{cases} iq_t + q_{xx} - i(qrq)_x = 0, \\ ir_t - r_{xx} - i(rqr)_x = 0, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \quad (1.11)$$

which is known as the derivative NLS (nonlinear Schrödinger) system. Note that the subscripts in (1.11) denote the corresponding partial derivatives. The special case of (1.11) when $r(x) = \mp q(x)^*$, where the asterisk denotes complex conjugation, corresponds to the derivative NLS equation [1, 3, 14]

$$iq_t + q_{xx} \pm i(q|q|^2)_x = 0. \quad (1.12)$$

The initial-value problem for (1.12) was solved by Kaup and Newell [14] by using the method of inverse scattering transform. Tsuchida developed [15] a method to solve the initial-value problem for (1.11) by deriving an uncoupled system of two linear integral equations resembling our alternate Marchenko system.

Our emphasis in this paper is on the scattering and inverse scattering problems for the linear system (1.1) and not on the integrable nonlinear system (1.11) or its special case (1.12), whereas the emphasis in [14] and in [15] is on the solutions to the integrable systems given in (1.12) and (1.11), respectively. The most relevant references to our study are [14] and [15]. The transformation of the energy-dependent system (1.1) into the energy-independent systems (1.2) and (1.7) resembles the conversion of the energy-dependent Schrödinger equation

$$\frac{d^2\Psi(k,x)}{dx^2} + k^2 n(x)^2 \Psi = 0, \quad x \in \mathbb{R}, \quad (1.13)$$

into the standard Schrödinger equation

$$\frac{d^2\Phi(k,y)}{dy^2} + k^2 \Phi(k,y) = V(y) \Phi(k,y) = 0, \quad y \in \mathbb{R}, \quad (1.14)$$

by using the Liouville transformation

$$y := \int_0^x dz n(z), \quad \Phi(k,y) := \sqrt{n(x)} \Psi(k,x),$$

$$V(y) := \frac{n''(x)}{2n(x)^3} - \frac{3n'(x)^2}{4n(x)^3}.$$

When $n(x) > 0$, it is known [8] that there are no bound states associated with (1.13) or (1.14), whereas the bound states and their multiplicities are relevant for (1.1), (1.2), and (1.7). Furthermore, the independent variable x remains unchanged when (1.1) is transformed into (1.2) and (1.7), whereas the independent variable x in (1.13) is different from the independent variable y in (1.14).

Let us also mention [11], where Guillaume and Jaulent considered the generalization of (1.1) given by

$$\frac{d}{dx} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\zeta^2 & \sum_{j=0}^n \zeta^{j/n} q_j(x) \\ \sum_{j=0}^n \zeta^{j/n} r_j(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad x \in \mathbb{R}, \quad (1.15)$$

where $q_j(x)$ and $r_j(x)$ for $j = 0, 1, \dots, n$ are some scalar-valued functions. Note that when $n = 1$, $q_0(x) \equiv 0$, and $r_0(x) \equiv 0$ in (1.15), one obtains (1.1). In [11], assuming that the bound states are all simple, the inverse scattering problem for (1.15) is analyzed using the scattering data for (1.15) as well as the scattering data for the related system obtained by replacing ζ^2 in (1.15) by $-\zeta^2$. The method used in [11] is a generalization of the method used to analyze the inverse scattering problem for the energy-dependent Schrödinger equation

$$\frac{d^2\Psi(k,x)}{dx^2} + k^2 \Psi = [ikP(x) + Q(x)] \Psi(k,x), \quad x \in \mathbb{R}, \quad (1.16)$$

by using the scattering data for (1.16) as well as the scattering data for the related system obtained by replacing k in (1.16) by $-k$. For the analysis of the inverse scattering problem for (1.16) when $P(x)$ is purely imaginary and $Q(x)$ is real, we refer the reader to [12, 13] and refer to [7] when $P(x)$ and $Q(x)$ are both real valued. For further information on the scattering problem for (1.16) and further related references, the reader is referred to [6], where it was also shown by explicit examples that there may be bound states with multiplicities.

2. Preliminaries

In this section we establish our notation and provide the preliminaries needed in later sections. This is done by presenting a brief summary of the solution to the direct and inverse problems associated with the energy-independent systems (1.2) and (1.7), as these are needed to solve the corresponding inverse problem for (1.1). We state the results mainly for (1.2), as the results for (1.7) can be stated in a similar way. A proof is omitted because it can be found in [2] under the further assumption of simplicity of bound states, and the bound states with multiplicities can be handled by the method of [4, 5, 9].

Theorem 2.1. *Assume that the potentials $u(x)$ and $v(x)$ appearing in (1.2) belong to the Schwartz class. We then have the following:*

(a) *The system (1.2) has the so-called Jost solutions ψ , ϕ , $\bar{\psi}$, $\bar{\phi}$ satisfying the respective asymptotics*

$$\psi(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \lambda \in \mathbb{R}, \quad (2.1)$$

$$\phi(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \lambda \in \mathbb{R}, \quad (2.2)$$

$$\bar{\psi}(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \lambda \in \mathbb{R}, \quad (2.3)$$

$$\bar{\phi}(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \lambda \in \mathbb{R}. \quad (2.4)$$

We remark that an overbar does not denote complex conjugation. We further remark that our definition of the Jost solution $\bar{\phi}$ differs by a minus sign from that used in [2].

(b) *The six scattering coefficients T , R , L , \bar{T} , \bar{R} , \bar{L} associated with (1.2) are obtained from the large x -asymptotics of the Jost solutions as*

$$\psi(\lambda, x) = \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \lambda \in \mathbb{R}, \quad (2.5)$$

$$\phi(\lambda, x) = \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \lambda \in \mathbb{R}, \quad (2.6)$$

$$\bar{\psi}(\lambda, x) = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \lambda \in \mathbb{R}, \quad (2.7)$$

$$\bar{\phi}(\lambda, x) = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ 1 \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \lambda \in \mathbb{R}. \quad (2.8)$$

We remark that T and \bar{T} are known as the transmission coefficients, R and \bar{R} as the reflection coefficients from the right, and L and \bar{L} as the reflection coefficients from the left.

(c) The scattering coefficients are not all independent, and in fact we have

$$L(\lambda) = -\frac{\bar{R}(\lambda)T(\lambda)}{\bar{T}(\lambda)}, \quad \bar{L}(\lambda) = -\frac{R(\lambda)\bar{T}(\lambda)}{T(\lambda)}, \quad \lambda \in \mathbb{R}. \quad (2.9)$$

(d) The transmission coefficient $T(\lambda)$ has a meromorphic extension from the real λ -axis to the upper-half complex plane \mathbb{C}^+ . Similarly, the transmission coefficient $\bar{T}(\lambda)$ has a meromorphic extension from the real λ -axis to the lower-half complex plane \mathbb{C}^- . Furthermore, we have

$$T(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty \quad \text{in} \quad \lambda \in \overline{\mathbb{C}^+}, \quad (2.10)$$

$$\bar{T}(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty \quad \text{in} \quad \lambda \in \overline{\mathbb{C}^-}, \quad (2.11)$$

where we have defined $\overline{\mathbb{C}^\pm} := \mathbb{C}^\pm \cup \mathbb{R}$.

(e) The bound states for (1.2), i.e. column-vector solutions to (1.2) that are square integrable in $x \in \mathbb{R}$, occur at those λ -values at which $T(\lambda)$ has poles in \mathbb{C}^+ and at those λ -values at which $\bar{T}(\lambda)$ has poles in \mathbb{C}^- . The number of bound states is finite, and each bound state is not necessarily simple but has a finite multiplicity. We use $\{\lambda_j\}_{j=1}^N$ to denote the set of poles of $T(\lambda)$ in \mathbb{C}^+ , and we use m_j to denote the multiplicity of λ_j , i.e. we assume that the number of linearly-independent square-integrable column-vector solutions to (1.2) when $\lambda = \lambda_j$ is equal to m_j . Thus, the nonnegative integer N denotes the number of bound states associated with $T(\lambda)$ without counting multiplicities. In a similar way, we use $\{\bar{\lambda}_j\}_{j=1}^{\bar{N}}$ to denote the set of poles of $\bar{T}(\lambda)$ in \mathbb{C}^- , and we assume that each $\bar{\lambda}_j$ has multiplicity \bar{m}_j , i.e. the number of linearly-independent square-integrable column-vector solutions to (1.2) when $\lambda = \bar{\lambda}_j$ is equal to \bar{m}_j . Thus, the nonnegative integer \bar{N} denotes the number of bound states associated with $\bar{T}(\lambda)$ without counting multiplicities.

For the system (1.2), associated with each bound state at $\lambda = \lambda_j$ we have the bound-state norming constants c_{jk} , where a double index is used as a subscript with $j = 1, \dots, N$ and $k = 0, 1, \dots, m_j - 1$. Similarly, associated with each bound state at $\lambda = \bar{\lambda}_j$ we have the bound-state norming constants \bar{c}_{jk} , where $j = 1, \dots, \bar{N}$ and $k = 0, 1, \dots, \bar{m}_j - 1$. We refer the reader to (4.29) and (4.49) of [9] for the definitions of and elaborations for c_{jk} and \bar{c}_{jk} , respectively.

We recall that the direct scattering problem for (1.2) consists of the determination of the set of scattering coefficients

$$\left\{ T(\lambda), R(\lambda), L(\lambda), \bar{T}(\lambda), \bar{R}(\lambda), \bar{L}(\lambda) \right\}, \quad (2.12)$$

as well as the bound-state information

$$\left\{ \lambda_j, \{c_{jk}\}_{k=0}^{m_j-1} \right\}_{j=1}^N, \quad \left\{ \bar{\lambda}_j, \{\bar{c}_{jk}\}_{k=0}^{\bar{m}_j-1} \right\}_{j=1}^{\bar{N}}, \quad (2.13)$$

when the potentials $u(x)$ and $v(x)$ are given.

The solution to the direct scattering problem for (1.2) can be obtained as follows:

- Using the potentials $u(x)$ and $v(x)$ appearing in (1.2), when they belong to the Schwartz class, we uniquely determine the corresponding Jost solutions $\psi, \phi, \bar{\psi}, \bar{\phi}$ satisfying the asymptotics (2.1)–(2.4). The existence and uniqueness of the Jost solutions are established by converting (1.2) and (2.1)–(2.4) into the corresponding integral equations for those Jost solutions and then by solving those integral equations iteratively by representing their solutions as uniformly convergent infinite series.
- From the respective asymptotics (2.5)–(2.8) of the four constructed Jost solutions, we recover the scattering coefficients appearing in the set described in (2.12).
- Since the transmission coefficient $T(\lambda)$ has a meromorphic extension from \mathbb{R} to \mathbb{C}^+ and the transmission coefficient $\bar{T}(\lambda)$ has a meromorphic extension from \mathbb{R} to \mathbb{C}^- , by using the respective poles and multiplicities of those poles we determine $N, \bar{N}, \{\lambda_j\}_{j=1}^N, \{\bar{\lambda}_j\}_{j=1}^{\bar{N}}$ as well as the multiplicities m_j and \bar{m}_j .
- Having obtained the Jost solutions $\psi, \phi, \bar{\psi}, \bar{\phi}$, and the transmission coefficients $T(\lambda)$ and $\bar{T}(\lambda)$, we determine the norming constants c_{jk} and \bar{c}_{jk} appearing in (2.13) as described by (4.29) and (4.49) of [9].

Having outlined the solution to the direct scattering problem for (1.2), let us turn our attention to the corresponding inverse scattering problem. The inverse problem for (1.2) consists of the determination of the potentials $u(x)$ and $v(x)$ in terms of the input data consisting of the corresponding sets appearing in (2.12) and (2.13). As a result of (2.9), we can omit the left reflection coefficients $L(\lambda)$ and $\bar{L}(\lambda)$ from the set in (2.12). A solution to this inverse problem can be given as follows:

- Using the quantities appearing in (2.12) and (2.13), we form the 2×2 matrix-valued function $F(y)$ given by

$$F(y) := \begin{bmatrix} \mathbf{0} & \bar{\Omega}(y) \\ \Omega(y) & \mathbf{0} \end{bmatrix}, \quad (2.14)$$

where

$$\Omega(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda R(\lambda) e^{i\lambda y} + C e^{-Ay} B, \quad (2.15)$$

$$\bar{\Omega}(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \bar{R}(\lambda) e^{-i\lambda y} + \bar{C} e^{-\bar{A}y} \bar{B}. \quad (2.16)$$

Here $R(\lambda)$ and $\bar{R}(\lambda)$ are the reflection coefficients from the right appearing in (2.12) and

$$C_j := [c_{j(m_j-1)} \quad c_{j(m_j-2)} \quad \cdots \quad c_{j1} \quad c_{j0}], \quad (2.17)$$

$$\bar{C}_j := [\bar{c}_{j(\bar{m}_j-1)} \quad \bar{c}_{j(\bar{m}_j-2)} \quad \cdots \quad \bar{c}_{j1} \quad \bar{c}_{j0}], \quad (2.18)$$

where we observe that C_j is a row vector with m_j components and \bar{C}_j is a row vector with \bar{m}_j components. The square matrices A and \bar{A} and the column vectors B and \bar{B} appearing in (2.15) and (2.16) are constructed as follow. We first form the square matrices A_j and \bar{A}_j , with respective sizes $m_j \times m_j$ and $\bar{m}_j \times \bar{m}_j$, as

$$A_j := \begin{bmatrix} -i\lambda_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & -i\lambda_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & -i\lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -i\lambda_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & -i\lambda_j \end{bmatrix}, \quad (2.19)$$

$$\bar{A}_j := \begin{bmatrix} i\bar{\lambda}_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & i\bar{\lambda}_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & i\bar{\lambda}_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & i\bar{\lambda}_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & i\bar{\lambda}_j \end{bmatrix}. \quad (2.20)$$

We note that A_j and \bar{A}_j are in Jordan canonical forms. Next, we construct the column vector B_j so that its first $(m_j - 1)$ components are all zero and its last component, i.e. the m_j th component, is equal to one. Similarly, we construct the column vector \bar{B}_j so that its first $(\bar{m}_j - 1)$ components are all zero and its last component, i.e. the \bar{m}_j th component, is equal to one. Thus, we have

$$B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad j = 1, \dots, N, \quad (2.21)$$

$$\bar{B}_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad j = 1, \dots, \bar{N}. \quad (2.22)$$

The block matrices A and \bar{A} , the block row vectors C and \bar{C} , and the block column vectors B and \bar{B} are formed in terms of $A_j, \bar{A}_j, B_j, \bar{B}_j, C_j, \bar{C}_j$ as

$$A := \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & A_N \end{bmatrix}, \quad (2.23)$$

$$\bar{A} := \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{A}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{A}_{\bar{N}-1} & 0 \\ 0 & 0 & \cdots & 0 & \bar{A}_{\bar{N}} \end{bmatrix}, \quad (2.24)$$

$$B := \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \vdots \\ \bar{B}_{\bar{N}} \end{bmatrix}, \quad (2.25)$$

$$C := [C_1 \ C_2 \ \cdots \ C_N], \quad \bar{C} := [\bar{C}_1 \ \bar{C}_2 \ \cdots \ \bar{C}_{\bar{N}}]. \quad (2.26)$$

We refer the reader to [4, 5, 9] for further details of the construction of the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ in terms of the scattering data sets appearing in (2.12) and (2.13).

- (b) Having formed the matrix $F(y)$ appearing in (2.14), we use it as input to the linear 2×2 matrix-valued integral equation, known as the Marchenko system of integral equations,

$$K(x, y) + F(x + y) + \int_x^\infty dz K(x, z) F(z + y) = 0, \quad x < y, \quad (2.27)$$

and we obtain the solution $K(x, y)$ for $-\infty < x < y < +\infty$, where $K(x, y)$ is expressed in terms of its entries as

$$\begin{bmatrix} \bar{K}_1(x, y) & K_1(x, y) \\ \bar{K}_2(x, y) & K_2(x, y) \end{bmatrix} := K(x, y). \quad (2.28)$$

It is understood that $K(x, y) = 0$ when $x > y$, and hence (2.27) is valid only when $x < y$.

- (c) Having obtained $K(x, y)$, we use $K(x, x)$, which is defined as $K(x, x^+)$, in order to recover the potentials $u(x)$ and $v(x)$ via

$$u(x) = -2K_1(x, x), \quad (2.29)$$

$$v(x) = -2\bar{K}_2(x, x), \quad (2.30)$$

$$\int_x^\infty dz u(z) v(z) = 2\bar{K}_1(x, x), \quad (2.31)$$

$$\int_x^\infty dz u(z) v(z) = 2K_2(x, x). \quad (2.32)$$

To prevent any confusion, we can use the superscript (u, v) and the superscript (p, s) for the input data sets appearing in (2.12) and (2.13) in order to associate them with (1.2) and (1.7), respectively. In other words, instead of (2.12) and (2.13) we can respectively use

$$\left\{ T^{(u,v)}, R^{(u,v)}, L^{(u,v)}, \bar{T}^{(u,v)}, \bar{R}^{(u,v)}, \bar{L}^{(u,v)} \right\}, \quad (2.33)$$

$$\left\{ \lambda_j^{(u,v)}, \{c_{jk}^{(u,v)}\}_{k=0}^{m_j^{(u,v)}-1} \right\}_{j=1}^{N^{(u,v)}}, \quad \left\{ \bar{\lambda}_j^{(u,v)}, \{\bar{c}_{jk}^{(u,v)}\}_{k=0}^{\bar{m}_j^{(u,v)}-1} \right\}_{j=1}^{\bar{N}^{(u,v)}}. \quad (2.34)$$

We remark that the potentials $p(x)$ and $s(x)$ appearing in (1.7) can be recovered from the input data sets expressed as in (2.12) and (2.13), i.e.

$$\left\{ T^{(p,s)}, R^{(p,s)}, L^{(p,s)}, \bar{T}^{(p,s)}, \bar{R}^{(p,s)}, \bar{L}^{(p,s)} \right\}, \quad (2.35)$$

$$\left\{ \lambda_j^{(p,s)}, \{c_{jk}^{(p,s)}\}_{k=0}^{m_j^{(p,s)}-1} \right\}_{j=1}^{N^{(p,s)}}, \quad \left\{ \bar{\lambda}_j^{(p,s)}, \{\bar{c}_{jk}^{(p,s)}\}_{k=0}^{\bar{m}_j^{(p,s)}-1} \right\}_{j=1}^{\bar{N}^{(p,s)}}, \quad (2.36)$$

and this can be achieved by using the analogs of the steps (a)–(c) listed above and by starting with the input data set consisting of (2.35) and (2.36).

3. Transformations

In this section we provide the transformations of the energy-dependent system (1.1) into the two energy-independent systems given in (1.2) and (1.7), respectively. Thus, we deal with three first-order systems given in (1.1), (1.2), and (1.7). For each of these systems there are four Jost solutions $\psi, \phi, \bar{\psi}, \bar{\phi}$ and there are six scattering coefficients $T, R, L, \bar{T}, \bar{R}, \bar{L}$. From (1.1), (1.2), and (1.7), we see that the corresponding coefficient matrices for these three systems all have the same asymptotic limits as $x \rightarrow \pm\infty$, and hence for each of these three systems it is possible to define the corresponding Jost solutions $\psi, \phi, \bar{\psi}, \bar{\phi}$ in the same way by using the spatial asymptotics given in (2.1)–(2.4). Similarly, for each of these three systems it is possible to define the scattering coefficients $T, R, L, \bar{T}, \bar{R}, \bar{L}$ in the same way by using the spatial asymptotics given in (2.5)–(2.8).

By indicating the appropriate pair of potentials appearing in the off-diagonal entries in the corresponding coefficient matrix, we are able to uniquely identify the Jost solutions and the scattering coefficients for each of the three first-order systems given in (1.1), (1.2), and (1.7), respectively. For example, we use $\psi^{(\zeta q, \zeta r)}, \phi^{(\zeta q, \zeta r)}, \bar{\psi}^{(\zeta q, \zeta r)}, \bar{\phi}^{(\zeta q, \zeta r)}$ to denote the four Jost solutions corresponding to (1.1); we use $\psi^{(u,v)}, \phi^{(u,v)}, \bar{\psi}^{(u,v)}, \bar{\phi}^{(u,v)}$ to denote the four Jost solutions corresponding to (1.2), etc. Similarly, we use $T^{(\zeta q, \zeta r)}, R^{(\zeta q, \zeta r)}, L^{(\zeta q, \zeta r)}, \bar{T}^{(\zeta q, \zeta r)}, \bar{R}^{(\zeta q, \zeta r)}, \bar{L}^{(\zeta q, \zeta r)}$ to denote the scattering coefficients corresponding to (1.1); as in (2.33) we use $T^{(u,v)}, R^{(u,v)}, L^{(u,v)}, \bar{T}^{(u,v)}, \bar{R}^{(u,v)}, \bar{L}^{(u,v)}$ to denote the scattering coefficients corresponding to the system (1.2), etc.

In the next theorem we provide the relations among the Jost solutions to (1.1) and (1.2), respectively.

Theorem 3.1. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the first-order system (1.1) belong to the Schwartz class. Then, the system (1.1) can be transformed into the system (1.2), where the potential pair (u, v) is related to (q, r) as in (1.4) and (1.5), and it also follows that the potentials $u(x)$ and $v(x)$ belong to the Schwartz class. The four Jost solutions corresponding to the respective systems (1.1) and (1.2) are related to each other as*

$$\psi^{(\zeta q, \zeta r)} = e^{i\mu/2} \begin{bmatrix} \sqrt{\lambda} E & 0 \\ \frac{i}{2} r(x) E & E^{-1} \end{bmatrix} \psi^{(u,v)}, \quad (3.1)$$

$$\phi^{(\zeta q, \zeta r)} = \begin{bmatrix} E & 0 \\ \frac{i}{2\sqrt{\lambda}} r(x) E & \frac{1}{\sqrt{\lambda}} E^{-1} \end{bmatrix} \phi^{(u,v)}, \quad (3.2)$$

$$\bar{\psi}(\zeta q, \zeta r) = e^{-i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i}{2\sqrt{\lambda}} r(x) E & \frac{1}{\sqrt{\lambda}} E^{-1} \end{bmatrix} \bar{\psi}^{(u,v)}, \quad (3.3)$$

$$\bar{\phi}(\zeta q, \zeta r) = \begin{bmatrix} \sqrt{\lambda} E & 0 \\ \frac{i}{2} r(x) E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)}, \quad (3.4)$$

where the quantity $\sqrt{\lambda}$ is the same as the spectral parameter ζ appearing in (1.3), E is the quantity defined in (1.6), and the scalar constant μ is given by

$$\mu := \int_{-\infty}^{\infty} dy q(y) r(y). \quad (3.5)$$

Proof. Multiplying the second equation in (1.1) by ζ we obtain

$$\begin{bmatrix} \alpha \\ \zeta \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & q(x) \\ \zeta^2 r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \zeta \beta \end{bmatrix}, \quad x \in \mathbb{R}. \quad (3.6)$$

We try to write (3.6) in a way similar to (1.2) by letting

$$\begin{bmatrix} \alpha \\ \zeta \beta \end{bmatrix} = \Gamma \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad (3.7)$$

where Γ is a 2×2 matrix-valued function of x having the form

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} := \Gamma. \quad (3.8)$$

Using (3.7) in (3.6) we get

$$\Gamma \begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \left(-\Gamma' + \begin{bmatrix} -i\zeta^2 & q(x) \\ \zeta^2 r(x) & i\zeta^2 \end{bmatrix} \Gamma \right) \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad x \in \mathbb{R}. \quad (3.9)$$

Multiplying (1.2) on the left by Γ we obtain

$$\Gamma \begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \Gamma \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad x \in \mathbb{R}. \quad (3.10)$$

As a result of (1.3), we can replace ζ^2 in (3.9) by λ . Then, comparing the right-hand sides of (3.9) and (3.10), with the help of (3.8) we obtain a first-order system of differential equations for Γ , i.e.

$$\begin{bmatrix} \Gamma_1' & \Gamma_2' \\ \Gamma_3' & \Gamma_4' \end{bmatrix} = \begin{bmatrix} q\Gamma_3 - v\Gamma_2 & -2i\lambda\Gamma_2 + q\Gamma_4 - u\Gamma_1 \\ 2i\lambda\Gamma_3 + \lambda r\Gamma_1 - v\Gamma_4 & \lambda r\Gamma_2 - u\Gamma_3 \end{bmatrix}, \quad x \in \mathbb{R}, \quad (3.11)$$

which is also a linear polynomial equation in λ of degree one. From (3.11), because γ is assumed to depend on x but not on λ , by solving the polynomial equation we get six equations that are given by

$$\begin{cases} \Gamma'_1 = q\Gamma_3, & \Gamma_2 = 0, & q\Gamma_4 - u\Gamma_1 = 0, \\ \Gamma'_3 = -v\Gamma_4, & 2i\Gamma_3 + r\Gamma_1 = 0, & \Gamma'_4 = -u\Gamma_3. \end{cases} \quad (3.12)$$

We would like to solve (3.12) to obtain $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, u,$ and v in terms of q and r . From (3.12) we obtain

$$\begin{cases} \Gamma_1 = \delta E, & \Gamma_2 = 0, & \Gamma_3 = \delta \frac{ir}{2} E, \\ \frac{q\Gamma_4}{\Gamma_1} = -\frac{\Gamma'_4}{\Gamma_3}, & u = q \frac{\Gamma_4}{\Gamma_1}, & v = -\frac{\Gamma'_3}{\Gamma_4}, \end{cases} \quad (3.13)$$

where δ is a constant and E is the quantity appearing in (1.6). From (3.13) we get

$$\Gamma_4 = \delta_1 E^{-1}, \quad u = \frac{\delta_1}{\delta} q E^{-2}, \quad v = -\frac{\Gamma'_3}{\Gamma_4}, \quad (3.14)$$

where δ_1 is a constant. For simplicity we let $\delta_1 = \delta$, and hence (3.13) and (3.14) yield u and v as in (1.4) and (1.5), respectively, and

$$\Gamma = \delta \begin{bmatrix} E & 0 \\ \frac{ir}{2} E & E^{-1} \end{bmatrix}. \quad (3.15)$$

As indicated in the beginning of this section, the four Jost solutions to (1.1) satisfy the asymptotics similar to those given in (2.1)–(2.4), i.e.

$$\psi^{(\zeta q, \zeta r)} = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \zeta \in \mathbb{R}, \quad (3.16)$$

$$\phi^{(\zeta q, \zeta r)} = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \zeta \in \mathbb{R}, \quad (3.17)$$

$$\bar{\psi}^{(\zeta q, \zeta r)} = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \zeta \in \mathbb{R}, \quad (3.18)$$

$$\bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \zeta \in \mathbb{R}, \quad (3.19)$$

where we remark that ζ^2 can also be replaced by λ as a result of (1.3). From (3.7) we see that we can relate any solution to (1.1) to any solution to (1.2) as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \Gamma \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad (3.20)$$

where Γ is the matrix-valued quantity given in (3.15). From (1.6) and (3.5) we have

$$\lim_{x \rightarrow -\infty} E = 1, \quad \lim_{x \rightarrow +\infty} E = e^{i\mu/2}. \quad (3.21)$$

With the help of (3.20) and (3.21), by comparing the asymptotics of the Jost solutions to (1.1) given in (3.16)–(3.19) with the asymptotics of the Jost solutions to (1.2) given in (2.1)–(2.4), we obtain (3.1)–(3.4). \square

Next we provide the relations among the Jost solutions to (1.1) and to (1.7), respectively.

Theorem 3.2. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the first-order system (1.1) belong to the Schwartz class. Then, the system (1.1) can be transformed into the system (1.7), where the potential pair (p, s) is related to (q, r) as in (1.8) and (1.9), and it also follows that the potentials $p(x)$ and $s(x)$ belong to the Schwartz class. The four Jost solutions corresponding to the respective systems (1.1) and (1.7) are related to each other as*

$$\psi^{(\zeta q, \zeta r)} = e^{i\mu/2} \begin{bmatrix} \frac{1}{\sqrt{\lambda}} E & -\frac{i}{2\sqrt{\lambda}} q(x) E^{-1} \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)}, \quad (3.22)$$

$$\phi^{(\zeta q, \zeta r)} = \begin{bmatrix} E & -\frac{i}{2} q(x) E^{-1} \\ 0 & \sqrt{\lambda} E^{-1} \end{bmatrix} \phi^{(p,s)}, \quad (3.23)$$

$$\bar{\psi}^{(\zeta q, \zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i}{2} q(x) E^{-1} \\ 0 & \sqrt{\lambda} E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}, \quad (3.24)$$

$$\bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} E & -\frac{i}{2\sqrt{\lambda}} q(x) E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}, \quad (3.25)$$

where the quantity $\sqrt{\lambda}$ is the same as the spectral parameter ζ appearing in (1.3) and E and μ are the quantities appearing in (1.6) and (3.5), respectively.

Proof. The proof is similar to the proof of theorem 3.1, except that, in the beginning of the proof, instead of multiplying the second equation in (1.1) by ζ , we multiply the first equation in (1.1) by ζ . \square

In the next theorem we summarize the transformations among the scattering coefficients for the first-order systems (1.1), (1.2), and (1.7).

Theorem 3.3. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the first-order system (1.1) belong to the Schwartz class. Then, the scattering coefficients for (1.1) are related to the scattering coefficients for (1.2) and for (1.7) as*

$$T(\zeta q, \zeta r) = e^{-i\mu/2} T^{(u,v)} = e^{-i\mu/2} T^{(p,s)}, \quad (3.26)$$

$$\bar{T}(\zeta q, \zeta r) = e^{i\mu/2} \bar{T}^{(u,v)} = e^{i\mu/2} \bar{T}^{(p,s)}, \quad (3.27)$$

$$R(\zeta q, \zeta r) = \frac{e^{-i\mu}}{\sqrt{\lambda}} R^{(u,v)} = e^{-i\mu} \sqrt{\lambda} R^{(p,s)}, \quad (3.28)$$

$$\bar{R}(\zeta q, \zeta r) = e^{i\mu} \sqrt{\lambda} \bar{R}^{(u,v)} = \frac{e^{i\mu}}{\sqrt{\lambda}} \bar{R}^{(p,s)}, \quad (3.29)$$

$$L(\zeta q, \zeta r) = \sqrt{\lambda} L^{(u,v)} = \frac{1}{\sqrt{\lambda}} L^{(p,s)}, \quad (3.30)$$

$$\bar{L}(\zeta q, \zeta r) = \frac{1}{\sqrt{\lambda}} \bar{L}^{(u,v)} = \sqrt{\lambda} \bar{L}^{(p,s)}, \quad (3.31)$$

where μ is the quantity defined in (3.5), the quantity $\sqrt{\lambda}$ is the same as the spectral parameter ζ appearing in (1.3), and the superscripts for the scattering coefficients are as indicated in (2.33) and (2.35).

Proof. As indicated in the beginning of the section, the scattering coefficients for (1.1) are obtained from the large x -asymptotics of the Jost solutions to (1.1) as in (2.5)–(2.7), i.e.

$$\psi^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{L(\zeta q, \zeta r)}{T(\zeta q, \zeta r)} e^{-i\zeta^2 x} \\ \frac{1}{T(\zeta q, \zeta r)} e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \zeta \in \mathbb{R}, \quad (3.32)$$

$$\phi^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{1}{T(\zeta q, \zeta r)} e^{-i\zeta^2 x} \\ \frac{R(\zeta q, \zeta r)}{T(\zeta q, \zeta r)} e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \zeta \in \mathbb{R}, \quad (3.33)$$

$$\bar{\psi}^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{1}{\bar{T}(\zeta q, \zeta r)} e^{-i\zeta^2 x} \\ \frac{\bar{L}(\zeta q, \zeta r)}{\bar{T}(\zeta q, \zeta r)} e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad \zeta \in \mathbb{R}, \quad (3.34)$$

$$\bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{\bar{R}(\zeta q, \zeta r)}{\bar{T}(\zeta q, \zeta r)} e^{-i\zeta^2 x} \\ \frac{1}{\bar{T}(\zeta q, \zeta r)} e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad \zeta \in \mathbb{R}. \quad (3.35)$$

Using (1.3), (3.16)–(3.19), (3.21), and (2.1)–(2.8) for each of (1.2) and (1.7) in (3.1)–(3.4) as well as in (3.22)–(3.25), we obtain (3.26)–(3.31). \square

From (3.26) we see that the poles of $T^{(\zeta q, \zeta r)}$ and of $T^{(u,v)}$ are closely related to each other, and similarly from (3.27) we see that the poles of $\bar{T}^{(\zeta q, \zeta r)}$ and of $\bar{T}^{(u,v)}$ are closely related to each other. With the help of theorem 2.1(e), from (1.3) and (3.26) we conclude that the bound states for (1.1) related to $T^{(\zeta q, \zeta r)}$ occur at the ζ -values ζ_j and $-\zeta_j$ for $j = 1, \dots, N$, where $\pm\zeta_j$ and λ_j are related to each other as $\lambda_j = (\pm\zeta_j)^2$ and N is the number of poles of $T^{(u,v)}$ in \mathbb{C}^+ in the complex λ -plane. We remark that (3.26) also implies that $T^{(\zeta q, \zeta r)}$ and $T^{(u,v)}$ have the same number of poles in \mathbb{C}^+ in the complex λ -plane. In a similar way, with the help of theorem 2.1(e), from (1.3) and (3.27) we see that the bound states for (1.1) related to $\bar{T}^{(\zeta q, \zeta r)}$ occur at the ζ -values $\bar{\zeta}_j$ and $-\bar{\zeta}_j$ for $j = 1, \dots, \bar{N}$, where $\pm\bar{\zeta}_j$ and $\bar{\lambda}_j$ are related to each other as $\bar{\lambda}_j = (\pm\bar{\zeta}_j)^2$. We also remark that (3.27) implies that $\bar{T}^{(\zeta q, \zeta r)}$ and $\bar{T}^{(u,v)}$ have the same number of poles in \mathbb{C}^- in the complex λ -plane and hence we can use \bar{N} to denote that common number. Since ζ_j and $-\zeta_j$ correspond to the same λ_j , from (3.26) we see that each of ζ_j and $-\zeta_j$ has multiplicity m_j and in an analogous manner from (3.27) we see that each of $\bar{\zeta}_j$ and $-\bar{\zeta}_j$ has multiplicity \bar{m}_j .

As shown from the second terms appearing on the right-hand sides of (2.15) and (2.16), the bound states with multiplicities and their corresponding norming constants can be handled by using the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ given in (2.23)–(2.26). Due to the flexibility and ease of including the bound-state norming constants in the row vectors C and \bar{C} appearing

in (2.26), the treatment of the bound states in the solution to the corresponding inverse scattering problems becomes elegant.

4. The direct problem for the energy-dependent system

In this section we describe the solution to the direct scattering problem for (1.1). This involves the determination of the corresponding scattering coefficients $T^{(\zeta q, \zeta r)}$, $R^{(\zeta q, \zeta r)}$, $L^{(\zeta q, \zeta r)}$, $\bar{T}^{(\zeta q, \zeta r)}$, $\bar{R}^{(\zeta q, \zeta r)}$, $\bar{L}^{(\zeta q, \zeta r)}$, and the relevant bound-state information when the potential pair (q, r) in the Schwartz class is given. The solution is obtained as described below:

- (a) We solve the system (1.1) and uniquely determine the four Jost solutions $\psi^{(\zeta q, \zeta r)}$, $\phi^{(\zeta q, \zeta r)}$, $\bar{\psi}^{(\zeta q, \zeta r)}$, $\bar{\phi}^{(\zeta q, \zeta r)}$ satisfying the respective asymptotics (3.16)–(3.19). The existence and uniqueness of the corresponding Jost solutions can be proved in the standard manner when $q(x)$ and $r(x)$ belong to the Schwartz class. For further details we refer the reader to [10]. For example, (1.1) and (3.16) for $\psi^{(\zeta q, \zeta r)}$ can be used in order to obtain an integral equation for $\psi^{(\zeta q, \zeta r)}$. When $q(x)$ and $r(x)$ belong to the Schwartz class, one can prove that the resulting integral equation can be solved via iteration by representing its solution as a uniformly convergent infinite series. Alternatively, one can use the already known existence and uniqueness of the Jost solution $\psi^{(u, v)}$ associated with (1.2), and by using (3.1) one can conclude the existence and uniqueness of $\psi^{(\zeta q, \zeta r)}$ associated with (1.1). The existence and uniqueness of the remaining Jost solutions $\phi^{(\zeta q, \zeta r)}$, $\bar{\psi}^{(\zeta q, \zeta r)}$, and $\bar{\phi}^{(\zeta q, \zeta r)}$ can be established in a similar manner.
- (b) We then determine the scattering coefficients $T^{(\zeta q, \zeta r)}$, $R^{(\zeta q, \zeta r)}$, $L^{(\zeta q, \zeta r)}$, $\bar{T}^{(\zeta q, \zeta r)}$, $\bar{R}^{(\zeta q, \zeta r)}$, $\bar{L}^{(\zeta q, \zeta r)}$ from the Jost solutions $\psi^{(\zeta q, \zeta r)}$, $\phi^{(\zeta q, \zeta r)}$, $\bar{\psi}^{(\zeta q, \zeta r)}$, $\bar{\phi}^{(\zeta q, \zeta r)}$ constructed in the previous step, by using the respective asymptotics described in (3.32)–(3.35). Alternatively, those scattering coefficients can be obtained by using some Wronskian relations among the constructed Jost solutions. We remark that the Wronskian of any two column-vector solutions to (1.1) is independent of x , which is a consequence of the fact that the coefficient matrix in (1.1) has a zero trace. For any two column-vector solutions $\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ to (1.1), we recall that the Wronskian is defined as

$$\left[\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}; \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \right] := \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}, \quad (4.1)$$

where the absolute bars on the right-hand side in (4.1) denote the determinant of a 2×2 matrix. By evaluating certain Wronskians of the Jost solutions to (1.1) at $x = +\infty$ and $x = -\infty$ and by using (3.16)–(3.19) and (3.32)–(3.35), it can directly be verified that we have

$$[\psi^{(\zeta q, \zeta r)}; \phi^{(\zeta q, \zeta r)}] = -\frac{1}{T^{(\zeta q, \zeta r)}}, \quad (4.2)$$

$$[\bar{\psi}^{(\zeta q, \zeta r)}; \bar{\phi}^{(\zeta q, \zeta r)}] = \frac{1}{\bar{T}^{(\zeta q, \zeta r)}}, \quad (4.3)$$

$$[\psi^{(\zeta q, \zeta r)}; \bar{\phi}^{(\zeta q, \zeta r)}] = -\frac{\bar{R}^{(\zeta q, \zeta r)}}{\bar{T}^{(\zeta q, \zeta r)}} = \frac{L^{(\zeta q, \zeta r)}}{T^{(\zeta q, \zeta r)}}, \quad (4.4)$$

$$[\bar{\psi}^{(\zeta q, \zeta r)}; \phi^{(\zeta q, \zeta r)}] = \frac{R^{(\zeta q, \zeta r)}}{T^{(\zeta q, \zeta r)}} = -\frac{\bar{L}^{(\zeta q, \zeta r)}}{\bar{T}^{(\zeta q, \zeta r)}}. \quad (4.5)$$

Hence using (4.2)–(4.5) the scattering coefficients can be expressed in terms of the Wronskians of the constructed Jost solutions as

$$T^{(\zeta q, \zeta r)} = \frac{1}{[\phi^{(\zeta q, \zeta r)}; \psi^{(\zeta q, \zeta r)}]}, \quad (4.6)$$

$$\bar{T}^{(\zeta q, \zeta r)} = \frac{1}{[\bar{\psi}^{(\zeta q, \zeta r)}; \bar{\phi}^{(\zeta q, \zeta r)}]}, \quad (4.7)$$

$$R^{(\zeta q, \zeta r)} = \frac{[\phi^{(\zeta q, \zeta r)}; \bar{\psi}^{(\zeta q, \zeta r)}]}{[\bar{\psi}^{(\zeta q, \zeta r)}; \phi^{(\zeta q, \zeta r)}]}, \quad (4.8)$$

$$\bar{R}^{(\zeta q, \zeta r)} = \frac{[\bar{\phi}^{(\zeta q, \zeta r)}; \psi^{(\zeta q, \zeta r)}]}{[\psi^{(\zeta q, \zeta r)}; \bar{\phi}^{(\zeta q, \zeta r)}]}, \quad (4.9)$$

$$L^{(\zeta q, \zeta r)} = \frac{[\psi^{(\zeta q, \zeta r)}; \bar{\phi}^{(\zeta q, \zeta r)}]}{[\phi^{(\zeta q, \zeta r)}; \psi^{(\zeta q, \zeta r)}]}, \quad (4.10)$$

$$\bar{L}^{(\zeta q, \zeta r)} = \frac{[\phi^{(\zeta q, \zeta r)}; \bar{\psi}^{(\zeta q, \zeta r)}]}{[\bar{\psi}^{(\zeta q, \zeta r)}; \bar{\phi}^{(\zeta q, \zeta r)}]}. \quad (4.11)$$

- (c) By using (3.26) and theorem 2.1 one can prove that the transmission coefficient $T^{(\zeta q, \zeta r)}$ is meromorphic in $\lambda \in \mathbb{C}^+$ with a finite number of poles there corresponding to the bound states. Similarly, by using (3.27) and theorem 2.1 one can prove that the transmission coefficient $\bar{T}^{(\zeta q, \zeta r)}$ is meromorphic in $\lambda \in \mathbb{C}^-$ with a finite number of poles there corresponding to the bound states. The location and multiplicities of such bound-state poles in \mathbb{C}^+ and \mathbb{C}^- , respectively, are determined as a result of the uniqueness of the meromorphic extensions from \mathbb{R} to \mathbb{C}^+ and \mathbb{C}^- , respectively. The norming constants c_{jk} and \bar{c}_{jk} can then be constructed in a similar way as in (4.29) and (4.49) of [9].

5. The inverse scattering problem for the energy-dependent system

In this section we describe a solution to the inverse scattering problem for (1.1). This involves the recovery of the potentials $q(x)$ and $r(x)$ appearing in (1.1) from the input data consisting of the set of scattering coefficients

$$\left\{ T^{(\zeta q, \zeta r)}, R^{(\zeta q, \zeta r)}, L^{(\zeta q, \zeta r)}, \bar{T}^{(\zeta q, \zeta r)}, \bar{R}^{(\zeta q, \zeta r)}, \bar{L}^{(\zeta q, \zeta r)} \right\}, \quad (5.1)$$

and the bound-state information

$$\left\{ \pm \zeta_j, \left\{ c_{jk}^{(\zeta q, \zeta r)} \right\}_{k=0}^{m_j^{(\zeta q, \zeta r)} - 1} \right\}_{j=1}^{N^{(\zeta q, \zeta r)}}, \tag{5.2}$$

$$\left\{ \pm \bar{\zeta}_j, \left\{ \bar{c}_{jk}^{(\zeta q, \zeta r)} \right\}_{k=0}^{\bar{m}_j^{(\zeta q, \zeta r)} - 1} \right\}_{j=1}^{\bar{N}^{(\zeta q, \zeta r)}}.$$

From (1.3), (3.26), and (3.27) we see that ζ appears as ζ^2 in $T^{(\zeta q, \zeta r)}$ and in $\bar{T}^{(\zeta q, \zeta r)}$. From (3.26) it follows that the number of poles for each of $T^{(\zeta q, \zeta r)}$, $T^{(u,v)}$, and $T^{(p,s)}$ in \mathbb{C}^+ of the complex λ -plane is the same, i.e. we have

$$N^{(\zeta q, \zeta r)} = N^{(u,v)} = N^{(p,s)}, \tag{5.3}$$

and we use N to denote their common value. Similarly, from (3.27) it follows that the number of poles for each of $\bar{T}^{(\zeta q, \zeta r)}$, $\bar{T}^{(u,v)}$, and $\bar{T}^{(p,s)}$ in \mathbb{C}^- of the complex λ -plane is the same, i.e.

$$\bar{N}^{(\zeta q, \zeta r)} = \bar{N}^{(u,v)} = \bar{N}^{(p,s)}, \tag{5.4}$$

and we use \bar{N} to denote their common value. Note that the value of $e^{-i\mu/2}$, where μ is the constant appearing in (3.5), can uniquely be determined from the large ζ -asymptotics of $T^{(\zeta q, \zeta r)}$ via

$$T^{(\zeta q, \zeta r)} = e^{-i\mu/2} \left[1 + O\left(\frac{1}{\zeta^2}\right) \right], \quad \zeta \rightarrow \pm\infty, \tag{5.5}$$

which directly follows from (1.3), (2.10), and (3.26). Alternatively, $e^{i\mu/2}$ can be determined from the large ζ -asymptotics of $\bar{T}^{(\zeta q, \zeta r)}$ via

$$\bar{T}^{(\zeta q, \zeta r)} = e^{i\mu/2} \left[1 + O\left(\frac{1}{\zeta^2}\right) \right], \quad \zeta \rightarrow \pm\infty, \tag{5.6}$$

which directly follows from (1.3), (2.11), and (3.27). Let us remark that from (4.4) and (4.5) we have

$$L^{(\zeta q, \zeta r)} = -\frac{\bar{R}^{(\zeta q, \zeta r)} T^{(\zeta q, \zeta r)}}{\bar{T}^{(\zeta q, \zeta r)}}, \quad \bar{L}^{(\zeta q, \zeta r)} = -\frac{R^{(\zeta q, \zeta r)} \bar{T}^{(\zeta q, \zeta r)}}{T^{(\zeta q, \zeta r)}}, \tag{5.7}$$

and hence, in order to solve the inverse problem, in the input data set, instead of (5.1), we can use its subset

$$\left\{ T^{(\zeta q, \zeta r)}, R^{(\zeta q, \zeta r)}, \bar{T}^{(\zeta q, \zeta r)}, \bar{R}^{(\zeta q, \zeta r)} \right\}. \tag{5.8}$$

We recall that $\pm\zeta_j$ and $\pm\bar{\zeta}_j$ appearing in (5.2) are related to λ_j and $\bar{\lambda}_j$ appearing in (2.13) as

$$(\pm\zeta_j)^2 = \lambda_j, \quad j = 1, \dots, N, \tag{5.9}$$

$$(\pm\bar{\zeta}_j)^2 = \bar{\lambda}_j, \quad j = 1, \dots, \bar{N}, \tag{5.10}$$

where we use λ_j to denote the common value of $\lambda_j^{(u,v)}$ and $\lambda_j^{(p,s)}$, i.e.

$$\lambda_j = \lambda_j^{(u,v)} = \lambda_j^{(p,s)}, \quad j = 1, \dots, N,$$

which is a consequence of (3.26). Similarly, we use $\bar{\lambda}_j$ to denote the common value of $\bar{\lambda}_j^{(u,v)}$ and $\bar{\lambda}_j^{(p,s)}$, i.e.

$$\bar{\lambda}_j = \bar{\lambda}_j^{(u,v)} = \bar{\lambda}_j^{(p,s)}, \quad j = 1, \dots, \bar{N},$$

which is a consequence of (3.27). From (3.26) it also follows that the multiplicity of $\zeta_j^{(\zeta q, \zeta r)}$ and the multiplicity of λ_j are equal, i.e. we have

$$m_j^{(\zeta q, \zeta r)} = m_j^{(u, v)} = m_j^{(p, s)}, \quad j = 1, \dots, N. \tag{5.11}$$

We use m_j to denote the common value appearing in (5.11). Similarly, from (3.27) it follows that the multiplicity of $\bar{\zeta}_j^{(\zeta q, \zeta r)}$ and the multiplicity of $\bar{\lambda}_j$ are equal, i.e. we have

$$\bar{m}_j^{(\zeta q, \zeta r)} = \bar{m}_j^{(u, v)} = \bar{m}_j^{(p, s)}, \quad j = 1, \dots, \bar{N}. \tag{5.12}$$

We use \bar{m}_j to denote the common value appearing in (5.12).

We can solve the inverse scattering problem for (1.1) by using the following steps:

- (a) From $T^{(\zeta q, \zeta r)}$ appearing in the data set (5.1) we can recover the value of $e^{i\mu/2}$ via (5.5).
- (b) Using the already recovered value of $e^{i\mu/2}$, from (5.1) we get the auxiliary data subset

$$\left\{ R^{(u, v)}, \bar{R}^{(u, v)}, R^{(p, s)}, \bar{R}^{(p, s)} \right\}, \tag{5.13}$$

where we have

$$R^{(u, v)} = \sqrt{\lambda} e^{i\mu} R^{(\zeta q, \zeta r)}, \quad \bar{R}^{(u, v)} = \frac{1}{\sqrt{\lambda}} e^{-i\mu} \bar{R}^{(\zeta q, \zeta r)}, \tag{5.14}$$

$$R^{(p, s)} = \frac{1}{\sqrt{\lambda}} e^{i\mu} R^{(\zeta q, \zeta r)}, \quad \bar{R}^{(p, s)} = \sqrt{\lambda} e^{-i\mu} \bar{R}^{(\zeta q, \zeta r)}, \tag{5.15}$$

which are obtained from (3.28) and (3.29).

- (c) Using (5.9)–(5.12) we obtain from (5.2) the auxiliary bound-state information

$$\left\{ \lambda_j, \left\{ c_{jk}^{(u, v)}, c_{jk}^{(p, s)} \right\}_{k=0}^{m_j-1} \right\}_{j=1}^N, \quad \left\{ \bar{\lambda}_j, \left\{ \bar{c}_{jk}^{(u, v)}, \bar{c}_{jk}^{(p, s)} \right\}_{k=0}^{\bar{m}_j-1} \right\}_{j=1}^{\bar{N}}, \tag{5.16}$$

where we remark that the norming constants appearing in (5.16) can be obtained from the norming constants appearing in (5.2) with the help of (4.29) and (4.49) of [9] by using the relations among the Jost solutions given in (3.1)–(3.4).

- (d) Since the individual norming constants appearing in (5.16) can be combined into the row vectors $C^{(u, v)}$, $\bar{C}^{(u, v)}$, $C^{(p, s)}$, $\bar{C}^{(p, s)}$ as in (2.17), (2.18), (2.26), we can conclude that the input data set consisting of (5.1) and (5.2) yields the auxiliary input data set consisting of

$$\left\{ R^{(u, v)}(\lambda), \bar{R}^{(u, v)}(\lambda), R^{(p, s)}(\lambda), \bar{R}^{(p, s)}(\lambda) \right\}, \tag{5.17}$$

$$\left\{ \left\{ \lambda_j \right\}_{j=1}^N, \left\{ \bar{\lambda}_j \right\}_{j=1}^{\bar{N}}, C^{(u, v)}, \bar{C}^{(u, v)}, C^{(p, s)}, \bar{C}^{(p, s)} \right\}. \tag{5.18}$$

In fact, with the help of (2.19), (2.20), (5.11), and (5.12) the sets $\{\lambda_j\}_{j=1}^N$ and $\{\bar{\lambda}_j\}_{j=1}^{\bar{N}}$ and the multiplicities m_j and \bar{m}_j yield the matrices A and \bar{A} appearing in (2.23) and (2.24), respectively. Knowing the multiplicities m_j and \bar{m}_j , with the help of (2.21) and (2.22) we are also able to construct the column vectors B and \bar{B} appearing in (2.25). Thus, the input data set consisting of (5.1) and (5.2) yields the auxiliary input data consisting of the two sets

$$\left\{ R^{(u,v)}, \bar{R}^{(u,v)}, (A, B, C^{(u,v)}), (\bar{A}, \bar{B}, \bar{C}^{(u,v)}) \right\}, \quad (5.19)$$

$$\left\{ R^{(p,s)}, \bar{R}^{(p,s)}, (A, B, C^{(p,s)}), (\bar{A}, \bar{B}, \bar{C}^{(p,s)}) \right\}, \quad (5.20)$$

where the set in (5.19) is related to (1.2) with the potential pair (u, v) and the set in (5.20) is related to (1.7) with the potential pair (p, s) . We remark that the matrices A, B, \bar{A}, \bar{B} appearing in (5.19) coincide with the respective matrices A, B, \bar{A}, \bar{B} appearing in (5.20). Furthermore, as seen from (3.28) and (3.29), we have

$$R^{(u,v)} = \lambda R^{(p,s)}, \quad \bar{R}^{(p,s)} = \lambda \bar{R}^{(u,v)}, \quad (5.21)$$

which, by using (2.14) and (2.15), yields

$$C^{(u,v)} = i C^{(p,s)} A, \quad \bar{C}^{(p,s)} = -i \bar{C}^{(u,v)} \bar{A}. \quad (5.22)$$

- (e) Having constructed the auxiliary input data set given in (5.19) with the help of (2.15) and (2.16), we form the matrix-valued function $F^{(u,v)}(y)$ as in (2.14). We then use $F^{(u,v)}(y)$ as input into the Marchenko integral equation (2.27) and obtain the solution $K^{(u,v)}(x, y)$ to (2.27). From (2.28) we then observe that we have the four scalar entries $\bar{K}_1^{(u,v)}(x, y)$, $\bar{K}_2^{(u,v)}(x, y)$, $K_1^{(u,v)}(x, y)$, $K_2^{(u,v)}(x, y)$. As seen from (2.29)–(2.32) we have

$$u(x) = -2 K_1^{(u,v)}(x, x), \quad (5.23)$$

$$v(x) = -2 \bar{K}_2^{(u,v)}(x, x), \quad (5.24)$$

$$\int_x^\infty dz u(z) v(z) = 2 \bar{K}_1^{(u,v)}(x, x), \quad (5.25)$$

$$\int_x^\infty dz u(z) v(z) = 2 K_2^{(u,v)}(x, x). \quad (5.26)$$

- (f) We now repeat the previous step by using the auxiliary input data set given in (5.20). In other words, with the help of (2.15) and (2.16) we form $F^{(p,s)}(y)$ as in (2.14), use it as input into the Marchenko integral equation (2.27), and obtain the solution $K^{(p,s)}(x, y)$ to (2.27). As seen from (2.28) we then have the four scalar entries $\bar{K}_1^{(p,s)}(x, y)$, $\bar{K}_2^{(p,s)}(x, y)$, $K_1^{(p,s)}(x, y)$, $K_2^{(p,s)}(x, y)$. Then, as seen from (2.29)–(2.32) we get

$$p(x) = -2 K_1^{(p,s)}(x, x), \quad (5.27)$$

$$s(x) = -2 \bar{K}_2^{(p,s)}(x, x), \quad (5.28)$$

$$\int_x^\infty dz p(z) s(z) = 2 \bar{K}_1^{(p,s)}(x, x), \quad (5.29)$$

$$\int_x^\infty dz p(z) s(z) = 2 K_2^{(p,s)}(x, x). \quad (5.30)$$

(g) Next, we construct the function E appearing in (1.6). This is done as follows. Note that from (1.4) and (1.5) we have

$$u(x) v(x) = -\frac{i}{2} q(x) r'(x) + \frac{1}{4} q(x)^2 r(x)^2, \quad (5.31)$$

and from (1.8) and (1.9) we have

$$p(x) s(x) = \frac{i}{2} q'(x) r(x) + \frac{1}{4} q(x)^2 r(x)^2. \quad (5.32)$$

From (5.31) and (5.32) we get

$$p(x) s(x) - u(x) v(x) = \frac{i}{2} [q(x) r(x)]', \quad (5.33)$$

which yields

$$\int_x^\infty dz [p(z) s(z) - u(z) v(z)] = -\frac{i}{2} q(x) r(x). \quad (5.34)$$

Comparing (5.34) with (5.26) and (5.30) we see that we can write (5.34) as

$$2 \left(K_2^{(u,v)}(x, x) - K_2^{(p,s)}(x, x) \right) = \frac{i}{2} q(x) r(x), \quad (5.35)$$

which yields

$$2 \int_x^\infty dz \left(K_2^{(u,v)}(z, z) - K_2^{(p,s)}(z, z) \right) = \frac{i}{2} \int_x^\infty dz q(z) r(z). \quad (5.36)$$

Using (1.6) and (3.5) we can write the right-hand side of (5.36) as

$$\frac{i}{2} \int_x^\infty dz q(z) r(z) = \frac{i}{2} \mu - \frac{i}{2} \int_{-\infty}^x dz q(z) r(z). \quad (5.37)$$

From (5.36) and (5.37) we obtain

$$2 \int_x^\infty dz \left(K_2^{(u,v)}(z, z) - K_2^{(p,s)}(z, z) \right) = \frac{i}{2} \mu - \frac{i}{2} \int_{-\infty}^x dz q(z) r(z). \quad (5.38)$$

Comparing (5.38) with (1.6) we see that E can be constructed via

$$E = e^{i\mu/2} \exp \left(2 \int_x^\infty dz \left(K_2^{(p,s)}(z, z) - K_2^{(u,v)}(z, z) \right) \right), \quad (5.39)$$

where we recall that $e^{i\mu/2}$ is already constructed via (5.5) or (5.6).

(h) Having already constructed $u(x)$, $s(x)$, and E in the previous steps, we then use (1.4) and (1.9) in order to recover $q(x)$ and $r(x)$ via

$$q(x) = u(x) E^2, \quad (5.40)$$

$$r(x) = s(x) E^{-2}. \quad (5.41)$$

Thus, we have completed the construction of $q(x)$ and $r(x)$ from the input data set consisting of (5.1) and (5.2), which completes the solution to the inverse scattering problem for (1.1).

6. The alternate Marchenko method

As indicated in section 2, the solution to the inverse scattering problems for (1.2) and (1.7) can be obtained from the solution to the system of linear integral equations given in (2.27), which is known as the Marchenko system. In section 5 we have seen that the solution to the inverse scattering problem for (1.1) can be obtained with the help of the solution to the Marchenko systems associated with (1.2) and (1.7), respectively. In this section we present another method to solve the inverse problem for (1.1) by formulating a system of linear integral equations directly associated with (1.1). We refer to our method as the alternate Marchenko method, and we call the resulting system of linear integral equations the alternate Marchenko system, due to its similarity to the standard Marchenko system of integral equations.

The motivation behind our alternate Marchenko method comes from the analysis by Tsuchida [15], who presented a system of linear integral equations resembling our alternate Marchenko system. There are also some differences between our alternate Marchenko system and that of Tsuchida's; for example, Tsuchida's system lacks the symmetry between the x and y variables that exists in the standard Marchenko system. Tsuchida's main interest in developing his method was to solve initial-value problems for certain integrable evolution equations, in particular the derivative NLS equation and related equations. We have found Tsuchida's formulation not very intuitive and not easy to comprehend because it is not clear how the scattering theory is used in the derivation and how Tsuchida's gauge transformations are implemented. Nevertheless, we would like to emphasize the importance of Tsuchida's contribution. Since our main concentration is on the analysis of the direct and inverse scattering problems for (1.1), our work presented here may help to clarify the idea behind Tsuchida's formulation, by providing a clearer method to derive the alternate Marchenko system. For further details and elaborations on our method, we refer the reader to [10].

In order to derive the alternate Marchenko system, we first present some auxiliary results.

Proposition 6.1. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the first-order system (1.1) belong to the Schwartz class. Let $u(x)$, $v(x)$, $p(x)$, $s(x)$ be the potentials defined as in (1.4), (1.5), (1.8), (1.9), respectively. Then, the zero-energy Jost solutions $\psi^{(u,v)}(0, x)$ and $\bar{\psi}^{(u,v)}(0, x)$ to (1.2) and the zero-energy Jost solutions $\psi^{(p,s)}(0, x)$ and $\bar{\psi}^{(p,s)}(0, x)$ to (1.7) are given by*

$$\begin{bmatrix} \psi_1^{(u,v)}(0, x) \\ \psi_2^{(u,v)}(0, x) \end{bmatrix} := \psi^{(u,v)}(0, x) = \begin{bmatrix} -e^{-i\mu/2} E^{-1} \int_x^\infty dy q(y) \\ e^{-i\mu/2} E \left(1 + \frac{i}{2} r(x) \int_x^\infty dy q(y)\right) \end{bmatrix}, \quad (6.1)$$

$$\begin{bmatrix} \bar{\psi}_1^{(u,v)}(0, x) \\ \bar{\psi}_2^{(u,v)}(0, x) \end{bmatrix} := \bar{\psi}^{(u,v)}(0, x) = \begin{bmatrix} e^{i\mu/2} E^{-1} \\ -\frac{i}{2} e^{i\mu/2} r(x) E \end{bmatrix}, \quad (6.2)$$

$$\begin{bmatrix} \psi_1^{(p,s)}(0,x) \\ \psi_2^{(p,s)}(0,x) \end{bmatrix} := \psi^{(p,s)}(0,x) = \begin{bmatrix} \frac{i}{2} e^{-i\mu/2} q(x) E^{-1} \\ e^{-i\mu/2} E \end{bmatrix}, \quad (6.3)$$

$$\begin{bmatrix} \bar{\psi}_1^{(p,s)}(0,x) \\ \bar{\psi}_2^{(p,s)}(0,x) \end{bmatrix} := \bar{\psi}^{(p,s)}(0,x) = \begin{bmatrix} e^{i\mu/2} E^{-1} \left(1 - \frac{i}{2} q(x) \int_x^\infty dy r(y)\right) \\ -e^{i\mu/2} E \int_x^\infty dy r(y) \end{bmatrix}, \quad (6.4)$$

where E and μ are the quantities defined in (1.6) and (3.5), respectively.

Proof. One can directly verify that the quantities given in (6.1) and (6.2) with the respective asymptotics in (2.1) and (2.3) with $\lambda = 0$ satisfy (1.2) with $\lambda = 0$. Similarly, one can directly verify that the quantities given in (6.3) and (6.4) with the respective asymptotics in (2.1) and (2.3) with $\lambda = 0$ satisfy (1.7) with $\lambda = 0$. Alternatively, one can directly solve (1.2) with $\lambda = 0$ using the respective asymptotics (2.1) and (2.3) with $\lambda = 0$ and directly obtain the zero-energy Jost solutions given in (6.1) and (6.2). In a similar way, one can directly construct the zero-energy Jost solutions to (1.7) with $\lambda = 0$ and obtain (6.3) and (6.4). \square

We remark that from (6.1)–(6.4) we obtain

$$q(x) = e^{i\mu} \frac{d}{dx} \left[\frac{\psi_1^{(u,v)}(0,x)}{\bar{\psi}_1^{(u,v)}(0,x)} \right], \quad (6.5)$$

$$r(x) = e^{-i\mu} \frac{d}{dx} \left[\frac{\bar{\psi}_2^{(p,s)}(0,x)}{\psi_2^{(p,s)}(0,x)} \right], \quad (6.6)$$

where we recall that the subscripts 1 and 2 in (6.1)–(6.6) refer to the first and second components in the respective Jost solutions.

It is already known [1–3] that the entries appearing in (2.28) are related to the entries of the corresponding Jost solutions as

$$\psi^{(u,v)}(\lambda, x) = \begin{bmatrix} \psi_1^{(u,v)}(\lambda, x) \\ \psi_2^{(u,v)}(\lambda, x) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^\infty dy \begin{bmatrix} K_1^{(u,v)}(x, y) \\ K_2^{(u,v)}(x, y) \end{bmatrix} e^{i\lambda y}, \quad (6.7)$$

$$\bar{\psi}^{(u,v)}(\lambda, x) = \begin{bmatrix} \bar{\psi}_1^{(u,v)}(\lambda, x) \\ \bar{\psi}_2^{(u,v)}(\lambda, x) \end{bmatrix} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^\infty dy \begin{bmatrix} \bar{K}_1^{(u,v)}(x, y) \\ \bar{K}_2^{(u,v)}(x, y) \end{bmatrix} e^{-i\lambda y}, \quad (6.8)$$

$$\psi^{(p,s)}(\lambda, x) = \begin{bmatrix} \psi_1^{(p,s)}(\lambda, x) \\ \psi_2^{(p,s)}(\lambda, x) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^\infty dy \begin{bmatrix} K_1^{(p,s)}(x, y) \\ K_2^{(p,s)}(x, y) \end{bmatrix} e^{i\lambda y}, \quad (6.9)$$

$$\bar{\psi}^{(p,s)}(\lambda, x) = \begin{bmatrix} \bar{\psi}_1^{(p,s)}(\lambda, x) \\ \bar{\psi}_2^{(p,s)}(\lambda, x) \end{bmatrix} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^\infty dy \begin{bmatrix} \bar{K}_1^{(p,s)}(x, y) \\ \bar{K}_2^{(p,s)}(x, y) \end{bmatrix} e^{-i\lambda y}. \quad (6.10)$$

In the next proposition we relate the potentials $q(x)$ and $r(x)$ given in (6.5) and (6.6), respectively, to the quantities appearing in the integrands in (6.7)–(6.10).

Proposition 6.2. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the first-order system (1.1) belong to the Schwartz class. Let $u(x)$, $v(x)$, $p(x)$, $s(x)$ be the potentials defined*

as in (1.4), (1.5), (1.8), (1.9), respectively. Then, the integrands appearing in (6.7)–(6.10) are related to the components of the respective zero-energy Jost solutions to (1.2) and (1.7) as

$$\psi_1^{(u,v)}(0, x) = \int_x^\infty dy K_1^{(u,v)}(x, y), \quad (6.11)$$

$$\bar{\psi}_1^{(u,v)}(0, x) = 1 + \int_x^\infty dy \bar{K}_1^{(u,v)}(x, y), \quad (6.12)$$

$$\psi_2^{(p,s)}(0, x) = 1 + \int_x^\infty dy K_2^{(p,s)}(x, y), \quad (6.13)$$

$$\bar{\psi}_2^{(p,s)}(0, x) = \int_x^\infty dy \bar{K}_2^{(p,s)}(x, y). \quad (6.14)$$

Proof. By letting $\lambda = 0$ in (6.7)–(6.10) we directly obtain (6.11)–(6.14). \square

Next we present the derivation of our alternate Marchenko system. For the motivation and further details we refer the reader to [10]. Inspired by the right-hand sides of (6.5) and (6.6) we define the scalar quantities $\mathcal{H}(x, y)$ and $\bar{\mathcal{H}}(x, y)$ as

$$\mathcal{H}(x, y) := \frac{\int_y^\infty dz K_1^{(u,v)}(x, z)}{1 + \int_x^\infty dz \bar{K}_1^{(u,v)}(x, z)}, \quad x < y, \quad (6.15)$$

$$\bar{\mathcal{H}}(x, y) := \frac{\int_y^\infty dz \bar{K}_2^{(p,s)}(x, z)}{1 + \int_x^\infty dz K_2^{(p,s)}(x, z)}, \quad x < y, \quad (6.16)$$

with the understanding that

$$\mathcal{H}(x, y) = 0, \quad \bar{\mathcal{H}}(x, y) = 0, \quad x > y. \quad (6.17)$$

We remark that the y -dependence of $\mathcal{H}(x, y)$ and $\bar{\mathcal{H}}(x, y)$ occurs only in the numerators in (6.15) and (6.16). Using $\mathcal{H}(x, x)$ and $\bar{\mathcal{H}}(x, x)$ to denote $\mathcal{H}(x, x^+)$ and $\bar{\mathcal{H}}(x, x^+)$, respectively, with the help of (6.11)–(6.16) we obtain

$$\mathcal{H}(x, x) = \frac{\int_x^\infty dz K_1^{(u,v)}(x, z)}{1 + \int_x^\infty dz \bar{K}_1^{(u,v)}(x, z)} = \frac{\psi_1^{(u,v)}(0, x)}{\bar{\psi}_1^{(u,v)}(0, x)}, \quad (6.18)$$

$$\bar{\mathcal{H}}(x, x) = \frac{\int_x^\infty dz \bar{K}_2^{(p,s)}(x, z)}{1 + \int_x^\infty dz K_2^{(p,s)}(x, z)} = \frac{\bar{\psi}_2^{(p,s)}(0, x)}{\psi_2^{(p,s)}(0, x)}. \quad (6.19)$$

Comparing (6.5), (6.6), (6.18), and (6.19) we observe that

$$q(x) = e^{i\mu} \frac{d\mathcal{H}(x, x)}{dx}, \quad (6.20)$$

$$r(x) = e^{-i\mu} \frac{d\bar{\mathcal{H}}(x, x)}{dx}. \quad (6.21)$$

We would like to show that the scalar quantities $\mathcal{H}(x, y)$ and $\bar{\mathcal{H}}(x, y)$ defined in (6.15)–(6.17) satisfy the alternate Marchenko integral system

$$\begin{aligned} \mathcal{H}(x, y) + \bar{G}^{(u,v)}(x + y) \\ + \int_x^\infty dz \int_x^\infty dw \mathcal{K}_w(x, w) G^{(u,v)}(w + z) \bar{G}_z^{(u,v)}(z + y) = 0, \quad x < y, \end{aligned} \quad (6.22)$$

$$\begin{aligned} \bar{\mathcal{H}}(x, y) + G^{(p,s)}(x + y) \\ + \int_x^\infty dz \int_x^\infty dw \bar{\mathcal{K}}_w(x, w) \bar{G}^{(p,s)}(w + z) G_z^{(p,s)}(z + y) = 0, \quad x < y, \end{aligned} \quad (6.23)$$

where the subscripts denote the respective partial derivatives and we have defined

$$G^{(u,v)}(y) := \int_y^\infty dz \Omega^{(u,v)}(z), \quad \bar{G}^{(u,v)}(y) := \int_y^\infty dz \bar{\Omega}^{(u,v)}(z), \quad (6.24)$$

$$G^{(p,s)}(y) := \int_y^\infty dz \Omega^{(p,s)}(z), \quad \bar{G}^{(p,s)}(y) := \int_y^\infty dz \bar{\Omega}^{(p,s)}(z), \quad (6.25)$$

with the scalar functions $\Omega^{(u,v)}(z)$, $\bar{\Omega}^{(u,v)}(z)$, $\Omega^{(p,s)}(z)$, $\bar{\Omega}^{(p,s)}(z)$ defined as in (2.15) and (2.16) for the potentials pairs (u, v) and (p, s) , respectively. Note that using the input data sets given in (5.19) and (5.20) for the potentials pairs (u, v) and (p, s) , respectively, from (2.15) and (2.16) it follows that

$$\Omega^{(u,v)}(y) := \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda R^{(u,v)}(\lambda) e^{i\lambda y} + C^{(u,v)} e^{-Ay} B, \quad (6.26)$$

$$\bar{\Omega}^{(u,v)}(y) := \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \bar{R}^{(u,v)}(\lambda) e^{-i\lambda y} + \bar{C}^{(u,v)} e^{-\bar{A}y} \bar{B}, \quad (6.27)$$

$$\Omega^{(p,s)}(y) := \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda R^{(p,s)}(\lambda) e^{i\lambda y} + C^{(p,s)} e^{-Ay} B, \quad (6.28)$$

$$\bar{\Omega}^{(p,s)}(y) := \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \bar{R}^{(p,s)}(\lambda) e^{-i\lambda y} + \bar{C}^{(p,s)} e^{-\bar{A}y} \bar{B}. \quad (6.29)$$

From (5.21) and (5.22), in order to have consistency in (6.26) and (6.28) and consistency in (6.27) and (6.29), we must have

$$\Omega^{(u,v)} = -i\Omega_y^{(p,s)}, \quad \bar{\Omega}^{(p,s)} = i\bar{\Omega}_y^{(u,v)}.$$

In the next theorem we outline the derivation of (6.22) and (6.23).

Theorem 6.3. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the first-order system (1.1) belong to the Schwartz class. Let $u(x)$, $v(x)$, $p(x)$, $s(x)$ be the potentials defined as in (1.4), (1.5), (1.8), (1.9), respectively. Let $\mathcal{H}(x, y)$ and $\bar{\mathcal{H}}(x, y)$ be scalar quantities defined as in (6.15)–(6.17) and let $G^{(u,v)}(y)$, $\bar{G}^{(u,v)}(y)$, $G^{(p,s)}(y)$, $\bar{G}^{(p,s)}(y)$ be the quantities defined as in (6.24)–(6.29). Then, $\mathcal{H}(x, y)$ and $\bar{\mathcal{H}}(x, y)$ satisfy the alternate Marchenko equations given in (6.22) and (6.23), respectively.*

Proof. With the help of (2.28) we can write the 2×2 Marchenko system in (2.27) as four coupled scalar equations as

$$\bar{K}_1(x, w) + \int_x^\infty dz K_1(x, z) \Omega(z + w) = 0, \quad x < w, \quad (6.30)$$

$$K_1(x, w) + \bar{\Omega}(x + w) + \int_x^\infty dz \bar{K}_1(x, z) \bar{\Omega}(z + w) = 0, \quad x < w, \quad (6.31)$$

$$\bar{K}_2(x, w) + \Omega(x + w) + \int_x^\infty dz K_2(x, z) \Omega(z + w) = 0, \quad x < w, \quad (6.32)$$

$$K_2(x, w) + \int_x^\infty dz \bar{K}_2(x, z) \bar{\Omega}(z + w) = 0, \quad x < w. \quad (6.33)$$

In order to get an integral equation involving $\mathcal{K}(x, y)$, from the numerator of the right-hand side of (6.15), we see that we need to write (6.31) for the potential pair (u, v) and integrate the resulting equation over $w \in (y, +\infty)$. This yields

$$\begin{aligned} & \int_y^\infty dw K_1^{(u,v)}(x, w) + \int_y^\infty dw \bar{\Omega}^{(u,v)}(x + w) \\ & + \int_y^\infty dw \int_x^\infty dz \bar{K}_1^{(u,v)}(x, z) \bar{\Omega}^{(u,v)}(z + w) = 0, \quad x < z < y. \end{aligned} \quad (6.34)$$

Similarly, in order to get an integral equation involving $\bar{\mathcal{K}}(x, y)$, from the numerator of the right-hand side of (6.16), we see that we need to write (6.32) for the potential pair (p, s) and integrate the resulting equation over $w \in (y, +\infty)$. This yields

$$\begin{aligned} & \int_y^\infty dw \bar{K}_2^{(p,s)}(x, w) + \int_y^\infty dw \Omega^{(p,s)}(x + w) \\ & + \int_y^\infty dw \int_x^\infty dz K_2^{(p,s)}(x, z) \Omega^{(p,s)}(z + w) = 0, \quad x < z < y. \end{aligned} \quad (6.35)$$

By taking the derivatives of (6.24) and (6.25), we see that we can express $\Omega^{(u,v)}$, $\bar{\Omega}^{(u,v)}$, $\Omega^{(p,s)}$, $\bar{\Omega}^{(p,s)}$ in terms of $G^{(u,v)}$, $\bar{G}^{(u,v)}$, $G^{(p,s)}$, $\bar{G}^{(p,s)}$ as

$$\Omega^{(u,v)}(y) = -G_y^{(u,v)}(y), \quad \bar{\Omega}^{(u,v)}(y) = -\bar{G}_y^{(u,v)}(y), \quad (6.36)$$

$$\Omega^{(p,s)}(y) = -G_y^{(p,s)}(y), \quad \bar{\Omega}^{(p,s)}(y) = -\bar{G}_y^{(p,s)}(y), \quad (6.37)$$

where we recall that we use a subscript to indicate the corresponding derivative. From (6.15) and (6.16) we get

$$\mathcal{K}(x, y) = \left[\int_y^\infty dw K_1^{(u,v)}(x, w) \right] \left[1 + \int_x^\infty dz \bar{K}_1^{(u,v)}(x, z) \right]^{-1}, \quad x < y, \quad (6.38)$$

$$\bar{\mathcal{K}}(x, y) = \left[\int_y^\infty dw \bar{K}_2^{(p,s)}(x, w) \right] \left[1 + \int_x^\infty dz K_2^{(p,s)}(x, z) \right]^{-1}, \quad x < y. \quad (6.39)$$

With the help of (6.36) and (6.38), we write (6.34) as an integral equation involving $\mathcal{K}(x, y)$, $G^{(u,v)}(y)$, and $\bar{G}^{(u,v)}(y)$. Applying integration by parts on the resulting integrals and finally dividing the resulting equation by the denominator of the right-hand side of (6.15), we obtain

the integral equation (6.22) in the alternate Marchenko system. The derivation of (6.23) is obtained in a similar manner. First, by using (6.37) and (6.39) we write (6.35) in terms of $\bar{\mathcal{K}}(x, y)$, $G^{(p,s)}(y)$, and $\bar{G}^{(p,s)}(y)$. Applying integration by parts in the resulting integrals and finally dividing the resulting equation by the denominator of the right-hand side of (6.16) we obtain the integral equation (6.23) in the alternate Marchenko system. \square

Acknowledgment

The second author is grateful to Prof T Tsuchida for his help.

ORCID iDs

T Aktosun  <https://orcid.org/0000-0003-1094-5676>

References

- [1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
- [2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform-Fourier analysis for nonlinear problems *Stud. Appl. Math.* **53** 249–315
- [3] Ablowitz M J and Segur H 1981 *Solitons and The Inverse Scattering Transform* (Philadelphia, PA: SIAM)
- [4] Aktosun T, Busse T, Demontis F and van der Mee C 2010 Symmetries for exact solutions to the nonlinear Schrödinger equation *J. Phys. A: Math. Theor.* **43** 025202
- [5] Aktosun T, Demontis F and van der Mee C 2007 Exact solutions to the focusing nonlinear Schrödinger equation *Inverse Problems* **23** 2171–95
- [6] Aktosun T, Klaus M and van der Mee C 1998 Wave scattering in one dimension with absorption *J. Math. Phys.* **39** 1957–92
- [7] Aktosun T and van der Mee C 1991 Scattering and inverse scattering for the 1-D Schrödinger equation with energy-dependent potentials *J. Math. Phys.* **32** 2786–801
- [8] Aktosun T, Klaus M and van der Mee C 1998 Inverse scattering in one-dimensional nonconservative media *Integr. Equ. Oper. Theory* **30** 279–316
- [9] Busse T N 2008 Generalized inverse scattering transform for the nonlinear Schrödinger equation *PhD Thesis* The University of Texas at Arlington
- [10] Ercan R 2018 Scattering and inverse scattering on the line for a first-order system with energy-dependent potentials *PhD Thesis* The University of Texas at Arlington
- [11] Guillaume M O and Jaulent M 1982 A Zakharov–Shabat inverse scattering problem with a polynomial spectral dependence in the potential *Lett. Math. Phys.* **6** 189–98
- [12] Jaulent M and Jean C 1976 The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. I *Ann. Inst. Henri Poincaré A* **25** 105–18 (http://www.numdam.org/item/AIHPA_1976__25_2_105_0)
- [13] Jaulent M and Jean C 1976 The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. II *Ann. Inst. Henri Poincaré A* **25** 119–37 (http://www.numdam.org/item/AIHPA_1976__25_2_119_0)
- [14] Kaup D J and Newell A C 1978 An exact solution for a derivative nonlinear Schrödinger equation *J. Math. Phys.* **19** 798–801
- [15] Tsuchida T 2010 New reductions of integrable matrix partial differential equations: $Sp(m)$ -invariant systems *J. Math. Phys.* **51** 053511
- [16] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys. JETP* **34** 62–9 (<http://jetp.ac.ru/cgi-bin/e/index/e/34/1/p62?a=list>)