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DARBOUX TRANSFORMATION FOR THE DISCRETE SCHRÖDINGER EQUATION

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Abstract. The discrete Schrödinger equation on a half-line lattice with the Dirichlet boundary condition is considered when the potential is real valued, is summable, and has a finite first moment. The Darboux transformation formulas are derived from first principles showing how the potential and the wave function change when a bound state is added to or removed from the discrete spectrum of the corresponding Schrödinger operator without changing the continuous spectrum. This is done by explicitly evaluating the change in the spectral density when a bound state is added or removed and also by determining how the continuous part of the spectral density changes. The theory presented is illustrated with some explicit examples.

1. Introduction

Our goal in this article is to analyze the Darboux transformation for the discrete Schrödinger equation on the half-line lattice with the Dirichlet boundary condition. In the Darboux transformation, the continuous part of the corresponding Schrödinger operator is unchanged and only the discrete part of the spectrum is changed by adding or removing a finite number of discrete eigenvalues to the spectrum. We can view the process of adding or removing discrete eigenvalues as changing the "unperturbed" potential and the "unperturbed" wavefunction into the "perturbed" potential and the "perturbed" wavefunction, respectively. Hence, our goal is to present the Darboux transformation formulas at the potential level and at the wavefunction level, by expressing the change in the potential and in the wavefunction in terms of quantities related to the perturbation and the unperturbed quantities.

The Darboux transformation was termed to honor the work of French mathematician Gaston Darboux [9], and it is useful for various reasons. For example, it allows us to produce explicit solutions to differential or difference equations by perturbing an already known explicit solution. As another example, we can mention that Darboux transformations for certain nonlinear partial differential equations or nonlinear partial differential-difference equations yield so-called soliton solutions, which have important applications [16] in wave propagation of electromagnetic waves and surface water waves. We refer the reader to the existing literature [5, 10, 16, 17, 18] on the wide applications of Darboux transformation, and in our

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paper we concentrate on the mathematical aspects of the Darboux transformation for the Schrödinger equation on the half-line lattice with the Dirichlet boundary condition.

On the half-line lattice the discrete Schrödinger equation is given by

$$-\psi_{n+1}+2\psi_n-\psi_{n-1}+V_n\psi_n=\lambda\,\psi_n,\quad n\ge 1, \tag{1.1}$$

where λ is the spectral parameter, n is the spacial independent variable taking positive integer values, and the subscripts are used to denote the dependence on n. Thus, ψ_n denotes the value of the wavefunction at n and V_n denotes the value of the potential at n. The point $n=0$ corresponds to the boundary. We remark that (1.1) is the analog of the half-line Schrödinger equation

$$-\psi''+V(x)\,\psi=\lambda\,\psi,\quad x>0, \tag{1.2}$$

where λ is the spectral parameter, the prime denotes the x-derivative, ψ is the wavefunction, and $V(x)$ is the potential. The point $x=0$ corresponds to the boundary. In analogy to (1.2), we can use (1.1) to describe [19] the behavior of a quantum mechanical particle on a half-line lattice (such as a crystal) experiencing the force at each lattice point n resulting from the potential value V_n.

In order to determine the spectrum of the corresponding Schrödinger operator related to (1.1) and to identify a square-summable solution in n as an eigenfunction, we must impose a boundary condition on square-summable wavefunctions at $n=0$. In applications related to quantum mechanics, it is appropriate to use the Dirichlet boundary condition at $x=0$ for (1.2), i.e.

$$\psi(0)=0,$$

and hence we impose the Dirichlet boundary condition at $n=0$ for (1.1), i.e.

$$\psi_0=0. \tag{1.3}$$

The spectrum of the corresponding operator for (1.2) is well understood when the potential $V(x)$ is real valued and satisfies the so-called L^1_1-condition [5, 10, 11] given by

$$\int_0^\infty dx\,(1+x)\,|V(x)|<+\infty. \tag{1.4}$$

Similarly, we assume that V_n is real valued and satisfies the analog of (1.4) given by

$$\sum_{n=1}^\infty (1+n)\,|V_n|<+\infty. \tag{1.5}$$

Clearly, (1.5) is equivalent to

$$\sum_{n=1}^\infty n\,|V_n|<+\infty. \tag{1.6}$$

The class of real-valued potentials $V(x)$ satisfying (1.4) is usually known [5, 10, 11] as the Faddeev class. Similarly, we refer to the set of real-valued potentials V_n satisfying (1.5), or equivalently (1.6), as the Faddeev class. The existence of the first moments in (1.4) and (1.5) assures that the number of discrete eigenvalues for each of the corresponding Schrödinger operators is finite.

Our paper is organized as follows. In Section 2 we present the appropriate preliminaries involving the Jost solution and the regular solution to (1.1); the Schrödinger operator, the scattering states, the bound states, the Jost function, the scattering matrix, the phase shift, and the spectral density associated with (1.1) and (1.3); the exceptional and generic cases that are related to $\lambda = 0$ and $\lambda = 4$ for the Schrödinger operator; Levinson's theorem; and the Gel'fand-Levitan procedure associated with (1.1) and (1.3). In Section 3 we present the Darboux transformation formulas when a bound state is added to the spectrum of the Schrödinger operator. In Theorem 3.1 we prove that the matrix inverses appearing in the relevant Darboux transformation formulas in Section 3 are well defined. In Section 4 we present the Darboux transformation formulas when a bound state is removed from the spectrum of the Schrödinger operator. In Theorem 4.1 we prove that the matrix inverses appearing in the relevant Darboux transformation formulas in Section 4 are well defined. Finally, in Section 5 we present some illustrative examples for better understanding of the results introduced and also make a contrast between (1.1) and (1.2) for certain results [2] related to compactly-supported potentials.

The most relevant reference for our paper is [3], and in the current paper we use the notation used in [3]. The results in [3] were presented under the assumption that the potential is compactly supported, i.e. $V_n = 0$ for $n > b$ for some positive integer b. In Section 2 we present the corresponding results when V_n belongs to the Faddeev class and does not necessarily have a compact support. Another relevant reference for our paper is the classic work by Case and Kac [4]. Even though [4] is more related to the Jacobi operator and not to the Schrödinger operator, the treatment of the spectral density in [4] is useful. We remark that the Darboux transformation results related to the Jacobi operators do not reduce to the Darboux transformation results for the Schrödinger operator. Hence, in our paper we use the Gel'fand-Levitan theory [4, 5, 12] and an appropriate formula for the spectral density for the corresponding Schrödinger operator with bound states, and we derive the Darboux transformation from first principles.

2. Preliminaries

In this section, associated with (1.1) and (1.3) we introduce various quantities such as the Jost solution f_n, the regular solution φ_n, the Jost function f_0, the scattering matrix S, and the spectral measure $d\rho$. We also present the basic properties of such quantities relevant to our analysis of Darboux transformations.

When the potential in (1.1) belongs to the Faddeev class, the Schrödinger operator corresponding to (1.1) and to the Dirichlet boundary condition (1.3) is a selfadjoint operator acting on the class of square-summable functions. The spectrum of the corresponding operator is well understood [3, 4, 7, 8, 13, 14, 15]. Let us use $\mathbb{R}$ to denote the real axis $(-\infty, +\infty)$. The continuous spectrum corresponds to $\lambda \in [0, 4]$, and the discrete spectrum consists of at most a finite number of discrete eigenvalues in $\mathbb{R} \setminus [0, 4]$, i.e. $\lambda \in (-\infty, 0) \cup (4, +\infty)$. For each λ-value in the interval $(0, 4)$, there are two linearly independent solutions to (1.1). There is only one linearly independent solution satisfying both (1.1) and (1.3), and such a solution is usually identified as a physical solution. Let us assume that the discrete spectrum consists of N eigenvalues given by $\{\lambda_s\}_{s=1}^N$, where $N = 0$ corresponds to the absence of the discrete spectrum. When $\lambda = \lambda_s$, there is only one linearly independent square-summable solution satisfying (1.1) and (1.3). For each of $\lambda = 0$ and $\lambda = 4$, there exists one linearly independent solution satisfying (1.1) and (1.3), and such a solution may be either bounded in n or it may grow as $O(n)$ as $n \to +\infty$. For $\lambda = 0$, one says that the exceptional case occurs if a solution satisfying (1.1) and

(1.3) is bounded in n and that the generic case occurs if a solution satisfying (1.1) and (1.3) is not bounded in n. Similarly, for $\lambda = 4$, the exceptional case occurs if a solution satisfying (1.1) and (1.3) is bounded in n and that the generic case occurs if a solution satisfying (1.1) and (1.3) is not bounded in n.

In quantum mechanics, it is customary to interpret the discrete spectrum associated with (1.1) and (1.3) as the bound states. Hence, the λ_s-values in the discrete spectrum can be called the bound-state energies and the corresponding square-summable solutions can be called bound-state wavefunctions. The solutions to (1.1) when $\lambda \in (0, 4)$ can be referred to as scattering solutions.

Associated with (1.1), instead of λ, it is convenient at times to use another spectral parameter related to λ, usually denoted by z, given by

$$z := 1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda - 4)}, \tag{2.1}$$

where the square root is used to denote the principal branch of the complex square-root function. Note that (2.1) yields

$$\lambda = 2 - z - z^{-1}. \tag{2.2}$$

Let us use $\mathbf{T}$ for the unit circle $|z| = 1$ in the complex plane $\mathbb{C}$, $\mathbf{T}^+$ for the upper portion of $\mathbf{T}$ given by $z = e^{i\theta}$ with $\theta \in (0, \pi)$, and $\overline{\mathbf{T}^+}$ for the closure of $\mathbf{T}^+$ given by $z = e^{i\theta}$ with $\theta \in [0, \pi]$. Under the transformation from $\lambda \in \mathbb{C}$ to $z \in \mathbb{C}$, the real interval $\lambda \in (0, 4)$ is mapped to $z \in \mathbf{T}^+$, the real half line $\lambda \in (-\infty, 0)$ is mapped to the real interval $z \in (0, 1)$, the real interval $\lambda \in (4, +\infty)$ is mapped to the real interval $z \in (-1, 0)$, the point $\lambda = 0$ is mapped to $z = 1$, and the point $\lambda = 4$ is mapped to $z = -1$. Using (2.2) it is convenient to write (1.1) as

$$\psi_{n+1} + \psi_{n-1} = (z + z^{-1} + V_n)\,\psi_n, \quad n \geq 1. \tag{2.3}$$

Let us now consider certain particular solutions to (1.1). A relevant solution to (1.1) or equivalently to (2.3) is the so-called regular solution φ_n satisfying the initial conditions

$$\varphi_0 = 0, \quad \varphi_1 = 1. \tag{2.4}$$

From (2.3) and (2.4) it follows that φ_n remains unchanged if we replace z with z^{-1} in φ_n.

The result presented in the following theorem is already known and its proof is omitted. A proof in our own notation can be obtained as in the proof of [3, Theorem 2.6].

Theorem 2.1. *Assume that the potential V_n belongs to the Faddeev class. Then, for $n \geq 1$ the regular solution φ_n to* (1.1) *with the initial values* (2.4) *is a polynomial in λ of degree $n - 1$ and is given by*

$$\varphi_n = \sum_{j=0}^{n-1} B_{nj}\lambda^j, \tag{2.5}$$

where, for each fixed positive integer n, the set of coefficients $\{B_{nj}\}_{j=0}^{n-1}$ are real valued and uniquely determined by the ordered set $\{V_1, V_2, \dots, V_{n-1}\}$ of potential values. In particular, we have

$$B_{n(n-1)} = (-1)^{n-1}, \quad B_{n(n-2)} = (-1)^{n-2}\Big[2(n-1) + \sum_{j=1}^{n-1} V_j\Big].$$

We remark that Theorem 2.1 holds even when the potential V_n does not belong to the Faddeev class. If the potential values are allowed to be complex, then the coefficients B_{nj} appearing in (2.5) are complex valued.

From (2.5) it is clear that the λ-domain of φ_n is the entire complex λ-plane. With the help of (2.2), we can conclude that the z-domain of φ_n corresponds to the punctured complex z-plane with the point $z = 0$ removed.

Another relevant solution to (1.1) or equivalently to (2.3) is the Jost solution f_n satisfying the asymptotic condition

$$f_n = z^n \left[1 + o(1)\right], \quad n \to +\infty. \tag{2.6}$$

On the unit circle $z \in \mathbf{T}$ we have $z^{-1} = z^*$, where we use an asterisk to denote complex conjugation. Let us use $f_n(z)$ to denote the value of f_n when $z \in \overline{\mathbf{T}^+}$. From (2.3) and (2.6) it follows that we have

$$f_n(z^{-1}) = f_n(z^*) = f_n(z)^*, \quad z \in \overline{\mathbf{T}^+}, \tag{2.7}$$

and hence the domain of $f_n(z)$ can be extended from $z \in \overline{\mathbf{T}^+}$ to $z \in \mathbf{T}$ by using (2.7). We will see in Theorem 2.2 that, when the potential V_n belongs to the Faddeev class, the domain of $f_n(z)$ can be extended from $z \in \mathbf{T}$ to the unit disc $|z| \le 1$.

Let us define g_n as the quantity f_n but by replacing z by z^{-1} there, i.e.

$$g_n(z) := f_n(z^{-1}), \quad z \in \mathbf{T}. \tag{2.8}$$

From (2.8) it follows that the domain of $g_n(z)$ is originally given as $z \in \mathbf{T}$ and it can be extended to $|z| \ge 1$ when the potential V_n in (1.1) belongs to the Faddeev class. With the help of (2.3) we see that g_n is also a solution to (1.1), and from (2.6) it follows that g_n satisfies the asymptotic condition

$$g_n = z^{-n} \left[1 + o(1)\right], \quad n \to +\infty. \tag{2.9}$$

The quantity f_0, which is obtained from the Jost solution f_n with $n = 0$, is known as the Jost function. Let us remark that the Jost solution f_n is determined by the potential V_n alone and is unaffected by the choice of the Dirichlet boundary condition (1.3). On the other hand, the Dirichlet boundary condition (1.3) is used when naming f_0 as the Jost function. For a non-Dirichlet boundary condition the Jost function is not defined as f_0 and it corresponds to an appropriate linear combination of f_0 and f_1. In this paper we do not deal with the Jost function in the non-Dirichlet case.

The Jost function $f_0(z)$ is used to define the scattering matrix S as

$$S(z) := \frac{f_0(z)^*}{f_0(z)}, \quad z \in \mathbf{T}. \tag{2.10}$$

Even though $S(z)$ is scalar valued, it is customary to refer to it as the scattering matrix. With the help of (2.7) and (2.8) we see that we can write (2.10) in various equivalent forms such as

$$S(z) = \frac{g_0(z)}{f_0(z)} = \frac{f_0(z^{-1})}{f_0(z)}, \quad z \in \mathbf{T}. \tag{2.11}$$

Let us write the Jost function in the polar form as

$$f_0(z) = |f_0(z)|\, e^{-i\,\phi(z)}, \quad z \in \mathbf{T}. \tag{2.12}$$

The real-valued quantity $\phi(z)$ appearing in (2.12) is usually called the phase shift. Its domain consists of $z \in \mathbf{T}$. Using (2.7) in (2.12) we see that the phase shift satisfies

$$\phi(z^{-1}) = \phi(z^*) = -\phi(z), \quad z \in \mathbf{T}. \tag{2.13}$$

From (2.10) we see that the scattering matrix can be expressed in terms of the phase shift as

$$S(z) = e^{2i\,\phi(z)}, \quad z \in \mathbf{T}. \tag{2.14}$$

The relevant properties of the Jost solution f_n and the Jost function f_0 are summarized in the following theorem.

Theorem 2.2. *Assume that the potential V_n in* (1.1) *belongs to the Faddeev class. Then:*

(a) *For each fixed $n \geq 0$, the Jost solution f_n satisfying* (1.1) *and* (2.6) *is analytic in z in $|z| < 1$ and continuous in z in $|z| \leq 1$. It has the representation*

$$f_n(z) = \sum_{m=n}^{\infty} K_{nm}\, z^m, \quad |z| \leq 1, \tag{2.15}$$

where each double-indexed coefficient K_{nm} is real valued and uniquely determined by the potential values in the ordered set $\{V_m\}_{m=n+1}^{\infty}$. In particular, we have

$$K_{nn} = 1, \quad K_{n(n+1)} = \sum_{j=n+1}^{\infty} V_j, \quad K_{n(n+2)} = \sum_{n+1 \leq j < l \leq +\infty} V_j\, V_l. \tag{2.16}$$

(b) *The Jost function $f_0(z)$ is analytic in $|z| < 1$ and continuous in $|z| \leq 1$. It has the representation*

$$f_0(z) = \sum_{m=0}^{\infty} K_{0m}\, z^m, \quad |z| \leq 1, \tag{2.17}$$

where each coefficient K_{0m} is uniquely determined by the set $\{V_n\}_{n=1}^{\infty}$ of potential values. In particular, we have

$$K_{00} = 1, \quad K_{01} = \sum_{j=1}^{\infty} V_j, \quad K_{02} = \sum_{1 \leq j < l \leq +\infty} V_j\, V_l. \tag{2.18}$$

(c) *For each fixed $n \geq 0$, the solution $g_n(z)$ satisfying* (1.1) *and* (2.9) *is analytic in $|z| > 1$ and continuous in $|z| \geq 1$. It has the representation*

$$g_n(z) = \sum_{m=n}^{\infty} K_{nm}\, z^{-m}, \quad |z| \geq 1.$$

(d) *The solutions f_n and g_n are linearly independent when $z \in \mathbf{T} \setminus \{-1, 1\}$. In particular, the regular solution φ_n appearing in* (2.4) *can be expressed in terms of f_n and g_n as*

$$\varphi_n = \frac{1}{z - z^{-1}}\, (g_0 f_n - f_0\, g_n)\,. \tag{2.19}$$

Proof. It is enough to prove the analyticity in $|z| < 1$ and the continuity in $|z| \leq 1$ for $f_n(z)$. The remaining results in (a)-(c) can be obtained with the help of [3, Proposition 2.4]. Note that (2.19) is the same as [3, (2.42)] and the linear independence of f_n and g_n is established by using (2.6) and (2.9). Let us then prove the aforementioned analyticity and continuity. In fact, for the analyticity in $|z| < 1$, it is enough to use the summability in (1.5) without the need for the first moment of the potential. The first moment in (1.5) is needed to prove the continuity at $z = \pm 1$. We can prove the analyticity by modifying the proof of [10, Lemma 1] so that it is applicable to the discrete Schrödinger equation. We only provide the key steps and let the reader work out the details. Letting

$$m_n := z^{-n} f_n, \tag{2.20}$$

from (2.6) we see that

$$m_n = 1 + o(1), \quad n \to +\infty,$$

for each fixed $z \in \mathbf{T}$. With the help of (2.3) and (2.20) we see that m_n satisfies the discrete equation given by

$$m_n = 1 + \frac{1}{z - z^{-1}} \sum_{j=n+1}^{\infty} \big(z^{2(j-n)} - 1\big)\, V_j\, m_j. \tag{2.21}$$

Note that (2.21) is the discrete analog of the second displayed formula on [10, p. 130]. Next we solve (2.21) iteratively by letting

$$m_n(z) = \sum_{p=0}^{\infty} m_n^{(p)}(z), \quad |z| < 1, \tag{2.22}$$

where we have defined

$$m_n^{(0)}(z) := 1, \quad |z| < 1, \tag{2.23}$$

$$m_n^{(p)}(z) := \frac{1}{z - z^{-1}} \sum_{j=n+1}^{\infty} \big(z^{2(j-n)} - 1\big)\, V_j\, m_j^{(p-1)}(z), \quad |z| < 1, \quad p \ge 1. \tag{2.24}$$

Each iterate $m_n^{(p)}(z)$ is analytic in $|z| < 1$, and the left-hand side of (2.22) is analytic in $|z| < 1$ if we can show that the series on the right-hand side of (2.22) converges uniformly in every compact subset of $|z| < 1$. When $|z| \le 1$, we have

$$|z^{2(j-n)} - 1| \le 2, \quad j \ge n + 1. \tag{2.25}$$

Furthermore, from (1.5) we have

$$\sum_{j=n+1}^{\infty} |V_j| \le \sum_{j=1}^{\infty} |V_j| < +\infty. \tag{2.26}$$

The uniform convergence is established by using the estimates in (2.25) and (2.26). Hence, $m_n(z)$ is analytic in $|z| < 1$ for each fixed nonnegative integer n. From (2.20) it then follows that $f_n(z)$ is analytic in $|z| < 1$ for each fixed $n \ge 0$. In order to prove the continuity of $m_n(z)$ in $|z| \le 1$, we need to show that each iterate $m_n^{(p)}(z)$ is continuous in $|z| \le 1$ and that the series in (2.22) converges absolutely in $|z| \le 1$. The factor $z - z^{-1}$ appearing in the denominator of (2.24) becomes troublesome at $z = \pm 1$. As a remedy, we use the identity

$$\frac{z^{2(j-n)} - 1}{z - z^{-1}} = z\, \frac{z^{2j-2n} - 1}{z^2 - 1} = z \sum_{k=0}^{j-n-1} z^{2k}, \quad j \ge n + 1. \tag{2.27}$$

From (2.27) it follows that for $|z| \le 1$ we have

$$\Big| \frac{z^{2(j-n)} - 1}{z - z^{-1}} \Big| \le j - n, \quad j \ge n + 1. \tag{2.28}$$

With the help of (1.5), (2.23), (2.24), and (2.28), one establishes the uniform convergence in $|z| \leq 1$ for the series on the right-hand side of (2.22). Furthermore, with the help of (2.24) and (2.27) we establish the continuity of each iterate $m_n^{(p)}(z)$ in $|z| \leq 1$. Then, it follows that $m_n(z)$ appearing on the left-hand side (2.22) is continuous in $|z| \leq 1$. Finally, from (2.20) it follows that $f_n(z)$ is continuous in $|z| \leq 1$ for each fixed value of n. □

Let us remark that, from (2.17) and (2.18) we see that the value of the Jost function $f_0(z)$ at $z = 0$ is given by

$$f_0(0) = 1. \tag{2.29}$$

From the second equality of (2.16) it follows that

$$V_n = K_{(n-1)n} - K_{n(n+1)}, \quad n \geq 1.$$

The results in following theorem clarifies the generic and exceptional cases encountered at the endpoints of the continuous spectrum, i.e. at $\lambda = 0$ and $\lambda = 4$.

Theorem 2.3. *Assume that the potential V_n in* (1.1) *belongs to the Faddeev class. Let λ and z be the spectral parameters appearing in* (1.1) *and* (2.1)*, respectively, and let φ_n and f_n be the corresponding regular solution and the Jost solution to* (1.1) *appearing in* (2.4) *and* (2.6)*, respectively. Let f_0 be the corresponding Jost function. Then:*

(a) *The Jost function $f_0(z)$ is nonzero when $z \in \mathbf{T} \setminus \{-1, 1\}$.*

(b) *At $\lambda = 0$, or equivalently at $z = 1$, the regular solution φ_n either grows linearly in n as $n \to +\infty$, which corresponds to the generic case, or it is bounded in n, which corresponds to the exceptional case. Hence, $\lambda = 0$ never corresponds to a bound state for* (1.1) *with the Dirichlet boundary condition* (1.3)*. In the generic case, $f_0 \neq 0$ at $z = 1$. In the exceptional case, f_0 has a simple zero at $z = 1$.*

(c) *At $\lambda = 4$, or equivalently at $z = -1$, the regular solution φ_n generically grows linearly in n as $n \to +\infty$, and in the exceptional case the regular solution φ_n is bounded in n. Hence, $\lambda = 4$ never corresponds to a bound state for* (1.1) *with the Dirichlet boundary condition* (1.3)*. In the generic case we have $f_0 \neq 0$ at $z = -1$. In the exceptional case, f_0 has a simple zero at $z = -1$.*

Proof. The proofs (b) and (c) can be obtained as in the proof of [3, Theorem 2.5]. The proof of (a) can be given as follows. Assume, on the contrary, that $f_0(z)$ vanished at some point $z = z_0$, where z_0 is located on the unit circle $\mathbf{T}$ and $z_0 \neq \pm 1$. From (2.7) and (2.8) it follows that $f_0(z_0) = 0$ implies that $g_0(z_0) = 0$. Using these values in (2.19) we would then get $\varphi_n \equiv 0$ for any positive integer n when $z = z_0$. On the other hand, by the second equality in (2.4) we know that φ_1 must be equal to 1 when $z = z_0$. This contradiction shows that f_0 cannot vanish on the unit circle, except perhaps at $z = \pm 1$. □

The following theorem shows that the Jost function $f_0(z)$ cannot vanish at any z-value inside the unit circle when the imaginary part of that z-value is nonzero.

Theorem 2.4. *Assume that the potential V_n appearing in* (1.1) *belongs to the Faddeev class. Let z be the spectral parameter appearing in* (2.1)*, $f_n(z)$ be the corresponding Jost solution appearing in* (2.15)*, and $f_0(z)$ be the corresponding Jost function appearing in* (2.17)*. Then, $f_0(z) \neq 0$ for any z satisfying $|z| < 1$ with* *the imaginary part* $\mathrm{Im}[z]$ *is nonzero. The zeros of* $f_0(z)$ *in the interior of the unit circle can only occur when* $z \in (-1,0)\cup(0,1)$.

Proof. From (2.29) we see that $f_0(0)=1$, and hence it is enough to prove that $f_0(z)\neq 0$ when $|z|<1$ with $z_{\mathrm{I}}\neq 0$, where we use the decomposition $z := z_{\mathrm{R}} + i\,z_{\mathrm{I}}$, with z_{R} and z_{I} denoting the real and imaginary parts of z, respectively. For simplicity, let us use f_n to denote $f_n(z)$. Since f_n satisfies (2.3) we have

$$f_{n+1} + f_{n-1} = (z + z^{-1} + V_n)\, f_n, \quad n \geq 1. \tag{2.30}$$

Taking the complex conjugate of both sides of (2.30) and using the fact that V_n is real, we obtain

$$f^*_{n+1} + f^*_{n-1} = \left[z^* + (z^*)^{-1} + V_n\right]\, f^*_n, \quad n \geq 1. \tag{2.31}$$

Let us multiply both sides of (2.30) with f^*_n and multiply both sides of (2.31) with f_n and subtract the resulting equations side by side. This yields

$$\begin{aligned} &f^*_n\, f_{n+1} + f^*_n\, f_{n-1} - f^*_{n+1}\, f_n - f^*_{n-1}\, f_n \\ &= [z - z^* + z^{-1} - (z^*)^{-1}]\, |f_n|^2, \quad n \geq 1. \end{aligned} \tag{2.32}$$

Note that

$$\mathrm{Im}[z^{-1}] = \mathrm{Im}\left[\frac{1}{z_{\mathrm{R}} + i\, z_{\mathrm{I}}}\right] = \frac{-z_{\mathrm{I}}}{z_{\mathrm{R}}^2 + z_{\mathrm{I}}^2}. \tag{2.33}$$

We have

$$z - z^* + z^{-1} - (z^*)^{-1} = 2i\,\mathrm{Im}[z] + 2i\,\mathrm{Im}[z^{-1}], \tag{2.34}$$

and using (2.33) in (2.34) we obtain

$$z - z^* + z^{-1} - (z^*)^{-1} = 2i\, z_{\mathrm{I}} - 2i\, \frac{z_{\mathrm{I}}}{z_{\mathrm{R}}^2 + z_{\mathrm{I}}^2},$$

or equivalently

$$z - z^* + z^{-1} - (z^*)^{-1} = 2i\, z_{\mathrm{I}}\, \frac{z_{\mathrm{R}}^2 + z_{\mathrm{I}}^2 - 1}{z_{\mathrm{R}}^2 + z_{\mathrm{I}}^2}. \tag{2.35}$$

Let us take the summation over n on both sides of (2.32) and use (2.35) in the resulting summation, which yields

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[f^*_n\, f_{n+1} - f^*_{n-1}\, f_n\right] + \sum_{n=1}^{\infty}\left[f^*_n\, f_{n-1} - f^*_{n+1}\, f_n\right] \\ &= 2i\, z_{\mathrm{I}}\, \frac{z_{\mathrm{R}}^2 + z_{\mathrm{I}}^2 - 1}{z_{\mathrm{R}}^2 + z_{\mathrm{I}}^2} \sum_{n=1}^{\infty} |f_n|^2. \end{aligned} \tag{2.36}$$

When $|z|<1$, the two series on the left-hand side of (2.36) are both telescoping, and using (2.6) in (2.36) we obtain

$$-\, f^*_0\, f_1 + f^*_1\, f_0 = -2i\, z_{\mathrm{I}}\, \frac{1 - |z|^2}{|z|^2} \sum_{n=1}^{\infty} |f_n|^2. \tag{2.37}$$

When $|z|<1$ with $z_{\mathrm{I}}\neq 0$, the right-hand side of (2.37) cannot vanish unless $f_n(z)=0$ for $n\geq 1$. However, because of (2.6) we cannot have $f_n(z)=0$ for all $n\geq 1$ at such a z-value. Thus, the right-hand side of (2.37) cannot be zero for any z-value satisfying $|z|<1$ with $z_{\mathrm{I}}\neq 0$. On the other hand, if we had $f_0(z)=0$ for some z-value satisfying $|z|<1$ with $z_{\mathrm{I}}\neq 0$, then we would also have $f_0(z)^*=0$ at the same z-value, and hence we would have the left-hand side of (2.37) vanishing at that z-value. This contradiction proves that $f_0(z)\neq 0$ for any

z-value satisfying $|z| < 1$ with $z_{\rm I} \neq 0$. Since we have already seen that $f_0(0) \neq 0$, we conclude that the zeros of $f_0(z)$ in the interior of the unit circle can only occur when $z \in (-1, 0) \cup (0, 1)$. □

In the next theorem, we summarize the facts relevant to the bound states of (1.1) with the Dirichlet boundary condition (1.3). Recall that the bound states correspond to the λ-values at which (1.1) has square-summable solutions satisfying the boundary condition (1.3).

Theorem 2.5. *Assume that the potential V_n in* (1.1) *belongs to the Faddeev class. Let λ and z be the spectral parameters appearing in* (1.1) *and* (2.1), *respectively, and let f_n, φ_n, and f_0 be the corresponding Jost solution appearing in* (2.6), *the regular solution appearing in* (2.4), *and the Jost function appearing in* (2.12), *respectively. Then:*

(a) *A bound state can only occur when $\lambda \in (-\infty, 0)$ or $\lambda \in (4, +\infty)$. Equivalently, a bound state can only occur when $z \in (-1, 0)$ or $z \in (0, 1)$.*

(b) *At a bound state, φ_n and f_n are both real valued for every $n \geq 1$. At a bound state, φ_n and f_n are linearly dependent and each is square summable in n.*

(c) *At a bound state the Jost function f_0 has a simple zero in λ and in z. At a bound state the value of the Jost solution at $n = 1$ cannot vanish, i.e. $f_1 \neq 0$ at a bound state.*

(d) *The number of bound states, denoted by N, is finite. In particular, we have $N = 0$ when $V_n \equiv 0$.*

Proof. The proofs for (a)-(c) can be obtained by slightly modifying the proof of [3, Theorem 2.5] as follows. At a bound state, (1.1) must have a square-summable solution satisfying the Dirichlet boundary condition (1.3). Note that (1.1) has two linearly independent solutions, and only one of those two linearly independent solutions can satisfy (1.3). We know from the first equality in (2.4) that the regular solution φ_n appearing in (2.5) satisfies (1.3). Thus, any bound-state solution to (1.1) must be linearly dependent on φ_n. Since the corresponding Schrödinger operator is selfadjoint, the bound states can only occur when the spectral parameter λ is real. From (2.1) we know that the λ-values in the interval $\lambda \in (0, 4)$ correspond to the z-values on $\mathbf{T}^+$, the upper portion of the unit circle $\mathbf{T}$. For such z-values, from (2.6) and (2.9) we conclude that neither of the two linearly independent solutions f_n and g_n can vanish as $n \to +\infty$, where we recall that g_n is the solution appearing in (2.8). Furthermore, by (b) and (c) of Theorem 2.3 we know that neither $\lambda = 0$ nor $\lambda = 4$ can correspond to a bound state. Thus, the bound states can only occur when $\lambda \in (-\infty, 0)$ or $\lambda \in (4, +\infty)$. Equivalently, with the help of (2.1) we conclude that a bound state can only occur when $z \in (-1, 0)$ or $z \in (0, 1)$. This completes the proof of (a). Let us now prove (b). From Theorem 2.1 we know that the coefficients B_{nj} appearing in (2.5) are real valued, and hence (2.5) implies that at any λ-value in the interval $\lambda \in (-\infty, 0)$ or $\lambda \in (4, +\infty)$ the corresponding φ_n is real valued for every $n \geq 1$. Similarly, we know from Theorem 2.2(a) that the coefficients K_{nm} appearing in (2.15) are real valued, and hence (2.15) implies that f_n for every $n \geq 1$ is real valued at any z-value occurring in $z \in (-1, 0) \cup (0, 1)$. In the proof of (a) we have already indicated the linear dependence of φ_n and f_n and we have also indicated that their square integrability follows from the definition of a bound-state solution. Thus, the proof of (b) is complete. Let us now turn to the proof of (c).

This follows by proceeding as in [3, (2.67)–(2.69)] and hence by concluding that at a bound state the Jost function f_0 must have a simple zero in λ and a simple zero in z and that f_1 cannot vanish at a bound state. This concludes the proof of (c). Let us now prove (d). The finiteness of the number of bound states can be proved as follows. From Theorem 2.2 we know that $f_0(z)$ is analytic in $|z| < 1$ and continuous in $|z| \le 1$. From (2.29) we know that $f_0(0) = 1$. Furthermore, from (a) and (c) above we know that the bound states can only occur at the zeros of $f_0(z)$ when $z \in (-1, 0) \cup (0, 1)$ and such zeros are simple. Thus, the bound-state zeros of $f_0(z)$ could only accumulate at $z = \pm 1$. On the other hand, Theorem 2.3 indicates that $f_0(z)$ can at most have simple zeros at $z = \pm 1$. Thus, $f_0(z)$ is analytic in $z \in (-1, 1)$ with no accumulation points in $z \in [-1, 1]$. Consequently, the number of bound-state zeros of $f_0(z)$ must be finite. □

For further elaborations on the finiteness of the number of bound states, we refer the reader to [7, 8] and the references therein.

Let us assume that the bound states occur at $\lambda = \lambda_s$ for $s = 1, \dots, N$. Let us also assume that the corresponding z_s-values are obtained via using (2.1), and hence the bound states occur at $z = z_s$ for $s = 1, \dots, N$. From (2.2) we see that

$$\lambda_s = 2 - z_s - z_s^{-1}, \quad s = 1, \dots, N. \tag{2.38}$$

From Theorem 2.5(b) we know that $\varphi_n(\lambda_s)$ is real valued and the quantity C_s defined as

$$C_s := \Big(\sum_{n=1}^{\infty} \varphi_n(\lambda_s)^2 \Big)^{-1/2}, \quad s = 1, \dots, N, \tag{2.39}$$

is a finite nonzero number. It is appropriate to refer to the positive number C_s as the Gel'fand-Levitan norming constant at $\lambda = \lambda_s$. Thus, the quantity $C_s \varphi_n(\lambda_s)$ is a normalized bound-state solution to (1.1) at the bound state $\lambda = \lambda_s$. Similarly, from Theorem 2.5(b) we know that $f_n(z_s)$ is real valued and the quantity c_s defined as

$$c_s := \Big(\sum_{n=1}^{\infty} f_n(z_s)^2 \Big)^{-1/2}, \quad s = 1, \dots, N, \tag{2.40}$$

is a finite nonzero number. It is appropriate to refer to the positive number c_s as the Marchenko norming constant at $z = z_s$. Thus, the quantity $c_s f_n(z_s)$ is a normalized bound-state solution to (1.1) at the bound state $z = z_s$. We then get

$$C_s^2 \left[\varphi_n(\lambda_s)\right]^2 = c_s^2 \left[f_n(z_s)\right]^2, \quad s = 1, \dots, N. \tag{2.41}$$

Using the second equality of (2.4) in (2.41) we see that the Gel'fand-Levitan norming constant C_s and the Marchenko norming constant c_s are related to each other as

$$C_s^2 = c_s^2 \left[f_1(z_s)\right]^2, \quad s = 1, \dots, N. \tag{2.42}$$

Let us use a circle above a quantity to emphasize that it corresponds to the trivial potential $V_n \equiv 0$ in (1.1). Hence, $\mathring{\varphi}_n$ denotes the regular solution, $\mathring{f}_n$ is the Jost solution, $\mathring{g}_n$ is related to $\mathring{f}_n$ as in (2.8), $\mathring{f}_0$ is the Jost function, and $\mathring{S}$ is the scattering matrix. We have [3]

$$\mathring{f}_n = z^n, \quad \mathring{g}_n = z^{-n}, \quad \mathring{\varphi}_n = \frac{z^n - z^{-n}}{z - z^{-1}}, \quad n \ge 1,$$
$$\mathring{f}_0(z) \equiv 1, \quad \mathring{g}_0(z) \equiv 1, \quad \mathring{S}(z) \equiv 1.$$

Let us use $d\rho$ to denote the spectral density corresponding to the Schrödinger equation (1.1) with the Dirichlet boundary condition (1.3). The spectral density is normalized, i.e. its integral over the real-λ axis is equal to one. Let us use $d\mathring{\rho}$ to denote the spectral density when the potential is zero. From [3, (4.1)] we have

$$d\mathring{\rho} = \begin{cases} 0, & \lambda < 0, \\ \dfrac{1}{2\pi}\,\sqrt{\lambda(4-\lambda)}\,d\lambda, & 0 \le \lambda \le 4, \\ 0, & \lambda > 4. \end{cases} \tag{2.43}$$

From (2.43) we see that, when the potential is zero, the discrete part of the spectral measure, i.e. the part corresponding to $\mathbb{R}\setminus[0,4]$ is zero. Thus, the continuous part of the spectral density in (2.43) has its integral over $\lambda\in[0,4]$ equal to one. Using (2.2) in (2.43), we can express [3] the continuous part of $d\mathring{\rho}$ in terms of z as

$$d\mathring{\rho} = -\frac{1}{2\pi i}\left(z - z^{-1}\right)^2 \frac{dz}{z}, \quad z \in \overline{\mathbf{T}^+},$$

where we recall that $\overline{\mathbf{T}^+}$ denotes the closure of the upper portion of the unit circle $\mathbf{T}$.

In the absence of bound states, the spectral density $d\rho$ associated with (1.1) and (1.3) is given by

$$d\rho = \begin{cases} \dfrac{d\mathring{\rho}}{|f_0(z)|^2}, & \lambda \in [0,4], \\ 0, & \lambda \in \mathbb{R}\setminus[0,4], \end{cases} \tag{2.44}$$

where we recall that $\lambda\in[0,4]$ corresponds to $z\in\overline{\mathbf{T}^+}$. Thus, the discrete part of the spectral density $d\rho$ is zero and the continuous part of the spectral density is obtained by dividing $d\mathring{\rho}$ by the absolute square of the Jost function $f_0(z)$. When there are N bound states at $\lambda=\lambda_s$ with the corresponding Gel'fand-Levitan norming constants C_s appearing in (2.39), one can evaluate the spectral density $d\rho$ as

$$d\rho = \begin{cases} \left(\dfrac{1-\sum_{s=1}^N C_s^2}{\prod_{k=1}^N z_k^2}\right)\dfrac{d\mathring{\rho}}{|f_0(z)|^2}, & \lambda \in [0,4], \\ \sum_{s=1}^N C_s^2\,\delta(\lambda-\lambda_s)\,d\lambda, & \lambda \in \mathbb{R}\setminus[0,4], \end{cases} \tag{2.45}$$

where $f_0(z)$ is the corresponding Jost function and each z_s corresponds to λ_s via (2.38). We remark that $\lambda\in[0,4]$ in (2.45) corresponds to $z\in\overline{\mathbf{T}^+}$. Note that, in the absence of bound states, i.e. when $N=0$, the spectral density given in (2.45) reduces to the expression given in (2.44). In the evaluation of (2.45) we have used the facts that

$$\int_{\lambda\in\mathbb{R}} d\rho = 1, \quad \int_{\lambda\in\mathbb{R}\setminus[0,4]} d\rho = \sum_{s=1}^N C_s^2, \quad \int_{\lambda\in[0,4]} d\rho = 1 - \sum_{s=1}^N C_s^2. \tag{2.46}$$

With the help of (2.46) we see that the first line of (2.45) yields

$$\int_{\lambda\in[0,4]} \frac{d\mathring{\rho}}{|f_0(z)|^2} = \prod_{k=1}^N z_k^2.$$

In order to understand the Darboux transformation, we need to establish the Gel'fand-Levitan formalism related to (1.1) and (1.3). Let V_n and $\tilde{V}_n$ be the unperturbed and perturbed potentials, respectively. Let φ_n and $\tilde{\varphi}_n$ be the respective corresponding regular solutions, and let $d\rho$ and $d\tilde{\rho}$ be the respective corresponding spectral densities. From Theorem 2.1 it follows that the set $\{\varphi_j\}_{j=0}^n$ forms a basis for polynomials in λ of degree $n-1$, and hence we can write

$$\tilde{\varphi}_n = \begin{cases} \varphi_n, & n = 1, \\ \varphi_n + \sum_{m=1}^{n-1} A_{nm}\, \varphi_m, & n \geq 2, \end{cases} \tag{2.47}$$

where A_{nm} are some double-indexed real coefficients to be determined. Let us define the double-indexed real-valued scalars G_{nm} as

$$G_{nm} := \int_{\lambda \in \mathbb{R}} \varphi_n \, [d\tilde{\rho} - d\rho] \, \varphi_m. \tag{2.48}$$

We already have [3, 4] the orthonormality

$$\int_{\lambda \in \mathbb{R}} \varphi_n \, d\rho \, \varphi_m = \delta_{nm}, \tag{2.49}$$

with δ_{nm} denoting the Kronecker delta. Proceeding as in [3, (4.13)–(4.17)] we obtain the Gel'fand-Levitan system

$$A_{nm} + G_{nm} + \sum_{j=1}^{n-1} A_{nj} G_{jm} = 0, \quad 1 \leq m < n. \tag{2.50}$$

Analogous to [3, (2.84)], we obtain

$$\tilde{V}_n - V_n = A_{(n+1)n} - A_{n(n-1)}, \quad n \geq 1, \tag{2.51}$$

with the understanding that $A_{10} = 0$.

For each integer $n \geq 2$, let $\mathbf{G}_{n-1}$ be the $(n-1)\times(n-1)$ matrix whose (k,l)-entry is equal to G_{kl} evaluated as in (2.48), i.e.

$$\mathbf{G}_{n-1} := \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1(n-2)} & G_{1(n-1)} \\ G_{21} & G_{22} & \cdots & G_{2(n-2)} & G_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ G_{(n-2)1} & G_{(n-2)2} & \cdots & G_{(n-2)(n-2)} & G_{(n-2)(n-1)} \\ G_{(n-1)1} & G_{(n-1)2} & \cdots & G_{(n-1)(n-2)} & G_{(n-1)(n-1)} \end{bmatrix}. \tag{2.52}$$

From (2.48) and (2.52) we see that $\mathbf{G}_{n-1}$ is a real symmetric matrix. For each integer $n \geq 2$, we can write the Gel'fand-Levitan system (2.50) in the matrix form as

$$(I_{n-1} + \mathbf{G}_{n1}) \begin{bmatrix} A_{n1} \\ A_{n2} \\ \vdots \\ A_{n(n-2)} \\ A_{n(n-1)} \end{bmatrix} = - \begin{bmatrix} G_{n1} \\ G_{n2} \\ \vdots \\ G_{n(n-2)} \\ G_{n(n-1)} \end{bmatrix}, \tag{2.53}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Let $\mathbf{g}_{n-1}$ be the column vector with $(n-1)$ components appearing on the right-hand side of (2.53), i.e.

$$\mathbf{g}_{n-1} := \begin{bmatrix} G_{n1} & G_{n2} & \cdots & G_{n(n-2)} & G_{n(n-1)} \end{bmatrix}^\dagger, \tag{2.54}$$

where the dagger denotes the matrix adjoint. Using (2.54) in (2.53) we obtain

$$\begin{bmatrix} A_{n1} \\ A_{n2} \\ \vdots \\ A_{n(n-2)} \\ A_{n(n-1)} \end{bmatrix} = -(I_{n-1} + \mathbf{G}_{n-1})^{-1}\,\mathbf{g}_{n-1}. \tag{2.55}$$

Thus, A_{nm} can be explicitly expressed in terms of the coefficients of $\mathbf{G}_{n-1}$ as

$$A_{nm} = -\hat{1}_m^\dagger\,(I_{n-1} + \mathbf{G}_{n-1})^{-1}\,\mathbf{g}_{n-1}, \quad 1 \le m < n, \tag{2.56}$$

where $\hat{1}_m$ is the column vector with $(n-1)$ components with all the entries being zero except for the mth entry being one. Note that the right-hand side of (2.56) contains a binomial form for a matrix inverse. Using [6, (15) on p. 12], the binomial form in (2.56) can be expressed as a ratio of two determinants, yielding

$$A_{nm} = \frac{\det \begin{bmatrix} 0 & \hat{1}_m^\dagger \\ \mathbf{g}_{n-1} & (I_{n-1} + \mathbf{G}_{n-1}) \end{bmatrix}}{\det[I_{n-1} + \mathbf{G}_{n-1}]}, \quad 1 \le m < n, \tag{2.57}$$

where the matrix in the numerator is a block matrix of size $n \times n$. Using (2.57) in (2.47) and (2.51) we obtain $\tilde{\varphi}_n$ and $\tilde{V}_n$ in terms of the unperturbed quantities.

Let us refer to the data set $\{\lambda_s, C_s\}_{s=1}^N$, which consists of all the bound-state energies and the corresponding Gel'fand-Levitan norming constants given in (2.39), as the bound-state data set. In general, the scattering matrix $S(z)$ defined in (2.10) and the bound-state data set are independent. This is because the domain of $S(z)$ consists of the unit circle $z \in \mathbf{T}$ and the bound-state energies correspond to the z_s-values inside the unit circle. Let us consider the case where the nontrivial potential V_n is compactly supported, i.e. when $V_n = 0$ for $n > b$ and $V_b \neq 0$ for some positive integer b. Thus, we use b to signify the compact support of V_n given by $\{1, 2, \dots, b\}$. For such potentials, it is known [3] that $S(z)$ has a meromorphic extension from $z \in \mathbf{T}$ to the region $|z| < 1$ and the z_s-values correspond to the poles of $S(z)$ in $|z| < 1$. Furthermore, for such potentials the corresponding C_s-values can be determined [3] in terms of certain residues evaluated at the z_s-values. In general, without a compact support, the values of z_s and C_s cannot be determined from the scattering matrix $S(z)$. On the other hand, even without a compact support, when the potential V_n belongs to the Faddeev class, the scattering matrix corresponding (1.1) and (1.3) contains some information related to the number of bound states N. This result is known as Levinson's theorem, and mathematically it can be viewed as an argument principle related to the integral of the logarithmic derivative of the scattering matrix along the unit circle $\mathbf{T}$ in the complex z-plane.

In the next theorem, we present Levinson's theorem associated with (1.1) and (1.3). For this purpose it is appropriate to introduce the constants μ_+ and μ_- as

$$\mu_+ := \begin{cases} 1, & f_0(1) = 0, \\ 0, & f_0(1) \neq 0, \end{cases} \tag{2.58}$$

$$\mu_- := \begin{cases} 1, & f_0(-1) = 0, \\ 0, & f_0(-1) \neq 0, \end{cases} \tag{2.59}$$

where we recall that $f_0(z)$ is the Jost function appearing in (2.12). Let us elaborate on (2.58) and (2.59). From Theorem 2.3(b), we know that $\mu_+ = 1$ if we have the exceptional case at $z = 1$ and we have $\mu_+ = 0$ if we have the generic case at $z = 1$. Similarly, from (2.59) and Theorem 2.3(c) we conclude that $\mu_- = 1$ if we have the exceptional case at $z = -1$ and we have $\mu_- = 0$ if we have the generic case at $z = -1$.

Let $\Delta_{\mathbf{T}}$ acting on a function of z denote the change in that function when the z-value moves along the unit circle $\mathbf{T}$ once in the counterclockwise direction in the sense of the Cauchy principal value. By the sense of the Cauchy principal value, we mean that in the evaluation of the change by using an integral along $\mathbf{T}$, we interpret the corresponding integral as a Cauchy principal value. In the theorem given below, that amounts to integrating along the unit circle $z = e^{i\theta}$ for $\theta \in (0^+, \pi - 0^+) \cup (\pi + 0^+, 2\pi - 0^+)$ because the only singularities for the integrand may occur at $z = 1$ or $z = -1$.

Theorem 2.6. *Assume that the potential V_n appearing in* (1.1) *belongs to the Faddeev class. Let $f_0(z)$ appearing in* (2.12)*, $S(z)$ appearing in* (2.10)*, $\phi(z)$ appearing in* (2.12)*, and N appearing in* (2.39) *be the respective Jost function, the scattering matrix, the phase shift, and the number of bound states corresponding to* (1.1) *and* (1.3)*. Let $\Delta_{\mathbf{T}}$ denote the change when the z-value moves along the unit circle $\mathbf{T}$ once in the counterclockwise direction in the sense of the Cauchy principal value. We then have the following:*

(a) *The change in the phase shift $\phi(z)$ when z moves along $\mathbf{T}$ in the counterclockwise direction once is given by*

$$\Delta_{\mathbf{T}}[\phi(z)] = -\pi \left[2N + \mu_+ + \mu_-\right], \tag{2.60}$$

where μ_+ and μ_- are the constants defined in (2.58) *and* (2.59)*, respectively.*

(b) *The change in the phase shift $\phi(z)$ when z moves along $\mathbf{T}^+$ from $z = 1$ to $z = -1$ is given by*

$$\Delta_{\mathbf{T}^+}[\phi(z)] = -\pi \left[N + \frac{\mu_+}{2} + \frac{\mu_-}{2}\right]. \tag{2.61}$$

(c) *The change in the argument of $S(z)$ when z moves along $\mathbf{T}^+$ from $z = 1$ to $z = -1$ is given by*

$$\Delta_{\mathbf{T}^+}[\arg[S(z)]] = -\pi\big[2N + \mu_+ + \mu_-\big]. \tag{2.62}$$

(d) *The change in the argument of $f_0(z)$ when z moves along $\mathbf{T}^+$ from $z = 1$ to $z = -1$ is given by*

$$\Delta_{\mathbf{T}^+}[\arg[f_0(z)]] = \pi \left[N + \frac{\mu_+}{2} + \frac{\mu_-}{2}\right]. \tag{2.63}$$

Proof. From Theorem 2.2(b) we know that f_0 is analytic in $|z| < 1$ and continuous in $|z| \le 1$. Thus, f_0 has no singularities in $|z| \le 1$. On the other hand, from Theorem 2.4 and Theorem 2.5(c) we know that the only zeros of f_0 in $|z| < 1$ occur at the bound states, those zeros are simple and can only occur when $z \in (-1, 0)$ or $z \in (0, 1)$, the number of such zeros is finite, and we use N to denote the nonnegative integer specifying the number of bound states. From Theorem 2.3 we know that the only zeros of f_0 on $z \in \mathbf{T}$ may occur at $z = \pm 1$, such zeros are simple, and the number of such zeros is equal to $\mu_+ + \mu_-$. Applying the argument principle to

$f_0(z)$ along the unit circle, we see that the change in the argument of $f_0(z)$ as z moves along the unit circle once in the counterclockwise direction is given by

$$\Delta_{\mathbf{T}}\left[\arg[f_0(z)]\right] = 2\pi\left[N + \frac{\mu_+}{2} + \frac{\mu_-}{2}\right], \tag{2.64}$$

where we have used the fact that the contribution from a zero of $f_0(z)$ on $z \in \mathbf{T}$ is half of the contribution from a zero in $|z| < 1$. Using (2.12) and (2.64) we obtain (2.60). Using (2.13) in (2.60) we obtain (2.61). Using (2.14) in (2.61) we obtain (2.62). Using (2.13) in (2.64) we have (2.63). □

3. Darboux transformation in adding a bound state

In this section we determine the effect of adding a bound state to the discrete spectrum of the Schrödinger operator corresponding to (1.1) and (1.3). For clarity, we use the notation $V_n(N)$ for V_n to indicate that the Schrödinger operator contains exactly N bound states occurring at $\lambda = \lambda_s$ for $s = 1, \dots, N$. Hence, we order the bound states by assuming that we start with the potential $V_n(0)$ containing no bound states. Then, we add one bound state at $\lambda = \lambda_1$ with some Gel'fand-Levitan norming constant and obtain the potential $V_n(1)$. Next, we add one bound state at $\lambda = \lambda_2$ with some Gel'fand-Levitan norming constant and obtain the potential $V_n(2)$. Continuing in this manner we recursively add all the bound states with $\lambda = \lambda_s$ for $s = 1, \dots, N$ and obtain the potential $V_n(N)$. Note that (2.38) establishes a one-to-one correspondence between λ_s and z_s, and hence we can equivalently say that the bound states of the potential $V_n(N)$ occur at $z = z_s$ for $s = 1, \dots, N$. We remark that the ordering of λ_s is completely arbitrary and that ordering does not have to have λ_s in an ascending or descending order.

To the "unperturbed" potential $V_n(N)$ let us add one bound state at $\lambda = \lambda_{N+1}$ with the Gel'fand-Levitan norming constant C_{N+1}. We then get the "perturbed" potential $V_n(N+1)$. Equivalently stated, we add one bound states at $z = z_{N+1}$, where z_{N+1} and λ_{N+1} are related to each other via (2.38) and $z_{N+1} \in (-1, 0) \cup (0, 1)$. The Jost function for the unperturbed problem is denoted by $f_0(z; N)$ and the Jost function for the perturbed problem is denoted by $f_0(z; N+1)$. In the analog of adding a bound state for the Schrödinger equation (1.2), we can uniquely express the perturbed Jost function in terms of the unperturbed Jost function by requiring that the absolute value of the Jost function in the continuous spectrum remains unchanged [5]. However, this is no longer the case for the discrete Schrödinger equation. Let us elaborate on this matter. We would like $f_0(z; N+1)$ to be obtained from $f_0(z; N)$ via

$$f_0(z; N+1) = \Big(1 - \frac{z}{z_{N+1}}\Big)\, Q(z)\, f_0(z; N), \quad |z| \le 1, \tag{3.1}$$

where $Q(z)$ is analytic in $|z| < 1$, continuous in $|z| \le 1$, and satisfies $Q(0) = 1$. The constraints on $Q(z)$ are determined by the fact that both $f_0(z; N+1)$ and $f_0(z; N)$ must be analytic in $|z| < 1$, continuous in $|z| \le 1$, and take the value of 1 at $z = 0$, as required by Theorem 2.2(b). Furthermore, $f_0(z; N+1)$ must have a simple zero at $z = z_{N+1}$ and $f_0(z; N)$ must be nonzero when $z = z_{N+1}$. The further requirement

$$|f_0(z; N+1)| = |f_0(z; N)|, \quad z \in \mathbf{T}, \tag{3.2}$$

combined with the maximum modulus principle would yield

$$\Big(1-\frac{z}{z_{N+1}}\Big)\,Q(z)\equiv 1,\quad |z|\le 1. \tag{3.3}$$

The result in (3.3) would follow from the fact that an analytic function in a bounded domain must take its maximum modulus value somewhere on the boundary, and it can be obtained as follows. The left-hand side of (3.3) is already equal to one at the interior point $z=0$ and hence (3.3) must hold for all z-values on the unit disk $|z|\le 1$. On the other hand, (3.3) is not acceptable because it requires $Q(z)$ to have a pole at $z=z_{N+1}$, contradicting the requirement that $Q(z)$ is analytic in $|z|<1$. Thus, in adding a bound state, we must use (3.1) without requiring (3.2).

In establishing a Darboux transformation, the choice of $Q(z)$ appearing in (3.1) is not unique. We find it convenient to choose a particular $Q(z)$ as

$$Q(z)=\frac{1}{1-z_{N+1}z},\quad |z|\le 1. \tag{3.4}$$

One could argue that the simplest choice $Q(z)\equiv 1$ would be a better choice than the one given in (3.4). It turns out that the choice in (3.4) has a few important advantages over other choices. For example, with the choice of $Q(z)$ given in (3.4) we obtain

$$|f_0(z;N+1)|^2=\frac{1}{z_{N+1}^2}\,|f_0(z;N)|^2,\quad z\in\mathbf{T}, \tag{3.5}$$

which greatly simplifies evaluations involving the spectral density given in (2.45). On the other hand, the choice $Q(z)\equiv 1$ yields

$$|f_0(z;N+1)|^2=\Big|1-\frac{z}{z_{N+1}}\Big|^2\,|f_0(z;N)|^2,\quad z\in\mathbf{T},$$

which hinders evaluations involving the spectral density. Another advantage of the choice of $Q(z)$ given in (3.4) is that the pole of $Q(z)$ at $z=1/z_{N+1}$ can be considered as a real-valued resonance for the discrete Schrödinger equation (1.1), where we recall that $z_{N+1}\in(-1,0)\cup(0,1)$. Consider the special case of a compactly-supported potential, where $z=z_{N+1}$ is a real-valued resonance for $V_n(N)$, i.e. $f_0(z;N)$ has a simple zero at $z=1/z_{N+1}$. We may then be able to convert that resonance into a bound state by adding a bound state to $V_n(N)$ at $z=z_{N+1}$ in such a way that $V_n(N+1)$ contains a bound state. We refer the reader to [2] for the analogous problem for (1.2) of converting a resonance into a bound state without affecting the compact support property of the potentials. For the discrete Schrödinger operator associated with (1.1) and (1.3), in some of the examples in Section 5 we illustrate converting a resonance into a bound state and determine how the compact-support property is impacted.

In our paper we exclusively use the choice in (3.4) in adding a bound state. Hence, as seen from (3.1) and (3.4), the Darboux transformation formula for the Jost function in adding one bound state at $z=z_{N+1}$ with $z_{N+1}\in(-1,0)\cup(0,1)$ yields

$$f_0(z;N+1)=\left(\frac{1-\frac{z}{z_{N+1}}}{1-z_{N+1}\,z}\right)f_0(z;N),\quad |z|\le 1. \tag{3.6}$$

Let $S(z;N)$ and $S(z;N+1)$ denote the scattering matrices for the unperturbed and perturbed problems, respectively. From (2.11) we obtain

$$S(z;N)=\frac{f_0(z^{-1};N)}{f_0(z;N)},\quad S(z;N+1)=\frac{f_0(z^{-1};N+1)}{f_0(z;N+1)},\quad z\in\mathbf{T}. \tag{3.7}$$

Using (3.6) in (3.7), after some simplification, we obtain the Darboux transformation for the scattering matrix as

$$S(z;N+1)=\Big(\frac{1-z_{N+1}\,z}{z-z_{N+1}}\Big)^2 S(z;N),\quad z\in\mathbf{T}. \tag{3.8}$$

One can directly verify that

$$\Big|\frac{1-z_{N+1}\,z}{z-z_{N+1}}\Big|^2=1,\quad z\in\mathbf{T},$$

and hence, with the help of (2.14), we see that the Darboux transformation for the phase shift is given by

$$\phi(z;N+1)=\phi(z;N)-\frac{i}{2}\,\log\Big(\frac{1-z_{N+1}\,z}{z-z_{N+1}}\Big)^2,\quad z\in\mathbf{T}. \tag{3.9}$$

Next, let us determine the Darboux transformation for the spectral density. Let $d\rho(\lambda;N)$ and $d\rho(\lambda;N+1)$ denote the unperturbed and perturbed spectral densities, respectively. From (2.45) we see that

$$d\rho(\lambda;N)=\begin{cases}\Big(\dfrac{1-\sum_{s=1}^N C_s^2}{\prod_{k=1}^N z_k^2}\Big)\dfrac{d\mathring{\rho}}{|f_0(z;N)|^2}, & \lambda\in[0,4],\\ \sum_{s=1}^N C_s^2\,\delta(\lambda-\lambda_s)\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4],\end{cases} \tag{3.10}$$

$$d\rho(\lambda;N+1)=\begin{cases}\Big(\dfrac{1-\sum_{s=1}^{N+1} C_s^2}{\prod_{k=1}^{N+1} z_k^2}\Big)\dfrac{d\mathring{\rho}}{|f_0(z;N+1)|^2}, & \lambda\in[0,4],\\ \sum_{s=1}^{N+1} C_s^2\,\delta(\lambda-\lambda_s)\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4],\end{cases} \tag{3.11}$$

where we recall that $\lambda\in[0,4]$ corresponds to $z\in\overline{\mathbf{T}^+}$. Using (3.5) in (3.11) we see that

$$d\rho(\lambda;N+1)=\begin{cases}\Big(\dfrac{1-\sum_{s=1}^{N+1} C_s^2}{\prod_{k=1}^{N} z_k^2}\Big)\dfrac{d\mathring{\rho}}{|f_0(z;N)|^2}, & \lambda\in[0,4],\\ \sum_{s=1}^{N+1} C_s^2\,\delta(\lambda-\lambda_s)\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4],\end{cases} \tag{3.12}$$

and hence from (3.10) and (3.12) we obtain the Darboux transformation for the spectral density as

$$d\rho(\lambda;N+1)-d\rho(\lambda;N)=\begin{cases}-\Big(\dfrac{C_{N+1}^2}{1-\sum_{s=1}^N C_s^2}\Big)d\rho(\lambda;N), & \lambda\in[0,4],\\ C_{N+1}^2\,\delta(\lambda-\lambda_{N+1})\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4].\end{cases} \tag{3.13}$$

Our next goal is to determine the Darboux transformation formula for the regular solution. In other words, we would like to determine the relationship between $\varphi_n(\lambda;N)$ and $\varphi_n(\lambda;N+1)$, where the former is the regular solution for the unperturbed problem and the latter is the regular solution for the perturbed problem.

Let us now use the Gel'fand-Levitan procedure in the special case with $V_n(N+1)$ denoting $\tilde V_n$ and $V_n(N)$ denoting V_n. In that special case $d\rho$ and $d\tilde\rho$ appearing in

(2.47) correspond to $d\rho(\lambda;N)$ and $d\rho(\lambda;N+1)$, respectively, appearing on the left-hand side of (3.13). The unperturbed and perturbed regular solutions φ_n and $\tilde\varphi_n$ appearing in (2.47) correspond to $\varphi_n(\lambda;N)$ and $\varphi_n(\lambda;N+1)$, respectively. From the second line of (3.10) we obtain

$$\int_{\lambda\in\mathbb{R}\setminus[0,4]} \varphi_n(\lambda;N)\,d\rho(\lambda;N)\,\varphi_m(\lambda;N)=\sum_{s=1}^{N} C_s^2\,\varphi_n(\lambda_s;N)\,\varphi_m(\lambda_s;N). \tag{3.14}$$

With the help of (2.49) and (3.14) we obtain

$$\int_{\lambda\in[0,4]} \varphi_n(\lambda;N)\,d\rho(\lambda;N)\,\varphi_m(\lambda;N)=\delta_{nm}-\sum_{s=1}^{N} C_s^2\,\varphi_n(\lambda_s;N)\,\varphi_m(\lambda_s;N), \tag{3.15}$$

where we recall that δ_{nm} denotes the Kronecker delta. Using (3.13) in (2.48) we obtain

$$\begin{aligned} G_{nm}=&-\Big(\frac{C_{N+1}^2}{1-\sum_{k=1}^N C_k^2}\Big)\int_{\lambda\in[0,4]} \varphi_n(\lambda;N)\,d\rho(\lambda;N)\,\varphi_m(\lambda;N)\\ &+C_{N+1}^2\,\varphi_n(\lambda_{N+1};N)\,\varphi_m(\lambda_{N+1};N). \end{aligned} \tag{3.16}$$

The integral in (3.16) is equal to the right-hand side of (3.15). Thus, from (3.15) and (3.16) we obtain

$$\begin{aligned} &G_{nm}\\ &=-\Big(\frac{C_{N+1}^2}{1-\sum_{k=1}^N C_k^2}\Big)\delta_{nm}+\Big(\frac{C_{N+1}^2}{1-\sum_{k=1}^N C_k^2}\Big)\sum_{s=1}^{N} C_s^2\,\varphi_n(\lambda_s;N)\,\varphi_m(\lambda_s;N)\\ &\quad+C_{N+1}^2\,\varphi_n(\lambda_{N+1};N)\,\varphi_m(\lambda_{N+1};N). \end{aligned} \tag{3.17}$$

Having obtained G_{nm} as in (3.17) in terms of the unperturbed quantities related to $V_n(N)$, one can then use G_{nm} in (2.52), (2.54), and (2.55) and determine the values of A_{nm}. One then uses those values of A_{nm} in (2.47) and in (2.51) in order to determine $\varphi_n(\lambda;N+1)$ and $V_n(N+1)$, respectively.

Alternatively, in order to obtain $\varphi_n(\lambda;N+1)$ and $V_n(N+1)$, we can proceed as follows. Let us write (3.17) in terms of the real-valued $(N+1)\times(N+1)$ diagonal matrix E_N and the real-valued column vector ξ_n with $N+1$ entries as

$$G_{nm}=-\Big(\frac{C_{N+1}^2}{1-\sum_{k=1}^N C_k^2}\Big)\delta_{nm}+\xi_n^\dagger\,E_N\,\xi_m, \tag{3.18}$$

where we have defined

$$E_N:=\operatorname{diag}\Big\{\frac{C_1^2\,C_{N+1}^2}{1-\sum_{k=1}^N C_k^2},\frac{C_2^2\,C_{N+1}^2}{1-\sum_{k=1}^N C_k^2},\dots,\frac{C_N^2\,C_{N+1}^2}{1-\sum_{k=1}^N C_k^2},C_{N+1}^2\Big\}, \tag{3.19}$$

$$\xi_n:=\begin{bmatrix}\varphi_n(\lambda_1;N) & \varphi_n(\lambda_2;N) & \cdots & \varphi_n(\lambda_N;N) & \varphi_n(\lambda_{N+1};N)\end{bmatrix}^\dagger. \tag{3.20}$$

We recall that the dagger in (3.20) can also be replaced by the matrix transpose since the column vector ξ_n is real valued. From (3.18) we see that G_{nm} is separable in n and m. Thus, we can solve the Gel'fand-Levitan system (2.50) explicitly by seeking A_{nm} in the form

$$A_{nm}=\beta_n^\dagger\,\xi_m,\quad 1\le m<n, \tag{3.21}$$

where the column vector β_n has $N+1$ components that are to be determined. Using (3.18) and (3.21) in (2.50) we observe that $\beta_n^\dagger$ satisfies

$$\beta_n^\dagger + \xi_n^\dagger\, E_N + \beta_n^\dagger\Big(-\Big(\frac{C_{N+1}^2}{1-\sum_{k=1}^N C_k^2}\Big) I_{N+1} + \sum_{j=1}^{n-1} \xi_j\, \xi_j^\dagger\, E_N\Big) = 0, \tag{3.22}$$

where we recall that I_{N+1} denotes the $(N+1)\times(N+1)$ identity matrix. From (3.22) we obtain

$$\beta_n^\dagger = -\xi_n^\dagger E_N\Big(I_{N+1} - \Big(\frac{C_{N+1}^2}{1-\sum_{k=1}^N C_k^2}\Big) I_{N+1} + \sum_{j=1}^{n-1} \xi_j\, \xi_j^\dagger\, E_N\Big)^{-1}, \quad n\ge 2, \tag{3.23}$$

which simplifies to

$$\beta_n^\dagger = -\xi_n^\dagger\Big(\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) E_N^{-1} + \sum_{j=1}^{n-1} \xi_j\, \xi_j^\dagger\Big)^{-1}, \quad n\ge 2. \tag{3.24}$$

From (3.21) and (3.24) we see that

$$A_{nm} = -\xi_n^\dagger\Big(\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) E_N^{-1} + \sum_{j=1}^{n-1} \xi_j\, \xi_j^\dagger\Big)^{-1} \xi_m, \quad 1\le m < n. \tag{3.25}$$

Hence, for $n\ge 2$, from (2.51) and (3.25) we obtain the Darboux transformation at the potential level as

$$\begin{aligned} V_n(N+1) - V_n(N) =& \xi_n^\dagger\Big(\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) E_N^{-1} + \sum_{j=1}^{n-1} \xi_j\, \xi_j^\dagger\Big)^{-1} \xi_{n-1} \\ & - \xi_{n+1}^\dagger\Big(\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) E_N^{-1} + \sum_{j=1}^{n} \xi_j\, \xi_j^\dagger\Big)^{-1} \xi_n. \end{aligned} \tag{3.26}$$

Since $A_{10}=0$, for $n=1$, instead of (3.26) we need to use

$$V_1(N+1) - V_1(N) = -\xi_2^\dagger\Big(\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) E_N^{-1} + \xi_1\, \xi_1^\dagger\Big)^{-1} \xi_1, \tag{3.27}$$

which is obtained from (3.26) by replacing the first term on the right-hand side by zero and by using $n=1$ in the second term. Note that $\xi_1\,\xi_1^\dagger$ appearing in (3.27) is the $(N+1)\times(N+1)$ matrix with all entries being equal to one.

Let us remark that (3.25)–(3.27) contain some binomial forms for the inverse of a matrix. Using [6, (15) on p. 12], such binomial forms can be expressed as ratios of two determinants. For example, we can write the right-hand side of (3.25) as

$$A_{nm} = \frac{\text{num}}{\text{den}}, \tag{3.28}$$

where we have defined num as the determinant of the $(N+2)\times(N+2)$ block matrix given by

$$\text{num} := \det\begin{bmatrix} 0 & \xi_n^\dagger \\ \xi_m & \Big(\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) E_N^{-1} + \sum_{j=1}^{n-1} \xi_j\, \xi_j^\dagger\Big) \end{bmatrix}, \tag{3.29}$$

and we have defined den as the determinant of the $(N+1)\times(N+1)$ matrix given by

$$\text{den} := \det\Big[\Big(\frac{1-\sum_{s=1}^{N+1} C_s^2}{1-\sum_{k=1}^{N} C_k^2}\Big)E_N^{-1} + \sum_{j=1}^{n-1} \xi_j\,\xi_j^\dagger\Big]. \tag{3.30}$$

The following theorem shows that the matrix inverses appearing in (3.23)–(3.27) are well defined and hence the Darboux transformation formulas at the potential level given in (3.26) and (3.27) are valid.

Theorem 3.1. *Assume that the potential V_n appearing in* (1.1) *belongs to the Faddeev class and that the discrete Schrödinger operator associated with* (1.1) *and* (1.3) *has N bound states with the corresponding Gel'fand-Levitan norming constants C_s defined in* (2.39) *for $s=1,\dots,N$. Assume that an additional bound state is added at $\lambda=\lambda_{N+1}$ with the Gel'fand-Levitan norming constant C_{N+1}. Furthermore, assume that $\sum_{s=1}^{N+1} C_s^2<1$. Then, the matrix inverse appearing in* (3.25) *exists for any $n\geq 2$.*

Proof. From (3.19) we see that E_N is a diagonal matrix with positive entries, and hence E_N^{-1} is also a diagonal matrix with positive entries. Then, from (3.25) we see that the matrix whose inverse needs to be established is given by the sum of a diagonal matrix with positive entries and the matrix $\sum_{j=1}^{n-1}\xi\xi^\dagger$. Let us now consider the hermitian form for that sum with any nonzero vector $v\in\mathbb{C}^{N+1}$. Because the first matrix in the summation is diagonal with positive entries, the corresponding hermitian form is strictly positive. The following argument shows that the hermitian form for the second matrix in the summation is nonnegative. This is established by using

$$v^\dagger \sum_{j=1}^{n-1} \xi_j\,\xi_j^\dagger\, v = \sum_{j=1}^{n-1} \big(\xi_j^\dagger\, v\big)^\dagger \big(\xi_j^\dagger\, v\big) = \sum_{j=1}^{n-1} \big|\xi_j^\dagger v\big|^2, \tag{3.31}$$

which shows that the right-hand side must be nonnegative. Thus, the hermitian form with any nonzero vector $v\in\mathbb{C}^{N+1}$ associated with the matrix whose inverse is used in (3.25) is positive, which proves that the matrix itself is positive and hence is invertible. Thus, the right-hand side in (3.25) is well defined when we have $\sum_{s=1}^{N+1} C_s^2<1$. □

Let us remark that the case $\sum_{s=1}^{N+1} C_s^2=1$ cannot happen, and hence it is not considered in Theorem 3.1. This can be seen as follows. If we had $\sum_{s=1}^{N+1} C_s^2=1$, then (3.12) would imply that $d\rho(\lambda;N+1)=0$ for $\lambda\in[0,4]$ and hence the corresponding discrete Schrödinger operator, which is a selfadjoint operator, would only have the discrete spectrum consisting of a finite number of eigenvalues and no continuous spectrum. The absence of generalized eigenfunctions as a result of the absence of the continuous spectrum and the presence of only a finite number of eigenfunctions related to the discrete spectrum would be incompatible for the selfadjoint discrete Schrödinger operator. From the spectral theory we know that the eigenfunctions and the generalized eigenfunctions must form a complete set acting as an orthogonal basis for the infinite-dimensional space of square-summable functions on the half-line lattice, and this cannot be done by using only a finite number of eigenfunctions.

Let us now evaluate the Darboux transformation for the regular solution. Using (3.21) in (2.47) we obtain

$$\varphi_n(\lambda; N+1) = \begin{cases} \varphi_n(\lambda; N), & n = 1, \\ \varphi_n(\lambda; N) + \beta_n^\dagger \sum_{m=1}^{n-1} \xi_m \, \varphi_m(\lambda; N), & n \geq 2. \end{cases} \tag{3.32}$$

As the next proposition shows, the summation term in (3.32) can be written as a linear combination of $\varphi_{n-1}(\lambda; N)$ and $\varphi_n(\lambda; N)$. Let us define the real-valued column vector $\alpha_n(\lambda)$ for $n \geq 1$ with $N+1$ components as

$$\alpha_n(\lambda) := \begin{bmatrix} \frac{\varphi_n(\lambda_1; N)}{\lambda - \lambda_1} & \frac{\varphi_n(\lambda_2; N)}{\lambda - \lambda_2} & \dots & \frac{\varphi_n(\lambda_N; N)}{\lambda - \lambda_N} & \frac{\varphi_n(\lambda_{N+1}; N)}{\lambda - \lambda_{N+1}} \end{bmatrix}^\dagger. \tag{3.33}$$

Proposition 3.2. *Assume that the potential V_n, also denoted by $V_n(N)$, appearing in* (1.1) *belongs to the Faddeev class and the discrete Schrödinger operator corresponding to* (1.1) *and* (1.3) *has N bound states at $\lambda = \lambda_s$ with $s = 1, \dots, N$. Let φ_n, also denoted by $\varphi_n(\lambda; N)$, be the corresponding regular solution appearing in* (2.4)*. Let ξ_n be the real-valued column vector in* (3.20) *with $N+1$ components. We then have the following:*

(a) *The summation term in* (3.32) *can be simplified and we have*

$$\sum_{m=1}^{n-1} \xi_m \, \varphi_m(\lambda; N) = \alpha_n(\lambda) \, \varphi_{n-1}(\lambda; N) - \alpha_{n-1}(\lambda) \, \varphi_n(\lambda; N), \quad n \geq 2, \tag{3.34}$$

where $\alpha_n(\lambda)$ is the real-valued column vector defined in (3.33) *with $N+1$ components.*

(b) *The $(N+1) \times (N+1)$ matrix consisting of the summation term in* (3.24) *can be simplified and its (k, l)-entry for $n \geq 2$ is given by*

$$\begin{aligned} &\Big(\sum_{j=1}^{n-1} \xi_j \, \xi_j^\dagger \Big)_{kl} \\ &= \begin{cases} \dfrac{\varphi_{n-1}(\lambda_k; N) \, \varphi_n(\lambda_l; N) - \varphi_n(\lambda_k; N) \, \varphi_{n-1}(\lambda_l; N)}{\lambda_k - \lambda_l}, & k \neq l, \\ \varphi_n(\lambda_k; N) \, \dot{\varphi}_{n-1}(\lambda_k; N) - \varphi_{n-1}(\lambda_k; N) \, \dot{\varphi}_n(\lambda_k; N), & k = l, \end{cases} \end{aligned} \tag{3.35}$$

where the dot over a quantity denotes the λ-derivative of that quantity.

Proof. Since $\varphi_n(\lambda; N)$ satisfies (1.1) we have

$$\varphi_{m+1}(\lambda; N) + \varphi_{m-1}(\lambda; N) = (2 + V_m - \lambda) \, \varphi_m(\lambda; N), \quad m \geq 1, \tag{3.36}$$

$$\varphi_{m+1}(\lambda_s; N) + \varphi_{m-1}(\lambda_s; N) = (2 + V_m - \lambda_s) \, \varphi_m(\lambda_s; N), \quad m \geq 1. \tag{3.37}$$

Let us multiply (3.36) by $-\varphi_m(\lambda_s; N)$ and multiply (3.37) by $\varphi_m(\lambda; N)$ and add the resulting equations and then apply the summation over m from $m = 1$ to $m = n-1$. After some simplifications and using the first equality in (2.4), we obtain

$$\begin{aligned} &\varphi_n(\lambda_s; N) \, \varphi_{n-1}(\lambda; N) - \varphi_{n-1}(\lambda_s; N) \, \varphi_n(\lambda; N) \\ &= (\lambda - \lambda_s) \sum_{m=1}^{n-1} \varphi_m(\lambda_s; N) \, \varphi_m(\lambda; N), \end{aligned}$$

or equivalently

$$\begin{aligned}&\sum_{m=1}^{n-1}\varphi_m(\lambda_s;N)\,\varphi_m(\lambda;N)\\&=\frac{\varphi_n(\lambda_s;N)}{\lambda-\lambda_s}\,\varphi_{n-1}(\lambda;N)-\frac{\varphi_{n-1}(\lambda_s;N)}{\lambda-\lambda_s}\,\varphi_n(\lambda;N).\end{aligned}\tag{3.38}$$

Note that (3.38) corresponds to the sth component of the vector relation given in (3.34). Thus, the proof of (a) is complete. Let us now turn the proof of (b). From (3.20) and the fact that ξ_j is real, we see that the (k,l)-entry of the matrix $\xi_j\xi_j^\dagger$ is given by

$$\big(\xi_j\xi_j^\dagger\big)_{kl}=\varphi_j(\lambda_k;N)\,\varphi_j(\lambda_l;N).\tag{3.39}$$

From (3.38) and (3.39) we see that, when $k\neq l$, we have

$$\Big(\sum_{m=1}^{n-1}\xi_m\,\xi_m^\dagger\Big)_{kl}=\frac{\varphi_n(\lambda_k;N)}{\lambda_l-\lambda_k}\,\varphi_{n-1}(\lambda_l;N)-\frac{\varphi_{n-1}(\lambda_k;N)}{\lambda_l-\lambda_k}\,\varphi_n(\lambda_l;N),\quad k\neq l,$$

yielding the first line of (3.35). When $k=l$, we can use the limit $\lambda\to\lambda_s$ in (3.38), which gives us

$$\sum_{m=1}^{n-1}\varphi_m(\lambda_s;N)\,\varphi_m(\lambda_s;N)=\varphi_n(\lambda_s;N)\,\dot\varphi_{n-1}(\lambda_s;N)-\varphi_{n-1}(\lambda_s;N)\,\dot\varphi_n(\lambda_s;N),$$

yielding the second line of (3.35). $\square$

Using (3.34) in (3.32) we obtain the Darboux transformation for the regular solution as

$$\begin{aligned}&\varphi_n(\lambda;N+1)\\&=\begin{cases}\varphi_n(\lambda;N), & n=1,\\ \big[1-\beta_n^\dagger\,\alpha_{n-1}(\lambda)\big]\,\varphi_n(\lambda;N)+\beta_n^\dagger\,\alpha_n(\lambda)\,\varphi_{n-1}(\lambda;N), & n\geq 2,\end{cases}\end{aligned}\tag{3.40}$$

where we recall that $\beta_n^\dagger$ is the real-valued row vector in (3.24), $\alpha_n(\lambda)$ is the real-valued column vector given in (3.33), and ξ_n is the real-valued column vector given in (3.20).

Note that the results presented in this section remain valid when $N=0$. In that case we interpret the summation $\sum_{k=1}^N C_k^2$ as zero in all the relevant formulas in this section.

4. Darboux transformation in removing a bound state

In this section we determine the effect of removing a bound state from the discrete spectrum of the Schrödinger operator corresponding to (1.1) and (1.3). For clarity, we use the notation introduced in Section 3. We have the unperturbed potential $V_n(N)$ containing N bound states at $\lambda=\lambda_s$ for $s=1,\dots,N$. We then remove the bound state at $\lambda=\lambda_N$ with the Gel'fand-Levitan norming constant C_N in order to obtain the perturbed potential $V_n(N-1)$ containing $N-1$ bound states. As in Section 3, we know from (2.38) that there is a one-to-one correspondence between λ_s and z_s, and hence we can equivalently say that the bound states of the potential $V_n(N)$ occur at $z=z_s$ for $s=1,\dots,N$, and we remove the bound state at $z=z_N$.

The Darboux transformation for the Jost function in going from $f_0(z;N)$ to $f_0(z;N-1)$ can be obtained via (3.6) as

$$f_0(z;N-1)=\Big(\frac{1-z_N z}{1-\frac{z}{z_N}}\Big)f_0(z;N),\quad |z|\le 1. \tag{4.1}$$

Similarly, the Darboux transformation for the scattering matrix in going from $S(z;N)$ to $S(z;N-1)$ can be obtained via (3.8) as

$$S(z;N-1)=\Big(\frac{z-z_N}{1-z_N z}\Big)^2 S(z;N),\quad z\in\mathbf{T}.$$

With the help of (3.9) we see that the Darboux transformation for the phase shift in going from $\phi(z;N)$ to $\phi(z;N-1)$ can be obtained via (3.9) as

$$\phi(z;N-1)=\phi(z;N)+\frac{i}{2}\,\log\Big(\frac{1-z_N\,z}{z-z_N}\Big)^2,\quad z\in\mathbf{T}.$$

Let us now determine the Darboux transformation for the spectral density in going from $d\rho(\lambda;N)$ to $d\rho(\lambda;N-1)$. From (3.10) we see that

$$d\rho(\lambda;N-1)=\begin{cases}\Big(\frac{1-\sum_{s=1}^{N-1}C_s^2}{\prod_{k=1}^{N-1}z_k^2}\Big)\frac{d\mathring{\rho}}{|f_0(z;N-1)|^2}, & \lambda\in[0,4],\\ \sum_{s=1}^{N-1}C_s^2\,\delta(\lambda-\lambda_s)\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4].\end{cases} \tag{4.2}$$

On the other hand, from (3.5) we have

$$|f_0(z;N-1)|^2=z_N^2\,|f_0(z;N)|^2,\quad z\in\mathbf{T}. \tag{4.3}$$

Using (4.3) in (4.2) we obtain

$$d\rho(\lambda;N-1)=\begin{cases}\Big(\frac{1-\sum_{s=1}^{N-1}C_s^2}{\prod_{k=1}^{N}z_k^2}\Big)\frac{d\mathring{\rho}}{|f_0(z;N)|^2}, & \lambda\in[0,4],\\ \sum_{s=1}^{N-1}C_s^2\,\delta(\lambda-\lambda_s)\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4].\end{cases} \tag{4.4}$$

We recall that $\lambda\in[0,4]$ in (4.2) and (4.4) corresponds to $z\in\overline{\mathbf{T}^+}$. Thus, from (3.10) and (4.4) we obtain

$$d\rho(\lambda;N-1)-d\rho(\lambda;N)=\begin{cases}\Big(\frac{C_N^2}{1-\sum_{s=1}^{N}C_s^2}\Big)d\rho(\lambda;N), & \lambda\in[0,4],\\ -C_N^2\,\delta(\lambda-\lambda_N)\,d\lambda, & \lambda\in\mathbb{R}\setminus[0,4].\end{cases} \tag{4.5}$$

Next, we determine the Darboux transformation for the regular solution in going from $\varphi_n(\lambda;N)$ to $\varphi_n(\lambda;N-1)$. In the Gel'fand-Levitan formalism outlined in (2.47)–(2.51), we have

$$\varphi_n(\lambda;N-1)=\begin{cases}\varphi_n(\lambda;N), & n=1,\\ \varphi_n(\lambda;N)+\sum_{m=1}^{n-1}A_{nm}\,\varphi_m(\lambda;N), & n\ge 2,\end{cases}$$

$$G_{nm}:=\int_{\lambda\in\mathbb{R}}\varphi_n(\lambda;N)\,[d\rho(\lambda;N-1)-d\rho(\lambda;N)]\,\varphi_m(\lambda;N), \tag{4.6}$$

where the constants A_{nm} are to be determined from (2.50) by using (4.6) as input. In this case, from (2.51) we obtain

$$V_n(N-1)-V_n(N)=A_{(n+1)n}-A_{n(n-1)},\quad n\ge 1,$$

again with the understanding that $A_{10} = 0$. Using (4.5) in (4.6) we obtain

$$\begin{aligned} G_{nm} =& \Big(\frac{C_N^2}{1-\sum_{k=1}^N C_k^2}\Big) \int_{\lambda\in[0,4]} \varphi_n(\lambda;N)\, d\rho(\lambda;N)\, \varphi_m(\lambda;N) \\ &- C_N^2\, \varphi_n(\lambda_N;N)\, \varphi_m(\lambda_N;N). \end{aligned} \tag{4.7}$$

Using (3.15) in (4.7), after some simplification we obtain

$$\begin{aligned} G_{nm} =& \Big(\frac{C_N^2}{1-\sum_{k=1}^N C_k^2}\Big)\delta_{nm} - \Big(\frac{C_N^2}{1-\sum_{k=1}^N C_k^2}\Big) \sum_{s=1}^{N-1} C_s^2\, \varphi_n(\lambda_s;N)\, \varphi_m(\lambda_s;N) \\ &- C_N^2 \Big(\frac{1-\sum_{s=1}^{N-1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) \varphi_n(\lambda_N;N)\, \varphi_m(\lambda_N;N). \end{aligned} \tag{4.8}$$

Proceeding as in (3.18)–(3.20) we can write G_{nm} given in (4.8) as

$$G_{nm} = \Big(\frac{C_N^2}{1-\sum_{k=1}^N C_k^2}\Big)\delta_{nm} + \theta_n^\dagger\, F_N\, \theta_m, \tag{4.9}$$

where F_N is the $N \times N$ diagonal matrix with real entries given by

$$\begin{aligned} F_N := \operatorname{diag}\Big\{ &\frac{-C_1^2\, C_N^2}{1-\sum_{k=1}^N C_k^2}, \frac{-C_2^2\, C_N^2}{1-\sum_{k=1}^N C_k^2}, \cdots, \\ &\frac{-C_{N-1}^2\, C_N^2}{1-\sum_{k=1}^N C_k^2}, \frac{-C_N^2\Big(1-\sum_{s=1}^{N-1} C_s^2\Big)}{1-\sum_{k=1}^N C_k^2} \Big\}, \end{aligned} \tag{4.10}$$

and θ_n is the column vector with N entries given by

$$\theta_n := \begin{bmatrix} \varphi_n(\lambda_1;N) & \varphi_n(\lambda_2;N) & \cdots & \varphi_n(\lambda_{N-1};N) & \varphi_n(\lambda_N;N) \end{bmatrix}^\dagger. \tag{4.11}$$

Comparing (3.20) and (4.11) we observe that the first N entries of the column vectors θ_n and ξ_n are identical and that ξ_n has an additional $(N+1)$st entry. As in Section 3, the quantity G_{nm} given in (4.9) is separable in n and m, and hence the Gel'fand-Levitan system (2.50) is explicitly solvable by using the analog of (3.21), i.e. by letting

$$A_{nm} = \gamma_n^\dagger\, \theta_m, \quad 1 \le m < n, \tag{4.12}$$

where the column vector γ_n has N components to be determined. Proceeding as in (3.22)–(3.25) we determine $\gamma_n^\dagger$ as

$$\gamma_n^\dagger = -\theta_n^\dagger \Big(\Big(\frac{1-\sum_{s=1}^{N-1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) F_N^{-1} + \sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger \Big)^{-1}. \tag{4.13}$$

From (4.12) and (4.13) we see that

$$A_{nm} = -\theta_n^\dagger \Big(\Big(\frac{1-\sum_{s=1}^{N-1} C_s^2}{1-\sum_{k=1}^N C_k^2}\Big) F_N^{-1} + \sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger \Big)^{-1} \theta_m, \quad 1 \le m < n. \tag{4.14}$$

The analogs of (3.28)–(3.30) also apply in this case. Since the right-hand side of (4.12) is a binomial for a matrix inverse, we can write A_{nm} given in (4.12) as the ratio of two determinants as

$$A_{nm}=\frac{\det\begin{bmatrix}0 & \theta_n^\dagger\\ \theta_m & \left(\left(\dfrac{1-\sum_{s=1}^{N-1}C_s^2}{1-\sum_{k=1}^{N}C_k^2}\right)F_N^{-1}+\sum_{j=1}^{n-1}\theta_j\,\theta_j^\dagger\right)\end{bmatrix}}{\det\left[\left(\dfrac{1-\sum_{s=1}^{N-1}C_s^2}{1-\sum_{k=1}^{N}C_k^2}\right)F_N^{-1}+\sum_{j=1}^{n-1}\theta_j\,\theta_j^\dagger\right]},\quad 1\le m<n. \tag{4.15}$$

As in Proposition 3.2(b), for $n\ge 2$ we can simplify the $N\times N$ matrix-valued quantity $\sum_{j=1}^{n-1}\theta_j\theta_j^\dagger$ appearing in (4.13)–(4.15) and find that its (k,l)-entry is given by

$$\begin{aligned}&\Big(\sum_{j=1}^{n-1}\theta_j\,\theta_j^\dagger\Big)_{kl}\\ &=\begin{cases}\dfrac{\varphi_{n-1}(\lambda_k;N)\,\varphi_n(\lambda_l;N)-\varphi_n(\lambda_k;N)\,\varphi_{n-1}(\lambda_l;N)}{\lambda_k-\lambda_l}, & k\ne l,\\ \varphi_n(\lambda_k;N)\,\dot\varphi_{n-1}(\lambda_k;N)-\varphi_{n-1}(\lambda_k;N)\,\dot\varphi_n(\lambda_k;N), & k=l.\end{cases}\end{aligned} \tag{4.16}$$

Let us remark that the matrix in (3.35) has $N+1$ rows and $N+1$ columns, and the matrix in (4.16) has N rows and N columns. If we delete the $(N+1)$st row and $(N+1)$st column from the matrix in (3.35) we obtain the matrix in (4.16).

The analog of (3.26) in this case is obtained by using (4.14) in (2.51), and for $n\ge 2$ we obtain the Darboux transformation in going from $V_n(N)$ to $V_n(N-1)$ given by

$$\begin{aligned}V_n(N-1)-V_n(N)=&\theta_n^\dagger\left(\left(\frac{1-\sum_{s=1}^{N-1}C_s^2}{1-\sum_{k=1}^{N}C_k^2}\right)F_N^{-1}+\sum_{j=1}^{n-1}\theta_j\,\theta_j^\dagger\right)^{-1}\theta_{n-1}\\ &-\theta_{n+1}^\dagger\left(\left(\frac{1-\sum_{s=1}^{N-1}C_s^2}{1-\sum_{k=1}^{N}C_k^2}\right)F_N^{-1}+\sum_{j=1}^{n}\theta_j\,\theta_j^\dagger\right)^{-1}\theta_n.\end{aligned} \tag{4.17}$$

For $n=1$, instead of (4.17) we use the analog of (3.27) and get

$$V_1(N-1)-V_1(N)=-\theta_2^\dagger\left(\left(\frac{1-\sum_{s=1}^{N-1}C_s^2}{1-\sum_{k=1}^{N}C_k^2}\right)F_N^{-1}+\theta_1\,\theta_1^\dagger\right)^{-1}\theta_1. \tag{4.18}$$

The analog of (3.32) in this case is

$$\varphi_n(\lambda;N-1)=\begin{cases}\varphi_n(\lambda;N), & n=1,\\ \varphi_n(\lambda;N)+\gamma_n^\dagger\sum_{m=1}^{n-1}\theta_m\,\varphi_m(\lambda;N), & n\ge 2,\end{cases}$$

and the analog of (3.40) in this case is

$$\varphi_n(\lambda;N-1)=\begin{cases}\varphi_n(\lambda;N), & n=1,\\ [1-\gamma_n^\dagger\,\epsilon_{n-1}(\lambda)]\varphi_n(\lambda;N)+\gamma_n^\dagger\,\epsilon_n(\lambda)\,\varphi_{n-1}(\lambda;N), & n\ge 2,\end{cases}$$

where $\epsilon_n(\lambda)$ for $n\ge 1$ is the column vector with N components and it is defined as

$$\epsilon_n(\lambda):=\begin{bmatrix}\dfrac{\varphi_n(\lambda_1;N)}{\lambda-\lambda_1} & \dfrac{\varphi_n(\lambda_2;N)}{\lambda-\lambda_2} & \dots & \dfrac{\varphi_n(\lambda_{N-1};N)}{\lambda-\lambda_{N-1}} & \dfrac{\varphi_n(\lambda_N;N)}{\lambda-\lambda_N}\end{bmatrix}^\dagger. \tag{4.19}$$

We remark that the column vector $\epsilon_n(\lambda)$ given in (4.19) has N components, and the column vector $\alpha_n(\lambda)$ given in (3.33) has $N+1$ components. In fact, $\epsilon_n(\lambda)$ is obtained from $\alpha_n(\lambda)$ by omitting the last entry.

In the following theorem we present the analog of the result presented in Theorem 3.1, i.e. we prove that the matrix inverse appearing in (4.14) is well defined and hence the Darboux transformation formulas at the potential level given in (4.17) and (4.18) are valid. Let us remark that the matrix in (3.25) whose inverse is established in Theorem 3.1 consists of the sum of a diagonal matrix with positive entries and a nonnegative hermitian matrix. In contrast, the matrix in (4.14) whose inverse is established in the next theorem consists of the sum of a diagonal matrix with negative entries and a nonnegative hermitian matrix.

Theorem 4.1. *Assume that the potential V_n appearing in* (1.1) *belongs to the Faddeev class and that the discrete Schrödinger operator associated with* (1.1) *and* (1.3) *has N bound states with the corresponding Gel'fand-Levitan norming constants C_s defined in* (2.39) *for $s=1,\dots,N$. Assume that the bound state at $\lambda=\lambda_N$ with the Gel'fand-Levitan norming constant C_N is removed from the discrete spectrum. Furthermore, assume that $\sum_{s=1}^N C_s^2<1$. Then, the matrix inverse appearing in* (4.14) *exists for any $n\geq 2$.*

Proof. As a result of the assumption $\sum_{s=1}^N C_s^2<1$, from (4.10) we observe that each entry of the diagonal matrix F_N given in (4.10) is negative and hence F_N^{-1} is also a diagonal matrix with negative entries. We can write the matrix in (4.14) whose inverse is to be established as $-H_N+\sum_{j=1}^{n-1}\theta_j\,\theta_j^\dagger$, where we have defined

$$H_N:=-\Big(\frac{1-\sum_{s=1}^{N-1}C_s^2}{1-\sum_{k=1}^N C_k^2}\Big)F_N^{-1}. \tag{4.20}$$

Using (4.10) in (4.20) we obtain

$$H_N=\Big(\frac{1-\sum_{s=1}^{N-1}C_s^2}{C_N^2}\Big)\operatorname{diag}\Big\{\frac{1}{C_1^2},\frac{1}{C_2^2},\cdots,\frac{1}{C_{N-1}^2},\frac{1}{1-\sum_{k=1}^{N-1}C_k^2}\Big\}. \tag{4.21}$$

We let

$$\varepsilon_N:=\frac{1-\sum_{s=1}^N C_s^2}{C_N^2}. \tag{4.22}$$

and observe that ε_N is a positive number as a result of $\sum_{s=1}^N C_s^2<1$. Note that

$$\frac{1-\sum_{k=1}^{N-1}C_k^2}{C_N^2}=\frac{C_N^2+1-\sum_{k=1}^N C_k^2}{C_N^2}=1+\frac{1-\sum_{s=1}^N C_s^2}{C_N^2}. \tag{4.23}$$

With the help of (4.22) and (4.23) we write (4.21) as

$$H_N=\operatorname{diag}\Big\{\frac{1+\varepsilon_N}{C_1^2},\frac{1+\varepsilon_N}{C_2^2},\cdots,\frac{1+\varepsilon_N}{C_{N-1}^2},\frac{1}{C_N^2}\Big\}. \tag{4.24}$$

Let v be a nonzero vector in $\mathbb{C}^N$ given by

$$v=\begin{bmatrix}v_1\\ \vdots\\ v_{N+1}\end{bmatrix}. \tag{4.25}$$

The hermitian form of H_N with the vector v given in (4.25) is obtained from (4.23) as

$$v^\dagger H_N\, v = \frac{(1+\varepsilon_N)\,|v_1|^2}{C_1^2} + \frac{(1+\varepsilon_N)\,|v_2|^2}{C_2^2} + \dots + \frac{(1+\varepsilon_N)\,|v_{N-1}|^2}{C_{N-1}^2} + \frac{|v_N|^2}{C_N^2}. \quad (4.26)$$

Since $\varepsilon_N > 0$, from (4.26) we obtain

$$v^\dagger H_N\, v \geq \frac{|v_1|^2}{C_1^2} + \frac{|v_2|^2}{C_2^2} + \dots + \frac{|v_{N-1}|^2}{C_{N-1}^2} + \frac{|v_N|^2}{C_N^2}. \quad (4.27)$$

We evaluate the hermitian form of $\sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger$ with the vector v given in (4.25) as in (3.31) and obtain

$$v^\dagger \sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger v = \sum_{j=1}^{n-1} |\theta_j^\dagger v|^2. \quad (4.28)$$

From (4.28) we conclude that

$$v^\dagger \sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger v < \sum_{j=1}^{\infty} |\theta_j^\dagger v|^2, \quad (4.29)$$

where we have used the fact that we cannot have $\theta_j^\dagger v = 0$ for all $j \geq n$. Using (4.11) and (4.25) we obtain

$$\theta_j^\dagger\, v = \varphi_j(\lambda_1; N)\, v_1 + \varphi_j(\lambda_2; N)\, v_2 + \dots + \varphi_j(\lambda_N; N)\, v_N, \quad (4.30)$$

where we recall that each entry in (4.11) is real. From (4.30) we obtain

$$|\theta_j^\dagger v|^2 = \sum_{k=1}^{N} \varphi_j(\lambda_k; N)^2\, |v_k|^2 + 2 \sum_{1=k<l\leq N} \varphi_j(\lambda_k; N)\, \varphi_j(\lambda_l; N)\, v_k^*\, v_l. \quad (4.31)$$

Since the discrete Schrödinger operator associated with (1.1) and (1.3) is selfadjoint, its eigenvectors corresponding to distinct eigenvalues are orthogonal and we have

$$\sum_{j=1}^{\infty} \varphi_j(\lambda_k; N)\, \varphi_j(\lambda_l; N) = 0, \quad k \neq l. \quad (4.32)$$

Thus, with the help of (4.32), from (4.31) we obtain

$$\sum_{j=1}^{\infty} |\theta_j^\dagger\, v|^2 = \sum_{k=1}^{N} \Big(\sum_{j=1}^{\infty} \varphi_j(\lambda_k; N)^2 \Big)\, |v_k|^2. \quad (4.33)$$

Using (2.39) in (4.33) we obtain

$$\sum_{j=1}^{\infty} |\theta_j^\dagger v|^2 = \sum_{k=1}^{N} \frac{|v_k|^2}{C_k^2}. \quad (4.34)$$

Thus, from (4.29) and (4.34) we obtain

$$v^\dagger \sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger v < \frac{|v_1|^2}{C_1^2} + \frac{|v_2|^2}{C_2^2} + \dots + \frac{|v_N|^2}{C_N^2}. \quad (4.35)$$

Combining (4.27) and (4.35) we obtain

$$v^\dagger \Big(-H_N + \sum_{j=1}^{n-1} \theta_j\, \theta_j^\dagger \Big) v < 0. \quad (4.36)$$

From (4.36) we conclude that the matrix whose inverse appears in (4.14) is negative and hence that matrix must be invertible. □

5. Some explicit examples

In this section we illustrate the results of the previous sections with some explicit examples. We also make some contrasts between the Darboux transformation for (1.1) and the Darboux transformation for (1.2) when the potentials are compactly supported.

Let us consider the case where the potential V_n in (1.1) is nontrivial and compactly supported, i.e. assume that $V_n = 0$ for $n > b$ and $V_b \neq 0$ for some positive integer b. The corresponding Jost function f_0 appearing in (2.10) is then a polynomial in z of degree $2b-1$ and, as [3, (2.50)] indicates, is given by

$$f_0 = 1 + z\sum_{j=1}^{b} V_j + \cdots + z^{2b-2}\sum_{j=1}^{b-1} V_b\, V_j + z^{2b-1} V_b. \tag{5.1}$$

For a compactly-supported potential, the Marchenko norming constant c_s defined in (2.40) is obtained [3] from the residue of S/z at the bound-state value z_s as

$$c_s^2 = \operatorname{Res}\Big[\frac{S}{z}, z_s\Big], \quad s = 1, \dots, N, \tag{5.2}$$

where S is the scattering matrix defined in (2.10). Consequently, the corresponding Gel'fand-Levitan norming constant C_s can be obtained by using (2.42).

In some of the examples in this section, we illustrate that not every polynomial in z of degree $2b-1$ necessarily corresponds to the Jost function f_0 of a compactly-supported potential vanishing for $n > b$. This is not surprising because the coefficients in such a polynomial must agree with the coefficients given in (5.1). There are b potential values that need to correspond to the $(2b-1)$ coefficients on the right-hand side of (5.1). For example, when $b = 2$ from (5.1) we obtain

$$f_0 = 1 + (V_1 + V_2)z + V_1 V_2 z^2 + V_2 z^3, \tag{5.3}$$

and the same quantity must also have the form

$$f_0 = \Big(1 - \frac{z}{\alpha_1}\Big)\Big(1 - \frac{z}{\alpha_2}\Big)\Big(1 - \frac{z}{\alpha_3}\Big), \tag{5.4}$$

for some nonzero constants α_1, α_2, α_3 satisfying

$$\begin{cases} V_1 + V_2 = -\Big(\dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} + \dfrac{1}{\alpha_3}\Big), \\ V_1 V_2 = \dfrac{1}{\alpha_1\,\alpha_2} + \dfrac{1}{\alpha_1\,\alpha_3} + \dfrac{1}{\alpha_2\,\alpha_3}, \\ V_2 = -\dfrac{1}{\alpha_1\,\alpha_2\,\alpha_3}. \end{cases} \tag{5.5}$$

In case the system (5.5) is inconsistent, the quantity given on the right-hand side of (5.4) cannot be the Jost function of a compactly-supported potential.

For the half-line Schrödinger equation (1.2) with a compactly-supported potential $V(x)$, the following property is known [2]. If we remove a bound state from such a potential, then the transformed potential is also compactly supported and the transformed potential is guaranteed to vanish outside the support of the original potential. In some of the examples in this section, we illustrate that the aforementioned support property does not necessarily hold for the discrete Schrödinger equation (1.1). We show that the property holds in one example but does not hold in another example.

For the half-line Schrödinger equation (1.2) with a compactly-supported potential $V(x)$, also the following second property holds [2]. If we add a bound state to a compactly-supported potential, then the transformed potential is also compactly supported (and the transformed potential is guaranteed to vanish outside the support of the original potential) if and only if the two conditions specified in [2, Theorem 3.5] are satisfied. The first condition is that the added bound-state λ_s-value must come from an "eligible" resonance [2] and the second condition is that the corresponding Gel'fand-Levitan norming constant C_s must have a specific positive value. In some of the examples in this section, we illustrate that the aforementioned support property does not necessarily hold for the discrete Schrödinger equation (1.1). We show that the property holds in one example but does not hold in another example.

In the next example, we add a bound state at $z = z_1$ with the Gel'fand-Levitan norming constant C_1 to a compactly-supported potential with $b = 1$. The example shows that the Darboux transformation on the compactly-supported potential results in a compactly-supported potential if the values for z_1 and C_1 are chosen appropriately.

Example 5.1. Consider the compactly-supported potential V_n with $b = 1$ and hence $V_n = 0$ for $n \geq 2$. Let us assume that $0 < |V_1| \leq 1$. From (5.1) we see that the Jost function is given by

$$f_0 = 1 + V_1 z. \tag{5.6}$$

Using (2.4) in (2.3), we obtain the corresponding regular solution φ_n as a function of z as

$$\varphi_n = \frac{z^n - z^{-n}}{z - z^{-1}} + V_1 \frac{z^{n-1} - z^{1-n}}{z - z^{-1}}, \quad n \geq 1. \tag{5.7}$$

Since the bound states correspond to the zeros of f_0 when $z \in (-1, 0) \cup (0, 1)$, from (5.6) we see that there are no bound states and hence we have $N = 0$. Let us now add one bound state at $z = z_1$ with the Gel'fand-Levitan norming constant C_1. Let us choose $z_1 = -V_1$, and hence impose the further restriction $0 < |V_1| < 1$. Let us use $\tilde{f}_0$ and $\tilde{V}_n$ to denote the corresponding Jost function and potential, respectively, when the bound state is added. From (3.6) and (5.6) we see that

$$\tilde{f}_0 = 1 + \frac{z}{V_1}. \tag{5.8}$$

Using (5.7) and $z_1 = -V_1$ in (3.20), we obtain

$$\xi_n = (-V_1)^{1-n}, \quad n \geq 1.$$

The quantity E_N defined in (3.19) with $N = 0$ is given by $E_0 = C_1^2$. Then, (3.27) and (3.26) respectively yield

$$\tilde{V}_1 = V_1 + \frac{C_1^2}{V_1}, \tag{5.9}$$

$$\tilde{V}_n = \frac{-C_1^2 V_1^{2n+1}(1 - V_1^2)^2 (C_1^2 - 1 + V_1^2)}{C_1^2 V_1^6 - C_1^2 V_1^{2n+2}(1 + V_1^2)(C_1^2 - 1 + V_1^2) + V_1^{4n}(C_1^2 - 1 + V_1^2)^2}, \tag{5.10}$$

for $n \geq 2$. From (5.10) we see that $\tilde{V}_n$ is compactly supported if and only if we have

$$C_1^2 = 1 - V_1^2. \tag{5.11}$$

In fact, with the special choice of the Gel'fand-Levitan norming constant in (5.11), from (5.9) we obtain $\tilde{V}_1 = 1/V_1$. In the presence of one bound state for the compactly-supported potential $\tilde{V}_n$, the corresponding Gel'fand-Levitan norming constant C_1 can be evaluated with the help of (2.41), (5.2), (5.8), and the fact that $\tilde{f}_1 = z$, yielding the value of C_1^2 given in (5.11).

In the following example, we illustrate that a polynomial in z of degree $2b-1$ may or may not correspond to the Jost function of a compactly-supported potential.

Example 5.2. Consider the Jost function

$$f_0 = (1+2z)(1-2z)\Big(1-\frac{z}{\sqrt{5}}\Big). \tag{5.12}$$

Comparing (5.12) with (5.3)–(5.5), we see that one solution to the corresponding system (5.5) results in

$$b = 2, \quad V_1 = -\sqrt{5}, \quad V_2 = \frac{4}{\sqrt{5}}. \tag{5.13}$$

From (5.12) we see that f_0 has two zeros when $z \in (-1,0) \cup (0,1)$, and hence it has two bound-state zeros given by $z_1 = -1/2$ and $z_2 = 1/2$. From (2.46) we see that the corresponding Gel'fand-Levitan norming constants C_1 and C_2 must satisfy $0 < C_1^2 + C_2^2 \le 1$. Corresponding to a compactly-supported potential we must [3] have $f_n = z^n$ for $n \ge b$. Hence, in our example, corresponding to (5.12) we have $f_2 = z^2$ and $f_3 = z^3$. Then, from (2.3) with $n = 2$ we obtain $f_1(z) = z + V_2 z^2$. With the help of (2.41), (2.42), and (5.2), we obtain

$$C_1^2 = \frac{3(12-5\sqrt{5})}{76} = 0.032355\bar{5}, \quad C_2^2 = \frac{3(12+5\sqrt{5})}{76} = 0.915013\bar{3}, \tag{5.14}$$

where the bar over a digit indicates a round off. We note that (5.14) is compatible with the constraint $0 < C_1^2 + C_2^2 \le 1$. Thus, we have confirmed that $z_1 = -1/2$ and $z_2 = 1/2$ do indeed correspond to bound states of the compactly-supported potential described in (5.13). In (5.4), if we choose $\alpha_j = 1$ for $j = 1,2,3$, then the system in (5.5) becomes inconsistent and hence there are no values V_1 and V_2 satisfying (5.5). Thus, the corresponding expression in (5.4) does not yield a compactly-supported potential. On the other hand, if we let $V_1 = -\sqrt{2}$ and $V_2 = 1/\sqrt{2}$ in (5.3), we obtain a solution to (5.5) with $\alpha_1 = -1$, $\alpha_2 = 1$, and $\alpha_3 = \sqrt{2}$, and hence the Jost solution obtained from (5.4) does not contain any zeros in $z \in (-1,0) \cup (0,1)$, yielding $N = 0$. Choosing $V_1 = -(7+\sqrt{10})/6$ and $V_2 = -(1+\sqrt{10})/2$ in (5.3), we obtain a solution to (5.5) given by

$$\alpha_1 = \frac{3}{2(1+\sqrt{10})} = 0.36038\bar{8}, \quad \alpha_2 = \frac{2}{1+\sqrt{2}i}, \quad \alpha_3 = \frac{2}{1-\sqrt{2}i},$$

which indicates that the corresponding f_0 in (5.4) has one bound state at $z_1 = \alpha_1$ with the corresponding Gel'fand-Levitan norming constant C_1, evaluated with the help of (2.40), (2.42), and (5.2), as

$$C_1^2 = \frac{625+128\sqrt{10}}{3489} = 0.295148\bar{8}.$$

We remark that it is impossible to have a compactly-supported potential with $b = 2$ having three bound states. This can be seen as follows. Assume that for some choice of V_1 and V_2 in (5.3) we had $-1 < \alpha_1 < \alpha_2 < \alpha_3 < 1$ for nonzero α_j

values. Using (5.4) in (2.10) and (5.2) we would get the corresponding Marchenko norming constants as

$$\begin{cases} c_1^2=\dfrac{(1-\alpha_1^2)(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_3)}{\alpha_1^4(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}, \\ c_2^2=\dfrac{(1-\alpha_1\alpha_2)(1-\alpha_2^2)(1-\alpha_2\alpha_3)}{\alpha_2^4(\alpha_1-\alpha_2)(\alpha_3-\alpha_2)}, \\ c_3^2=\dfrac{(1-\alpha_1\alpha_3)(1-\alpha_2\alpha_3)(1-\alpha_3^2)}{\alpha_3^4(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}. \end{cases} \tag{5.15}$$

From the three equations in (5.15) we see that we would have $c_1^2>0$, $c_2^2<0$, $c_3^2>0$, and hence it is impossible to have $N=3$. From Example 5.1 we know that $0\le N\le b$ when $b=1$, and from (5.15) we know that $0\le N\le b$ when $b=2$. From (5.1) it is clear that the number of zeros of $f_0(z)$ in $z\in(-1,0)\cup(0,1)$ cannot exceed $2b-1$. This naturally leads to the following question, which can perhaps be answered with the help of a generalization of (5.15) from $b=2$ to an arbitrary positive integer b : For any given positive integer b, what is the maximal number of bound states for the corresponding Schrödinger operator associated with (1.1) and (1.4), if the potential V_n has a compact support with $V_n=0$ for $n>b$? The answer to this question turns out to be the integer b itself and a proof can be found in [1].

The regular solution φ_n to (1.1) corresponding to (5.3) can be obtained recursively with the help of (2.4). We have

$$\varphi_1=1,\quad \varphi_2=-\lambda+2+V_1, \tag{5.16}$$

$$\varphi_3=\lambda^2-(4+V_1+V_2)\lambda+3+2V_1+2V_2+V_1V_2, \tag{5.17}$$

$$\begin{aligned}\varphi_4=&-\lambda^3+(6+V_1+V_2)\lambda^2\\ &-(10+4V_1+4V_2+V_1V_2)\lambda+4+3V_1+4V_2+2V_1V_2,\end{aligned} \tag{5.18}$$

$$\begin{aligned}\varphi_5=&\lambda^4-(8+V_1+V_2)\lambda^3+(21+6V_1+6V_2+V_1V_2)\lambda^2\\ &-(20+10V_1+11V_2+4V_1V_2)\lambda+5+4V_1+6V_2+3V_1V_2.\end{aligned} \tag{5.19}$$

In the next two examples, we show that if we remove a bound state from a compactly-supported potential then the resulting potential may or may not be compactly supported.

Example 5.3. Consider the compactly-supported potential V_n with $b=1$ and hence $V_n=0$ for $n\ge 2$. The corresponding Jost function is given by (5.6). Since the bound states correspond to the zeros of f_0 when $z\in(-1,0)\cup(0,1)$, from (5.6) we see that there exists one bound state if $|V_1|>1$. We assume that $|V_1|>1$ so that we have exactly one bound state at $z=z_1$, where $z_1=-1/V_1$. From (2.10) and (5.6) we see that the corresponding scattering matrix is given by

$$S(z)=\frac{V_1+z}{z+V_1z^2},\quad z\in\mathbf{T}. \tag{5.20}$$

In this case, the Jost solution satisfies $f_n=z^n$ for $n\ge 1$. In the presence of one bound state, the corresponding Gel'fand-Levitan norming constant C_1 is evaluated with the help of (2.42), (5.2), (5.20), and $f_1=z$, yielding

$$C_1^2=V_1^2-1. \tag{5.21}$$

From (2.46) we see that we must have $0 < C_1^2 \le 1$ and hence we must use the restriction $0 < |V_1| \le \sqrt{2}$. Let us now remove the bound state with $z_1 = -1/V_1$. The transformed Jost function $\tilde f_0$ is obtained via (4.1) and is given by $\tilde f_0 = 1 + z/V_1$. In this case, using (4.11) and (5.7) we obtain

$$\theta_n = \Big(-\frac{1}{V_1}\Big)^{n-1}, \quad n \ge 1. \tag{5.22}$$

Using (5.21) with $N = 1$, we obtain the quantity F_N given in (4.10) as

$$F_1 = 1 - V_1^2. \tag{5.23}$$

Using (5.22) and (5.23) in (4.17) and (4.18) we obtain $\tilde V_n = 0$ for $n \ge 2$ and $\tilde V_1 = 1/V_1$.

Example 5.4. Consider the compactly-supported potential V_n described by (5.13) in Example 5.2. We know from Example 5.2 that there are two bound states with $z_1 = -1/2$ and $z_2 = 1/2$ with the respective corresponding Gel'fand-Levitan norming constants C_1 and C_2 as in (5.14). Hence, we have $N = 2$. We now demonstrate that if we remove the bound state at $z = z_2$ by using the Darboux transformation formulas given in Section 4 then the transformed potential is no longer compactly supported. From (2.38) we see that the values λ_1 and λ_2 corresponding z_1 and z_2, respectively, are given by

$$z_1 = -\frac{1}{2}, \quad \lambda_1 = \frac{9}{2}, \quad z_2 = \frac{1}{2}, \quad \lambda_2 = -\frac{1}{2}. \tag{5.24}$$

Using (5.16)–(5.20) and (5.24) in (4.11) we obtain

$$\theta_n = \Big(\frac{1}{2}\Big)^{n-1} \begin{bmatrix} (-1)^{n-1}(5+2\sqrt{5}) \\ (5-2\sqrt{5}) \end{bmatrix}, \quad n \ge 1. \tag{5.25}$$

Using (5.14) with $N = 2$ in (4.10) we obtain

$$F_2 = \begin{bmatrix} -\frac{9}{10} & 0 \\ 0 & -\frac{15}{16}\,(9+4\sqrt{5}) \end{bmatrix}. \tag{5.26}$$

With the help of (5.14), (5.25), and (5.26), from (4.17) and (4.18) we can evaluate the transformed potential $\tilde V_n$ for all $n \ge 1$. We list the first few values below and mention that $\tilde V_n$ is not compactly supported:

$$\tilde V_1 = \frac{5(3-2\sqrt{5})}{16}, \quad \tilde V_2 = \frac{1125 + 21826\sqrt{5}}{119120}, \quad \tilde V_3 = \frac{270(14781 + 6364\sqrt{5})}{15975481},$$
$$\tilde V_4 = \frac{1080(231681 + 102364\sqrt{5})}{1284143281}, \quad \tilde V_5 = \frac{4320(3691281 + 163364\sqrt{5})}{204372438481}.$$

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