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# EXPLICIT WIENER-HOPF FACTORIZATION FOR CERTAIN NON-RATIONAL MATRIX FUNCTIONS

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Explicit Wiener-Hopf factorizations are obtained for a certain class of nonrational $2 \times 2$ matrix functions that are related to the scattering matrices for the 1-D Schrödinger equation. The diagonal elements coincide and are meromorphic and nonzero in the upper-half complex plane and either they vanish linearly at the origin or they do not vanish. The most conspicuous nonrationality consists of imaginary exponential factors in the off-diagonal elements.

## 1. INTRODUCTION

In this article we obtain explicit Wiener-Hopf factorizations of certain nonrational $2 \times 2$ matrix functions which arise as (modified) scattering matrices for the 1-D Schrödinger equation **[20,21,22]** and some related Schrödinger-type equations **[6,8]**. These matrix functions have the form

$$\mathbf{G}(k,x) = \begin{pmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{-2ikx} & T(k) \end{pmatrix}, \tag{1.1}$$

where, for any real parameter $x$,

1. $T(k)$ is nonzero on $\overline{\mathbf{C}^+}\setminus\{0\}$,[1] is meromorphic on $\mathbf{C}^+$ with continuous boundary values on the extended real axis, either $T(0) \neq 0$ or $T(k)$ vanishes linearly at $k = 0$, and $T(\infty) = 1$,
2. $R(k)$ and $L(k)$ are meromorphic on $\mathbf{C}^+$ with continuous boundary values on the extended real axis and vanish as $k \to \infty$ in $\overline{\mathbf{C}^+}$,
3. $\mathbf{G}(k,x)^{-1} = \mathbf{q}\mathbf{G}(-k,x)\mathbf{q}$ for $k \in \mathbf{R}$, where $\mathbf{q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and
4. $\mathbf{G}(k,x)$, as a function of $k \in \mathbf{R}$, belongs to a suitable Banach algebra of $2 \times 2$ matrix functions within which Wiener-Hopf factorization is possible. This may be the Wiener algebra or the algebra of functions $\mathbf{f}(k)$ such that $\mathbf{f}^\star(\xi) = \mathbf{f}(i\frac{1+\xi}{1-\xi})$ is Hölder continuous with exponent $\alpha$ on the unit circle where $\alpha \in (0,1)$. We will define these algebras shortly.

Wiener-Hopf factorization problems of the above type arise as an offshoot of the inverse scattering problems for the 1-D Schrödinger equation **[20,21,22]** and some related

[1] Throughout this article we denote by $\mathbf{C}^+$ and $\mathbf{C}^-$ the open upper and lower half-planes and by $\overline{\mathbf{C}^+}$ and $\overline{\mathbf{C}^-}$ the closed upper and lower half-planes including infinity.

Schrödinger-type equations [**6,8**]. $T(k)$ is usually called the transmission coefficient, $R(k)$ and $L(k)$ the reflection coefficients from the right and the left, respectively, and

$$\mathbf{S}(k) = \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix} \tag{1.2}$$

is the scattering matrix. The solution of the inverse scattering problem is achieved by obtaining the potential of the Schrödinger equation when the scattering matrix is known. Such an inverse scattering problem can be posed [**20,21,22**] as a Riemann-Hilbert problem which can be solved by various means, such as the methods due to Gel'fand and Levitan, Marchenko, Faddeev, and Newton [**11,12,13,20,21,22**], where the Riemann-Hilbert problem is transformed into a nonhomogeneous Fredholm integral equation. When the reflection coefficients have meromorphic extensions to $\mathbf{C}^+$, the resulting integral equation has a separable kernel and thus its solution can be obtained explicitly by solving a system of linear algebraic equations. It is then possible to obtain the solution of this Riemann-Hilbert problem by a contour integration [**1**] without solving the Fredholm integral equation when $T(0) \neq 0$; if $T(0) = 0$, one can find a scattering matrix $\mathbf{S}_\epsilon(k)$ such that its transmission coefficient does not vanish at $k = 0$ and $\mathbf{S}_\epsilon(k) \to S(k)$ as $\epsilon \to 0$. Then the Riemann-Hilbert problem can be solved using $\mathbf{S}_\epsilon(k)$ as the input matrix, and then letting $\epsilon \to 0$ one obtains the solution of the Riemann-Hilbert problem where the input matrix is $\mathbf{S}(k)$ [**1,2**]. When $T(k)$ has a zero at $k = 0$, the factorization of $\mathbf{G}(k, x)$ becomes noncanonical; in this case the solution of the inverse scattering problem becomes nonunique unless $R(0) = L(0) = -1$ and the zero of $T(k)$ at $k = 0$ is a simple one. Explicit examples of nonuniqueness of the solution of the inverse scattering problem for the 1-D Schrödinger equation can be found in [**3,4,5,7,10**].

For many years it has been customary to view explicit Wiener-Hopf factorization of nonrational matrix functions as a Herculean task well-nigh impossible to carry out. In recent years there have appeared some papers [**16,17,19,24**] in which nonrational $2 \times 2$ matrix functions within special classes are factorized explicitly. The present article is devoted to a completely different class of $2 \times 2$ matrix functions and our factorization method differs significantly from the ones adopted in [**16,17,19,24**]. In this paper we will obtain the Wiener-Hopf factors of the matrix $\mathbf{G}(k, x)$ given in (1.1) by the contour integration method.

This article is organized as follows. In Section 2 we give the preliminary results needed for the factorization. In Section 3, assuming $T(0) \neq 0$, we pose the inverse scattering problem for the 1-D Schrödinger equation as a matrix Riemann-Hilbert problem and obtain the canonical Wiener-Hopf factors of $\mathbf{G}(k, x)$ by solving the Riemann-Hilbert problem posed. In Section 4 explicit canonical factorizations of $\mathbf{G}(k, x)$ are obtained by the contour integration method when the reflection coefficients have meromorphic extension to $\mathbf{C}^+$ with continuous boundary values as $k$ approaches the extended real axis. In Section 5 we treat the case $T(0) = 0$ and the case where the extension of $T(k)$ to $\mathbf{C}^+$ is meromorphic, and we obtain the noncanonical Wiener-Hopf factorization of $\mathbf{G}(k, x)$. In Section 6 some instructive examples are presented. Finally, in the Appendix some special functions needed in Section 4 are defined.

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## 2. PRELIMINARY RESULTS

A $2\times 2$ matrix function $\mathbf{W}(k)$ for $k \in \mathbf{R}$ has a (right) Wiener-Hopf factorization if there exist matrix functions $\mathbf{W}_+(k)$ and $\mathbf{W}_-(k)$, complementary rank-one projections $\mathbf{Q}_+$ and $\mathbf{Q}_-$, and integers $\rho_1$ and $\rho_2$ such that

1. $\mathbf{W}_\pm(k)$ can be extended to a matrix function that is continuous and invertible on $\overline{\mathbf{C}^\pm}$,
2. the extension of $\mathbf{W}_\pm(k)$ is analytic on $\mathbf{C}^\pm$, and
3. the equality

$$\mathbf{W}(k) = \mathbf{W}_-(k)\left[\left(\frac{k-i}{k+i}\right)^{\rho_1}\mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)^{\rho_2}\mathbf{Q}_-\right]\mathbf{W}_+(k), \qquad k\in\mathbf{R}, \tag{2.1}$$

holds true.

The partial indices $\rho_1$ and $\rho_2$ are uniquely determined by $\mathbf{W}(k)$. Their sum, the sum index, is the winding number of $\det \mathbf{W}(k)$ with respect to $+i$. If $\rho_1 = \rho_2 = 0$ so that (2.1) reduces to $\mathbf{W}(k) = \mathbf{W}_-(k)\,\mathbf{W}_+(k)$, the factorization (2.1) is called (right) canonical. It is possible to choose the factorization (2.1) in such a way that $\mathbf{Q}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{Q}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For general information on Wiener-Hopf factorization of matrix functions, we refer the reader to **[9,14]**.

In inverse scattering theory the matrix function $\mathbf{W}(k)$ usually satisfies $\mathbf{W}(\infty) = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix. In that case, we will require that $\mathbf{W}_+(\infty) = \mathbf{W}_-(\infty) = \mathbf{I}$. It may then no longer be possible to choose $\mathbf{Q}_+$ and $\mathbf{Q}_-$ as the coordinate projections. Instead, we will choose $\mathbf{Q}_\pm$ as in (5.1).

The Wiener algebra $\mathcal{W}^{p\times q}$ is defined as the Banach space of all complex $p\times q$ matrix functions $\mathbf{f}(k)$ for $k\in\mathbf{R}$ of the form

$$\mathbf{f}(k) = \mathbf{f}(\infty) + \int_{-\infty}^{\infty} dy\, e^{iky}\hat{\mathbf{f}}(y),$$

where $\int_{-\infty}^{\infty} dy\,\|\hat{\mathbf{f}}(y)\|$ is finite, endowed with the norm

$$\|\mathbf{f}\|_{\mathcal{W}^{p\times q}} = \|\mathbf{f}(\infty)\| + \int_{-\infty}^{\infty} dy\,\|\hat{\mathbf{f}}(y)\|. \tag{2.2}$$

Further, for $0<\alpha<1$, $\mathcal{H}_\alpha^{p\times q}$ denotes the Banach space of all complex $p\times q$ matrix functions $\mathbf{f}(k)$ for $k\in\mathbf{R}$ such that $\mathbf{f}^\star(\xi) = \mathbf{f}(i\frac{1+\xi}{1-\xi})$ is Hölder continuous on the unit circle $\mathbf{T}$ with exponent $\alpha$, endowed with the norm

$$\|\mathbf{f}\|_{\mathcal{H}_\alpha^{p\times q}} = \max_{\xi\in\mathbf{T}}\|\mathbf{f}^\star(\xi)\| + \sup_{\xi_1\neq\xi_2\in\mathbf{T}} \frac{\|\mathbf{f}^\star(\xi_1)-\mathbf{f}^\star(\xi_2)\|}{|\xi_1-\xi_2|^\alpha}. \tag{2.3}$$

In (2.2) and (2.3), $\|\cdot\|$ is a suitable $p \times q$ vector norm. We write $\mathcal{W}$ and $\mathcal{H}_\alpha$ for $\mathcal{W}^{1\times 1}$ and $\mathcal{H}_\alpha^{1\times 1}$, respectively.

Let $\mathbf{W}(k) \in \mathcal{H}_\alpha^{2\times 2}$ for some $\alpha \in (0,1)$. Then if $\mathbf{W}(k)$ is invertible for all $k \in \mathbf{R} \cup \{\infty\}$, $\mathbf{W}(k)$ has a (right) Wiener-Hopf factorization of the form (2.1) where $\mathbf{W}_+^\star(\xi) = \mathbf{W}_+(i\frac{1+\xi}{1-\xi})$ is Hölder continuous of exponent $\alpha$ on $\mathbf{T}_+$ and $\mathbf{W}_-^\star(\xi) = \mathbf{W}_-(i\frac{1+\xi}{1-\xi})$ is Hölder continuous of exponent $\alpha$ on $\mathbf{T}_-$ ([9], Theorem II 6.2). Here $\mathbf{T}_+$ is the set of all $\xi \in \mathbf{C}$ with $|\xi| \le 1$, and $\mathbf{T}_-$ is the set of all $\xi \in \mathbf{C}$ with $|\xi| \ge 1$ including $\infty$. Similarly, if $\mathbf{W}(k)$ is invertible for all $k \in \mathbf{R} \cup \{\infty\}$ and $\mathbf{W} \in \mathcal{W}^{2\times 2}$, $\mathbf{W}(k)$ has a (right) Wiener-Hopf factorization of the form (2.1) where $\mathbf{W}_+(k)$, $\mathbf{W}_-(k)$, and their inverses belong to $\mathcal{W}^{2\times 2}$ ([9], Theorem II 6.3).

PROPOSITION 2.1. *Suppose*

$$\mathbf{W}(k) = \begin{pmatrix} 1 & q(k) \\ -\overline{q(k)} & 1 \end{pmatrix}, \qquad k \in \mathbf{R}, \tag{2.4}$$

*where* $q(\infty) = 0$, *and* $q \in \mathcal{W}$ *or* $q \in \mathcal{H}_\alpha$ *for some* $\alpha \in (0,1)$. *Then* $\mathbf{W}(k)$ *has a unique (right) canonical factorization*

$$\mathbf{W}(k) = \mathbf{W}_-(k)\,\mathbf{W}_+(k), \qquad k \in \mathbf{R},$$

*where* $\mathbf{W}_\pm \in \mathcal{W}^{2\times 2}$ *or* $\mathbf{W}_\pm \in \mathcal{H}_\alpha^{2\times 2}$, *respectively, and* $\mathbf{W}_\pm(\infty) = \mathbf{I}$.

**Proof**: Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the usual inner product and $L^2$-norm on $\mathbf{C}^2$, respectively. Then

$$\operatorname{Re}\langle \mathbf{W}(k)\eta, \eta\rangle = \|\eta\|^2, \qquad \eta \in \mathbf{C}^2,\ k \in \mathbf{R},$$

which, according to Lemma 1.1 of [15], implies that $\sup_{k\in\mathbf{R}} \|\gamma \mathbf{W}(k) - \mathbf{I}\| < 1$ for all $k \in \mathbf{R}$ and a suitable constant $\gamma$. Since $\mathbf{W} \in \mathcal{W}^{2\times 2}$ or $\mathbf{W} \in \mathcal{H}_\alpha^{2\times 2}$ for some $\alpha \in (0,1)$, the result is clear from Theorem 1.1 (for $\mathcal{W}^{2\times 2}$) and Theorem 5.1 (for $\mathcal{H}_\alpha^{2\times 2}$) of [15]. ∎

COROLLARY 2.2. *Suppose*

$$\mathbf{G}(k,x) = \begin{pmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{-2ikx} & T(k) \end{pmatrix}$$

*is a unitary matrix for all* $k \in \mathbf{R}$ *such that*

1. $T(k)$ *is nonzero for all* $k \in \mathbf{R}$,
2. $T(k)$ *can be continued to a meromorphic function on* $\mathbf{C}^+$ *with continuous boundary values on the extended real axis, and* $T(\infty) = 1$,
3. $T(k)$, $R(k)$, *and* $L(k)$ *belong to either* $\mathcal{W}$ *or* $\mathcal{H}_\alpha$ *for some* $\alpha \in (0,1)$.

*Then* $\mathbf{G}(k,x)$ *has a (right) Wiener-Hopf factorization with equal indices* $\rho_1 = \rho_2 = \rho$, *where* $\rho$ *is the number of zeros minus the number of poles of* $T(k)$ *in* $\mathbf{C}^+$.

**Proof**: From the unitarity of $\mathbf{G}(k,x)$ it follows that $\overline{T(k)}R(k) = -T(k)\overline{L(k)}$, and thus $\mathbf{G}(k)/T(k)$ coincides with the matrix function (2.4) with $q(k) = -R(k)e^{2ikx}/T(k)$. So $\mathbf{G}(k,x)/T(k)$ has the (right) canonical factorization

$$\frac{\mathbf{G}(k,x)}{T(k)} = \mathbf{W}_-(k,x)\,\mathbf{W}_+(k,x), \qquad k \in \mathbf{R},$$

where $\mathbf{W}_{\pm}(\infty,x)=\mathbf{I}$. Also, the scalar function $T(k)$ has the Wiener-Hopf factorization

$$T(k)=T_-(k)\left(\frac{k-i}{k+i}\right)^{\rho}T_+(k),\qquad k\in\mathbf{R},$$

where $T_{\pm}(\infty)=1$. Hence,

$$\mathbf{G}(k,x)=\mathbf{W}_-(k,x)\,T_-(k)\cdot\left(\frac{k-i}{k+i}\right)^{\rho}\mathbf{I}\cdot\mathbf{W}_+(k,x)\,T_+(k),\qquad k\in\mathbf{R},$$

is a Wiener-Hopf factorization of $\mathbf{G}(k,x)$. ∎

In the scattering theory for the 1-D Schrödinger equation one has a more special case than that given in Corollary 2.2, namely $T_-(k)=1$ and $T(k)$ does not have any zeros in $\mathbf{C}^+$; the poles of $T(k)$ in $\mathbf{C}^+$ correspond to the bound state energies for the Schrödinger equation (3.1).

## 3. INVERSE SCATTERING PROBLEM

Consider the 1-D Schrödinger equation

$$-\psi''(k,x)+V(x)\,\psi(k,x)=k^2\psi(k,x),\qquad x\in\mathbf{R},\tag{3.1}$$

where the prime denotes differentiation with respect to $x$, $k^2$ is energy, and $V(x)$ is the (real) potential assumed to satisfy $\int_{-\infty}^{\infty}dx\,(1+|x|)|V(x)|<\infty$ and is allowed to contain delta distributions. Being a second-order differential equation, (3.1) has two linearly independent solutions, which we will call $\psi_l(k,x)$ and $\psi_r(k,x)$, satisfying the boundary conditions

$$\psi_l(k,x)=\begin{cases}T(k)e^{ikx}+o(1), & x\to+\infty\\ e^{ikx}+L(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases}\tag{3.2}$$

$$\psi_r(k,x)=\begin{cases}e^{-ikx}+R(k)e^{ikx}+o(1), & x\to+\infty\\ T(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases}\tag{3.3}$$

where $T$ is the transmission coefficient, and $L$ and $R$ are the reflection coefficients. The inverse scattering problem is to obtain the potential $V(x)$ from the scattering data $\mathbf{S}(k)$. Let $m_l(k,x)=\frac{1}{T(k)}\,e^{-ikx}\psi_l(k,x)$ and $m_r(k,x)=\frac{1}{T(k)}\,e^{ikx}\psi_r(k,x)$. We will call $m_l(k,x)$ and $m_r(k,x)$ Faddeev solutions of the Schrödinger equation. They satisfy the differential equations

$$m_l''(k,x)+2ikm_l'(k,x)=V(x)\,m_l(k,x),$$

$$m_r''(k,x)-2ikm_r'(k,x)=V(x)\,m_r(k,x),$$

with boundary conditions

$$m_l(k,x) = 1 + o(1), \quad m_l'(k,x) = o(1), \qquad x \to +\infty,$$

$$m_r(k,x) = 1 + o(1), \quad m_r'(k,x) = o(1), \qquad x \to -\infty.$$

In the Schrödinger equation (3.1), $k$ appears as $k^2$ and hence $\{\psi_l(k,x), \psi_r(k,x)\}$ and $\{\psi_l(-k,x), \psi_r(-k,x)\}$ each form an independent set of solutions of (3.1). Thus, the first set can be written as a linear combination of the second set. As a result we are led to **[21,22]**

$$\begin{pmatrix} \psi_l(k,x) \\ \psi_r(k,x) \end{pmatrix} = \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix} \begin{pmatrix} \psi_r(-k,x) \\ \psi_l(-k,x) \end{pmatrix}, \qquad k \in \mathbf{R}. \tag{3.4}$$

In terms of the Faddeev solutions $\mathbf{m}(k,x) = \begin{pmatrix} m_l(k,x) \\ m_r(k,x) \end{pmatrix}$ and $\mathbf{G}(k,x)$ defined in (1.1), we can write (3.4) as

$$\mathbf{m}(-k,x) = \mathbf{G}(k,x)\,\mathbf{q}\,\mathbf{m}(k,x), \qquad k \in \mathbf{R}. \tag{3.5}$$

When $T(k)$ has analytic extension to $\mathbf{C}^+$, for the class of potentials specified in the beginning of this section there is a one-to-one correspondence between that class and a class of scattering matrices **[12,18]**, and it follows that if $T(0) \neq 0$, then $\mathbf{m}(k,x)$ is continuous on $\overline{\mathbf{C}^+}$, analytic on $\mathbf{C}^+$, and $\mathbf{m}(k,x) = \hat{1} + O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, where $\hat{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If $T(k)$ vanishes linearly at $k = 0$, these properties of $\mathbf{m}(k,x)$ are retained except for the continuity at $k = 0$; however, when $R(0) = L(0) = -1$, the continuity of $\mathbf{m}(k,x)$ is also valid at $k = 0$ **[12,18]**. The vector $\mathbf{m}(k,x)$ can then be obtained uniquely by solving (3.5) provided $T(k)$ is analytic in $\mathbf{C}^+$. Hence, if $T(k)$ has analytic extension to $\mathbf{C}^+$ and is nonzero in $\overline{\mathbf{C}^+}$, the Riemann-Hilbert problem

$$\mathbf{n}(-k,x) = \mathbf{J}\,\mathbf{G}(k,x)\,\mathbf{J}\,\mathbf{q}\,\mathbf{n}(k,x), \qquad k \in \mathbf{R}, \tag{3.6}$$

where $\mathbf{J} = \operatorname{diag}(1,-1)$, is also uniquely solvable for the vector $\mathbf{n}(k,x)$ possessing the same analyticity and continuity properties as $\mathbf{m}(k,x)$. In fact, defining

$$\mathbf{M}(k,x) = \frac{1}{2} \begin{pmatrix} m_l(k,x) + n_l(k,x) & m_l(k,x) - n_l(k,x) \\ m_r(k,x) - n_r(k,x) & m_r(k,x) + n_r(k,x) \end{pmatrix}, \tag{3.7}$$

where

$$\mathbf{m}(k,x) = \begin{pmatrix} m_l(k,x) \\ m_r(k,x) \end{pmatrix} = \mathbf{M}(k,x)\hat{1}, \quad \mathbf{n}(k,x) = \begin{pmatrix} n_l(k,x) \\ n_r(k,x) \end{pmatrix} = \mathbf{J}\,\mathbf{M}(k,x)\,\hat{e}, \tag{3.8}$$

and $\hat{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, from (3.5) and (3.6) one obtains the matrix Riemann-Hilbert problem

$$\mathbf{M}(-k,x) = \mathbf{G}(k,x)\,\mathbf{q}\,\mathbf{M}(k,x)\,\mathbf{q}, \qquad k \in \mathbf{R}. \tag{3.9}$$

Hence, if $T(k)$ has analytic extension to $\mathbf{C}^+$ and is nonzero in $\overline{\mathbf{C}^+}$, (3.9) is uniquely solvable and the solution matrix $\mathbf{M}(k,x)$ is continuous on $\overline{\mathbf{C}^+}$, analytic on $\mathbf{C}^+$, and $\mathbf{M}(k,x) = \mathbf{I} + O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Since the scattering matrix $\mathbf{S}(k)$ satisfies the property $\mathbf{S}(-k) = \mathbf{q}\mathbf{S}(k)^{-1}\mathbf{q}$, it follows that

$$\det \mathbf{G}(k,x) = \det \mathbf{S}(k) = \frac{T(k)}{T(-k)}. \tag{3.10}$$

Hence, from (3.9) and (3.10) we obtain

$$T(-k) \det \mathbf{M}(-k,x) = T(k) \det \mathbf{M}(k,x), \qquad k \in \mathbf{R},$$

and from Liouville's theorem it follows that

$$\det \mathbf{M}(k,x) = \frac{1}{T(k)}, \qquad k \in \overline{\mathbf{C}^+}.$$

Thus, if $T(k)$ is nonzero in $\overline{\mathbf{C}^+}$, the matrix $\mathbf{M}(k,x)^{-1}$ is also continuous on $\overline{\mathbf{C}^+}$, analytic on $\mathbf{C}^+$, and $\mathbf{M}(k,x)^{-1} = \mathbf{I} + O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Hence, from (3.9) it follows that $\mathbf{G}(k,x)$ has the canonical factorization

$$\mathbf{G}(k,x) = \mathbf{M}(-k,x)\, \mathbf{q}\, \mathbf{M}(k,x)^{-1}\, \mathbf{q} \tag{3.11}$$

with factors $\mathbf{G}_+(k,x) = \mathbf{M}(-k,x)$ and $\mathbf{G}_-(k,x) = \mathbf{q}\, \mathbf{M}(k,x)^{-1}\, \mathbf{q}$. Thus, the Wiener-Hopf factorization of $\mathbf{G}(k,x)$ given in (1.1) can be achieved by solving the inverse scattering problem for the scattering matrix $\mathbf{S}(k)$ given in (1.2).

Let us start the process of evaluating of $\mathbf{M}(k,x)$ when $\mathbf{G}(k,x)$ is given. Defining

$$\mathbf{B}(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-iky}[\mathbf{M}(k,x) - \mathbf{I}], \tag{3.12}$$

from the analyticity properties of $\mathbf{M}(k,x)$ it follows that $\mathbf{B}(x,y) = 0$ for $y < 0$ and hence

$$\mathbf{M}(k,x) = \mathbf{I} + \int_0^{\infty} dy\, e^{iky}\mathbf{B}(x,y).$$

Writing (3.9) in the form

$$\mathbf{M}(-k,x) - \mathbf{I} = [\mathbf{G}(k,x) - \mathbf{I}]\mathbf{q}\, \mathbf{M}(k,x)\, \mathbf{q} + \mathbf{q}[\mathbf{M}(k,x) - \mathbf{I}]\mathbf{q}, \qquad k \in \mathbf{R},$$

and using (3.12), we are led to

$$\mathbf{B}(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{iky}[\mathbf{G}(k,x) - \mathbf{I}]\mathbf{q}\, \mathbf{M}(k,x)\, \mathbf{q}, \qquad y > 0. \tag{3.13}$$

Using (3.8), from (3.13) we obtain

$$\begin{aligned}\mathbf{b}(x,y)=\begin{pmatrix} b_l(x,y)\\ b_r(x,y)\end{pmatrix}&=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{iky}\,[\mathbf{G}(k,x)-\mathbf{I}]\,\mathbf{q}\,\mathbf{m}(k,x)\\ &=-\int_{-\infty}^{\infty}\frac{dk}{2\pi}\begin{pmatrix}0 & R(k)e^{ik(2x+y)}\\ L(k)e^{ik(-2x+y)} & 0\end{pmatrix}\mathbf{q}\,\mathbf{m}(k,x),\qquad y>0,\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\mathbf{c}(x,y)=\begin{pmatrix} c_l(x,y)\\ c_r(x,y)\end{pmatrix}&=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{iky}\,[\mathbf{J}\,\mathbf{G}(k,x)\,\mathbf{J}-\mathbf{I}]\,\mathbf{q}\,\mathbf{n}(k,x)\\ &=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\begin{pmatrix}0 & R(k)e^{ik(2x+y)}\\ L(k)e^{ik(-2x+y)} & 0\end{pmatrix}\mathbf{q}\,\mathbf{n}(k,x),\qquad y>0,\end{aligned}\tag{3.15}$$

where we have defined $\mathbf{b}(x,y)$ and $\mathbf{c}(x,y)$ as

$$\mathbf{b}(x,y)=\mathbf{B}(k,x)\hat{1}=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{-iky}[\mathbf{m}(k,x)-\hat{1}],$$

$$\mathbf{c}(x,y)=\mathbf{J}\mathbf{B}(k,x)\hat{e}=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{-iky}[\mathbf{n}(k,x)-\hat{1}].$$

In the special case where $R(k)$ extends to a function meromorphic on $\mathbf{C}^+$ with continuous boundary values on the extended real line and only simple poles, say $N_R$ of them, $b_l(x,y)$ and $c_l(x,y)$ for $x\geq 0$ can be computed from (3.14) and (3.15) by performing a contour integration and solving a linear system of order $N_R$. Fourier transformation then yields $m_l(k,x)$ and $n_l(k,x)$ by using

$$\mathbf{m}(k,x)=\hat{1}+\int_0^{\infty}dy\,e^{iky}\mathbf{b}(x,y),\tag{3.16}$$

$$\mathbf{n}(k,x)=\hat{1}+\int_0^{\infty}dy\,e^{iky}\mathbf{c}(x,y).\tag{3.17}$$

On the other hand, if $L(k)$ extends to a function meromorphic on $\mathbf{C}^+$ with continuous boundary values on the extended real line and only simple poles, say $N_L$ of them, then $b_r(x,y)$ and $c_r(x,y)$ for $x\leq 0$ can be computed from (3.14) and (3.15) in a similar fashion. From (3.16) and (3.17) one then finds $m_r(k,x)$ and $n_r(k,x)$. From (3.5) and (3.6) it follows that

$$m_l(k,x)=\{m_r(-k,x)+e^{-2ikx}L(k)m_r(k,x)\}/T(k),\tag{3.18}$$

$$n_l(k,x)=\{n_r(-k,x)-e^{-2ikx}L(k)n_r(k,x)\}/T(k),\tag{3.19}$$

$$m_r(k,x) = \{m_l(-k,x) + e^{2ikx}R(k)m_l(k,x)\}/T(k), \tag{3.20}$$

$$n_r(k,x) = \{n_l(-k,x) - e^{2ikx}R(k)n_l(k,x)\}/T(k). \tag{3.21}$$

Thus, using (3.18)-(3.21) one obtains $\mathbf{M}(k,x)$ for $x \in \mathbf{R}$. In the next section we will use this procedure to obtain $\mathbf{M}(k,x)$ explicitly when the reflection coefficients have meromorphic extensions to $\mathbf{C}^+$, and thus the canonical Wiener-Hopf factorization of $\mathbf{G}(k,x)$ will be obtained as in (3.11).

## 4. EXPLICIT FACTORIZATION

In this section we will obtain explicit expressions for the Faddeev solutions $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ of the Riemann-Hilbert problems (3.5) and (3.6) for a certain class of $\mathbf{G}(k,x)$. Then, the canonical Wiener-Hopf factors of $\mathbf{G}(k,x)$ can be determined as in (3.11). The function $F_{t-s}(k,x,\kappa)$ appearing in (4.1)-(4.4) below will be defined in the Appendix.

THEOREM 4.1. *Suppose*

1. *$T(k)$ is nonzero for all $k \in \mathbf{R}$, $T(\infty)=1$, $R(\infty)=0$ and $L(\infty)=0$,*
2. *$T(k)$ is continuous on $\overline{\mathbf{C}^+}$ and analytic on $\mathbf{C}^+$,*
3. *$T(k)$, $R(k)$ and $L(k)$ belong to either $\mathcal{W}$ or $\mathcal{H}_\alpha$ for some $\alpha \in (0,1)$,*
4. *$\mathbf{S}(-k) = \mathbf{q}\mathbf{S}(k)^{-1}\mathbf{q}$, $k \in \mathbf{R}$, where $\mathbf{S}(k)$ is the matrix defined in (1.2),*

*and $\mathbf{G}(k,x)$ is defined by (1.1). In addition, assume that $R(k)$ is meromorphic on $\mathbf{C}^+$ with principal parts $\sum_{s=0}^{p_j-1}(k-i\kappa_j)^{-(s+1)}R_{j,s}$ at the poles $i\kappa_j$ ($j=1,\cdots,N_R$) and with continuous boundary values on the extended real axis. Suppose $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ are solutions of the Riemann-Hilbert problems (3.5) and (3.6) which are continuous on $\overline{\mathbf{C}^+}$, are analytic on $\mathbf{C}^+$ and approach $\hat{1}$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Then for $x \ge 0$*

$$m_l(k,x) = 1 + \sum_{j=1}^{N_R}\sum_{s=0}^{p_j-1}\frac{1}{s!}\left[\left(\frac{d}{dk}\right)^s m_l(k,x)\right]_{k=i\kappa_j}\sum_{t=s}^{p_j-1} i^{t-s}R_{j,t}F_{t-s}(k,x,\kappa_j), \tag{4.1}$$

$$n_l(k,x) = 1 - \sum_{j=1}^{N_R}\sum_{s=0}^{p_j-1}\frac{1}{s!}\left[\left(\frac{d}{dk}\right)^s n_l(k,x)\right]_{k=i\kappa_j}\sum_{t=s}^{p_j-1} i^{t-s}R_{j,t}F_{t-s}(k,x,\kappa_j). \tag{4.2}$$

*Similarly, if $L(k)$ is meromorphic on $\mathbf{C}^+$ with principal parts $\sum_{s=0}^{q_j-1}(k-i\lambda_j)^{-(s+1)}L_{j,s}$ at the poles $i\lambda_j$ ($j=1,\cdots,N_L$) and with continuous boundary values on the extended real axis, then for $x \le 0$*

$$m_r(k,x) = 1 + \sum_{j=1}^{N_L}\sum_{s=0}^{q_j-1}\frac{1}{s!}\left[\left(\frac{d}{dk}\right)^s m_r(k,x)\right]_{k=i\lambda_j}\sum_{t=s}^{q_j-1} i^{t-s}L_{j,t}F_{t-s}(k,-x,\lambda_j), \tag{4.3}$$

$$(4.4) \qquad n_r(k,x) = 1 - \sum_{j=1}^{N_L} \sum_{s=0}^{q_j-1} \frac{1}{s!} \left[ \left( \frac{d}{dk} \right)^s n_r(k,x) \right]_{k=i\lambda_j} \sum_{t=s}^{q_j-1} i^{t-s} L_{j,t} F_{t-s}(k,-x,\lambda_j).$$

*Conversely, any pair of vector functions* $\mathbf{m}(k,x)$ *and* $\mathbf{n}(k,x)$ *satisfying* (4.1), (4.2), (3.20) *and* (3.21) *[for* $x \geq 0$*], or* (4.3), (4.4), (3.18) *and* (3.19) *[for* $x \leq 0$*] are solutions of the Riemann-Hilbert problems* (3.5) *and* (3.6) *and are analytic in* $\mathbf{C}^+$.

**Proof:** By calculus of residues, we get from (3.14) and (3.15)

$$(4.5) \qquad b_l(x,y) = (-i) \sum_{j=1}^{N_R} \operatorname*{Res}_{k=i\kappa_j} \{R(k)\, m_l(k,x)\, e^{ik(y+2x)}\}, \qquad x \geq 0,$$

$$(4.6) \qquad c_l(x,y) = (+i) \sum_{j=1}^{N_R} \operatorname*{Res}_{k=i\kappa_j} \{R(k)\, n_l(k,x)\, e^{ik(y+2x)}\}, \qquad x \geq 0,$$

$$(4.7) \qquad b_r(x,y) = (-i) \sum_{j=1}^{N_L} \operatorname*{Res}_{k=i\lambda_j} \{L(k)\, m_r(k,x)\, e^{ik(y-2x)}\}, \qquad x \leq 0,$$

$$(4.8) \qquad c_r(x,y) = (+i) \sum_{j=1}^{N_L} \operatorname*{Res}_{k=i\lambda_j} \{L(k)\, n_r(k,x)\, e^{ik(y-2x)}\}, \qquad x \leq 0.$$

Further,

$$\begin{aligned}
&\operatorname*{Res}_{k=i\kappa_j} \{R(k)\, m_l(k,x)\, e^{ik(y+2x)}\} \\
&= \sum_{s=0}^{p_j-1} R_{j,s} \frac{1}{s!} \left[ \left( \frac{d}{dk} \right)^s \{m_l(k,x)\, e^{ik(y+2x)}\} \right]_{k=i\kappa_j} \\
&= e^{-\kappa_j(y+2x)} \sum_{s=0}^{p_j-1} R_{j,s} \frac{1}{s!} \sum_{t=0}^{s} \frac{s!}{t!(s-t)!}\, i^{s-t} (y+2x)^{s-t} \left[ \left( \frac{d}{dk} \right)^t m_l(k,x) \right]_{k=i\kappa_j} \\
&= e^{-\kappa_j(y+2x)} \sum_{t=0}^{p_j-1} \frac{1}{t!} \left[ \left( \frac{d}{dk} \right)^t m_l(k,x) \right]_{k=i\kappa_j} \sum_{s=t}^{p_j-1} R_{j,s} \frac{i^{s-t}(y+2x)^{s-t}}{(s-t)!}.
\end{aligned}$$

Using (4.5)-(4.8) in (3.16) and (3.17) and using (A.2), we find (4.1)-(4.4).

Conversely, let $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ be vector functions satisfying (4.1), (4.2), (3.20) and (3.21) [for $x \geq 0$], or (4.3), (4.4), (3.18) and (3.19) [for $x \leq 0$]. Then $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ are continuous on $\overline{\mathbf{C}^+}$, are meromorphic on $\mathbf{C}^+$ and approach $\hat{1}$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$.

Further, $m_l(k,x)$ and $n_l(k,x)$ do not have poles in $\mathbf{C}^+$ for $x \geq 0$ and $m_r(k,x)$ and $n_r(k,x)$ do not have poles in $\mathbf{C}^+$ for $x \leq 0$. We now compute

$$\begin{aligned}
&m_r(-k,x) + e^{-2ikx} L(k)\, m_r(k,x) \\
&= \frac{\left[m_l(k,x) + e^{-2ikx} R(-k)\, m_l(-k,x)\right]}{T(-k)} + e^{-2ikx} \frac{L(k)}{T(k)} \left[m_l(-k,x) + e^{2ikx} R(k)\, m_l(k,x)\right] \\
&= \left(\frac{1}{T(-k)} + \frac{L(k)\, R(k)}{T(k)}\right) m_l(k,x) + e^{-2ikx} \left(\frac{R(-k)}{T(-k)} + \frac{L(k)}{T(k)}\right) m_l(-k,x) \\
&= T(k)\, m_l(k,x),
\end{aligned}$$

which yields (3.18). Here we have employed condition 4 of the statement of this theorem. In a similar way we prove (3.19).

It remains to prove that $m_r(k,x)$ and $n_r(k,x)$ [for $x \geq 0$] and $m_l(k,x)$ and $n_l(k,x)$ [for $x \leq 0$] do not have poles in $\mathbf{C}^+$. For example, let us prove that, for $x \geq 0$, $m_r(k,x)T(k)$ does not have a pole at $i\kappa_j$, by showing that the coefficients of $(k - i\kappa_j)^{-(u+1)}$ ($u = 0, 1, \cdots, p_j - 1$) in the Laurent series of $m_r(k,x)T(k)$ all vanish. Using (4.1) one verifies that, for $u = 0, 1, \cdots, p_j - 1$, the coefficients of $(k - i\kappa_j)^{-(u+1)}$ in the Laurent series of $-m_l(-k,x)$ and $R(k)\, e^{2ikx} m_l(k,x)$ both equal

$$\sum_{s=0}^{p_j-1} \frac{1}{s!} \left[\left(\frac{d}{dk}\right)^s m_l(k,x)\right]_{k=i\kappa_j} \sum_{t=s+u}^{p_j-1} R_{j,t+1} \frac{e^{-2\kappa_j x}}{t-s+1} \frac{(2ix)^{t-s-u}}{(t-s-u)!},$$

which, in view of (3.20), shows that $m_r(k,x)\, T(k)$ is analytic at $k = i\kappa_j$. The same reasoning may be applied to $n_r(k,x)\, T(k)$ [for $x \geq 0$] and to $m_l(k,x)\, T(k)$ and $n_l(k,x)\, T(k)$ [for $x \leq 0$]. ∎

If all poles $\kappa_1, \cdots, \kappa_{N_R}$ of $R(k)$ in $\mathbf{C}^+$ are simple and $R_j = \lim_{k \to i\kappa_j} (k - i\kappa_j)\, R(k)$, we get

$$m_l(k,x) = 1 + \sum_{j=1}^{N_R} m_l(i\kappa_j, x)\, R_j \frac{e^{-2\kappa_j x}}{k + i\kappa_j}, \qquad x \geq 0, \tag{4.9}$$

$$n_l(k,x) = 1 - \sum_{j=1}^{N_R} n_l(i\kappa_j, x)\, R_j \frac{e^{-2\kappa_j x}}{k + i\kappa_j}, \qquad x \geq 0. \tag{4.10}$$

If all poles $\lambda_1, \cdots, \lambda_{N_L}$ of $L(k)$ in $\mathbf{C}^+$ are simple and $L_j = \lim_{k \to i\lambda_j} (k - i\lambda_j)\, L(k)$, we have [Cf. (A.3)]

$$m_r(k,x) = 1 + \sum_{j=1}^{N_L} m_r(i\lambda_j, x)\, L_j \frac{e^{2\lambda_j x}}{k + i\lambda_j}, \qquad x \leq 0, \tag{4.11}$$

$$n_r(k,x) = 1 - \sum_{j=1}^{N_L} n_r(i\lambda_j, x)\, L_j \frac{e^{2\lambda_j x}}{k + i\lambda_j}, \qquad x \le 0. \tag{4.12}$$

Substituting $k = i\kappa_1, \cdots, \kappa_{N_R}$ in (4.9) and (4.10), we obtain two systems of $N_R$ linear equations for $m_l(i\kappa_j, x)$ and $n_l(i\kappa_j, x)$ $(j = 1, \cdots, N_R)$, respectively. After we have found $m_l(k,x)$ and $n_l(k,x)$ for $x \ge 0$, $m_r(k,x)$ and $n_r(k,x)$ for $x \ge 0$ are obtained with the help of (3.20) and (3.21), respectively. Substituting (4.9) and (4.10) in (3.20) and (3.21) we find for $x \ge 0$

(4.13)

$$m_r(k,x)\, T(k) = 1 - \sum_{j=1}^{N_R} m_l(i\kappa_j, x)\, R_j \frac{e^{-2\kappa_j x}}{k - i\kappa_j} + e^{2ikx} R(k) \left[ 1 + \sum_{j=1}^{N_R} m_l(i\kappa_j, x)\, R_j \frac{e^{-2\kappa_j x}}{k + i\kappa_j} \right],$$

(4.14)

$$n_r(k,x)\, T(k) = 1 + \sum_{j=1}^{N_R} m_l(i\kappa_j, x)\, R_j \frac{e^{-2\kappa_j x}}{k - i\kappa_j} - e^{2ikx} R(k) \left[ 1 - \sum_{j=1}^{N_R} m_l(i\kappa_j, x)\, R_j \frac{e^{-2\kappa_j x}}{k + i\kappa_j} \right].$$

Hence, from (4.9), (4.10), (4.13), and (4.14), we obtain

$$\lim_{k \to i\kappa_j} (k - i\kappa_j) m_r(k,x) T(k) = R_j e^{-2\kappa_j x} \left[ -m_l(i\kappa_j, x) + 1 + \sum_{s=1}^{N_R} \frac{e^{-2\kappa_s x} m_l(i\kappa_s, x) R_s}{i(\kappa_j + \kappa_s)} \right] = 0,$$

$$\lim_{k \to i\kappa_j} (k - i\kappa_j) n_r(k,x) T(k) = -R_j e^{-2\kappa_j x} \left[ -n_l(i\kappa_j, x) + 1 - \sum_{s=1}^{N_R} \frac{e^{-2\kappa_s x} n_l(i\kappa_s, x) R_s}{i(\kappa_j + \kappa_s)} \right] = 0,$$

which implies the analyticity of $m_r(k,x)$ and $n_r(k,x)$ on $\mathbf{C}^+$ for $x \ge 0$. Analogously, substituting $k = i\lambda_1, \cdots, i\lambda_{N_L}$ in (4.11) and (4.12), we obtain two systems of $N_L$ linear equations for $m_r(i\lambda_j, x)$ and $n_r(i\lambda_j, x)$ $(j = 1, \cdots, N_L)$. After finding $m_r(k,x)$ and $n_r(k,x)$ for $x \le 0$, $m_l(k,x)$ and $n_l(k,x)$ for $x \le 0$ are computed with the help of (3.18) and (3.19), respectively. The analyticity of $m_l(k,x)$ and $n_l(k,x)$ on $\mathbf{C}^+$ for $x \le 0$ is proved in an analogous manner.

If the poles of $R(k)$ and $L(k)$ in $\mathbf{C}^+$ are simple, it is straightforward to write down expressions for $m_l(k,x)$ and $n_l(k,x)$ if $x \le 0$ and for $m_r(k,x)$ and $n_r(k,x)$ if $x \ge 0$. Indeed, defining $\ell(k) = L(k) - \sum_{j=1}^{N_L} (k - i\lambda_j)^{-1} L_j$ as the nonprincipal part of $L(k)$ in $\mathbf{C}^+$, using (4.11) in (3.18) and using (4.12) in (3.19), we obtain for $x \le 0$

$$\begin{aligned} m_l(k,x) = \frac{1}{T(k)} \Bigg[ 1 + e^{-2ikx} \ell(k)\, m_r(k,x) + \sum_{j=1}^{N_L} \frac{L_j\, e^{2\lambda_j x}}{k - i\lambda_j} \Bigg( e^{-2i(k - i\lambda_j)x} - 1 \\ + \sum_{s=1}^{N_L} L_s\, m_r(i\lambda_s, x)\, e^{2\lambda_s x} \left\{ \frac{e^{-2i(k - i\lambda_j)x}}{k + i\lambda_s} - \frac{1}{i(\lambda_j + \lambda_s)} \right\} \Bigg) \Bigg], \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
n_l(k,x) = \frac{1}{T(k)} \Bigg[ 1 - e^{-2ikx}\ell(k)\, n_r(k,x) + \sum_{j=1}^{N_L} \frac{L_j\, e^{2\lambda_j x}}{k - i\lambda_j} \Bigg( 1 - e^{-2i(k-i\lambda_j)x} \\
+ \sum_{s=1}^{N_L} L_s\, n_r(i\lambda_s, x)\, e^{2\lambda_s x} \left\{ \frac{e^{-2i(k-i\lambda_j)x}}{k + i\lambda_s} - \frac{1}{i(\lambda_j + \lambda_s)} \right\} \Bigg) \Bigg].
\end{aligned}
\tag{4.16}
$$

Note that the analyticity of $m_l(k,x)$ and $n_l(k,x)$ for $k \in \mathbf{C}^+$ when $x \leq 0$ is also apparent from (4.15) and (4.16). Further, defining $\rho(k) = R(k) - \sum_{j=1}^{N_R} (k - i\kappa_j)^{-1} R_j$ as the non-principal part of $R(k)$ in $\mathbf{C}^+$, using (4.9) in (3.20) and using (4.10) in (3.21), we obtain for $x \geq 0$

$$
\begin{aligned}
m_r(k,x) = \frac{1}{T(k)} \Bigg[ 1 + e^{2ikx}\rho(k)\, m_l(k,x) + \sum_{j=1}^{N_R} \frac{R_j\, e^{-2\kappa_j x}}{k - i\kappa_j} \Bigg( e^{2i(k-i\kappa_j)x} - 1 \\
+ \sum_{s=1}^{N_R} R_s\, m_l(i\kappa_s, x)\, e^{-2\kappa_s x} \left\{ \frac{e^{2i(k-i\kappa_j)x}}{k + i\kappa_s} - \frac{1}{i(\kappa_j + \kappa_s)} \right\} \Bigg) \Bigg]
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
n_r(k,x) = \frac{1}{T(k)} \Bigg[ 1 - e^{2ikx}\rho(k)\, n_l(k,x) + \sum_{j=1}^{N_R} \frac{R_j\, e^{-2\kappa_j x}}{k - i\kappa_j} \Bigg( 1 - e^{2i(k-i\kappa_j)x} \\
+ \sum_{s=1}^{N_R} R_s\, n_l(i\kappa_s, x)\, e^{-2\kappa_s x} \left\{ \frac{e^{2i(k-i\kappa_j)x}}{k + i\kappa_s} - \frac{1}{i(\kappa_j + \kappa_s)} \right\} \Bigg) \Bigg].
\end{aligned}
\tag{4.18}
$$

Note that the analyticity of $m_r(k,x)$ and $n_r(k,x)$ for $k \in \mathbf{C}^+$ when $x \geq 0$ is also apparent from (4.17) and (4.18).

Once $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ have been determined for $x \in \mathbf{R}$, $\mathbf{M}(k,x)$ follows with the help of (3.7). As a result of (3.7) and (3.8), the first part of the next corollary is immediate from Theorem 4.1. The part pertaining to unitary matrix functions follows with the help of Corollary 2.2.

COROLLARY 4.2. *Suppose the hypotheses of Theorem* 4.1 *are fulfilled. Then* $\mathbf{G}(k,x)$ *has a (right) canonical factorization if and only if the two systems of linear equations determining the unspecified constants in* (4.1) *and* (4.2) *[for* $x \geq 0$*] or* (4.3) *and* (4.4) *[for* $x \leq 0$*] are both uniquely solvable. In particular, if* $\mathbf{S}(k)$ *is a unitary matrix, then these two systems of linear equations are uniquely solvable.*

## 5. ADAPTATIONS IN THE CASE $T(0) = 0$

In this section we adapt the construction of the Wiener-Hopf factorization performed in Sections 3 and 4 to scattering matrices of the form (1.2) where $T(k)$ vanishes linearly at $k = 0$. We will also treat the case where the extension of $T(k)$ to $\mathbf{C}^+$ is meromorphic.

Suppose a $2 \times 2$ matrix function $\mathbf{W}(k)$ for $k \in \mathbf{R}$ has a (right) Wiener-Hopf factorization of the form (2.1), while $\mathbf{W}(\infty) = \mathbf{I}$ and

$$\mathbf{W}(-k) = \mathbf{q}\mathbf{W}(k)^{-1}\mathbf{q}, \qquad k \in \mathbf{R}.$$

Then it is possible to choose the factorization in such a way that

$$\mathbf{W}_{\pm}(-k) = \mathbf{q}\mathbf{W}_{\mp}(k)^{-1}\mathbf{q}, \ \mathbf{q}\mathbf{Q}_{\pm} = \mathbf{Q}_{\pm}\mathbf{q}.$$

Since $\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$ is the only pair of complementary rank-one projections on $\mathbf{C}^2$ commuting with $\mathbf{q}$, we may choose, with no loss of generality,

$$\mathbf{Q}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{Q}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{5.1}$$

As we now have $\mathbf{W}_{\pm}(\infty) = \mathbf{q}\mathbf{W}_{\mp}(\infty)^{-1}\mathbf{q}$, $\mathbf{W}_{\pm}(\infty)$ must commute with the projections $\mathbf{Q}_+$ and $\mathbf{Q}_-$ in (5.1) and hence the factorization (2.1) may be adjusted in such a way that (1) $\mathbf{Q}_{\pm}$ are as in (5.1), (2) $\mathbf{W}_{\pm}(-k) = \mathbf{q}\mathbf{W}_{\mp}(k)^{-1}\mathbf{q}$ for $k \in \mathbf{R}$, and (3) $\mathbf{W}_{\pm}(\infty) = \mathbf{I}$. We shall henceforth call such Wiener-Hopf factorizations special. In **[23]** such factorizations were called Jost function factorizations.

PROPOSITION 5.1. *Suppose*

$$\mathbf{S}(k) = \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix}$$

*is a $2 \times 2$ matrix for all $k \in \mathbf{R}$ such that*

1. *$T(k)$ is nonzero for all $k \in \mathbf{R} \setminus \{0\}$, $T(\infty) = 1$, $R(\infty) = 0$ and $L(\infty) = 0$, and the order of the zero of $T(k)$ at $k = 0$ is finite,*
2. *$T(k)$ can be continued to a function continuous on $\overline{\mathbf{C}^+}$ and analytic on $\mathbf{C}^+$,*
3. *$\mathbf{S}(k)^{-1} = \mathbf{q}\mathbf{S}(-k)\mathbf{q}$, $k \in \mathbf{R}$,*
4. *$T(k)$, $R(k)$, and $L(k)$ belong to either $\mathcal{W}$ or $\mathcal{H}_\alpha$ for some $\alpha \in (0,1)$.*

*Then all special (right) Wiener-Hopf factorizations of $\mathbf{G}(k,x)$ of the form*

$$\mathbf{G}(k,x) = \mathbf{G}_-(k,x)\left[\left(\frac{k-i}{k+i}\right)^{\tau}\mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)^{\sigma}\mathbf{Q}_-\right]\mathbf{G}_+(k,x), \qquad k \in \mathbf{R}, \tag{5.2}$$

*are given by*

$$\mathbf{G}_-(k,x) = \widetilde{\mathbf{M}}(-k,x), \quad \mathbf{G}_+(k,x) = \mathbf{q}\,\widetilde{\mathbf{M}}(k,x)^{-1}\mathbf{q}, \tag{5.3}$$

*where*

(5.4)
$$\widetilde{\mathbf{M}}(k,x) = \begin{pmatrix} \widetilde{M}_1(k,x) & \widetilde{M}_2(k,x) \\ \widetilde{M}_3(k,x) & \widetilde{M}_4(k,x) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \widetilde{m}_l(k,x) + \widetilde{n}_l(k,x) & \widetilde{m}_l(k,x) - \widetilde{n}_l(k,x) \\ \widetilde{m}_r(k,x) - \widetilde{n}_r(k,x) & \widetilde{m}_r(k,x) + \widetilde{n}_r(k,x) \end{pmatrix}$$

*and* $\widetilde{\mathbf{m}}(k,x) = \begin{pmatrix} \widetilde{m}_l(k,x) \\ \widetilde{m}_r(k,x) \end{pmatrix}$ *and* $\widetilde{\mathbf{n}}(k,x) = \begin{pmatrix} \widetilde{n}_l(k,x) \\ \widetilde{n}_r(k,x) \end{pmatrix}$ *are continuous on* $\overline{\mathbf{C}^+}$, *are analytic on* $\mathbf{C}^+$, *and satisfy* $\widetilde{\mathbf{m}}(k,x) \to \hat{1}$ *and* $\widetilde{\mathbf{n}}(k,x) \to \hat{1}$ *as* $k \to \infty$ *in* $\overline{\mathbf{C}^+}$ *as well as the Riemann-Hilbert problems*

$$\widetilde{\mathbf{m}}(-k,x) = \left(\frac{k+i}{k-i}\right)^{\tau} \mathbf{G}(k,x)\,\mathbf{q}\,\widetilde{\mathbf{m}}(k,x), \qquad k \in \mathbf{R}, \tag{5.5}$$

$$\widetilde{\mathbf{n}}(-k,x) = \left(\frac{k+i}{k-i}\right)^{\sigma} \mathbf{J}\,\mathbf{G}(k,x)\,\mathbf{J}\,\mathbf{q}\,\widetilde{\mathbf{n}}(k,x), \qquad k \in \mathbf{R}. \tag{5.6}$$

**Proof**: The existence of the factorization is clear from [9], Theorem II 6.2 (for $\mathcal{H}_\alpha^{2\times 2}$) or Theorem II 6.3 (for $\mathcal{W}^{2\times 2}$). Suppose $\widetilde{\mathbf{m}}(k,x)$ and $\widetilde{\mathbf{n}}(k,x)$ are continuous on $\overline{\mathbf{C}^+}$, are analytic on $\mathbf{C}^+$, approach $\hat{1}$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$ and satisfy the respective Riemann-Hilbert problems (5.5) and (5.6), and let us define $\mathbf{G}_\pm(k,x)$ by (5.3) and (5.4). Writing $\mathbf{D}(k) = \left[\left(\frac{k-i}{k+i}\right)^{\tau}\mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)^{\sigma}\mathbf{Q}_-\right]$, we have

$$\begin{aligned}
&[\mathbf{G}_-(k,x)\,\mathbf{D}(k) - \mathbf{G}(k,x)\,\mathbf{G}_+(k,x)^{-1}]\hat{1} \\
&= \left(\frac{k-i}{k+i}\right)^{\tau}\left[\mathbf{G}_-(k,x) - \left(\frac{k+i}{k-i}\right)^{\tau}\mathbf{G}(k,x)\,\mathbf{G}_+(k,x)^{-1}\right]\hat{1} \\
&= \left(\frac{k-i}{k+i}\right)^{\tau}\left[\widetilde{\mathbf{M}}(-k,x) - \left(\frac{k+i}{k-i}\right)^{\tau}\mathbf{G}(k,x)\,\mathbf{q}\,\widetilde{\mathbf{M}}(k,x)\mathbf{q}\right]\hat{1} \\
&= \left(\frac{k-i}{k+i}\right)^{\tau}\left[\widetilde{\mathbf{m}}(-k,x) - \left(\frac{k+i}{k-i}\right)^{\tau}\mathbf{G}(k,x)\,\mathbf{q}\,\widetilde{\mathbf{m}}(k,x)\right]
\end{aligned}$$

and

$$\begin{aligned}
&[\mathbf{G}_-(k,x)\,\mathbf{D}(k) - \mathbf{G}(k,x)\,\mathbf{G}_+(k,x)^{-1}]\hat{e} \\
&= \left[\left(\frac{k-i}{k+i}\right)^{\sigma}\mathbf{G}_-(k,x) - \mathbf{G}(k,x)\,\mathbf{G}_+(k,x)^{-1}\right]\hat{e} \\
&= \left(\frac{k-i}{k+i}\right)^{\sigma}\mathbf{J}\left[\mathbf{J}\,\widetilde{\mathbf{M}}(-k,x) + \left(\frac{k+i}{k-i}\right)^{\sigma}\mathbf{J}\,\mathbf{G}(k,x)\,\mathbf{q}\,\widetilde{\mathbf{M}}(k,x)\right]\hat{e} \\
&= \left(\frac{k-i}{k+i}\right)^{\sigma}\mathbf{J}\left[\widetilde{\mathbf{n}}(-k,x) - \left(\frac{k+i}{k-i}\right)^{\sigma}\mathbf{J}\,\mathbf{G}(k,x)\,\mathbf{J}\,\mathbf{q}\widetilde{\mathbf{n}}(k,x)\right].
\end{aligned}$$

As a result, (5.5) and (5.6) imply (5.2), and (5.2) implies (5.5) and (5.6). The latter implication is most easily obtained with the help of the equalities

$$\widetilde{\mathbf{m}}(k,x) = \widetilde{\mathbf{M}}(k,x)\,\hat{1}, \quad \widetilde{\mathbf{n}}(k,x) = \mathbf{J}\,\widetilde{\mathbf{M}}(k,x)\,\hat{e}.$$

This completes the proof. ∎

If we define

$$\mathbf{m}(k,x) = \left(\frac{k+i}{k}\right)^{\tau} \widetilde{\mathbf{m}}(k,x), \quad \mathbf{n}(k,x) = \left(\frac{k+i}{k}\right)^{\sigma} \widetilde{\mathbf{n}}(k,x), \tag{5.7}$$

we obtain instead of (5.5) and (5.6) the Riemann-Hilbert problems

$$\mathbf{m}(-k,x) = \mathbf{G}(k,x)\,\mathbf{q}\,\mathbf{m}(k,x), \qquad k \in \mathbf{R},$$

$$\mathbf{n}(-k,x) = \mathbf{J}\mathbf{G}(k,x)\,\mathbf{J}\,\mathbf{q}\,\mathbf{n}(k,x), \qquad k \in \mathbf{R},$$

where $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ are continuous on $\overline{\mathbf{C}^+} \setminus \{0\}$, are analytic on $\mathbf{C}^+$, and approach $\hat{1}$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, while the limits of $k^{\tau}\mathbf{m}(k,x)$ and $k^{\sigma}\mathbf{n}(k,x)$ exist as $k \to 0$ in $\overline{\mathbf{C}^+}$. Defining

$$\mathbf{M}(k,x) = \widetilde{\mathbf{M}}(k,x) \left[ \left(\frac{k+i}{k}\right)^{\tau} \mathbf{Q}_+ + \left(\frac{k+i}{k}\right)^{\sigma} \mathbf{Q}_- \right], \tag{5.8}$$

we obtain the matrix Riemann-Hilbert problem (3.9) where $\mathbf{M}(k,x)$ is continuous on $\overline{\mathbf{C}^+} \setminus \{0\}$, is analytic on $\mathbf{C}^+$, and approach $\mathbf{I}$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, while the limits of $k^{\tau}\mathbf{M}(k,x)\mathbf{Q}_+$ and $k^{\sigma}\mathbf{M}(k,x)\mathbf{Q}_-$ exist as $k \to 0$ in $\overline{\mathbf{C}^+}$. The matrix function $\mathbf{M}(k,x)$ is related to $\mathbf{m}(k,x)$ and $\mathbf{n}(k,x)$ as in (3.7).

In the inverse scattering problem for (3.1) for the class of potentials specified in the beginning of Section 3, generically $T(k)$ vanishes linearly at $k=0$ and $R(0) = L(0) = -1$. In that case $\mathbf{m}(k,x)$ remains continuous at $k=0$. Hence if $T(k)$ is analytic in $\mathbf{C}^+$ from (5.7) we see that we have $\tau = 0$ in the above analysis. Letting $T(k) = \frac{k}{k+i}\widetilde{T}(k)$ we have $\widetilde{T}(k)$ analytic on $\mathbf{C}^+$ without zeros and $\widetilde{T}(0) \neq 0$, and from (3.10) we obtain

$$\det \mathbf{G}(k,x) = \frac{T(k)}{T(-k)} = \left(\frac{k-i}{k+i}\right) \frac{\widetilde{T}(k)}{\widetilde{T}(-k)}.$$

Thus, we see that the partial indices of $\mathbf{G}(k,x)$ add up to 1, and hence $\sigma = 1$ in the above analysis, and that

$$\det \widetilde{\mathbf{M}}(k,x) = \frac{1}{\widetilde{T}(k)}, \quad \det \mathbf{M}(k,x) = \frac{1}{T(k)}, \qquad k \in \overline{\mathbf{C}^+}. \tag{5.9}$$

COROLLARY 5.2. *Suppose*

$$\mathbf{S}(k) = \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix}$$

*is a unitary matrix for all $k \in \mathbf{R}$ such that*

1. *$T(k)$ is nonzero for all $k \in \mathbf{R} \setminus \{0\}$, $T(\infty) = 1$, $R(\infty) = 0$ and $L(\infty) = 0$,*
2. *$T(k)$ can be continued to a meromorphic function on $\mathbf{C}^+$ having finitely many zeros and continuous boundary values on the extended real axis, while $T(\infty) = 1$,*

3. *$T(k)$, $R(k)$, and $L(k)$ belong to either $\mathcal{W}$ or $\mathcal{H}_\alpha$ for some $\alpha \in (0,1)$, and*
4. $T(k) = \left(\frac{k}{k+i}\right) \widetilde{T}(k)$ *where* $\widetilde{T}(0) \neq 0$, and $R(0) = L(0) = -1$.

*Then $\mathbf{G}(k,x)$ defined in (1.1) has a (right) Wiener-Hopf factorization with partial indices $\rho$ and $\rho+1$, where $\rho$ is the number of zeros minus the number of poles of $T(k)$ in $\mathbf{C}^+$.*

**Proof**: Let us first assume that $T(k)$ is analytic and nonzero on $\mathbf{C}^+$. Then, inserting (5.8) with $\tau = 0$ and $\sigma = 1$ in (3.9) and using the fact the $\mathbf{Q}_\pm$ commute with $\mathbf{q}$, we obtain the explicit noncanonical factorization

$$\mathbf{G}(k,x) = \widetilde{\mathbf{M}}(-k,x)\left[\mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)\mathbf{Q}_-\right]\mathbf{q}\widetilde{\mathbf{M}}(k,x)^{-1}\,\mathbf{q}.$$

If the extension of $T(k)$ to $\mathbf{C}^+$ has zeros on $\mathbf{C}^+$ or is meromorphic instead of analytic on $\mathbf{C}^+$, the Wiener-Hopf factorization of $\mathbf{G}(k,x)$ can be obtained as follows. Assume $T(k)$ has poles at $k = \beta_j \in \mathbf{C}^+$ for $j = 1, \ldots, \mathcal{N}$ and zeros at $k = \gamma_s \in \mathbf{C}^+$ for $s = 1, \cdots, \mathcal{M}$ there. We can then factor $\mathbf{G}(k,x)$ into a scalar factor and a matrix as

$$\mathbf{G}(k,x) = \mathbf{H}(k,x)\prod_{j=1}^{\mathcal{N}}\frac{k+\beta_j}{k-\beta_j}\prod_{s=1}^{\mathcal{M}}\frac{k-\gamma_s}{k+\gamma_s}, \tag{5.10}$$

where

$$\mathbf{H}(k,x) = \mathbf{G}(k,x)\prod_{j=1}^{\mathcal{N}}\frac{k-\beta_j}{k+\beta_j}\prod_{s=1}^{\mathcal{M}}\frac{k+\gamma_s}{k-\gamma_s}. \tag{5.11}$$

The diagonal entries of $\mathbf{H}(k,x)$ have nonzero analytic extension to $\mathbf{C}^+$ and hence the Wiener-Hopf factorization of $\mathbf{H}(k,x)$ can be obtained by replacing $T(k)$ by $T(k)w(k)$, $R(k)$ by $R(k)w(k)$, $L(k)$ by $L(k)w(k)$ in (1.1), where $w(k) = \prod_{j=1}^{\mathcal{N}}\frac{k-\beta_j}{k+\beta_j}\prod_{s=1}^{\mathcal{M}}\frac{k+\gamma_s}{k-\gamma_s}$, and by employing the method we have presented. The scalar factor in (5.10) has the factorization (5.12)

$$\prod_{j=1}^{\mathcal{N}}\frac{k+\beta_j}{k-\beta_j}\prod_{s=1}^{\mathcal{M}}\frac{k-\gamma_s}{k+\gamma_s} = \left[\prod_{j=1}^{\mathcal{N}}\frac{k-i}{k-\beta_j}\prod_{s=1}^{\mathcal{M}}\frac{k-\gamma_s}{k-i}\right]\left(\frac{k-i}{k+i}\right)^{\mathcal{M}-\mathcal{N}}\left[\prod_{j=1}^{\mathcal{N}}\frac{k+\beta_j}{k+i}\prod_{s=1}^{\mathcal{M}}\frac{k+i}{k+\gamma_s}\right].$$

Hence, as seen from (5.12) the Wiener-Hopf factorization of $\mathbf{G}(k,x)$ then becomes noncanonical with partial indices $\mathcal{M}-\mathcal{N}$ and $\mathcal{M}-\mathcal{N}$ in case $T(0) \neq 0$, and in case $T(k)$ vanishes linearly at $k=0$ the partial indices are given by $\mathcal{M}-\mathcal{N}$ and $1+\mathcal{M}-\mathcal{N}$. The Wiener-Hopf factors of $\mathbf{G}(k,x)$ are then obtained by multiplying the factors of $\mathbf{H}(k,x)$ and those of $w(k)^{-1}$. ∎

## 6. EXAMPLES

**Example 6.1.** Consider the unitary matrix function (1.2) where for $0 < \epsilon < 1$ and $\gamma > 0$

$$T(k) = \frac{k+i\epsilon}{k+i}, \quad L(k) = \frac{-i\sqrt{1-\epsilon^2}}{k+i}\,\frac{k+i\gamma}{k-i\gamma}, \quad R(k) = \frac{-i\sqrt{1-\epsilon^2}}{k+i}\,\frac{k+i\epsilon}{k-i\epsilon}\frac{k-i\gamma}{k+i\gamma}.$$

Note that $T(k)$ does not have any poles in $\mathbf{C}^+$ and $T(0) \neq 0$. The canonical Wiener-Hopf factors of $\mathbf{G}(k,x)$ are then given as in (3.11). Letting $\mathbf{M}(k,x) = \begin{pmatrix} M_1(k,x) & M_2(k,x) \\ M_3(k,x) & M_4(k,x) \end{pmatrix}$, we find for $x \geq 0$

$$M_1(k,x) = 1 + \frac{2i\epsilon}{k+i\epsilon}\frac{s(x)^2}{1-s(x)^2},$$

$$M_2(k,x) = -\frac{2i\epsilon}{k+i\epsilon}\frac{s(x)}{1-s(x)^2},$$

$$M_3(k,x) = \frac{iA_1(x)}{k-i\epsilon}\left(1-e^{2ix(k-i\epsilon)}\right) + \frac{iA_2(x)}{k+i\epsilon}\left(1-e^{2ix(k+i\epsilon)}\right) + \frac{iA_3(x)}{k+i\gamma}e^{2ikx},$$

$$M_4(k,x) = 1 + \frac{iA_4(x)}{k-i\epsilon}\left(1-e^{2ix(k-i\epsilon)}\right) + \frac{iA_5(x)}{k+i\epsilon}\left(1-e^{2ix(k+i\epsilon)}\right) + \frac{iA_6(x)}{k+i\gamma}e^{2ikx},$$

and when $x \leq 0$ we have

$$M_1(k,x) = 1 + \frac{iA_7(x)}{k+i\epsilon} + \frac{iA_8(x)}{k+i\epsilon}e^{-2ikx} + \frac{iA_9(x)}{k-i\gamma}\left(1-e^{-2ix(k-i\gamma)}\right),$$

$$M_2(k,x) = \frac{iA_{10}(x)}{k+i\epsilon} + \frac{iA_{11}(x)}{k+i\epsilon}e^{-2ikx} + \frac{iA_{12}(x)}{k-i\gamma}\left(1-e^{-2ix(k-i\gamma)}\right),$$

$$M_3(k,x) = -\frac{2i\gamma}{k+i\gamma}\frac{t(x)}{1-t(x)^2},$$

$$M_4(k,x) = 1 + \frac{2i\gamma}{k+i\gamma}\frac{t(x)^2}{1-t(x)^2},$$

where we have defined

$$s(x) = \sqrt{\frac{1-\epsilon}{1+\epsilon}\frac{\epsilon-\gamma}{\epsilon+\gamma}}e^{-2\epsilon x},$$

$$t(x) = \frac{\sqrt{1-\epsilon^2}}{1+\gamma}e^{2\gamma x},$$

$$A_1(x) = \frac{(1+\epsilon)\,s(x)}{1-s(x)^2},$$

$$A_2(x) = -\frac{(1-\epsilon)\,s(x)}{1-s(x)^2},$$

$$A_3(x) = -\sqrt{1-\epsilon^2}\,\frac{2\gamma}{\epsilon+\gamma}\left[1+\frac{2\epsilon}{\epsilon-\gamma}\frac{s(x)^2}{1-s(x)^2}\right],$$

$$A_4(x) = -\frac{(1+\epsilon)s(x)^2}{1-s(x)^2},$$

$$A_5(x) = \frac{1-\epsilon}{1-s(x)^2},$$

$$A_6(x) = \sqrt{1-\epsilon^2}\frac{2\epsilon}{\epsilon+\gamma}\frac{2\gamma}{\epsilon-\gamma}\frac{s(x)}{1-s(x)^2},$$

$$A_7(x) = (1-\epsilon)\left[1+\frac{2\gamma}{\epsilon+\gamma}\frac{t(x)^2}{1-t(x)^2}\right],$$

$$A_8(x) = -\sqrt{1-\epsilon^2}\frac{2\gamma}{\epsilon+\gamma}\frac{t(x)}{1-t(x)^2},$$

$$A_9(x) = -\frac{2\gamma(1+\gamma)}{\epsilon+\gamma}\frac{t(x)^2}{1-t(x)^2},$$

$$A_{10}(x) = -\frac{2\gamma(1-\epsilon)}{\epsilon+\gamma}\frac{t(x)}{1-t(x)^2},$$

$$A_{11}(x) = \sqrt{1-\epsilon^2}\left[-1+\frac{2\gamma}{\epsilon+\gamma}\frac{1}{1-t(x)^2}\right],$$

$$A_{12}(x) = \frac{2\gamma(1+\gamma)}{\epsilon+\gamma}\frac{t(x)}{1-t(x)^2}.$$

**Example 6.2.** As our second example, consider the unitary matrix function (1.2) where

$$T(k) = \frac{k}{k+i}, \quad R(k) = L(k) = \frac{-i}{k+i}.$$

A Wiener-Hopf factorization of $\mathbf{G}(k,x)$ is then given by

$$\mathbf{G}_-(k,x) = \mathbf{M}(-k,x)\left[\mathbf{Q}_+ + \left(\frac{k}{k-i}\right)\mathbf{Q}_-\right],$$

$$\mathbf{D}(k) = \mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)\mathbf{Q}_-,$$

$$\mathbf{G}_+(k,x) = \left[\mathbf{Q}_+ + \left(\frac{k+i}{k}\right)\mathbf{Q}_-\right]\mathbf{q}\,\mathbf{M}(k,x)^{-1}\,\mathbf{q},$$

where $\mathbf{M}(k,x) = \begin{pmatrix} M_1(k,x) & M_2(k,x) \\ M_3(k,x) & M_4(k,x)\end{pmatrix}$, and we have for $x \geq 0$

$$M_1(k,x) = 1 + \frac{i}{k}\frac{1}{1+2x+2a},$$

$$M_2(k,x) = -\frac{i}{k}\frac{1}{1+2x+2a},$$

$$M_3(k,x) = -\frac{i}{k}e^{2ikx} + \left[\frac{i}{k} - \frac{1}{k^2} + \frac{1}{k^2}e^{2ikx}\right]\frac{1}{1+2x+2a},$$

$$M_4(k,x) = 1 + \frac{i}{k} + \left[ -\frac{i}{k} + \frac{1}{k^2} - \frac{1}{k^2} e^{2ikx} \right] \frac{1}{1+2x+2a},$$

where $a$ is an arbitrary positive parameter. When $x \le 0$, we have

$$M_1(k,x) = 1 + \frac{i}{k} + \left[ -\frac{i}{k} + \frac{1}{k^2} - \frac{e^{-2ikx}}{k^2} \right] \frac{2a}{1+2a-4ax},$$

$$M_2(k,x) = -\frac{i}{k} e^{-2ikx} + \left[ \frac{i}{k} - \frac{1}{k^2} + \frac{1}{k^2} e^{-2ikx} \right] \frac{2a}{1+2a-4ax},$$

$$M_3(k,x) = -\frac{i}{k} \frac{2a}{1+2a-4ax},$$

$$M_4(k,x) = 1 + \frac{i}{k} \frac{2a}{1+2a-4ax}.$$

## APPENDIX

In Section 3 we are using functions defined in terms of the polynomials

$$Q_0(z) \equiv 1, \quad Q_m(z) = \sum_{j=0}^{m} \frac{m!}{j!} z^j = z^m + m z^{m-1} + m(m-1) z^{m-2} + \cdots + (m!),$$

which satisfy the recurrence relation

$$Q_0(z) \equiv 1, \quad Q_{m+1}(z) = (z+m+1) Q_m(z) - z \frac{d}{dz} Q_m(z). \tag{A.1}$$

For all $\kappa$ with Re $\kappa > 0$, $k \in \mathbf{R}$ and $x > 0$ we have

$$F_m(k,x,\kappa) = (-i) \int_0^\infty dy\, e^{iky} \frac{(y+2x)^m}{m!} e^{-\kappa(y+2x)} = (-i) e^{-2ikx} G_m(\kappa - ik, x), \tag{A.2}$$

where

$$G_m(\beta,x) = \int_0^\infty dy\, \frac{(y+2x)^m}{m!} e^{-\beta(y+2x)}, \qquad \text{Re}\, \beta > 0.$$

We have

$$\frac{d}{d\beta} G_m(\beta,x) = -(m+1) G_{m+1}(\beta,x),$$

and hence using (A.1) we obtain

$$G_m(\beta,x) = \frac{Q_m(2\beta x)}{m! \beta^{m+1}} e^{-2\beta x}, \qquad \text{Re}\, \beta > 0.$$

Thus

$$F_m(k,x,\kappa) = \frac{(-i)e^{-2\kappa x}}{(\kappa - ik)^{m+1}} \sum_{j=0}^{m} \frac{[2(\kappa - ik)x]^{m-j}}{j!} = e^{-2\kappa x} \sum_{j=0}^{m} \frac{i^j (2x)^{m-j}}{j!(k+i\kappa)^{j+1}}.$$

In particular,

$$F_0(k,x,\kappa) = \frac{e^{-2\kappa x}}{k+i\kappa}. \tag{A.3}$$

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