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Changing the discrete spectrum of the half-line matrix Schrödinger operator

Abstract: We consider the matrix-valued Schrödinger operator on the half line with the general self-adjoint boundary condition. When the discrete spectrum is changed without changing the continuous spectrum, we present a review of the transformations of the relevant quantities including the regular solution, the Jost solution, the Jost matrix, the scattering matrix, and the boundary matrices used to describe the self-adjoint boundary condition. The changes in the discrete spectrum are considered when an existing bound state is removed, a new bound state is added, and the multiplicity of a bound state is decreased or increased without removing the bound state. We provide various explicit examples to illustrate the theoretical results presented.

Keywords: Matrix Schrödinger operator, self-adjoint boundary condition, bound states, bound-state multiplicity, spectral function, Gel'fand–Levitan method, Darboux transformation, removal of bound states, addition of bound states, dependency matrix

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1 Introduction

In [4] we have presented the Gel'fand–Levitan method for the matrix-valued Schrödinger equation on the half line with the general self-adjoint boundary condition. By using that method we have then provided the transformations of the relevant quantities when the discrete spectrum of the corresponding Schrödinger operator is changed without changing the continuous spectrum. In particular, we have presented the transformation formulas for the relevant quantities when the discrete spectrum is changed in four ways, which are the complete removal of a bound state, the decrease of the multiplicity of an existing bound state without removing that bound state, the addition of a new bound state, and the increase of the multiplicity of an existing bound state, respectively. These four types of elementary operations can be applied successively in any order to change the discrete spectrum by removing or adding any number of bound states or by decreasing or increasing of the multiplicity of any existing bound state.

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The goal of the present work is to illustrate the primary results of [4] by providing various explicit examples. Such examples are expected to help the reader understand the theory better by clarifying the theoretical results presented in [4]. Our chapter is organized as follows. In Section 2, we present the preliminaries related to the matrix-valued Schrödinger equation on the half line with the general self-adjoint boundary condition. In Section 3, we provide the basic material on the bound states of the relevant Schrödinger operator. In Section 4, we describe the perturbation initiated by a change in the spectral measure, we introduce the Gel'fand–Levitan system of integral equations, and we describe how the solution to the Gel'fand–Levitan system yields the corresponding perturbation in the potential, in the regular solution, in the Jost solution, in the Jost matrix, in the scattering matrix, and in the boundary matrices. In Section 5, we briefly describe the method to remove a bound state from the discrete spectrum with the help of the Gel'fand–Levitan method. In Section 6, we provide a summary of the method to decrease the multiplicity of an existing bound state with the help of the Gel'fand–Levitan procedure. In Section 7, we summarize the process to add a new bound state by using the Gel'fand–Levitan method. In Section 8, we briefly describe the method to increase the multiplicity of an existing bound state by the Gel'fand–Levitan procedure. In Section 9, we illustrate the evaluation of various relevant quantities described in Sections 2 and 3 associated with the Schrödinger operator related to (2.1) and (2.7). Finally, in Section 10, we illustrate the mathematical procedures described in Sections 5–8. Except for the proofs of Theorems 8.1, 9.9, and 9.10, we do not include any proofs in our paper. For the proofs of the mathematical techniques used here we refer the reader to [4]. The reader can consult [4] also for a brief review of the literature on transformations to remove and to add bound states and for a discussion of important applications of matrix-valued Schrödinger operators.

2 The matrix Schrödinger operator on the half line

In this section, we present the preliminaries related to the matrix-valued Schrödinger equation on the half line with the self-adjoint boundary condition. We refer the reader to [3] for any elaborations on the material presented here.

We consider the half-line Schrödinger equation

$$-\psi'' + V(x)\psi = k^2\psi, \quad x \in \mathbb{R}^+, \quad (2.1)$$

where x is the independent variable, the prime denotes the x -derivative, k^2 is the spectral parameter, $\mathbb{R}^+$ denotes the interval $(0, +\infty)$, and the potential V is an $n \times n$ self-adjoint matrix-valued function of x , with n being a fixed positive integer. The wave function ψ is either an $n \times n$ matrix or a column vector with n components. The self-adjointness is expressed as

$$V(x)^\dagger = V(x), \quad x \in \mathbb{R}^+, \quad (2.2)$$

with the dagger denoting the matrix adjoint (matrix transpose and complex conjugation). We use the terms self-adjoint and Hermitian interchangeably. We use both V and $V(x)$ to refer to the potential with the latter use for emphasis on the x -dependence. We let $\mathbb{C}^+$ denote the upper-half complex plane, use $\mathbb{R}$ for the real axis, and let $\overline{\mathbb{C}^+} := \mathbb{C}^+ \cup \mathbb{R}$.

We always assume that the potential V is at least integrable, i.e. we have

$$\int_0^\infty dx |V(x)| < +\infty, \quad (2.3)$$

where $|V(x)|$ denotes the operator norm of the $n \times n$ matrix $V(x)$. We mention explicitly when we need a stronger condition on the potential. For example, at times we require that the potential V also has the first moment, i.e. we require that we have

$$\int_0^\infty dx (1+x) |V(x)| < +\infty. \quad (2.4)$$

When (2.4) is satisfied, we say that the potential V belongs to $L_1^1(\mathbb{R}^+)$. In general, we say that the potential V belongs to $L_\epsilon^1(\mathbb{R}^+)$ for some nonnegative constant ϵ if we have

$$\int_0^\infty dx (1+x)^\epsilon |V(x)| < +\infty. \quad (2.5)$$

Thus, the potential V is integrable when $\epsilon = 0$ in (2.5), in which case we also say that V belongs to $L^1(\mathbb{R}^+)$. Because all matrix norms are equivalent in a finite-dimensional vector space, any matrix norm can be used in (2.3)–(2.5) instead of the operator norm.

We obtain our self-adjoint Schrödinger operator on the half line by supplementing (2.1) with a self-adjoint boundary condition at $x = 0$. We find it the most convenient to describe that self-adjoint boundary condition in terms of the two constant $n \times n$ matrices A and B used in the definition of the regular solution $\varphi(k, x)$ to (2.1) satisfying the initial conditions

$$\varphi(k, 0) = A, \quad \varphi'(k, 0) = B. \quad (2.6)$$

We then state the self-adjoint boundary condition as

$$-B^\dagger \psi(0) + A^\dagger \psi'(0) = 0, \quad (2.7)$$

with the understanding that the matrices A and B satisfy

$$-B^\dagger A + A^\dagger B = 0, \quad (2.8)$$

$$A^\dagger A + B^\dagger B > 0. \quad (2.9)$$

We refer to A and B as the boundary matrices. We use the positivity of a matrix to mean the positive definiteness, and we use the nonnegativity of a matrix to mean the positive semidefiniteness. We recall that an $n \times n$ matrix is positive if and only if all its eigenvalues are positive and that it is nonnegative if and only if all its eigenvalues are real and nonnegative. The boundary condition (2.7) is uniquely determined by the boundary matrices A and B . On the other hand, the boundary matrices A and B can be simultaneously multiplied on the right by an $n \times n$ invertible matrix T without affecting (2.7)–(2.9). In fact, this is the only freedom we have in choosing the boundary matrices A and B .

By using $A = 0$ and $B = I$ in the initial conditions (2.6), we obtain the Dirichlet boundary condition $\psi(0) = 0$. We remark that we use 0 for the scalar zero, the zero vector with n components, and the $n \times n$ zero matrix, depending on the context, and we use I to denote the $n \times n$ identity matrix. We use the terms Dirichlet and purely Dirichlet interchangeably, where the latter is used to make a contrast with a boundary condition consisting of a mixture of Dirichlet and non-Dirichlet components.

We recall that we always assume that the potential V is integrable and satisfies (2.2). In that case, for each $k \in \overline{\mathbb{C}^+} \setminus \{0\}$ the Schrödinger equation (2.1) has a unique $n \times n$ matrix-valued particular solution, known as the Jost solution $f(k, x)$, satisfying the spacial asymptotics

$$f(k, x) = e^{ikx}[I + o(1)], \quad f'(k, x) = e^{ikx}[ikI + o(1)], \quad x \rightarrow +\infty. \quad (2.10)$$

The regular solution $\varphi(k, x)$ to (2.1) satisfying the initial conditions given in (2.6) is entire in $k \in \mathbb{C}$ for each fixed $x \in [0, +\infty)$. In general, the Jost solution $f(k, x)$ does not satisfy the boundary condition (2.7), but the regular solution $\varphi(k, x)$ satisfies (2.7) at any $k \in \mathbb{C}$. The Jost matrix $J(k)$ associated with (2.1) and (2.7) is defined as

$$J(k) := f(-k^*, 0)^\dagger B - f'(-k^*, 0)^\dagger A, \quad k \in \mathbb{R} \setminus \{0\}, \quad (2.11)$$

where the asterisk denotes complex conjugation. The $n \times n$ matrix-valued $J(k)$ has an extension from $k \in \mathbb{R} \setminus \{0\}$ to $k \in \mathbb{C}^+ \setminus \{0\}$, and in fact we use the asterisk in (2.11) to indicate how that extension occurs. The Jost matrix $J(k)$ is analytic in $k \in \mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+} \setminus \{0\}$. The scattering matrix $S(k)$ associated with (2.1) and (2.7) is defined as

$$S(k) := -J(-k)J(k)^{-1}, \quad k \in \mathbb{R} \setminus \{0\}, \quad (2.12)$$

and it is continuous in $k \in \mathbb{R} \setminus \{0\}$. The physical solution $\Psi(k, x)$ associated with (2.1) and (2.7) is defined as

$$\Psi(k, x) := f(-k, x) + f(k, x)S(k), \quad k \in \mathbb{R} \setminus \{0\}. \quad (2.13)$$

For each fixed $x \in \mathbb{R}^+$, the physical solution $\Psi(k, x)$ has an extension from $k \in \mathbb{R} \setminus \{0\}$ to $k \in \mathbb{C}^+$, and that extension is meromorphic in $k \in \mathbb{C}^+$.

If the potential V is further assumed to belong to $L^1_1(\mathbb{R}^+)$, then the Jost solution $f(k, x)$, the Jost matrix $J(k)$, the scattering matrix $S(k)$, and the physical solution $\Psi(k, x)$ are also defined at $k = 0$, and those four quantities are continuous at $k = 0$. On the other hand, if the potential V satisfies (2.2) and (2.3) but not (2.4), as illustrated in Example 9.11, those four quantities may not have continuous extensions to $k = 0$.

The regular solution $\varphi(k, x)$ can be expressed in terms of the physical solution $\Psi(k, x)$ and the Jost matrix $J(k)$ as

$$\varphi(k, x) = -\frac{1}{2ik} \Psi(k, x) J(k), \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}, \quad (2.14)$$

or in terms of the Jost solution $f(k, x)$ and the Jost matrix $J(k)$ as

$$\varphi(k, x) = \frac{1}{2ik} [f(k, x) J(-k) - f(-k, x) J(k)], \quad k \in \mathbb{R} \setminus \{0\}. \quad (2.15)$$

When the potential V is locally integrable for $x \in [0, +\infty)$, it follows [3] that the regular solution is defined for $k \in \mathbb{C}$ and in fact it is entire in k for each $x \in [0, +\infty)$. Thus, when V belongs to $L^1(\mathbb{R}^+)$ but not to $L^1_1(\mathbb{R}^+)$, even though the zero-energy Jost solution $f(0, x)$ or the zero-energy physical solution $\Psi(0, x)$ may not exist and the right-hand sides of (2.14) and (2.15) may not be defined at $k = 0$, the regular solution $\varphi(k, x)$ is still well defined at $k = 0$.

3 The bound states of the Schrödinger operator

In this section, we assume that the potential is integrable and satisfies (2.2). We present the relevant preliminary material related to the bound states of the half-line matrix Schrödinger operator.

A bound-state solution corresponds to a square-integrable column-vector solution to (2.1) satisfying the boundary condition (2.7). We let $\lambda := k^2$. When the potential satisfies (2.2) and (2.4), there are no bound states when $\lambda \geq 0$ and there are at most a finite number of bound states when $\lambda < 0$ occurring at the k -values on the positive imaginary axis of the complex k -plane. The bound-state k -values correspond to the zeros of the determinant $\det[J(k)]$ on the positive imaginary axis. We use N to denote the number of zeros of $\det[J(k)]$ occurring when $k = i\kappa_j$ for $1 \leq j \leq N$, with the κ_j -values being distinct positive constants. We note that N may be zero, finite, or infinite. If there are no bound states, then we have $N = 0$. Thus, $\det[J(i\kappa_j)] = 0$, and we use m_j to denote the number of linearly independent vectors in $\mathbb{C}^n$ spanning the kernel $\text{Ker}[J(i\kappa_j)]$. We note that m_j is a positive integer satisfying $1 \leq m_j \leq n$, and it corresponds to the multiplicity of the bound state at $k = i\kappa_j$. Thus, N describes the number of bound states without counting the multiplicities. The total number of bound states including the multiplicities, denoted by $\mathcal{N}$, is given by

$$\mathcal{N} := \sum_{j=1}^N m_j.$$

The $n \times n$ matrices $J(i\kappa_j)$ and $J(i\kappa_j)^\dagger$ each have the rank equal to $n - m_j$. When $n = 1$, we must have $m_j = 1$, in which case $J(i\kappa_j)$ and $J(i\kappa_j)^\dagger$ are both equal to the scalar zero. If $n \geq 2$, then the kernels $\text{Ker}[J(i\kappa_j)]$ and $\text{Ker}[J(i\kappa_j)^\dagger]$ are in general different from each other, but they have the same dimension equal to m_j . We use Q_j and P_j to denote the orthogonal projections onto $\text{Ker}[J(i\kappa_j)]$ and $\text{Ker}[J(i\kappa_j)^\dagger]$, respectively. The definition of the orthogonal projection indicates that we have

$$P_j^2 = P_j^\dagger = P_j, \quad Q_j^2 = Q_j^\dagger = Q_j. \quad (3.1)$$

The bound-state solutions to (2.1) can be described either via the Marchenko theory or the Gel'fand–Levitan theory. In the Marchenko theory the normalization matrices for the bound states are obtained by normalizing the Jost solution $f(k, x)$ at the bound states, whereas in the Gel'fand–Levitan theory the normalization matrices for the bound states are obtained by normalizing the regular solution $\phi(k, x)$ at the bound states. We use M_j and $\Psi_j(x)$ to denote the $n \times n$ Marchenko normalization matrix and the corresponding $n \times n$ normalized matrix solution at the bound state with $k = i\kappa_j$, respectively. Similarly, we use C_j and $\Phi_j(x)$ to denote the Gel'fand–Levitan normalization matrix and the corresponding normalized matrix solution at the bound state with $k = i\kappa_j$, respectively.

At the bound state $k = i\kappa_j$, the Schrödinger equation (2.1) is given by

$$-\psi'' + V(x)\psi = -\kappa_j^2 \psi, \quad x \in \mathbb{R}^+, \quad 1 \leq j \leq N. \quad (3.2)$$

In general, the solution $f(i\kappa_j, x)$ to (3.2) does not satisfy the boundary condition (2.7). On the other hand, $f(i\kappa_j, x)$ is exponentially decaying asymptotically, i.e. we have

$$f(i\kappa_j, x) = e^{-\kappa_j x} [I + o(1)], \quad x \rightarrow +\infty, \quad 1 \leq j \leq N. \quad (3.3)$$

Hence, for large x -values, the n columns of $f(i\kappa_j, x)$ are linearly independent and they are each exponentially decaying solutions to (3.2). However, those columns do not, in general, satisfy the boundary condition (2.7). We construct an $n \times n$ matrix solution to (3.2) satisfying the boundary condition (2.7) and exponentially decaying as $x \rightarrow +\infty$. Such a matrix solution, denoted by $\Psi_j(x)$, is obtained with the help of $f(i\kappa_j, x)$ and the $n \times n$ Marchenko normalization matrix M_j , and it is given by

$$\Psi_j(x) := f(i\kappa_j, x) M_j, \quad 1 \leq j \leq N. \quad (3.4)$$

We refer to the $n \times n$ matrix $\Psi_j(x)$ as the Marchenko normalized bound-state solution to the Schrödinger equation (3.2) at $k = i\kappa_j$. Although each column of the $n \times n$ matrix $\Psi_j(x)$ is a solution to (3.2) and satisfies (2.7), among its n columns we only have m_j linearly independent column-vector solutions to (3.2) satisfying the boundary condition (2.7). Any nontrivial square-integrable column-vector solution to (3.2) satisfying (2.7) can be writ-

ten as $\Psi_j(x)v$ for a constant nonzero column vector $v \in \mathbb{C}^n$. The Marchenko normalized bound-state solutions $\Psi_j(x)$ for $1 \leq j \leq N$ satisfy the normalization condition given by

$$\int_0^\infty dx \Psi_j(x)^\dagger \Psi_j(x) = P_j, \quad 1 \leq j \leq N, \quad (3.5)$$

as well as the orthogonality condition given by

$$\int_0^\infty dx \Psi_j(x)^\dagger \Psi_l(x) = 0, \quad j \neq l, \quad (3.6)$$

where we have $1 \leq j \leq N$ and $1 \leq l \leq N$. For further information on the description of bound states in terms of the Marchenko normalization matrices, we refer the reader to [3] and [4].

The Marchenko normalization matrix M_j is constructed as follows. Using the orthogonal projection matrix P_j and the Jost solution $f(ik_j, x)$, we obtain the $n \times n$ Hermitian matrix A_j given by

$$A_j := \int_0^\infty dx P_j f(ik_j, x)^\dagger f(ik_j, x) P_j, \quad 1 \leq j \leq N. \quad (3.7)$$

Using A_j and P_j , we define the $n \times n$ Hermitian matrix B_j as

$$B_j := I - P_j + A_j, \quad 1 \leq j \leq N. \quad (3.8)$$

It is known that B_j is a positive matrix, and hence there exists a unique positive Hermitian matrix $B_j^{1/2}$ satisfying

$$B_j^{1/2} B_j^{1/2} = B_j, \quad 1 \leq j \leq N. \quad (3.9)$$

Since $B_j^{1/2}$ is positive, its inverse exists and is denoted by $B_j^{-1/2}$. It is known that each of B_j , $B_j^{1/2}$, and $B_j^{-1/2}$ commutes with P_j . The Marchenko normalization matrix M_j is defined in terms of $B_j^{-1/2}$ and P_j as

$$M_j := B_j^{-1/2} P_j, \quad 1 \leq j \leq N. \quad (3.10)$$

The Marchenko normalization matrix M_j is an $n \times n$ nonnegative matrix of rank m_j . Hence, exactly m_j of its eigenvalues are positive and the remaining $n - m_j$ eigenvalues are zero.

The description of the bound states pertinent to the Gel'fand–Levitan method is as follows. We construct the $n \times n$ matrix-valued Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$ with the help of the regular solution $\varphi(k, x)$ and the Gel'fand–Levitan normalization matrices C_j as

$$\Phi_j(x) := \varphi(ik_j, x) C_j, \quad 1 \leq j \leq N, \quad (3.11)$$

where $\varphi(ik_j, x)$ is the value of the regular solution evaluated at $k = ik_j$. We refer to the $n \times n$ matrix $\Phi_j(x)$ as the Gel'fand–Levitan normalized bound-state solution to the Schrödinger equation (3.2) at $k = ik_j$. Although each column of the regular solution $\varphi(ik_j, x)$ to (3.2) satisfies the boundary condition (2.7), unless we have $m_j = n$ those columns in general become unbounded as $x \rightarrow +\infty$, and we have

$$\varphi(ik_j, x) = O(e^{k_j x}), \quad x \rightarrow +\infty.$$

The presence of the $n \times n$ matrix C_j of rank m_j in (3.11) ensures that each column of $\Phi_j(x)$, besides being a solution to (3.2) and satisfying the boundary condition (2.7), decays exponentially as $O(e^{-k_j x})$ as $x \rightarrow +\infty$. There are exactly m_j linearly independent columns of $\Phi_j(x)$ among its n columns. The Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$ satisfy the normalization property given by

$$\int_0^\infty dx \Phi_j(x)^\dagger \Phi_j(x) = Q_j, \quad 1 \leq j \leq N, \quad (3.12)$$

as well as the orthogonalization property

$$\int_0^\infty dx \Phi_j(x)^\dagger \Phi_l(x) = 0, \quad j \neq l, \quad (3.13)$$

for $1 \leq j \leq n$ and $1 \leq l \leq N$.

The Gel'fand–Levitan normalization matrix C_j is constructed as follows. Using the orthogonal projection matrix Q_j and the regular solution $\varphi(ik_j, x)$ to (3.2), we form the $n \times n$ Hermitian matrix $\mathbf{G}_j$ as

$$\mathbf{G}_j := \int_0^\infty dx Q_j \varphi(ik_j, x)^\dagger \varphi(ik_j, x) Q_j, \quad 1 \leq j \leq N. \quad (3.14)$$

Then, using $\mathbf{G}_j$ and Q_j we form the $n \times n$ Hermitian matrix $\mathbf{H}_j$ as

$$\mathbf{H}_j := I - Q_j + \mathbf{G}_j, \quad 1 \leq j \leq N. \quad (3.15)$$

It is known that $\mathbf{H}_j$ is a positive matrix. Hence, there exists a unique positive Hermitian matrix $\mathbf{H}_j^{1/2}$ so that

$$\mathbf{H}_j^{1/2} \mathbf{H}_j^{1/2} = \mathbf{H}_j, \quad 1 \leq j \leq N. \quad (3.16)$$

Since the matrix $\mathbf{H}_j^{1/2}$ is positive, its inverse exists and is denoted by $\mathbf{H}_j^{-1/2}$. Each of $\mathbf{H}_j$, $\mathbf{H}_j^{1/2}$, and $\mathbf{H}_j^{-1/2}$ commutes with Q_j . The Gel'fand–Levitan normalization matrix C_j is constructed using $\mathbf{H}_j^{-1/2}$ and Q_j as

$$C_j := \mathbf{H}_j^{-1/2} Q_j, \quad 1 \leq j \leq N. \quad (3.17)$$

The $n \times n$ matrix C_j is Hermitian and nonnegative, and it has a rank equal to m_j , which is the same as the rank of Q_j .

The Marchenko normalized bound-state solution $\Psi_j(x)$ is not identical to the Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$, but each of them contains m_j linearly independent columns. Those two $n \times n$ matrix-valued bound-state solutions can be expressed in terms of each other with the help of the $n \times n$ dependency matrix D_j via

$$\Phi_j(x) = \Psi_j(x) D_j, \quad 1 \leq j \leq N. \quad (3.18)$$

In order to define D_j uniquely, we need to impose one further restriction on it. For example, we can require that D_j satisfies

$$D_j = P_j D_j, \quad 1 \leq j \leq N, \quad (3.19)$$

where we recall that the $n \times n$ matrix P_j is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)^\dagger]$ and appearing in (3.1). Then, the dependency matrix satisfying (3.18) and (3.19) is uniquely determined. In other words, when we are given the Schrödinger equation (2.1) with the potential V satisfying (2.2) and (2.3) and we are also given the self-adjoint boundary condition (2.7), then there exists a unique dependency matrix D_j satisfying (3.18) and (3.19) associated with each bound state at $k = i\kappa_j$ for $1 \leq j \leq N$. The dependency matrix D_j can be constructed by using the fact that there exists a bijection $\alpha \mapsto \beta$ from the kernel of $J(i\kappa_j)$ onto the kernel of $J(i\kappa_j)^\dagger$ in such a way that

$$\varphi(i\kappa_j, x) \alpha = f(i\kappa_j, x) \beta, \quad 1 \leq j \leq N.$$

For the explicit construction of D_j associated with the corresponding Schrödinger operator, we refer the reader to [4]. From (3.3), we know that the Jost solution $f(i\kappa_j, x)$ is an invertible $n \times n$ matrix when x is large. We can express D_j in terms of the Marchenko normalization matrix M_j , the Gel'fand–Levitan normalized bound-state wavefunction $\Phi_j(x)$, and the matrix quantity $f(i\kappa_j, x)$ as [4]

$$D_j = M_j^+ f(i\kappa_j, x)^{-1} \Phi_j(x), \quad 1 \leq j \leq N, \quad (3.20)$$

where we evaluate the right-hand side at any x -value at which the matrix $f(i\kappa_j, x)$ is invertible and we note that M_j^+ denotes [5] the Moore–Penrose inverse of M_j . As seen from (3.18), we can write (3.20) in the equivalent form as

$$D_j = \Psi_j(x)^+ \Phi_j(x), \quad 1 \leq j \leq N. \quad (3.21)$$

If the matrix $f(i\kappa_j, x)$ is not invertible at a particular x -value but the matrix $f'(i\kappa_j, x)$ is invertible there, then we can evaluate D_j by using

$$D_j = M_j^+ f'(i\kappa_j, x)^{-1} \Phi_j'(x), \quad 1 \leq j \leq N, \quad (3.22)$$

where we evaluate the right-hand side at any x -value at which the matrix $f'(i\kappa_j, x)$ is invertible. From (3.18), we see that we can write (3.22) in the equivalent form as

$$D_j = [\Psi_j'(x)]^+ \Phi_j(x), \quad 1 \leq j \leq N.$$

We remark that D_j satisfies various properties such as

$$D_j = D_j Q_j, \quad 1 \leq j \leq N, \quad (3.23)$$

where we recall that the $n \times n$ matrix Q_j is the orthogonal projection onto $\text{Ker}[J(i\kappa_j)]$. Each of the matrix products $D_j^\dagger D_j$ and $D_j D_j^\dagger$ is an orthogonal projection, and we have

$$D_j^\dagger D_j = Q_j, \quad D_j D_j^\dagger = P_j, \quad 1 \leq j \leq N. \quad (3.24)$$

The matrices D_j and $D_j^\dagger$ each have rank m_j , which is the same as the common rank of the orthogonal projection matrices P_j and Q_j . The adjoint matrix $D_j^\dagger$ coincides with the Moore–Penrose inverse of D_j , i.e. we have

$$D_j^+ = D_j^\dagger, \quad 1 \leq j \leq N. \quad (3.25)$$

Analogous to (3.18), where $\Phi_j(x)$ is expressed in terms of $\Psi_j(x)$, we have

$$\Psi_j(x) = \Phi_j(x) D_j^\dagger, \quad 1 \leq j \leq N, \quad (3.26)$$

which expresses $\Psi_j(x)$ in terms of $\Phi_j(x)$.

4 The Gel'fand–Levitan system of integral equations

In this section, we assume that the potential V satisfies (2.2) and at least (2.4). We present the basic relevant material related to the spectral measure for the half-line matrix Schrödinger operator described by (2.1) and (2.7). We also mention Parseval's equality involving the regular solution and the Gel'fand–Levitan normalized bound-state solutions as well as another version of Parseval's equality involving the physical solution and the Marchenko normalized bound-state solutions. We show how a perturbation of the spectral measure results in the perturbation of all relevant quantities related to the

corresponding Schrödinger operator. We also indicate how those perturbations can be expressed in terms of the solution to a system of integral equations, which we refer to the matrix-valued Gel'fand–Levitan system. For the details and further elaborations, we refer the reader to [4].

We recall that when the potential V in (2.1) satisfies (2.2) and (2.4), the Schrödinger operator associated with (2.1) and (2.7) has at most a finite number of distinct bound states occurring at $k = i\kappa_j$ for $1 \leq j \leq N$. It is understood that $N = 0$ if there are no bound states. The spectral measure $d\rho$ associated with the corresponding Schrödinger operator is given by

$$d\rho := \begin{cases} \frac{\sqrt{\lambda}}{\pi} (J(\sqrt{\lambda})^\dagger J(\sqrt{\lambda}))^{-1} d\lambda, & \lambda \geq 0, \\ \sum_{j=1}^N C_j^2 \delta(\lambda - \lambda_j) d\lambda, & \lambda < 0, \end{cases} \quad (4.1)$$

where $\delta(\cdot)$ denotes the Dirac delta distribution and we recall that $J(k)$ is the Jost matrix given in (2.11), the quantity C_j is the Gel'fand–Levitan normalization matrix defined in (3.17), the matrix C_j^2 is the matrix product of C_j with itself, and we let $\lambda := k^2$ and $\lambda_j := -\kappa_j^2$ for $1 \leq j \leq N$. Using the spectral measure $d\rho$, we express the completeness relation, known as Parseval's equality, as

$$\int_{\lambda \in \mathbb{R}^+} \varphi(k, x) d\rho \varphi(k, y)^\dagger + \sum_{j=1}^N \Phi_j(x) \Phi_j(y)^\dagger = \delta(x - y) I,$$

where the integration is from $\lambda = 0$ to $\lambda = +\infty$, and we recall that $\varphi(k, x)$ is the regular solution to (2.1) satisfying the initial conditions (2.6) and that $\Phi_j(x)$ is the $n \times n$ Gel'fand–Levitan normalized bound-state solution at $k = i\kappa_j$ and appearing in (3.11). We remark that Parseval's equality can also be expressed in terms of the physical solution $\Psi(k, x)$ appearing in (2.13) and the $n \times n$ Marchenko normalized bound-state solutions $\Psi_j(x)$ appearing in (3.4), and we have

$$\frac{1}{2\pi} \int_0^\infty dk \Psi(k, x) \Psi(k, y)^\dagger + \sum_{j=1}^N \Psi_j(x) \Psi_j(y)^\dagger = \delta(x - y) I.$$

Let us view $V(x)$, $\varphi(k, x)$, (A, B) , $f(k, x)$, $J(k)$, $S(k)$, and $d\rho$ as the potential, the regular solution, the pair of boundary matrices, the Jost solution, the Jost matrix, the scattering matrix, and the spectral measure, respectively, corresponding to the unperturbed problem. Let us assume that the unperturbed problem has N bound states at $k = i\kappa_j$ for $1 \leq j \leq N$ and that the quantities relevant to the bound state at $k = i\kappa_j$ are $\lambda_j := -\kappa_j^2$, the multiplicity m_j , the Gel'fand–Levitan normalization matrix C_j , the orthogonal projection Q_j onto $\text{Ker}[J(i\kappa_j)]$, the orthogonal projection P_j onto $\text{Ker}[J(i\kappa_j)^\dagger]$, and the Gel'fand–Levitan normalized bound-state solution $\Phi_j(x)$. Thus, the regular solution $\varphi(k, x)$ satisfies the unperturbed Schrödinger equation (2.1) and the unperturbed initial conditions (2.6).

Let us view $\tilde{V}(x)$, $\tilde{\varphi}(k, x)$, $(\tilde{A}, \tilde{B})$, $\tilde{f}(k, x)$, $\tilde{J}(k)$, $\tilde{S}(k)$, and $d\tilde{\rho}$ as the potential, the regular solution, the pair of boundary matrices, the Jost solution, the Jost matrix, the scattering matrix, and the spectral measure, respectively, corresponding to the perturbed problem. Let us assume that the perturbed problem has $\tilde{N}$ bound states at $k = i\tilde{\kappa}_j$ for $1 \leq j \leq \tilde{N}$ and that the quantities relevant to the bound state at $k = i\tilde{\kappa}_j$ are $\tilde{\lambda}_j := -\tilde{\kappa}_j^2$, the multiplicity $\tilde{m}_j$, the Gel'fand–Levitan normalization matrix $\tilde{C}_j$, the orthogonal projection $\tilde{Q}_j$ onto $\text{Ker}[\tilde{J}(i\tilde{\kappa}_j)]$, the orthogonal projection $\tilde{P}_j$ onto $\text{Ker}[\tilde{J}(i\tilde{\kappa}_j)^\dagger]$, and the Gel'fand–Levitan normalized bound-state solution $\tilde{\Phi}_j(x)$. We remark that the perturbed quantities are defined in the same way the unperturbed quantities are defined, but by using the relevant perturbed quantities as input to those definitions. Thus, the perturbed Schrödinger equation is given by

$$-\tilde{\psi}'' + \tilde{V}(x)\tilde{\psi} = k^2\tilde{\psi}, \quad x \in \mathbb{R}^+, \quad (4.2)$$

and the perturbed boundary condition is described by

$$-\tilde{B}^\dagger \tilde{\psi}(0) + \tilde{A}^\dagger \tilde{\psi}'(0) = 0,$$

where the constant $n \times n$ perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ satisfy

$$-\tilde{B}^\dagger \tilde{A} + \tilde{A}^\dagger \tilde{B} = 0, \quad (4.3)$$

$$\tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B} > 0. \quad (4.4)$$

The perturbed Jost solution $\tilde{f}(k, x)$ is the solution to (4.2) satisfying the spacial asymptotics

$$\tilde{f}(k, x) = e^{ikx}[I + o(1)], \quad \tilde{f}'(k, x) = e^{ikx}[ikI + o(1)], \quad x \rightarrow +\infty.$$

The perturbed Jost matrix $\tilde{J}(k)$ is defined as

$$\tilde{J}(k) := \tilde{f}(-k^*, 0)^\dagger \tilde{B} - \tilde{f}'(-k^*, 0)^\dagger \tilde{A}, \quad k \in \mathbb{R},$$

and the perturbed scattering matrix $\tilde{S}(k)$ is given by

$$\tilde{S}(k) := -\tilde{J}(-k)\tilde{J}(k)^{-1}, \quad k \in \mathbb{R}.$$

The perturbed Gel'fand–Levitan normalized bound-state solution at $k = i\tilde{\kappa}_j$ is defined as

$$\tilde{\Phi}_j(x) := \tilde{\varphi}(i\tilde{\kappa}_j, x)\tilde{C}_j, \quad 1 \leq j \leq \tilde{N},$$

and the perturbed spectral measure $d\tilde{\rho}$ is given by

$$d\tilde{\rho} := \begin{cases} \frac{\sqrt{\lambda}}{\pi} (\tilde{J}(\sqrt{\lambda})^\dagger \tilde{J}(\sqrt{\lambda}))^{-1} d\lambda, & \lambda \geq 0, \\ \sum_{j=1}^{\tilde{N}} \tilde{C}_j^2 \delta(\lambda - \tilde{\lambda}_j) d\lambda, & \lambda < 0. \end{cases} \quad (4.5)$$

The perturbed regular solution $\tilde{\varphi}(k, x)$ satisfies the perturbed Schrödinger equation (4.2) and also satisfies the perturbed initial conditions

$$\tilde{\varphi}(k, 0) = \tilde{A}, \quad \tilde{\varphi}'(k, 0) = \tilde{B}. \quad (4.6)$$

We assume that the unperturbed potential V and the perturbed potential $\tilde{V}$ both satisfy (2.2) and we also have $V \in L_1^1(\mathbb{R}^+)$ and $\tilde{V} \in L^1(\mathbb{R}^+)$. We suppose that the perturbation is initiated by the change $d\tilde{\rho} - d\rho$ in the spectral measure. We also assume that the perturbed regular solution $\tilde{\varphi}(k, x)$ is related to the unperturbed regular solution $\varphi(k, x)$ through an $n \times n$ matrix-valued quantity $\mathcal{A}(x, y)$ via the relation given by

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \int_0^x dy \mathcal{A}(x, y) \varphi(k, y) \quad (4.7)$$

and that the perturbed potential $\tilde{V}(x)$ is related to the unperturbed potential $V(x)$ as

$$\tilde{V}(x) = V(x) + 2 \frac{d\mathcal{A}(x, x)}{dx}, \quad (4.8)$$

where we use $\mathcal{A}(x, x)$ to denote $\mathcal{A}(x, x^-)$. By choosing $\mathcal{A}(x, y)$ appearing in (4.7) appropriately, we would like to relate the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ to the unperturbed boundary matrices A and B through the quantity $\mathcal{A}(x, y)$. We also would like to determine how all other relevant perturbed quantities are related to the corresponding unperturbed quantities. It turns out that the appropriate $\mathcal{A}(x, y)$ is determined as the solution to the Gel'fand–Levitan system of integral equations given by

$$\mathcal{A}(x, y) + G(x, y) + \int_0^x dz \mathcal{A}(x, z) G(z, y) = 0, \quad 0 \leq y < x, \quad (4.9)$$

where we have defined

$$G(x, y) := \int_{\lambda \in \mathbb{R}} \varphi(k, x) [d\tilde{\rho} - d\rho] \varphi(k, y)^\dagger, \quad (4.10)$$

with $d\rho$ and $d\tilde{\rho}$ denoting the unperturbed and perturbed spectral measures, respectively. and $\varphi(k, x)$ is the unperturbed regular solution satisfying (2.1) and the initial conditions given in (2.6). Then, the constant $n \times n$ matrices $\tilde{A}$ and $\tilde{B}$ appearing in (4.6) are related to the constant $n \times n$ matrices A and B appearing in (2.6) as

$$\tilde{A} = A, \quad \tilde{B} = B + \mathcal{A}(0, 0) A. \quad (4.11)$$

5 The removal of a bound state

In this section, we assume that the potential V satisfies (2.2) and at least (2.4). We describe the changes in the relevant quantities related to the Schrödinger operator associated with (2.1) and (2.7), when we perturb the spectrum of that operator by removing one of the bound states completely without changing the rest of the discrete spectrum and without changing the continuous spectrum. Without loss of generality, we remove the bound state at $k = i\kappa_N$ with the Gel'fand–Levitan normalization matrix C_N without changing the rest of the spectrum.

Toward our goal, we find it convenient to use the Gel'fand–Levitan system (4.9) and its solution $\mathcal{A}(x, y)$ by choosing the change in the spectral measure as

$$d\tilde{\rho} - d\rho = -C_N^2 \delta(\lambda - \lambda_N), \quad \lambda \in \mathbb{R}, \quad (5.1)$$

where we recall that $\lambda_N := -\kappa_N^2$. We remark that (5.1) is suggested by (4.1) and (4.5) by assuming that the Gel'fand–Levitan normalization matrices C_j for the remaining bound states at $k = i\kappa_j$ with $1 \leq j \leq N - 1$ do not change. In fact, it turns out that if we use (5.1) in the construction of the input matrix $G(x, y)$ defined in (4.10), solve the resulting Gel'fand–Levitan system (4.9), and use the corresponding solution $\mathcal{A}(x, y)$ to construct the perturbed quantities, then for $1 \leq j \leq N - 1$ each perturbed Gel'fand–Levitan normalization matrix $\tilde{C}_j$ coincides with the unperturbed Gel'fand–Levitan normalization matrix C_j .

Using (5.1) as input to (4.10), we obtain

$$G(x, y) = -\varphi(i\kappa_N, x) C_N^2 \varphi(i\kappa_N, y)^\dagger, \quad (5.2)$$

where we recall that $\varphi(k, x)$ is the unperturbed regular solution to (2.1) satisfying the initial conditions (2.6). Using (3.11) in (5.2), we express $G(x, y)$ in terms of the unperturbed Gel'fand–Levitan normalized bound-state solution $\Phi_N(x)$ as

$$G(x, y) = -\Phi_N(x) \Phi_N(y)^\dagger. \quad (5.3)$$

Since $G(x, y)$ given in (5.3) constitutes a separable kernel for (4.9), the solution $\mathcal{A}(x, y)$ to (4.9) is obtained by using the methods of linear algebra, and it is given by

$$\mathcal{A}(x, y) = \Phi_N(x) W_N(x)^\dagger \Phi_N(y)^\dagger, \quad 0 \leq y < x, \quad (5.4)$$

with $W_N(x)$ defined as

$$W_N(x) := \int_x^\infty dz \Phi_N(z)^\dagger \Phi_N(z), \quad (5.5)$$

where we recall that $W_N(x)^+$ denotes the Moore–Penrose inverse of $W_N(x)$. Using (5.4) in (4.7), we obtain the perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \Phi_N(x) W_N(x)^+ \int_0^x dy \Phi_N(y)^\dagger \varphi(k, y). \quad (5.6)$$

It is known [4] that (5.6) can be equivalently expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_N(x) W_N(x)^+ [\Phi_N'(x)^\dagger \varphi(k, x) - \Phi_N(x)^\dagger \varphi'(k, x)], \quad (5.7)$$

with the understanding that the term $(k^2 + \kappa_N^2)$ in the denominator causes only removable singularities. Using (5.4) in (4.8), we obtain the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) = V(x) + 2 \frac{d}{dx} [\Phi_N(x) W_N(x)^+ \Phi_N(x)^\dagger]. \quad (5.8)$$

Next, using (5.4) in (4.11), we express the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = A, \quad \tilde{B} = B + A C_N^2 A^\dagger. \quad (5.9)$$

In the removal of the bound state at $k = i\kappa_N$, we are interested in the asymptotic behavior of the potential increment $\tilde{V} - V$. For the details, we refer the reader to Theorem 6.3 of [4]. We recall that the unperturbed potential V satisfies (2.2) and (2.4), the boundary matrices A and B appearing in (2.7) satisfy (2.8) and (2.9), and the perturbed potential $\tilde{V}$ is given by (5.8). We have the following:

- (a) The perturbed potential $\tilde{V}$ also satisfies (2.2).
- (b) The potential increment $\tilde{V} - V$ has the asymptotic behavior

$$\tilde{V}(x) - V(x) = O\left(\int_x^\infty dy |V(y)|\right), \quad x \rightarrow +\infty. \quad (5.10)$$

- (c) If the unperturbed potential V is further restricted to $L_{1+\epsilon}^1(\mathbb{R}^+)$ for some fixed $\epsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L_\epsilon^1(\mathbb{R}^+)$. We recall that the potential class $L_\epsilon^1(\mathbb{R}^+)$ consists of potentials V satisfying (2.5).
- (d) If the unperturbed potential V is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (5.11)$$

for some positive constants α and x_0 and with c denoting a generic constant, then the potential increment $\tilde{V}(x) - V(x)$ also satisfies

$$|\tilde{V}(x) - V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0. \quad (5.12)$$

We recall that a generic constant c does not necessarily have the same value in different appearances. We remark that the estimates in (5.10)–(5.12) are not necessarily sharp.

- (e) If the support of the unperturbed potential V is contained in the interval $[0, x_0]$ for some positive x_0 , then the support of the perturbed potential $\tilde{V}$ is also contained in $[0, x_0]$.

We refer the reader to [4] for the construction of the remaining relevant quantities under the further sufficiency assumption that the unperturbed potential V belongs to $L^1_2(\mathbb{R}^+)$. We provide the results from [4] as follows. The perturbed Jost matrix $\tilde{J}(k)$ is related to the unperturbed Jost matrix $J(k)$ as

$$\tilde{J}(k) = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right] J(k), \quad k \in \overline{\mathbb{C}^+}, \quad (5.13)$$

and the determinant of the Jost matrix is changed as

$$\det[\tilde{J}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{m_N} \det[J(k)], \quad k \in \overline{\mathbb{C}^+}, \quad (5.14)$$

which indicates that the total number of bound states including the multiplicities is decreased by m_N . The perturbed scattering matrix $\tilde{S}(k)$ is related to the unperturbed scattering matrix $S(k)$ as

$$\tilde{S}(k) = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_N \right] S(k) \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_N \right], \quad k \in \mathbb{R}, \quad (5.15)$$

and the determinant of the scattering matrix is changed via

$$\det[\tilde{S}(k)] = \left(\frac{k - i\kappa_N}{k + i\kappa_N} \right)^{2m_N} \det[S(k)], \quad k \in \mathbb{R}. \quad (5.16)$$

The perturbed Jost solution $\tilde{f}(k, x)$ is related to the unperturbed Jost solution $f(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) - \Phi_N(x) W_N(x)^+ \int_x^\infty dy \Phi_N(y)^\dagger f(k, y) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right], \quad (5.17)$$

which can also be expressed in the alternate form as

$$\tilde{f}(k, x) = \left[f(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_N(x) W_N(x)^+ q_1(k, x) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_N \right], \quad (5.18)$$

where we have defined

$$q_1(k, x) := \Phi'_N(x)^\dagger f(k, x) - \Phi_N(x)^\dagger f'(k, x). \quad (5.19)$$

Finally, the orthogonal projection matrices Q_j and the Gel'fand–Levitan normalization matrices C_j for the remaining bound states do not change, i.e. we have

$$\tilde{Q}_j = Q_j, \quad \tilde{C}_j = C_j, \quad 1 \leq j \leq N-1.$$

6 Decreasing the multiplicity of a bound state

In this section, we assume that the potential V satisfies (2.2) and at least (2.4). We describe the changes in the relevant quantities related to the Schrödinger operator associated with (2.1) and (2.7), when we perturb the spectrum of our operator by reducing the multiplicity of a bound state without removing that bound state completely. We do not change the remaining discrete spectrum, and we do not change the continuous spectrum. We assume that the unperturbed operator has N bound states at $k = i\kappa_j$ with the multiplicity m_j and with the Gel'fand–Levitan normalization matrix C_j for $1 \leq j \leq N$. Without loss of generality, we assume that we reduce the multiplicity m_N of the bound state at $k = i\kappa_N$ without completely removing that bound state. Thus, it is implicitly assumed that m_N satisfies $2 \leq m_N \leq n$.

We assume that the multiplicity of the bound state at $k = i\kappa_N$ is reduced by m_{Nr} , and hence we define the positive integer representing the reduction in the multiplicity of the bound state at $k = i\kappa_N$ as

$$m_{Nr} := m_N - \tilde{m}_N. \quad (6.1)$$

The subscript Nr indicates that we refer to the N th bound state at $k = i\kappa_N$ and that we reduce its multiplicity. From (6.1), we see that m_{Nr} satisfies the inequality $1 \leq m_{Nr} \leq n-1$. The reduction of the multiplicity of the bound state at $k = i\kappa_N$ by m_{Nr} is achieved by introducing the matrix Q_{Nr} in such a way that Q_{Nr} is an orthogonal projection onto a proper subspace of $Q_N \mathbb{C}^n$, it has its rank equal to m_{Nr} , and it satisfies $Q_{Nr} \leq Q_N$. This last matrix inequality is equivalent to $Q_N - Q_{Nr} \geq 0$, which indicates that the matrix $Q_N - Q_{Nr}$ is a nonnegative matrix. In other words, the eigenvalues of the matrix $Q_N - Q_{Nr}$ are all real and nonnegative. For the construction of the matrix Q_{Nr} we refer the reader to [4].

When the multiplicity of the bound state at $k = i\kappa_N$ is reduced from m_N to $\tilde{m}_N$, the transformations of the relevant quantities are obtained as follows. For the details, we refer the reader to Sect. 7 of [4]. Since the matrix Q_{Nr} is an orthogonal projection onto a proper subspace of $Q_N \mathbb{C}^n$, it satisfies the matrix equalities

$$Q_{Nr} Q_N = Q_N Q_{Nr} = Q_{Nr}. \quad (6.2)$$

Since the columns of the matrix $\varphi(i\kappa_N, x) Q_N$ are square integrable in $x \in \mathbb{R}^+$, from (6.2) it follows that the columns of the matrix $\varphi(i\kappa_N, x) Q_{Nr}$ are also square integrable. Analogous to (3.14) and (3.15), we introduce the $n \times n$ matrices $\mathbf{G}_{Nr}$ and $\mathbf{H}_{Nr}$ as

$$\mathbf{G}_{N\Gamma} := \int_0^\infty dx \, Q_{N\Gamma} \varphi(i\kappa_N, x)^\dagger \varphi(i\kappa_N, x) Q_{N\Gamma}, \quad (6.3)$$

$$\mathbf{H}_{N\Gamma} := I - Q_{N\Gamma} + \mathbf{G}_{N\Gamma}. \quad (6.4)$$

One can prove that $\mathbf{H}_{N\Gamma}$ is positive, and hence it is invertible. Thus, we uniquely determine $\mathbf{H}_{N\Gamma}^{1/2}$ and its inverse $\mathbf{H}_{N\Gamma}^{-1/2}$ as the positive matrices satisfying the respective matrix equalities

$$\mathbf{H}_{N\Gamma}^{1/2} \mathbf{H}_{N\Gamma}^{1/2} = \mathbf{H}_{N\Gamma}, \quad \mathbf{H}_{N\Gamma}^{-1/2} \mathbf{H}_{N\Gamma}^{-1/2} = \mathbf{H}_{N\Gamma}^{-1}.$$

Analogous to (3.17), we introduce the $n \times n$ matrix $C_{N\Gamma}$ as

$$C_{N\Gamma} := \mathbf{H}_{N\Gamma}^{-1/2} Q_{N\Gamma}. \quad (6.5)$$

Inspired by (4.10), (5.1), and (5.2), with the help of $C_{N\Gamma}$ and the unperturbed regular solution $\varphi(k, x)$, we construct the quantity $G(x, y)$ as

$$G(x, y) := -\varphi(i\kappa_N, x) C_{N\Gamma}^2 \varphi(i\kappa_N, y)^\dagger. \quad (6.6)$$

In analogy with (3.11), we introduce the $n \times n$ matrix solution $\Phi_{N\Gamma}(x)$ to the Schrödinger equation (3.2) with $j = N$ as

$$\Phi_{N\Gamma}(x) := \varphi(i\kappa_N, x) C_{N\Gamma}. \quad (6.7)$$

Using (6.7) in (6.6), we write $G(x, y)$ in the equivalent form as

$$G(x, y) = -\Phi_{N\Gamma}(x) \Phi_{N\Gamma}(y)^\dagger. \quad (6.8)$$

The quantity $G(x, y)$ in (6.8) forms a separable kernel for the Gel'fand–Levitan system (4.9). Thus, we obtain the solution $\mathcal{A}(x, y)$ to (4.9) by using the methods of linear algebra, and we get

$$\mathcal{A}(x, y) = \Phi_{N\Gamma}(x) W_{N\Gamma}(x)^+ \Phi_{N\Gamma}(y)^\dagger, \quad 0 \leq y < x, \quad (6.9)$$

which is the analog of (5.4). We remark that the $n \times n$ matrix-valued quantity $W_{N\Gamma}(x)$ in (6.9) is defined as

$$W_{N\Gamma}(x) := \int_x^\infty dy \, \Phi_{N\Gamma}(y)^\dagger \Phi_{N\Gamma}(y), \quad (6.10)$$

which is the analog of (5.5). We recall that $W_{N\Gamma}(x)^+$ denotes the Moore–Penrose inverse of $W_{N\Gamma}(x)$. Using (6.9) in (4.8), the perturbed potential $\tilde{V}(x)$ is obtained as

$$\tilde{V}(x) = V(x) + 2 \frac{d}{dx} [\Phi_{Nr}(x) W_{Nr}(x)^+ \Phi_{Nr}(x)^\dagger], \quad (6.11)$$

which is the analog of (5.8). Using (6.9) in (4.7), we obtain the perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \Phi_{Nr}(x) W_{Nr}(x)^+ \int_0^x dy \Phi_{Nr}(y)^\dagger \varphi(k, y),$$

which is equivalently expressed as

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_{Nr}(x) W_{Nr}(x)^+ [\Phi_{Nr}'(x)^\dagger \varphi(k, x) - \Phi_{Nr}(x)^\dagger \varphi'(k, x)]. \quad (6.12)$$

We note that (6.12) is the analog of (5.7). Using (6.9) in (4.11), the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ are expressed in terms of the unperturbed boundary matrices A and B and the matrix C_{Nr} as

$$\tilde{A} = A, \quad \tilde{B} = B + A C_{Nr}^2 A^\dagger, \quad (6.13)$$

which is the analog of (5.9).

In decreasing the multiplicity of the bound state at $k = i\kappa_N$, it is relevant to estimate the asymptotic behavior of the potential increment $\tilde{V} - V$. For the details, we refer the reader to Sect. 7 of [4]. We recall that the unperturbed potential V satisfies (2.2) and (2.4), the boundary matrices A and B appearing in (2.7) satisfy (2.8) and (2.9), and the perturbed potential $\tilde{V}$ is given by (6.11). We have the following:

- (a) The perturbed potential $\tilde{V}$ also satisfies (2.2).
- (b) The asymptotics for the potential increment $\tilde{V} - V$ is given by

$$\tilde{V}(x) - V(x) = O\left(\int_x^\infty dy |V(y)|\right), \quad x \rightarrow +\infty, \quad (6.14)$$

which is the analog of (5.10). The asymptotic estimate in (6.14) is not necessarily sharp.

- (c) If the unperturbed potential V belongs to $L_{1+\epsilon}^1(\mathbb{R}^+)$ for some fixed $\epsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L_\epsilon^1(\mathbb{R}^+)$.
- (d) We have the analogs of (5.11) and (5.12). In other words, if the unperturbed potential V is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (6.15)$$

for some positive constants α and x_0 , then the potential increment $\tilde{V} - V$ satisfies

$$|\tilde{V}(x) - V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0. \quad (6.16)$$

Here, c denotes a generic constant not necessarily taking the same value in different appearances. We remark that (6.15) and (6.16) are the analogs of (5.11) and (5.12), respectively. We note that the estimates in (6.14)–(6.16) are not necessarily sharp.

- (e) If the support of V is contained in the interval $[0, x_0]$, then the support of $\tilde{V}$ is also contained in $[0, x_0]$.

By further using the sufficiency assumption that V belongs to $L_2^1(\mathbb{R}^+)$, we determine [4] the transformations of the remaining relevant quantities. The unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_{N_r} \right] J(k), \quad k \in \overline{\mathbb{C}^+}, \quad (6.17)$$

which is the analog of (5.13). The construction of the orthogonal projection matrix P_{N_r} is accomplished as follows. Using the already known orthogonal projection matrix Q_{N_r} , we determine an orthonormal basis $\{w_N^{(l)}\}_{l=1}^{m_{N_r}}$ for the subspace $Q_{N_r} \mathbb{C}^n$. For each unit vector $w_N^{(l)}$, we uniquely determine the vector $\beta_N^{(l)}$ satisfying

$$\varphi(i\kappa_N, x) w_N^{(l)} = f(i\kappa_N, x) \beta_N^{(l)}, \quad 1 \leq l \leq m_{N_r}. \quad (6.18)$$

We then obtain P_{N_r} as the orthogonal projection onto the subspace generated by the set $\{\beta_N^{(l)}\}_{l=1}^{m_{N_r}}$. We note that the set $\{\beta_N^{(l)}\}_{l=1}^{m_{N_r}}$ is in general not an orthogonal set and the vectors in that set are in general not unit vectors. It is ensured [4] that the m_{N_r} vectors in the set are linearly independent. For the details, we refer the reader to [4].

The determinant $\det[J(k)]$ of the unperturbed Jost matrix is transformed as

$$\det[\tilde{J}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{m_{N_r}} \det[J(k)], \quad k \in \overline{\mathbb{C}^+},$$

which is the analog of (5.14). We recall that m_{N_r} corresponds to the rank of the orthogonal projection Q_{N_r} . The scattering matrix $S(k)$ undergoes the transformation

$$\tilde{S}(k) = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_{N_r} \right] S(k) \left[I - \frac{2i\kappa_N}{k + i\kappa_N} P_{N_r} \right], \quad k \in \mathbb{R},$$

which is the analog of (5.15). The determinant $\det[S(k)]$ of the scattering matrix $S(k)$ is transformed as

$$\det[\tilde{S}(k)] = \left(\frac{k - i\kappa_N}{k + i\kappa_N} \right)^{2m_{N_r}} \det[S(k)], \quad k \in \mathbb{R},$$

which is the analog of (5.16). The unperturbed Jost solution $f(k, x)$ is transformed into the perturbed Jost solution $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) + \frac{1}{k^2 + \kappa_N^2} \Phi_{Nr}(x) W_{Nr}(x)^+ q_2(k, x) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_{Nr} \right], \quad (6.19)$$

which is the analog of (5.18). The quantity $q_2(k, x)$ in (6.19) is defined as

$$q_2(k, x) := \Phi'_{Nr}(x)^\dagger f(k, x) - \Phi_{Nr}(x)^\dagger f'(k, x),$$

which is the analog of (5.19). We can express (6.19) equivalently as

$$\tilde{f}(k, x) = \left[f(k, x) - \Phi_{Nr}(x) W_{Nr}(x)^+ \int_x^\infty dy \Phi_{Nr}(y)^\dagger f(k, y) \right] \left[I + \frac{2i\kappa_N}{k - i\kappa_N} P_{Nr} \right],$$

which is the analog of (5.17). Under the perturbation specified in (6.2), the orthogonal projection matrices Q_j and the Gel'fand–Levitan normalization matrices C_j for the remaining bound states do not change, i.e. we have

$$\tilde{Q}_j = Q_j, \quad \tilde{C}_j = C_j, \quad 1 \leq j \leq N-1.$$

7 The addition of a bound state

In this section, we assume that the potential V satisfies (2.2) and at least (2.4). We provide a description of the changes in the relevant quantities related to the Schrödinger operator associated with (2.1) and (2.7), when we perturb the spectrum of our operator by adding a new bound state without changing the rest of the discrete spectrum and without changing the continuous spectrum. We assume that the unperturbed Schrödinger operator has N bound states occurring at $k = i\kappa_j$ with the multiplicity m_j and with the Gel'fand–Levitan normalization matrix C_j for $1 \leq j \leq N$. We recall that N may be equal to zero, in which case there are no bound states. We add a new bound state at $k = i\tilde{\kappa}_{N+1}$ with the multiplicity $\tilde{m}_{N+1}$ and the Gel'fand–Levitan normalization matrix $\tilde{C}_{N+1}$, where the positive constant $\tilde{\kappa}_{N+1}$ is distinct from κ_j for $1 \leq j \leq N$. We remark that we cannot choose $\tilde{C}_{N+1}$ in an arbitrary manner. For example, it is necessary that $\tilde{C}_{N+1}$ is an $n \times n$ self-adjoint nonnegative matrix. The choice of $\tilde{C}_{N+1}$ must be compatible with the construction process given in (3.14)–(3.17) starting with the perturbed regular solution $\tilde{\varphi}(k, x)$ and the orthogonal projection matrix $\tilde{Q}_{N+1}$ onto the kernel of the perturbed Jost matrix $\tilde{J}(i\tilde{\kappa}_{N+1})$. We illustrate this compatibility issue in Example 10.4, where we verify that the choice of $\tilde{C}_{N+1}$ used in that example indeed agrees with the construction described in (3.14)–(3.17).

We construct $\tilde{C}_{N+1}$ as follows. We first choose an $n \times n$ orthogonal projection matrix $\tilde{Q}_{N+1}$ having the rank $\tilde{m}_{N+1}$. Next, we construct an $n \times n$ nonnegative Hermitian matrix $\tilde{G}_{N+1}$ satisfying the equalities

$$\tilde{G}_{N+1} \tilde{Q}_{N+1} = \tilde{Q}_{N+1} \tilde{G}_{N+1} = \tilde{G}_{N+1}, \quad (7.1)$$

in such a way that the restriction of $\tilde{\mathbf{G}}_{N+1}$ to $\tilde{Q}_{N+1} \mathbb{C}^n$ is invertible. We then define the matrix $\tilde{\mathbf{H}}_{N+1}$ as

$$\tilde{\mathbf{H}}_{N+1} := I - \tilde{Q}_{N+1} + \tilde{\mathbf{G}}_{N+1}. \quad (7.2)$$

From (7.2), we observe that $\tilde{\mathbf{H}}_{N+1}$ is Hermitian and nonnegative. It turns out [4] that $\tilde{\mathbf{H}}_{N+1}$ is positive and hence invertible. As in (3.16), the positivity of $\tilde{\mathbf{H}}_{N+1}$ allows the unique construction of the positive matrix $\tilde{\mathbf{H}}_{N+1}^{1/2}$ and in turn the construction of $\tilde{\mathbf{H}}_{N+1}^{-1/2}$. We then use the analog of (3.17) and obtain $\tilde{C}_{N+1}$ as

$$\tilde{C}_{N+1} := \tilde{\mathbf{H}}_{N+1}^{-1/2} \tilde{Q}_{N+1}. \quad (7.3)$$

We determine [4] that each of $\tilde{\mathbf{H}}_{N+1}$, $\tilde{\mathbf{H}}_{N+1}^{1/2}$, and $\tilde{\mathbf{H}}_{N+1}^{-1/2}$ commutes with $\tilde{Q}_{N+1}$. Consequently, from (7.3) we get

$$\tilde{C}_{N+1} \tilde{Q}_{N+1} = \tilde{Q}_{N+1} \tilde{C}_{N+1} = \tilde{C}_{N+1}.$$

It is convenient to use the Gel'fand–Levitan method to determine the resulting perturbation of the relevant quantities. We use the Gel'fand–Levitan system (4.9) and its solution $\mathcal{A}(x, y)$ to construct those relevant quantities. Toward our goal, we choose the change in the spectral measure as

$$d\tilde{\rho} - d\rho = \tilde{C}_{N+1}^2 \delta(\lambda - \tilde{\lambda}_{N+1}), \quad \lambda \in \mathbb{R}, \quad (7.4)$$

where we have let $\tilde{\lambda}_{N+1} := -\tilde{\kappa}_{N+1}^2$. Note that (7.4) is compatible with (4.1) and (4.5) by assuming that the Gel'fand–Levitan normalization matrices C_j for the existing bound states at $k = i\kappa_j$ with $1 \leq j \leq N$ do not change. It turns out that if we use (7.4) in the construction of the input matrix $G(x, y)$ appearing in (4.10), solve the resulting Gel'fand–Levitan system (4.9), and use the corresponding solution $\mathcal{A}(x, y)$ to construct the perturbed quantities, then for $1 \leq j \leq N$ each perturbed Gel'fand–Levitan normalization matrix $\tilde{C}_j$ coincides with the unperturbed Gel'fand–Levitan normalization matrix C_j , and the continuous part of the unperturbed spectral measure $d\rho$ remains unchanged.

Using (7.4) as input to (4.10), we get

$$G(x, y) = \varphi(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1}^2 \varphi(i\tilde{\kappa}_{N+1}, y)^\dagger, \quad (7.5)$$

where we recall that $\varphi(k, x)$ is the unperturbed regular solution to (2.1) satisfying the initial conditions (2.6). With the help of $\varphi(k, x)$ and $\tilde{C}_{N+1}$, we define the $n \times n$ matrix $\xi_{N+1}(x)$ as

$$\xi_{N+1}(x) := \varphi(i\tilde{\kappa}_{N+1}, x) \tilde{C}_{N+1}. \quad (7.6)$$

Using (7.6) in (7.5), we express $G(x, y)$ in the equivalent form as

$$G(x, y) = \xi_{N+1}(x) \xi_{N+1}(y)^\dagger. \quad (7.7)$$

We note that $G(x, y)$ forms a separable kernel for the Gel'fand–Levitan system (4.9), and hence the solution $\mathcal{A}(x, y)$ to (4.9) can be obtained by using the methods from linear algebra. We get $\mathcal{A}(x, y)$ explicitly as

$$\mathcal{A}(x, y) = -\xi_{N+1}(x) \Omega_{N+1}(x)^+ \xi_{N+1}(y)^\dagger, \quad 0 \leq y < x, \quad (7.8)$$

with the $n \times n$ matrix $\Omega_{N+1}(x)$ defined as

$$\Omega_{N+1}(x) := \bar{Q}_{N+1} + \int_0^x dy \xi_{N+1}(y)^\dagger \xi_{N+1}(y), \quad (7.9)$$

where we recall that $\Omega_{N+1}(x)^+$ denotes the Moore–Penrose inverse of $\Omega_{N+1}(x)$. Using (7.8) in (4.8), we obtain the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) = V(x) - 2 \frac{d}{dx} [\xi_{N+1}(x) \Omega_{N+1}(x)^+ \xi_{N+1}(x)^\dagger]. \quad (7.10)$$

Using (7.8) in (4.7), we get the perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \xi_{N+1}(x) \Omega_{N+1}(x)^+ \int_0^x dy \xi_{N+1}(y)^\dagger \varphi(k, y), \quad (7.11)$$

which can also be written in the equivalent form as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \frac{1}{k^2 + \tilde{\kappa}_{N+1}^2} \xi_{N+1}(x) \Omega_{N+1}(x)^+ [\xi'_{N+1}(x)^\dagger \varphi(k, x) - \xi_{N+1}(x)^\dagger \varphi'(k, x)]. \quad (7.12)$$

Next, using (7.8) in (4.11), we express the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ in terms of the unperturbed boundary matrices A and B as

$$\tilde{A} = A, \quad \tilde{B} = B - A \tilde{C}_{N+1}^2 A^\dagger. \quad (7.13)$$

In adding a new bound state at $k = i\tilde{\kappa}_{N+1}$ to the spectrum, we are interested in estimating the asymptotic behavior of the potential increment $\tilde{V} - V$. For the details we refer the reader to Theorem 8.5 of [4]. We recall that the unperturbed potential V satisfies (2.2) and (2.4), the boundary matrices A and B appearing in (2.7) satisfy (2.8) and (2.9), and the perturbed potential $\tilde{V}$ is given by (7.10). We have the following:

- (a) The perturbed potential $\tilde{V}$ also satisfies (2.2).
- (b) For every nonnegative constant a and every parameter ε satisfying $0 < \varepsilon < 1$, the potential increment $\tilde{V} - V$ has the asymptotic behavior

$$\tilde{V}(x) - V(x) = O(q_3(a, \varepsilon, x)), \quad x \rightarrow +\infty, \quad (7.14)$$

with the scalar quantity $q_3(a, \varepsilon, x)$ defined as

$$q_3(a, \varepsilon, x) := x e^{-2\tilde{\kappa}_{N+1}x} + \int_x^\infty dy |V(y)| + e^{-2\varepsilon\tilde{\kappa}_{N+1}x} \int_a^{(1-\varepsilon)x} dy |V(y)| \\ + \int_{(1-\varepsilon)x}^x dy e^{-2\tilde{\kappa}_{N+1}(x-y)} |V(y)|. \quad (7.15)$$

- (c) If the unperturbed potential V is further restricted to $L^1_{1+\varepsilon}(\mathbb{R}^+)$ for some fixed $\varepsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L^1_\varepsilon(\mathbb{R}^+)$.
 (d) If the unperturbed potential V is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (7.16)$$

for some positive constants α and x_0 and with c denoting a generic constant, then for every $0 < \varepsilon < 1$ the potential increment $\tilde{V} - V$ has the asymptotic behavior as $x \rightarrow +\infty$ given by

$$\tilde{V}(x) - V(x) = \begin{cases} O(e^{-\alpha x} + e^{-2\varepsilon\tilde{\kappa}_{N+1}x}), & \alpha \leq 2\tilde{\kappa}_{N+1}, \\ O(e^{-2\varepsilon\tilde{\kappa}_{N+1}x}), & \alpha > 2\tilde{\kappa}_{N+1}. \end{cases} \quad (7.17)$$

- (e) If the unperturbed potential V has compact support, then the perturbed potential $\tilde{V}$ has the asymptotic behavior

$$\tilde{V}(x) = O(x e^{-2\tilde{\kappa}_{N+1}x}), \quad x \rightarrow +\infty. \quad (7.18)$$

We remark that the estimates given in (7.14)–(7.18) are not necessarily sharp.

We refer the reader to [4] for the construction of the remaining relevant quantities under the further sufficiency assumption that the unperturbed potential V belongs to $L^1_2(\mathbb{R}^+)$. We provide the results from [4] as follows. The unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right] J(k), \quad k \in \overline{\mathbb{C}^+}. \quad (7.19)$$

The orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})$ coincides [4] with the orthogonal projection matrix $\tilde{Q}_{N+1}$ appearing in (7.1)–(7.3) and used in the definition of the Gel'fand–Levitan normalization matrix $\tilde{C}_{N+1}$. We note that the $n \times n$ matrix $\tilde{P}_{N+1}$ appearing in (7.19) is the orthogonal projection onto the kernel of $\tilde{J}(i\tilde{\kappa}_{N+1})^\dagger$, and it has rank $\tilde{m}_{N+1}$. We recall that $\tilde{m}_{N+1}$ is the rank of $\tilde{C}_{N+1}$. We have

$$\tilde{P}_{N+1} := L_{N+1} \tilde{C}_{N+1} [\tilde{C}_{N+1} L_{N+1}^\dagger L_{N+1} \tilde{C}_{N+1}]^\dagger \tilde{C}_{N+1} L_{N+1}^\dagger. \quad (7.20)$$

The constant $n \times n$ matrix L_{N+1} appearing in (7.20) is invertible, and it is obtained by using the corresponding coefficient matrix in the expansion

$$\varphi(i\tilde{\kappa}_{N+1}, x) = f(i\tilde{\kappa}_{N+1}, x) K_{N+1} + g(i\tilde{\kappa}_{N+1}, x) L_{N+1}. \quad (7.21)$$

We note that the quantity $f(k, x)$ in (7.21) is the Jost solution to (2.1) appearing in (2.10). The quantity $g(k, x)$ in (7.21) is the $n \times n$ matrix-valued solution to (2.1) constructed for $k \in \mathbb{C}^+ \setminus \{0\}$ and satisfying the spacial asymptotics

$$g(k, x) = e^{-ikx} [I + o(1)], \quad g'(k, x) = -e^{-ikx} [ikI + o(1)], \quad x \rightarrow +\infty. \quad (7.22)$$

For more information on $g(k, x)$, we refer the reader to [3] and [4].

Under the perturbation specified in (7.4), the determinant of the Jost matrix $J(k)$ is transformed as

$$\det[\tilde{J}(k)] = \left(\frac{k - i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \right)^{\tilde{m}_{N+1}} \det[J(k)], \quad k \in \overline{\mathbb{C}^+}, \quad (7.23)$$

from which we observe that the total number of bound states including the multiplicities is increased by $\tilde{m}_{N+1}$. The scattering matrix $S(k)$ is transformed into $\tilde{S}(k)$ as

$$\tilde{S}(k) = \left[I + \frac{2i\tilde{\kappa}_{N+1}}{k - i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right] S(k) \left[I + \frac{2i\tilde{\kappa}_N}{k - i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right], \quad k \in \mathbb{R}, \quad (7.24)$$

where we recall that $\tilde{P}_{N+1}$ is the matrix defined in (7.20). The determinant of the scattering matrix is transformed as

$$\det[\tilde{S}(k)] = \left(\frac{k + i\tilde{\kappa}_{N+1}}{k - i\tilde{\kappa}_{N+1}} \right)^{2\tilde{m}_{N+1}} \det[S(k)], \quad k \in \mathbb{R}. \quad (7.25)$$

The Jost solution $f(k, x)$ is transformed into $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) = \left[f(k, x) - \frac{1}{k^2 + \tilde{\kappa}_{N+1}^2} \xi_{N+1}(x) \Omega_{N+1}(x)^+ q_4(k, x) \right] \left[I - \frac{2i\tilde{\kappa}_{N+1}}{k + i\tilde{\kappa}_{N+1}} \tilde{P}_{N+1} \right], \quad (7.26)$$

where we have defined $q_4(k, x)$ as

$$q_4(k, x) := [\xi'_{N+1}(x)]^\dagger f(k, x) - \xi_{N+1}(x)^\dagger f'(k, x).$$

Furthermore, under the perturbation specified in (7.4), the existing orthogonal projections Q_j and the existing Gel'fand–Levitan normalization matrices C_j remain unchanged, i.e. we have

$$\tilde{Q}_j = Q_j, \quad \tilde{C}_j = C_j, \quad 1 \leq j \leq N.$$

8 Increasing the multiplicity of a bound state

In this section we assume that the potential V satisfies (2.2) and at least (2.4). We describe the changes in the relevant quantities related to the Schrödinger operator associated with (2.1) and (2.7), when we perturb the spectrum of our operator by increasing the multiplicity of a bound state. We do not change the remaining part of the discrete spectrum and we do not change the continuous spectrum. We assume that the unperturbed operator has N bound states at $k = ik_j$ with the multiplicity m_j and with the Gel'fand–Levitan normalization matrix C_j for $1 \leq j \leq N$. Without loss of generality, we increase the multiplicity of the bound state at $k = ik_N$ from m_N to $\tilde{m}_N$ by the positive integer m_{Ni} , which is defined as

$$m_{Ni} := \tilde{m}_N - m_N. \quad (8.1)$$

By using the subscript Ni in (8.1), we indicate that we refer to the N th bound state at $k = ik_N$ and that we increase its multiplicity. Since the multiplicity of a bound state cannot exceed n , it is implicitly assumed that we have $1 \leq m_N \leq n - 1$.

Changing the multiplicity of a bound state does not apply in the scalar case, i.e. when $n = 1$, because in the scalar case all bound states have the multiplicity equal to one. Hence, we assume that we are in the matrix case with $n \geq 2$ and that there exists at least one bound state, i.e. we have $N \geq 1$. As in the previous sections, we use a tilde to identify the perturbed quantities obtained after the multiplicity of the bound state is increased. We assume that our unperturbed potential V satisfies (2.2) and (2.4). In our notation, our unperturbed regular solution is $\varphi(k, x)$, unperturbed Jost solution is $f(k, x)$, unperturbed Jost matrix is $J(k)$, unperturbed scattering matrix is $S(k)$, unperturbed matrix Q_j corresponds to the orthogonal projection onto the kernel of $J(ik_j)$ for $1 \leq j \leq N$, and unperturbed boundary matrices are given by A and B . Our perturbed potential is $\tilde{V}(x)$, perturbed regular solution is $\tilde{\varphi}(k, x)$, perturbed Jost solution is $\tilde{f}(k, x)$, perturbed Jost matrix is $\tilde{J}(k)$, perturbed scattering matrix is $\tilde{S}(k)$, and perturbed boundary matrices are given by $\tilde{A}$ and $\tilde{B}$.

We recall that Q_N denotes the orthogonal projection onto the kernel of $J(ik_N)$, the positive integer m_N corresponds to the dimension of $\text{Ker}[J(ik_N)]$, and we use C_N to denote the Gel'fand–Levitan normalization matrix associated with the bound state at $k = ik_N$ in the unperturbed case. The increase of the multiplicity of the bound state at $k = ik_N$ by m_{Ni} is accomplished as follows. From (8.1), we see that $1 \leq m_{Ni} \leq n - m_N$. When the multiplicity of the bound state at $k = ik_N$ is increased from m_N to $\tilde{m}_N$, the transformations of the relevant quantities are obtained as follows. For the details we refer the reader to Sect. 9 of [4]. We introduce the $n \times n$ matrix $\tilde{Q}_{Ni}$ with the rank m_{Ni} in order to denote the orthogonal projection onto a subspace of the orthogonal complement in $\mathbb{C}^n$ of $Q_N \mathbb{C}^n$. Hence, we have

$$\tilde{Q}_{Ni} \mathbb{C}^n \subset (Q_N \mathbb{C}^n)^\perp. \quad (8.2)$$

We use an $n \times n$ nonnegative Hermitian matrix $\tilde{\mathbf{G}}_{N_i}$ satisfying the equalities

$$\tilde{\mathbf{G}}_{N_i} \tilde{Q}_{N_i} = \tilde{Q}_{N_i} \tilde{\mathbf{G}}_{N_i} = \tilde{\mathbf{G}}_{N_i}. \quad (8.3)$$

We further assume that the restriction of $\tilde{\mathbf{G}}_{N_i}$ to the subspace $\tilde{Q}_{N_i} \mathbb{C}^n$ is invertible. With the help of the matrices $\tilde{Q}_{N_i}$ and $\tilde{\mathbf{G}}_{N_i}$, we define the $n \times n$ matrix $\tilde{\mathbf{H}}_{N_i}$ as

$$\tilde{\mathbf{H}}_{N_i} := I - \tilde{Q}_{N_i} + \tilde{\mathbf{G}}_{N_i}, \quad (8.4)$$

which is analogous to (3.15). Since $\tilde{Q}_{N_i}$ and $\tilde{\mathbf{G}}_{N_i}$ are Hermitian, from (8.4) it follows that $\tilde{\mathbf{H}}_{N_i}$ is also Hermitian. With the help of (8.3) and (8.4) we observe that

$$\tilde{\mathbf{H}}_{N_i} \tilde{Q}_{N_i} = \tilde{Q}_{N_i} \tilde{\mathbf{H}}_{N_i}. \quad (8.5)$$

It is known [4] that $\tilde{\mathbf{H}}_{N_i}$ is a positive matrix, and hence its inverse $\tilde{\mathbf{H}}_{N_i}^{-1}$ is uniquely determined. Thus, the positive matrix $\tilde{\mathbf{H}}_{N_i}^{1/2}$ and its inverse $\tilde{\mathbf{H}}_{N_i}^{-1/2}$ are also uniquely determined, and they satisfy

$$\tilde{\mathbf{H}}_{N_i}^{1/2} \tilde{\mathbf{H}}_{N_i}^{1/2} = \tilde{\mathbf{H}}_{N_i}, \quad \tilde{\mathbf{H}}_{N_i}^{-1/2} \tilde{\mathbf{H}}_{N_i}^{-1/2} = \tilde{\mathbf{H}}_{N_i}^{-1}. \quad (8.6)$$

Analogous to (3.17), we define the $n \times n$ matrix $\tilde{C}_{N_i}$ as

$$\tilde{C}_{N_i} := \tilde{\mathbf{H}}_{N_i}^{-1/2} \tilde{Q}_{N_i}. \quad (8.7)$$

It follows that $\tilde{C}_{N_i}$ is Hermitian and nonnegative, its rank is equal to m_{N_i} , and it satisfies the equalities

$$\tilde{C}_{N_i} \tilde{Q}_{N_i} = \tilde{Q}_{N_i} \tilde{C}_{N_i} = \tilde{C}_{N_i}. \quad (8.8)$$

With the help of the unperturbed regular solution $\varphi(k, x)$ and the matrix $\tilde{C}_{N_i}$, we introduce the $n \times n$ matrix solution $\xi_{N_i}(x)$ to the Schrödinger equation (3.2) with $j = N$ as

$$\xi_{N_i}(x) := \varphi(i\kappa_N, x) \tilde{C}_{N_i}, \quad (8.9)$$

which is the analog of (7.6). Using (8.9) we construct the kernel $G(x, y)$ of the Gel'fand–Levitan system (4.9) as

$$G(x, y) = \xi_{N_i}(x) \xi_{N_i}(y)^\dagger, \quad (8.10)$$

which is the analog of (7.7). The quantity $G(x, y)$ in (8.10) forms a separable kernel for the Gel'fand–Levitan system (4.9). By using the methods of linear algebra, we obtain the solution $\mathcal{A}(x, y)$ to (4.9) as

$$\mathcal{A}(x, y) = -\xi_{N_i}(x) \Omega_{N_i}(x)^\dagger \xi_{N_i}(y)^\dagger, \quad 0 \leq y < x, \quad (8.11)$$

where the $n \times n$ matrix $\Omega_{N_i}(x)$ is defined as

$$\Omega_{N_i}(x) := \tilde{Q}_{N_i} + \int_0^x dy \, \xi_{N_i}(y)^\dagger \xi_{N_i}(y), \quad (8.12)$$

and we recall that $\Omega_{N_i}(x)^+$ denotes the Moore–Penrose inverse of $\Omega_{N_i}(x)$. We note that (8.11) is the analog of (7.8). Using (8.11) in (4.8), we obtain the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) := V(x) - 2 \frac{d}{dx} [\xi_{N_i}(x) \Omega_{N_i}(x)^+ \xi_{N_i}(x)^\dagger], \quad (8.13)$$

which is the analog of (7.10). Using (8.11) in (4.7), we obtain the perturbed regular solution $\tilde{\varphi}(k, x)$ in terms of the unperturbed regular solution $\varphi(k, x)$ as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \xi_{N_i}(x) \Omega_{N_i}(x)^+ \int_0^x dy \, \xi_{N_i}(y)^\dagger \varphi(k, y),$$

which is the analog of (7.11). In analogy with (7.12), we can express the perturbed regular solution also as

$$\tilde{\varphi}(k, x) = \varphi(k, x) - \frac{1}{k^2 + \kappa_N^2} \xi_{N_i}(x) \Omega_{N_i}(x)^+ [\xi'_{N_i}(x)^\dagger \varphi(k, x) - \xi_{N_i}(x)^\dagger \varphi'(k, x)], \quad (8.14)$$

where the singularities caused by the term $(k^2 + \kappa_N^2)$ in the denominator on the right-hand side of (8.14) are both removable, and hence the value of the right-hand side of (8.14) is understood to hold in the limiting sense at $k = \pm i\kappa_N$. The perturbed quantity $\tilde{\varphi}(k, x)$ satisfies the initial conditions (4.6), where the matrices $\tilde{A}$ and $\tilde{B}$ are expressed in terms of the unperturbed boundary matrices A and B and the nonnegative matrix $\tilde{C}_{N_i}$ appearing in (8.7) as

$$\tilde{A} = A, \quad \tilde{B} = B - A \tilde{C}_{N_i}^2 A^\dagger, \quad (8.15)$$

which is the analog of (7.13). The matrices $\tilde{A}$ and $\tilde{B}$ appearing in (8.15) indeed satisfy (4.3) and (4.4), which are the counterparts of (2.8) and (2.9), respectively, satisfied by the unperturbed boundary matrices A and B .

Analogous to (7.21), we express the matrix solution $\varphi(i\kappa_N, x) \tilde{Q}_{N_i}$ to the Schrödinger equation (2.1) with $k = i\kappa_N$ as

$$\varphi(i\kappa_N, x) \tilde{Q}_{N_i} = f(i\kappa_N, x) K_{N_i} \tilde{Q}_{N_i} + g(i\kappa_N, x) L_{N_i} \tilde{Q}_{N_i}, \quad (8.16)$$

for some $n \times n$ matrices K_{N_i} and L_{N_i} , where we recall that $g(k, x)$ is the matrix-valued solution to (2.1) appearing in (7.22). As shown in [4], the restriction of L_{N_i} to the subspace $\tilde{Q}_{N_i} \mathbb{C}^n$ is invertible.

In increasing the multiplicity of the bound state at $k = i\kappa_N$, it is relevant to estimate the asymptotic behavior of the potential increment $\tilde{V} - V$. For the details, we refer the reader to Sect. 9 of [4]. We recall that the unperturbed potential V satisfies (2.2) and (2.4), the boundary matrices A and B appearing in (2.7) satisfy (2.8) and (2.9), and the perturbed potential $\tilde{V}$ is given by (8.13). We have the following:

- (a) The perturbed potential $\tilde{V}$ also satisfies (2.2).
- (b) For any a and ε with $a \geq 0$ and $0 < \varepsilon < 1$, the corresponding potential increment $\tilde{V} - V$ has the asymptotic behavior

$$\tilde{V}(x) - V(x) = O(q_5(x, a, \varepsilon)), \quad x \rightarrow +\infty, \quad (8.17)$$

where we have defined the scalar quantity $q_5(x, a, \varepsilon)$ as

$$q_5(x, a, \varepsilon) := x e^{-2\kappa_N x} + \int_x^\infty dy |V(y)| + e^{-2\varepsilon\kappa_N x} \int_a^{(1-\varepsilon)x} dy |V(y)| + \int_{(1-\varepsilon)x}^x dy e^{-2\kappa_N(x-y)} |V(y)|. \quad (8.18)$$

- (c) If the unperturbed potential V belongs $L_{1+\epsilon}^1(\mathbb{R}^+)$ for some fixed $\epsilon \geq 0$, then the perturbed potential $\tilde{V}$ belongs to $L_\epsilon^1(\mathbb{R}^+)$.
- (d) Assume that the unperturbed potential V is further restricted to satisfy

$$|V(x)| \leq c e^{-\alpha x}, \quad x \geq x_0, \quad (8.19)$$

for some positive constants α and x_0 , where c denotes a generic constant. Then, for every $0 < \varepsilon < 1$, the potential increment $\tilde{V} - V$ has the asymptotic behavior as $x \rightarrow +\infty$ given by

$$\tilde{V}(x) - V(x) = \begin{cases} O(e^{-\alpha x} + e^{-2\varepsilon\kappa_N x}), & \alpha \leq 2\kappa_N, \\ O(e^{-2\varepsilon\kappa_N x}), & \alpha > 2\kappa_N. \end{cases} \quad (8.20)$$

- (e) If the unperturbed potential V has compact support, then the perturbed potential $\tilde{V}$ has the asymptotic behavior

$$\tilde{V}(x) = O(x e^{-2\kappa_N x}), \quad x \rightarrow +\infty. \quad (8.21)$$

We remark that the estimates given in (8.17)–(8.21) are not necessarily sharp.

We refer the reader to [4] for the construction of the remaining relevant quantities under the further sufficiency assumption that the unperturbed potential V belongs to $L_2^1(\mathbb{R}^+)$. We provide the results from [4] as follows. In analogy with (7.19), the unperturbed Jost matrix $J(k)$ is transformed into the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \left[I - \frac{2i\kappa_N}{k + i\kappa_N} \tilde{P}_{Ni} \right] J(k), \quad k \in \overline{\mathbb{C}^+}, \quad (8.22)$$

where we have defined

$$\tilde{P}_{N_1} := L_{N_1} \tilde{C}_{N_1} (\tilde{C}_{N_1} L_{N_1}^\dagger L_{N_1} \tilde{C}_{N_1})^\dagger \tilde{C}_{N_1} L_{N_1}^\dagger, \quad (8.23)$$

with L_{N_1} being the $n \times n$ matrix appearing in (8.16). The orthogonal projection $\tilde{Q}_{N_1}$ appearing in (8.2) projects into a subspace of dimension m_{N_1} of the kernel of $\tilde{J}(i\kappa_N)$. Moreover, $\tilde{P}_{N_1}$ is an orthogonal projection with the rank m_{N_1} , where we recall that m_{N_1} is also equal to the rank of the orthogonal projection $\tilde{Q}_{N_1}$ appearing in (8.2). The quantity $\tilde{P}_{N_1}$ projects onto a subspace of the kernel of $\tilde{J}(i\kappa_N)^\dagger$, where that subspace has dimension m_{N_1} . With the help of (8.8), we can write (8.23) as

$$\tilde{P}_{N_1} = (L_{N_1} \tilde{Q}_{N_1} \tilde{C}_{N_1}) [(L_{N_1} \tilde{Q}_{N_1} \tilde{C}_{N_1})^\dagger (L_{N_1} \tilde{Q}_{N_1} \tilde{C}_{N_1})]^\dagger (L_{N_1} \tilde{Q}_{N_1} \tilde{C}_{N_1})^\dagger. \quad (8.24)$$

Analogous to (7.23), the perturbation changes the determinant of the Jost matrix as

$$\det[\tilde{J}(k)] = \left(\frac{k - i\kappa_N}{k + i\kappa_N} \right)^{m_{N_1}} \det[J(k)], \quad k \in \overline{\mathbb{C}^+},$$

which indicates that the total number of bound states including the multiplicities is increased by m_{N_1} . Under the perturbation, the scattering matrix $S(k)$ undergoes the transformation

$$\tilde{S}(k) = \left[I + \frac{2i\kappa_N}{k - i\kappa_N} \tilde{P}_{N_1} \right] S(k) \left[I + \frac{2i\kappa_N}{k - i\kappa_N} \tilde{P}_{N_1} \right], \quad k \in \mathbb{R},$$

which is analogous to (7.24), and the determinant of the scattering matrix changes as

$$\det[\tilde{S}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N} \right)^{2m_{N_1}} \det[S(k)], \quad k \in \mathbb{R},$$

which is the analog of (7.25). The perturbation changes the Jost solution $f(k, x)$ into $\tilde{f}(k, x)$ given by

$$\tilde{f}(k, x) = \left[f(k, x) - \frac{1}{k^2 + \kappa_N^2} \xi_{N_1}(x) \Omega_{N_1}(x)^+ q_6(k, x) \right] \left[I - \frac{2i\kappa_N}{k + i\kappa_N} \tilde{P}_{N_1} \right], \quad (8.25)$$

where we have defined the $n \times n$ matrix-valued quantity $q_6(k, x)$ as

$$q_6(k, x) := \xi'_{N_1}(x)^\dagger f(k, x) - \xi_{N_1}(x)^\dagger f'(k, x),$$

and the singularity at $k = i\kappa_N$ appearing on the right-hand side of (8.25) is a removable singularity. We note that (8.25) is the analog of (7.26). Under the perturbation, the projection matrices Q_j and the Gel'fand–Levitan normalization matrices for $1 \leq j \leq N - 1$ remain unchanged, i.e. we have

$$\tilde{Q}_j = Q_j, \quad \tilde{C}_j = C_j, \quad 1 \leq j \leq N - 1.$$

Let us consider the unperturbed problem described in Section 7 with N bound states by assuming that the unperturbed potential V satisfies (2.2) and belongs to $L_3^1(\mathbb{R}^+)$. Let us add a bound state at $k = i\tilde{\kappa}_{N+1}$ with the multiplicity $\tilde{m}_{N+1}$ and with the Gel'fand–Levitan bound-state normalization matrix $\tilde{C}_{N+1}$, where the spectral measure changes from $d\rho$ to $d\tilde{\rho}$ as in (7.4). We would like to analyze the addition of the bound state in two different ways. The first way is to add the aforementioned bound state in one step, and the second way is to add the bound state in two steps. The next theorem shows that, in either way, we end up with the same perturbed quantities, i.e. each of the two ways yields the same perturbed regular solution, boundary matrices, potential, Jost solution, Jost matrix, scattering matrix, and physical solution. We remark that the assumption that the unperturbed potential V belongs to $L_3^1(\mathbb{R}^+)$ is a sufficiency assumption and is used to ensure that the perturbed potential obtained by adding the bound state in two steps belongs to $L_1^1(\mathbb{R}^+)$.

Theorem 8.1. *Consider the unperturbed problem described in Section 7 with N bound states. Assume that the unperturbed potential V satisfies (2.2) and belongs to $L_3^1(\mathbb{R}^+)$. Let us add a bound state at $k = i\tilde{\kappa}_{N+1}$ with the multiplicity $\tilde{m}_{N+1}$ and with the Gel'fand–Levitan bound-state normalization matrix $\tilde{C}_{N+1}$ in such a way that the spectral measure changes from $d\rho$ to $d\tilde{\rho}$ as in (7.4). Let $d\rho$ be the unperturbed spectral measure, $\tilde{\varphi}(k, x)$ denote the corresponding perturbed regular solution, $\tilde{A}$ and $\tilde{B}$ be the perturbed boundary matrices, $\tilde{V}(x)$ be the perturbed potential, $\tilde{f}(k, x)$ be the perturbed Jost solution, $\tilde{J}(k)$ be the perturbed Jost matrix, $\tilde{S}(k)$ be the perturbed scattering matrix, and $\tilde{\Psi}(k, x)$ be the perturbed physical solution. Alternatively, consider adding the same bound state with the same multiplicity and the same Gel'fand–Levitan normalization matrix in two steps, where the unperturbed potential changes from V to $\hat{V}$ in the first step and from $\hat{V}$ to $\hat{\hat{V}}$ in the second step. Let us use $d\hat{\rho}$, $\hat{\varphi}(k, x)$, $\hat{A}$, $\hat{B}$, $\hat{f}(k, x)$, $\hat{J}(k, x)$, $\hat{S}(k)$, $\hat{\Psi}(k, x)$ to denote the respective perturbed quantities after the first step and use $d\hat{\hat{\rho}}$, $\hat{\hat{\varphi}}(k, x)$, $\hat{\hat{A}}$, $\hat{\hat{B}}$, $\hat{\hat{f}}(k, x)$, $\hat{\hat{J}}(k, x)$, $\hat{\hat{S}}(k)$, $\hat{\hat{\Psi}}(k, x)$ to denote the respective perturbed quantities after the second step. We know that $d\tilde{\rho} = d\hat{\hat{\rho}}$. Then, the perturbed quantities $\tilde{V}(x)$, $\tilde{\varphi}(k, x)$, $\tilde{A}$, $\tilde{B}$, $\tilde{f}(k, x)$, $\tilde{J}(k)$, $\tilde{S}(k)$, $\tilde{\Psi}(k)$ coincides with their respective counterparts $\hat{\hat{V}}(x)$, $\hat{\hat{\varphi}}(k, x)$, $\hat{\hat{A}}$, $\hat{\hat{B}}$, $\hat{\hat{f}}(k, x)$, $\hat{\hat{J}}(k, x)$, $\hat{\hat{S}}(k)$, $\hat{\hat{\Psi}}(k, x)$.*

Proof. From Theorem 5.3 of [4], we know that the perturbed regular solution $\tilde{\varphi}(k, x)$ and the unperturbed regular solution $\varphi(k, x)$ are related to each other as in (4.7), where $\mathcal{A}(x, y)$ satisfies (4.9) with $G(x, y)$ expressed in terms of $d\tilde{\rho} - d\rho$ as in (4.10). Using the same theorem, we get

$$\tilde{\varphi}(k, x) = \varphi(k, x) + \int_0^x dy \mathcal{A}(x, y) \varphi(k, y), \quad (8.26)$$

$$\hat{\hat{\varphi}}(k, x) = \hat{\varphi}(k, x) + \int_0^x dy \hat{\hat{\mathcal{A}}}(x, y) \hat{\varphi}(k, y), \quad (8.27)$$

where the quantity $\hat{\mathcal{A}}(x, y)$ is related to $d\hat{\rho} - d\rho$ and the quantity $\hat{\hat{\mathcal{A}}}(x, y)$ is related to $d\hat{\hat{\rho}} - d\hat{\rho}$ in a similar manner as in (4.7), (4.9), and (4.10). From (8.26) and (8.27), we obtain

$$\hat{\phi}(k, x) = \phi(k, x) + \int_0^x dy \mathcal{B}(x, y) \phi(k, y), \quad (8.28)$$

where the quantity $\mathcal{B}(x, y)$ is given by

$$\mathcal{B}(x, y) := \hat{\mathcal{A}}(x, y) + \hat{\hat{\mathcal{A}}}(x, y) + \int_y^x dz \hat{\mathcal{A}}(x, z) \hat{\mathcal{A}}(z, y).$$

As indicated in Theorem 5.3 of [4], we can invert the Volterra integral equation (8.28) and express $\phi(k, x)$ in terms of $\hat{\phi}(k, x)$ as

$$\phi(k, x) = \hat{\phi}(k, x) + \int_0^x dy \mathcal{C}(x, y) \hat{\phi}(k, y), \quad (8.29)$$

for some appropriate quantity $\mathcal{C}(x, y)$ determined by $d\rho - d\hat{\rho}$. Using (8.29) on the right-hand side of (4.7), we obtain

$$\tilde{\phi}(k, x) = \hat{\phi}(k, x) + \int_0^x dy \mathcal{D}(x, y) \hat{\phi}(k, y), \quad (8.30)$$

where we have

$$\mathcal{D}(x, y) := \mathcal{C}(x, y) + \hat{\mathcal{A}}(x, y) + \int_y^x \hat{\mathcal{A}}(x, z) \mathcal{C}(z, y) dz.$$

Since $d\hat{\hat{\rho}} = d\tilde{\rho}$, using (8.30) and Theorem 5.3 of [4], we obtain $\mathcal{D}(x, y) = 0$. Consequently, from (8.30) we get

$$\tilde{\phi}(k, x) = \hat{\phi}(k, x), \quad k \in \mathbb{C}, \quad x \in \mathbb{R}^+. \quad (8.31)$$

From the analog of (4.8), we know that $\tilde{\phi}(k, x)$ is a solution to the perturbed Schrödinger equation with the potential $\tilde{V}$ and that $\hat{\phi}(k, x)$ is a solution to the perturbed Schrödinger equation with the potential $\hat{V}$. By using (8.31) in those two equations and by subtracting one equation from the other, we obtain

$$[\tilde{V}(x) - \hat{V}(x)] \tilde{\phi}(k, x) = 0, \quad k \in \mathbb{C}, \quad x \in \mathbb{R}^+. \quad (8.32)$$

The large k -asymptotics of the regular solution is known. Using (3.2.201) of [3], we have

$$e^{ikx} \tilde{\varphi}(k, x) = \frac{1}{2}(1 + e^{2ikx}) \tilde{A} + \frac{1}{2}(1 - e^{2ikx}) \tilde{B} + O\left(\frac{1}{k}\right), \quad k \rightarrow \pm\infty, \quad (8.33)$$

$$e^{ikx} \hat{\varphi}(k, x) = \frac{1}{2}(1 + e^{2ikx}) \hat{A} + \frac{1}{2}(1 - e^{2ikx}) \hat{B} + O\left(\frac{1}{k}\right), \quad k \rightarrow \pm\infty. \quad (8.34)$$

From (8.31), (8.33), and (8.34), we see that the pair of boundary matrices $(\tilde{A}, \tilde{B})$ agrees with the pair of boundary matrices $(\hat{A}, \hat{B})$, i.e. we have

$$\tilde{A} = \hat{A}, \quad \tilde{B} = \hat{B}.$$

Next, we multiply both sides of (8.32) by e^{ikx} . On the left-hand side of the resulting equality, we use the asymptotic expansion given in (8.33). Then, from the leading terms in the resulting asymptotic expansion as $k \rightarrow \pm\infty$, we conclude that either we must have $\tilde{V}(x) - \hat{V}(x) \equiv 0$ or we must have

$$\tilde{A} + \tilde{B} = 0, \quad \tilde{A} - \tilde{B} = 0. \quad (8.35)$$

The latter case, i.e. the satisfaction of the two equalities in (8.35), cannot happen because that would imply $(\tilde{A}, \tilde{B}) = (0, 0)$. We would then in turn have the implication that the perturbed regular solution $\tilde{\varphi}(k, x)$, as the solution to the Schrödinger equation with the zero initial conditions at $x = 0$, would be identically zero. That cannot happen because of the analog of (2.9) satisfied by $\tilde{A}$ and $\tilde{B}$. Thus, we must have $\tilde{V}(x) \equiv \hat{V}(x)$. From the uniqueness of the Jost solution to the Schrödinger equation, we then conclude that $\tilde{f}(k, x) \equiv \hat{f}(k, x)$. The agreement of the Jost solutions and the boundary matrices, as seen from (2.11), implies that the Jost matrices $\tilde{J}(k)$ and $\hat{J}(k)$ agree. Then, using (2.12), we conclude that the scattering matrices $\tilde{S}(k)$ and $\hat{S}(k)$ agree. Finally, using (2.14), we conclude that the physical solutions $\tilde{\Psi}(k, x)$ and $\hat{\Psi}(k, x)$ agree. $\square$

9 Illustrative examples of the relevant quantities

In this section, we illustrate the construction and usage of various quantities relevant to the half-line matrix-valued Schrödinger operator with a self-adjoint boundary condition. In order to present the relevant quantities in their simplest form, we at times display only their numerical values in case their exact expressions are too lengthy.

In the first example, we demonstrate a way to construct some explicit examples of 2×2 matrix-valued potentials on the half line and the associated Jost solutions by starting with some scalar-valued potentials and the corresponding Jost solutions.

Example 9.1. Assume that V_1 and V_2 are two real-valued potentials used in (2.1) in the scalar case, i.e. when $n = 1$, and let us use $f_1(k, x)$ and $f_2(k, x)$, respectively, to denote the associated Jost solutions satisfying (2.10). It can directly be verified that the 2×2 matrix-valued potential V defined as

$$V(x) = \frac{1}{2} \begin{bmatrix} V_1(x) + V_2(x) & V_1(x) - V_2(x) \\ V_1(x) - V_2(x) & V_1(x) + V_2(x) \end{bmatrix}, \quad (9.1)$$

has the associated Jost solution $f(k, x)$ given by

$$f(k, x) = \frac{1}{2} \begin{bmatrix} f_1(k, x) + f_2(k, x) & f_1(k, x) - f_2(k, x) \\ f_1(k, x) - f_2(k, x) & f_1(k, x) + f_2(k, x) \end{bmatrix}. \quad (9.2)$$

Using $f(k, x)$ and $f(-k, x)$ as a fundamental set of solutions to (2.1) for $k \in \mathbb{R} \setminus \{0\}$, we can then construct any other solution to (2.1). In the special case $V_2(x) \equiv 0$, we have $f_2(k, x) \equiv e^{ikx}$. In that case, from (9.1) and (9.2), we obtain the corresponding potential and its associated Jost solution as

$$V(x) = \frac{1}{2} \begin{bmatrix} V_1(x) & V_1(x) \\ V_1(x) & V_1(x) \end{bmatrix}, \quad f(k, x) = \frac{1}{2} \begin{bmatrix} f_1(k, x) + e^{ikx} & f_1(x) - e^{ikx} \\ f_1(k, x) - e^{ikx} & f_1(k, x) + e^{ikx} \end{bmatrix}. \quad (9.3)$$

In a similar way, we can construct 2×2 matrix-valued self-adjoint potentials that are not real valued. In that case, instead of using (9.1), we can let

$$V(x) = \frac{1}{2} \begin{bmatrix} V_1(x) + V_2(x) & i V_1(x) - i V_2(x) \\ -i V_1(x) + i V_2(x) & V_1(x) + V_2(x) \end{bmatrix},$$

and use the corresponding Jost solution $f(k, x)$ given by

$$f(k, x) = \frac{1}{2} \begin{bmatrix} f_1(k, x) + f_2(k, x) & i f_1(k, x) - i f_2(k, x) \\ -i f_1(k, x) + i f_2(k, x) & f_1(k, x) + f_2(k, x) \end{bmatrix}.$$

This procedure also yields $n \times n$ matrix valued potentials and their associated Jost solutions by using n scalar-valued potentials V_j and their associated Jost solutions $f_j(k, x)$, where $1 \leq j \leq n$. We refer the reader to Sect. 6.2 of [3] for the details of such constructions as well as some other ways of producing matrix-valued potentials and their associated Jost solutions.

In the next example, we present some particular scalar-valued potentials and their associated Jost solutions, which can be used as the starting elements to construct the corresponding matrix-valued quantities by the method of Example 9.1.

Example 9.2. We present the scalar-valued potential V_1 and its associated Jost solution $f_1(k, x)$ given by

$$V_1(x) = -\frac{8a\epsilon\beta^2 e^{2\beta x}}{(\alpha + \epsilon e^{2\beta x})^2}, \quad f_1(k, x) = e^{ikx} \left[1 - \frac{2ia\beta}{(k + i\beta)(\alpha + \epsilon e^{2\beta x})} \right], \quad x \in \mathbb{R}^+, \quad (9.4)$$

where α , ϵ , and β are positive parameters. Even though we use three parameters in (9.4), there are actually two independent parameters given by α/ϵ and β . Next, we introduce the scalar-valued potential V_2 and its associated Jost solution $f_2(k, x)$ given by

$$V_2(x) = \frac{2}{(x+a)^2}, \quad f_2(k, x) = e^{ikx} \left[1 + \frac{i}{k(x+a)} \right], \quad x \in \mathbb{R}^+,$$

where a is a positive parameter. With the help of (2.4) and (2.5) we observe that V_2 is integrable but does not belong to $L_1^1(\mathbb{R}^+)$.

In the next example, starting with a 2×2 matrix-valued potential V and a pair of 2×2 boundary matrices A and B , we illustrate the derivation of the corresponding relevant particular solutions, Jost matrix, and scattering matrix. For some of the evaluations, we have used the software Mathematica to do symbolic computations. For the convenience of displaying some of the results in their simplest forms, we use an overbar on a digit to indicate a roundoff in that digit.

Example 9.3. As the boundary matrices A and B appearing in (2.6) and (2.7), let us use

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 20 \\ 4 & -2 \end{bmatrix}, \quad (9.5)$$

which satisfy (2.7) and (2.8). The positivity of the matrix given in (2.8) can be verified by the fact that we have

$$A^\dagger A + B^\dagger B = \begin{bmatrix} 33 & 74 \\ 74 & 417 \end{bmatrix},$$

where the eigenvalues of the matrix on the right-hand side are evaluated as $430.76\bar{7}$ and $19.233\bar{1}$, which are both positive. We choose $V_2(x) \equiv 0$ and construct $V_1(x)$ by using the parameter values $\alpha = 2$, $\epsilon = 1$, and $\beta = 1/3$ in the first equality of (9.4). Using the resulting potential $V_1(x)$ in the first equality of (9.3), we obtain the 2×2 matrix-valued potential V given by

$$V(x) = -\frac{8e^{2x/3}}{9(2 + e^{2x/3})^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad x \in \mathbb{R}^+, \quad (9.6)$$

which satisfies (2.2) and (2.4). Using the corresponding parameter values $\alpha = 2$, $\epsilon = 1$, and $\beta = 1/3$ in the second equality of (9.4), we construct $f_1(k, x)$. Then, with the help of the second equality of (9.3), we obtain the associated Jost solution satisfying (2.1) and (2.10) as

$$f(k, x) = e^{ikx} \begin{bmatrix} 1 - \frac{2i}{(3k+i)(2+e^{2x/3})} & -\frac{2i}{(3k+i)(2+e^{2x/3})} \\ -\frac{2i}{(3k+i)(2+e^{2x/3})} & 1 - \frac{2i}{(3k+i)(2+e^{2x/3})} \end{bmatrix}, \quad x \in \mathbb{R}^+. \quad (9.7)$$

From (9.7), we get

$$f(k, 0) = \begin{bmatrix} \frac{9k+i}{3(3k+i)} & -\frac{2i}{3(3k+i)} \\ -\frac{2i}{3(3k+i)} & \frac{9k+i}{3(3k+i)} \end{bmatrix}, \quad (9.8)$$

$$f'(k, 0) = \begin{bmatrix} \frac{81ik^2-9k+4i}{27(3k+i)} & \frac{18k+4i}{27(3k+i)} \\ \frac{18k+4i}{27(3k+i)} & \frac{81ik^2-9k+4i}{27(3k+i)} \end{bmatrix}. \quad (9.9)$$

Using (9.5), (9.8), and (9.9) in (2.11), we construct the corresponding Jost matrix $J(k)$ as

$$J(k) = \begin{bmatrix} \frac{-81ik^2+333k-40i}{27(3k+i)} & \frac{-162ik^2+1584k+196i}{27(3k+i)} \\ \frac{306k-40i}{27(3k+i)} & \frac{-243ik^2-171k-398i}{27(3k+i)} \end{bmatrix}, \quad k \in \mathbb{R}. \quad (9.10)$$

From (9.10) we obtain

$$J(0) = \frac{1}{27} \begin{bmatrix} -40 & 196 \\ -40 & -398 \end{bmatrix},$$

with the eigenvalues of $J(0)$ given by $-13.872\bar{8}$ and $-2.3493\bar{8}$. We note that the determinant of $J(0)$ is given by

$$\det[J(0)] = \frac{880}{27},$$

which indicates that $J(k)$ is invertible at $k = 0$. Since $J(0)$ is invertible, this corresponds to a generic case [3]. The adjoint of the Jost matrix in (9.10) is evaluated as

$$J(k)^\dagger = \begin{bmatrix} \frac{81ik^2+333k+40i}{27(3k-i)} & \frac{306k+40i}{27(3k-i)} \\ \frac{162ik^2+1584k-196i}{27(3k-i)} & \frac{243ik^2-171k+398i}{27(3k-i)} \end{bmatrix}, \quad k \in \mathbb{R}.$$

Next, we evaluate the scattering matrix $S(k)$. From (9.10), we have

$$J(-k) = \begin{bmatrix} \frac{81ik^2+333k+40i}{27(3k-i)} & \frac{162ik^2+1584k-196i}{27(3k-i)} \\ \frac{306k+40i}{27(3k-i)} & \frac{243ik^2-171k+398i}{27(3k-i)} \end{bmatrix}, \quad k \in \mathbb{R}. \quad (9.11)$$

Hence, using (9.10) and (9.11) in (2.12), we obtain the scattering matrix as

$$S(k) = \begin{bmatrix} s_{11}(k) & s_{12}(k) \\ s_{12}(k) & s_{11}(k) \end{bmatrix}, \quad (9.12)$$

where we have defined

$$s_{11}(k) := \frac{729k^4 - 5346ik^3 - 18747k^2 - 594ik + 880}{(3k-i)(243k^3 + 135ik^2 + 7170k - 880i)}, \quad (9.13)$$

$$s_{12}(k) := \frac{-12ik(459k^2 + 794)}{(3k-i)(243k^3 + 135ik^2 + 7170k - 880i)}. \quad (9.14)$$

Using (9.12)–(9.14), it can be verified directly that the scattering matrix $S(k)$ satisfies

$$S(-k) = S(k)^\dagger = S(k)^{-1}, \quad k \in \mathbb{R},$$

confirming (2.5.4) of [3]. From (9.12)–(9.14), we obtain the large k -asymptotics of the scattering matrix as

$$S(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{9ik} \begin{bmatrix} 68 & 68 \\ 68 & -64 \end{bmatrix} + O\left(\frac{1}{k^2}\right), \quad k \rightarrow \pm\infty,$$

and we also get the small k -asymptotics of $S(k)$ as

$$S(k) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + O(k), \quad k \rightarrow 0 \text{ in } \mathbb{R}.$$

The physical solution $\Psi(k, x)$ is obtained by using (9.7) and (9.12)–(9.14) in (2.13), but it is too lengthy to display here. Using $x = 0$, from the explicitly evaluated physical solution $\Psi(k, x)$ and its x -derivative $\Psi'(k, x)$, we have

$$\Psi(k, 0) = \frac{1}{q_7(k)} \begin{bmatrix} 6k(81k^2 - 261ik + 106) & -12ik(153k + 46i) \\ -12ik(153k - 20i) & 6k(81k^2 + 333ik + 40) \end{bmatrix}, \quad (9.15)$$

$$\Psi'(k, 0) = \frac{1}{q_7(k)} \begin{bmatrix} 72k(27k^2 - 189ik + 22) & 72k(27k^2 + 9ik + 44) \\ 8k(243k^2 - 18ik + 418) & -4k(405k^2 + 3501ik - 352) \end{bmatrix}, \quad (9.16)$$

where we have defined

$$q_7(k) := 243k^3 + 135ik^2 + 7170k - 880i. \quad (9.17)$$

Using (9.5) and (9.15)–(9.17) on the left-hand side of (2.7), we confirm that the physical solution $\Psi(k, x)$ satisfies the boundary condition given in (2.7). Next, we evaluate the regular solution $\varphi(k, x)$ by using (2.14) with input from the physical solution $\Psi(k, x)$ and the Jost matrix $J(k)$. We also alternatively evaluate the regular solution by using (9.2), (9.10), and (9.11) in (2.15). We confirm that we obtain the same expression for $\varphi(k, x)$ from either evaluation. We get

$$\varphi(k, x) = \frac{1}{q_8(k, x)} \begin{bmatrix} q_9(k, x) - q_9(-k, x) & q_{10}(k, x) - q_{10}(-k, x) \\ q_{11}(k, x) - q_{11}(-k, x) & q_{12}(k, x) - q_{12}(-k, x) \end{bmatrix}, \quad (9.18)$$

where we have defined

$$q_8(k, x) := 54k(3k + i)(3k - i)(2 + e^{2x/3}), \quad (9.19)$$

$$q_9(k, x) := e^{ikx}(-80i - 372k - 1998ik^2 + 486k^3) + e^{ikx+2x/3}(40i + 453k - 918ik^2 + 243k^3), \quad (9.20)$$

$$q_{10}(k, x) := e^{ikx}(-796i - 834k - 9990ik^2 + 972k^3) + e^{ikx+2x/3}(-196i + 996k - 4590ik^2 + 486k^3), \quad (9.21)$$

$$q_{11}(k, x) := e^{ikx}(-80i - 426k - 1998ik^2) + e^{ikx+2x/3}(40i + 426k - 918ik^2), \quad (9.22)$$

$$q_{12}(k, x) := e^{ikx}(392i - 780k + 702ik^2 + 1458k^3) + e^{ikx+2x/3}(398i + 1023k + 756ik^2 + 729k^3). \quad (9.23)$$

By evaluating $\varphi(k, 0)$ and $\varphi'(k, 0)$ and using (9.5), we observe that the regular solution satisfies the initial conditions (2.6) with the boundary matrices A and B appearing in (9.5). As $x \rightarrow +\infty$ in (9.18), with the help of (9.19)–(9.23), we observe that, unless k is real, each of the four entries of $\varphi(k, x)$ blows up exponentially as $x \rightarrow +\infty$. We remark that the right-hand side of (9.18) is indeed entire in k . The three zeros of $q_8(k, x)$ in k all correspond to removable singularities for the right-hand side of (9.18).

Next, we consider the bound states associated with the potential V in (9.6) and the boundary matrices A and B appearing in (9.5). The matrix case is naturally more complicated than the scalar case. This is due to the fact that the bound states are simple in the scalar case, whereas their multiplicities must be taken into account in the matrix case. In the next example, we illustrate the construction of the relevant orthogonal projections used in the analysis of bound states in the matrix case.

Example 9.4. In this example, we continue using the potential V in (9.6) as well as the boundary matrices A and B given in (9.5). Hence, this example is the continuation of Example 9.3. We analyze the determinant of the Jost matrix $J(k)$. From (9.10) we have

$$\det[J(k)] = \frac{-243k^3 - 135ik^2 - 7170k + 880i}{27(3k + i)}. \quad (9.24)$$

We observe that $\det[J(k)]$ has three simple zeros that are given by

$$k = i\kappa_1, \quad k = i\kappa_2, \quad k = -5.77419i, \quad (9.25)$$

where we have let

$$\kappa_1 := 5.09554\bar{8}, \quad \kappa_2 := 0.12308204\bar{1}. \quad (9.26)$$

From (9.24)–(9.26) we see that, among the three zeros of $\det[J(k)]$, only two of them occur in $\mathbb{C}^+$. In fact, those two simple zeros listed in (9.26) occur on the positive imaginary axis in $\mathbb{C}^+$, as they must. Hence, we have two simple bound states occurring at $k = i\kappa_1$ and $k = i\kappa_2$, respectively. Let us evaluate $J(i\kappa_1)$ and $J(i\kappa_2)$ by using (9.26) in (9.10). We get

$$J(i\kappa_1) = \begin{bmatrix} 8.5504\bar{1} & 28.365\bar{9} \\ 3.4548\bar{6} & 11.461\bar{5} \end{bmatrix}, \quad J(i\kappa_2) = \begin{bmatrix} 0.059870\bar{8} & 10.641\bar{6} \\ -0.063211\bar{2} & -11.235\bar{3} \end{bmatrix}. \quad (9.27)$$

From (9.27), we see that

$$J(i\kappa_1)^\dagger = \begin{bmatrix} 8.5504\bar{1} & 3.4548\bar{6} \\ 28.365\bar{9} & 11.461\bar{5} \end{bmatrix}, \quad J(i\kappa_2)^\dagger = \begin{bmatrix} 0.059870\bar{8} & -0.063211\bar{2} \\ 10.641\bar{6} & -11.235\bar{3} \end{bmatrix}.$$

Next, we determine the kernels of the matrices $J(i\kappa_1)$, $J(i\kappa_1)^\dagger$, $J(i\kappa_2)$, and $J(i\kappa_2)^\dagger$. From (9.26) and (9.27), we observe that the two rows in each of those four matrices are proportional, and hence each of those four matrices has the rank equal to 1 and that the

nullity of each of those four matrices is $2 - 1$, which is equal to 1. We see that each of the following four column vectors forms a basis for the kernels of $J(i\kappa_1)$, $J(i\kappa_1)^\dagger$, $J(i\kappa_2)$, and $J(i\kappa_2)^\dagger$, respectively:

$$\begin{bmatrix} 28.3659 \\ -8.5504\bar{1} \end{bmatrix}, \quad \begin{bmatrix} 3.4548\bar{6} \\ -8.5504\bar{1} \end{bmatrix}, \quad \begin{bmatrix} 10.641\bar{6} \\ -0.059870\bar{8} \end{bmatrix}, \quad \begin{bmatrix} 0.063211\bar{2} \\ 0.059870\bar{8} \end{bmatrix}. \quad (9.28)$$

Hence, normalizing the vectors in (9.28), we obtain the unit column vectors w_1, v_1, w_2, v_2 , generating $\text{Ker}[J(i\kappa_1)]$, $\text{Ker}[J(i\kappa_1)^\dagger]$, $\text{Ker}[J(i\kappa_2)]$, and $\text{Ker}[J(i\kappa_2)^\dagger]$, respectively, where we have let

$$w_1 := \begin{bmatrix} 0.957448 \\ -0.28860\bar{6} \end{bmatrix}, \quad v_1 := \begin{bmatrix} 0.37463\bar{2} \\ -0.92717\bar{4} \end{bmatrix}, \quad (9.29)$$

$$w_2 := \begin{bmatrix} 0.99998\bar{4} \\ -0.0056260\bar{3} \end{bmatrix}, \quad v_2 := \begin{bmatrix} 0.7260\bar{3} \\ 0.68766\bar{3} \end{bmatrix}. \quad (9.30)$$

As indicated in (3.8) of [4], we have the orthogonal projection matrices Q_1, P_1, Q_2 , and P_2 onto the kernels of $J(i\kappa_1)$, $J(i\kappa_1)^\dagger$, $J(i\kappa_2)$, and $J(i\kappa_2)^\dagger$, respectively, given by

$$Q_1 = w_1 w_1^\dagger, \quad P_1 = v_1 v_1^\dagger, \quad Q_2 = w_2 w_2^\dagger, \quad P_2 = v_2 v_2^\dagger. \quad (9.31)$$

Using (9.29) and (9.30) in (9.31), we obtain the corresponding orthogonal projections Q_1, P_1, Q_2, P_2 as

$$Q_1 = \begin{bmatrix} 0.91670\bar{7} & -0.27632\bar{5} \\ -0.27632\bar{5} & 0.083293\bar{3} \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.14034\bar{9} & -0.34734\bar{9} \\ -0.34734\bar{9} & 0.85965\bar{1} \end{bmatrix}, \quad (9.32)$$

$$Q_2 = \begin{bmatrix} 0.999996\bar{8} & -0.0056259\bar{4} \\ -0.0056259\bar{4} & 0.000031652\bar{2} \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.52711\bar{9} & 0.49926\bar{4} \\ 0.49926\bar{4} & 0.47288\bar{1} \end{bmatrix}. \quad (9.33)$$

One can directly verify that the matrices appearing in (9.32) and (9.33) satisfy the appropriate equalities stated in (3.1). One can verify that P_1 and P_2 do not commute, as expected in general. In fact, we have

$$P_1 P_2 - P_2 P_1 = \begin{bmatrix} 0 & -0.34028\bar{2} \\ 0.34028\bar{2} & 0 \end{bmatrix}.$$

Similarly, as expected in general, P_1 and Q_1 do not commute because we have

$$P_1 Q_1 - Q_1 P_1 = \begin{bmatrix} 0 & 0.48824\bar{6} \\ -0.48824\bar{6} & 0 \end{bmatrix}.$$

Likewise, P_2 and Q_2 , as expected in general, do not commute, and we have

$$P_2 Q_2 - Q_2 P_2 = \begin{bmatrix} 0 & -0.49953\bar{8} \\ 0.49953\bar{8} & 0 \end{bmatrix}.$$

Similarly, one can verify that P_2 and Q_1 do not commute, as expected in general, and in fact we have

$$P_2 Q_1 - Q_1 P_2 = \begin{bmatrix} 0 & -0.43108\bar{1} \\ 0.43108\bar{1} & 0 \end{bmatrix}.$$

One can appreciate the simplicity of the treatment of the bound states in the scalar case, in which there exists only one orthogonal projection for each bound state, and that orthogonal projection is simply equal to the real number 1. In the scalar case, there is no need to make a distinction between the kernel of $J(k)$ and the kernel of $J(k)^\dagger$ at a bound-state k -value.

In the next example, we illustrate the determination of the Marchenko normalization matrices at the bound states. We continue to use the potential and the boundary matrices specified in Examples 9.3 and 9.4.

Example 9.5. In this example, we continue to use as input the potential V specified in (9.6) and the boundary matrices A and B specified in (9.5). We recall that the corresponding Jost solution is given in (9.7), and we also know that there are two simple bound states at $k = i\kappa_1$ and $k = i\kappa_2$, respectively, as indicated in (9.25). We next evaluate the Jost solution $f(k, x)$ at $k = i\kappa_1$ and $k = i\kappa_2$, where κ_1 and κ_2 are the positive constants given in (9.26). Using (9.26) in (9.7), we get

$$f(i\kappa_1, x) = e^{-\kappa_1 x} \begin{bmatrix} 1 - \frac{0.122\bar{8}}{2+e^{2x/3}} & -\frac{0.122\bar{8}}{2+e^{2x/3}} \\ -\frac{0.122\bar{8}}{2+e^{2x/3}} & 1 - \frac{0.122\bar{8}}{2+e^{2x/3}} \end{bmatrix}, \quad (9.34)$$

$$f(i\kappa_2, x) = e^{-\kappa_2 x} \begin{bmatrix} 1 - \frac{1.4606\bar{6}}{2+e^{2x/3}} & -\frac{1.4606\bar{6}}{2+e^{2x/3}} \\ -\frac{1.4606\bar{6}}{2+e^{2x/3}} & 1 - \frac{1.4606\bar{6}}{2+e^{2x/3}} \end{bmatrix}. \quad (9.35)$$

From (9.34) and (9.35), we observe that

$$f(i\kappa_1, x) = e^{-\kappa_1 x} [I + o(1)], \quad f(i\kappa_2, x) = e^{-\kappa_2 x} [I + o(1)], \quad x \rightarrow +\infty,$$

which are compatible with (3.3). Using (9.34) and (9.35) we get

$$\int_0^\infty dx f(i\kappa_1, x)^\dagger f(i\kappa_1, x) = \begin{bmatrix} 0.090584\bar{9} & -0.0075399\bar{2} \\ -0.0075399\bar{2} & 0.090584\bar{9} \end{bmatrix}, \quad (9.36)$$

$$\int_0^\infty dx f(i\kappa_2, x)^\dagger f(i\kappa_2, x) = \begin{bmatrix} 2.9955\bar{7} & -1.0667\bar{6} \\ -1.0667\bar{6} & 2.9955\bar{7} \end{bmatrix}. \quad (9.37)$$

Using (3.7), with the help of (9.36), (9.37), and the second equalities of (9.32) and (9.33), we evaluate the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ defined as

$$\mathbf{A}_1 := \int_0^\infty dx P_1 f(i\kappa_1, x)^\dagger f(i\kappa_1, x) P_1,$$

$$\mathbf{A}_2 := \int_0^\infty dx P_2 f(ik_2, x)^\dagger f(ik_2, x) P_2,$$

and we get

$$\mathbf{A}_1 = \begin{bmatrix} 0.013448\bar{6} & -0.03328\bar{4} \\ -0.03328\bar{4} & 0.082374\bar{3} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1.0175\bar{4} & 0.96376\bar{9} \\ 0.96376\bar{9} & 0.9128\bar{4} \end{bmatrix}. \quad (9.38)$$

Using (9.38) and the second equalities of (9.32) and (9.33), from (3.8) we obtain

$$\mathbf{B}_1 = \begin{bmatrix} 0.873\bar{1} & 0.31406\bar{5} \\ 0.31406\bar{5} & 0.22272\bar{3} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 1.4904\bar{2} & 0.46450\bar{5} \\ 0.46450\bar{5} & 1.4399\bar{6} \end{bmatrix}. \quad (9.39)$$

We observe that $\mathbf{B}_1$ and $\mathbf{B}_2$ are Hermitian. They are also positive, and this can be checked by evaluating their eigenvalues and confirming that those eigenvalues are all positive. We recall that we mean a positive definite matrix when we say that a matrix is positive. The eigenvalues of $\mathbf{B}_1$ are given by 1 and $0.095822\bar{9}$, and the eigenvalues of $\mathbf{B}_2$ are given by 1 and $1.9303\bar{8}$. Since $\mathbf{B}_1$ and $\mathbf{B}_2$ are positive matrices, we can uniquely determine $\mathbf{B}_1^{1/2}$ and $\mathbf{B}_2^{1/2}$ as positive matrices so that (3.9) is satisfied. We get

$$\mathbf{B}_1^{1/2} = \begin{bmatrix} 0.90309\bar{6} & 0.23982\bar{6} \\ 0.23982\bar{6} & 0.40645\bar{6} \end{bmatrix}, \quad \mathbf{B}_2^{1/2} = \begin{bmatrix} 1.2052\bar{5} & 0.19440\bar{4} \\ 0.19440\bar{4} & 1.1841\bar{3} \end{bmatrix},$$

which are compatible with (3.9) and (9.39). Because $\mathbf{B}_1^{1/2}$ and $\mathbf{B}_2^{1/2}$ are positive, they are invertible, and their inverses are given by

$$\mathbf{B}_1^{-1/2} = \begin{bmatrix} 1.3130\bar{4} & -0.7747\bar{5} \\ -0.7747\bar{5} & 2.9174\bar{2} \end{bmatrix}, \quad \mathbf{B}_2^{-1/2} = \begin{bmatrix} 0.85227\bar{2} & -0.13992\bar{1} \\ -0.13992\bar{1} & 0.86747\bar{3} \end{bmatrix}. \quad (9.40)$$

One can directly verify that each of $\mathbf{B}_1$, $\mathbf{B}_1^{1/2}$, $\mathbf{B}_1^{-1/2}$ is Hermitian and commutes with P_1 . Similarly, each of $\mathbf{B}_2$, $\mathbf{B}_2^{1/2}$, $\mathbf{B}_2^{-1/2}$ is Hermitian and commutes with P_2 . One can also verify that $\mathbf{B}_1$ and $\mathbf{B}_2$ do not commute, as expected in general, and in fact we have

$$\mathbf{B}_1 \mathbf{B}_2 - \mathbf{B}_2 \mathbf{B}_1 = \begin{bmatrix} 0 & 0.28625\bar{5} \\ -0.28625\bar{5} & 0 \end{bmatrix}.$$

Using the second equality in (9.32), the second equality in (9.33), and (9.40) in (3.10), we obtain the Marchenko normalization matrices M_1 and M_2 as

$$M_1 = \begin{bmatrix} 0.45339\bar{3} & -1.122\bar{1} \\ -1.122\bar{1} & 2.7770\bar{7} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.37939\bar{1} & 0.35934\bar{3} \\ 0.35934\bar{3} & 0.34035\bar{4} \end{bmatrix}. \quad (9.41)$$

Note that M_1 and M_2 are nonnegative and have their ranks equal to one as the multiplicity of each bound state is one. This can be verified by evaluating the eigenvalues of M_1

and M_2 . The eigenvalues of M_1 are given by 0 and $3.2304\bar{7}$, and the eigenvalues of M_2 are given by 0 and $0.71974\bar{5}$, confirming that M_1 and M_2 are nonnegative and that each have the rank equal to one.

In the next example, we illustrate the determination of the Marchenko normalized bound-state solutions. We continue to use the potential and the boundary matrices used as input to Examples 9.3–9.5.

Example 9.6. In this example, we continue to use the potential V specified in (9.6) and the boundary matrices A and B appearing in (9.5). Using (9.34), (9.35), and (9.41) in (3.4), we obtain the Marchenko normalized bound-state matrix solutions $\Psi_1(x)$ and $\Psi_2(x)$ as

$$\Psi_1(x) = \frac{e^{-\kappa_1 x}}{2 + e^{2x/3}} \begin{bmatrix} 0.98890\bar{3} + 0.45339\bar{3} e^{2x/3} & -2.4474\bar{3} - 1.122\bar{1} e^{2x/3} \\ -2.1620\bar{8} - 1.122\bar{1} e^{2x/3} & 5.3509\bar{2} + 2.7770\bar{7} e^{2x/3} \end{bmatrix}, \quad (9.42)$$

$$\Psi_2(x) = \frac{e^{-\kappa_2 x}}{2 + e^{2x/3}} \begin{bmatrix} -0.32025\bar{5} + 0.37939\bar{1} e^{2x/3} & -0.30333\bar{1} + 0.35934\bar{3} e^{2x/3} \\ -0.36035\bar{2} + 0.35934\bar{3} e^{2x/3} & -0.3413\bar{1} + 0.34035\bar{4} e^{2x/3} \end{bmatrix}. \quad (9.43)$$

From (9.42) and (9.43), we observe that

$$\Psi_1(x) = e^{-\kappa_1 x} \left(\begin{bmatrix} 0.45339\bar{3} & -1.122\bar{1} \\ -1.122\bar{1} & 2.7770\bar{7} \end{bmatrix} + o(1) \right), \quad x \rightarrow +\infty,$$

$$\Psi_2(x) = e^{-\kappa_2 x} \left(\begin{bmatrix} 0.37939\bar{1} & 0.35934\bar{3} \\ 0.35934\bar{3} & 0.34035\bar{4} \end{bmatrix} + o(1) \right), \quad x \rightarrow +\infty.$$

From (9.42) and (9.43), we obtain

$$\Psi_1(0) = \begin{bmatrix} 0.48076\bar{5} & -1.1898\bar{4} \\ -1.0947\bar{3} & 2.7093\bar{3} \end{bmatrix}, \quad \Psi_1'(0) = \begin{bmatrix} -2.4558\bar{4} & 6.0779\bar{5} \\ 5.5721\bar{5} & -13.790\bar{5} \end{bmatrix}, \quad (9.44)$$

$$\Psi_2(0) = \begin{bmatrix} 0.019712\bar{1} & 0.018670\bar{4} \\ -0.00033649\bar{4} & -0.00031871\bar{2} \end{bmatrix}, \quad \Psi_2'(0) = \begin{bmatrix} 0.077502\bar{5} & 0.073406\bar{9} \\ 0.079970\bar{1} & 0.075744\bar{2} \end{bmatrix}. \quad (9.45)$$

Using (9.5) and (9.44) in (2.7), we verify that $\Psi_1(x)$ satisfies the boundary condition (2.7). Similarly, using (9.5) and (9.45) in (2.7), we confirm that $\Psi_2(x)$ satisfies the boundary condition (2.7). Using P_1 given in the second equality in (9.32), P_2 given in the second equality of (9.33), $\Psi_1(x)$ in (9.42), and $\Psi_2(x)$ in (9.43), we confirm the orthonormality relations stated in (3.5) and (3.6), i.e. we verify that

$$\begin{aligned} \int_0^\infty dx \Psi_1(x)^\dagger \Psi_1(x) &= P_1, & \int_0^\infty dx \Psi_2(x)^\dagger \Psi_2(x) &= P_2, \\ \int_0^\infty dx \Psi_1(x)^\dagger \Psi_2(x) &= 0, & \int_0^\infty dx \Psi_2(x)^\dagger \Psi_1(x) &= 0. \end{aligned} \quad (9.46)$$

We remark that we use 0 to denote the zero matrix on the right-hand sides in (9.46).

In the next example, we illustrate the determination of the Gel'fand–Levitan normalization matrices at the bound states. We continue to use as input the potential V and the boundary matrices A and B from Examples 9.3–9.6.

Example 9.7. In this example, we continue to use the potential V specified in (9.6) and the boundary matrices A and B given in (9.5). Using (2.10) and (2.15), we see that

$$\varphi(k, x) = \frac{e^{ikx}}{2ik} [I + o(1)] J(-k) - \frac{e^{-ikx}}{2ik} [I + o(1)] J(k), \quad x \rightarrow +\infty, \quad k \in \mathbb{C}. \quad (9.47)$$

We remark that (9.47) would normally hold only for $k \in \mathbb{R}$ because in general the common domain of $J(k)$ and $J(-k)$ would be only $k \in \mathbb{R}$. In this example, (9.47) holds for $k \in \mathbb{C}$ because the corresponding Jost solution $f(k, x)$ appearing in (9.7) and Jost matrix $J(k)$ appearing in (9.10) each have an extension to $k \in \mathbb{C}$. Using (9.47), we evaluate the large x -asymptotics of the regular solution at the bound states with $k = i\kappa_1$ and $k = i\kappa_2$ as

$$\begin{aligned} \varphi(i\kappa_1, x) &= \frac{e^{\kappa_1 x}}{2\kappa_1} [I + o(1)] J(i\kappa_1), \quad x \rightarrow +\infty, \\ \varphi(i\kappa_2, x) &= \frac{e^{\kappa_2 x}}{2\kappa_2} [I + o(1)] J(i\kappa_2), \quad x \rightarrow +\infty, \end{aligned}$$

and hence we observe that $\varphi(i\kappa_1, x)$ blows up exponentially as $x \rightarrow +\infty$ unless $J(i\kappa_1) = 0$ and that $\varphi(i\kappa_2, x)$ blows up exponentially as $x \rightarrow +\infty$ unless $J(i\kappa_2) = 0$. We note that we would have $J(i\kappa_1) = 0$ in this example only when the kernel of $J(i\kappa_1)$ had the dimension 2. However, in this example, the bound state at $k = i\kappa_1$ is simple, and hence the kernel of $J(i\kappa_1)$ has the dimension 1. Similarly, because the bound state at $k = i\kappa_2$ is simple, it follows that the kernel of $J(i\kappa_2)$ has the dimension 1, and hence $J(i\kappa_2) \neq 0$ in this example. Thus, each of $\varphi(i\kappa_1, x)$ and $\varphi(i\kappa_2, x)$ is exponentially growing as $x \rightarrow +\infty$. Consequently, the integrals $I_1(b)$ and $I_2(b)$ defined as

$$I_1(b) := \int_0^b dx \, \varphi(i\kappa_1, x)^\dagger \varphi(i\kappa_1, x), \quad I_2(b) := \int_0^b dx \, \varphi(i\kappa_2, x)^\dagger \varphi(i\kappa_2, x),$$

do not converge as $b \rightarrow +\infty$. We can illustrate this numerically by observing that we have

$$\begin{aligned} I_1(1) &= \begin{bmatrix} 2417.8\bar{6} & 8021.1\bar{2} \\ 8021.1\bar{2} & 26610.7\bar{7} \end{bmatrix}, \quad I_2(1) = \begin{bmatrix} 14.975\bar{2} & 44.034\bar{3} \\ 44.034\bar{3} & 176.1\bar{8} \end{bmatrix}, \\ I_1(50) &= \begin{bmatrix} 1.5916\bar{4} \times 10^{220} & 4.5391\bar{9} \times 10^{220} \\ 4.5391\bar{9} \times 10^{220} & 1.7517\bar{3} \times 10^{221} \end{bmatrix}, \\ I_2(50) &= \begin{bmatrix} 11391\bar{1} & 2.0007\bar{9} \times 10^7 \\ 2.0007\bar{9} \times 10^7 & 3.5556\bar{9} \times 10^9 \end{bmatrix}. \end{aligned}$$

We note that we can prevent the exponential growths in $\varphi(ik_1, x)$ and $\varphi(ik_2, x)$ as $x \rightarrow +\infty$, and this can be achieved by using the matrices Q_1 and Q_2 , respectively, appearing in the first equalities of (9.32) and (9.33). Since Q_1 and Q_2 are the orthogonal projections onto the kernels of $J(ik_1)$ and $J(ik_2)$, respectively, we have

$$J(ik_1) Q_1 = 0, \quad J(ik_2) Q_2 = 0, \quad (9.48)$$

and hence, using (9.47) with $k = ik_1$ and $k = ik_2$, respectively, with the help of (9.48) we obtain

$$\varphi(ik_1, x) Q_1 = O(e^{-\kappa_1 x}), \quad \varphi(ik_2, x) Q_2 = O(e^{-\kappa_2 x}), \quad x \rightarrow +\infty.$$

Using the orthogonal projection matrices Q_1 and Q_2 from Example 9.4 and the regular solution $\varphi(k, x)$ constructed explicitly in Example 9.3, from (3.14) we obtain

$$\mathbf{G}_1 := \int_0^\infty dx Q_1 \varphi(ik_1, x)^\dagger \varphi(ik_1, x) Q_1 = \begin{bmatrix} 0.080478\bar{7} & -0.024258\bar{9} \\ -0.024258\bar{9} & 0.0073124\bar{2} \end{bmatrix}, \quad (9.49)$$

$$\mathbf{G}_2 := \int_0^\infty dx Q_2 \varphi(ik_2, x)^\dagger \varphi(ik_2, x) Q_2 = \begin{bmatrix} 1326.1\bar{3} & -7.4609\bar{5} \\ -7.4609\bar{5} & 0.041976\bar{2} \end{bmatrix}. \quad (9.50)$$

Using (9.49), (9.50), and the first equalities of (9.32) and (9.33), from (3.15) we obtain the constant positive matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ as

$$\mathbf{H}_1 = \begin{bmatrix} 0.16377\bar{2} & 0.25206\bar{6} \\ 0.25206\bar{6} & 0.92401\bar{9} \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 1326.1\bar{3} & -7.4553\bar{3} \\ -7.4553\bar{3} & 1.0419\bar{4} \end{bmatrix},$$

from which we get the positive matrices $\mathbf{H}_1^{1/2}$ and $\mathbf{H}_2^{1/2}$ appearing in (3.16) as

$$\mathbf{H}_1^{1/2} = \begin{bmatrix} 0.3549\bar{1} & 0.19445\bar{1} \\ 0.19445\bar{1} & 0.94138\bar{6} \end{bmatrix}, \quad \mathbf{H}_2^{1/2} = \begin{bmatrix} 36.415\bar{5} & -0.19925\bar{2} \\ -0.19925\bar{2} & 1.0011\bar{2} \end{bmatrix}.$$

We note that $\mathbf{H}_1$ and $\mathbf{H}_2$ are Hermitian and that they are positive. Their positivity is seen from the fact that $\mathbf{H}_1$ has the positive eigenvalues 1 and $0.087791\bar{2}$ and $\mathbf{H}_2$ has the positive eigenvalues 1 and $1326.1\bar{7}$. We also observe that $\mathbf{H}_1^{1/2}$ and $\mathbf{H}_2^{1/2}$ are Hermitian, and they are positive. Their positivity is also confirmed by the fact that $\mathbf{H}_1^{1/2}$ has the positive eigenvalues 1 and $0.29629\bar{6}$ and $\mathbf{H}_2^{1/2}$ has the positive eigenvalues 1 and $36.416\bar{6}$. Since $\mathbf{H}_1^{1/2}$ and $\mathbf{H}_2^{1/2}$ are positive, they are invertible and their inverses are given by

$$\mathbf{H}_1^{-1/2} = \begin{bmatrix} 3.1771\bar{8} & -0.65627\bar{4} \\ -0.65627\bar{4} & 1.1978\bar{2} \end{bmatrix}, \quad \mathbf{H}_2^{-1/2} = \begin{bmatrix} 0.027490\bar{8} & 0.0054714\bar{5} \\ 0.0054714\bar{5} & 0.99996\bar{9} \end{bmatrix}. \quad (9.51)$$

We directly verify that each of $\mathbf{H}_1$, $\mathbf{H}_1^{1/2}$, and $\mathbf{H}_1^{-1/2}$ is Hermitian and commutes with the orthogonal projection matrix Q_1 . Similarly, we directly verify that each of $\mathbf{H}_2$, $\mathbf{H}_2^{1/2}$, and

$\mathbf{H}_2^{-1/2}$ is Hermitian and commutes with Q_2 . With the help of Q_1 given in the first equality of (9.32), Q_2 appearing in the first equality of (9.33), and the two matrices $\mathbf{H}_1^{-1/2}$ and $\mathbf{H}_2^{-1/2}$ appearing in (9.51), from (3.17) we obtain the Gel'fand–Levitan normalization matrices C_1 and C_2 as

$$C_1 = \begin{bmatrix} 3.0938\bar{9} & -0.93259\bar{9} \\ -0.93259\bar{9} & 0.28111\bar{5} \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.027459\bar{1} & -0.00015448\bar{8} \\ -0.00015448\bar{8} & 8.6916\bar{8} \times 10^{-7} \end{bmatrix}. \quad (9.52)$$

From (9.52), we observe that C_1 and C_2 are Hermitian, the eigenvalues of C_1 are 0 and $3.3750\bar{1}$, and the eigenvalues of C_2 are 0 and $0.0274\bar{6}$. Thus, we confirm that C_1 and C_2 each are nonnegative and have their rank equal to one.

In the next example, we illustrate the determination of the Gel'fand–Levitan normalized bound-state solutions. We continue to use as input the same potential V and the same boundary matrices A and B we have used in Examples 9.3–9.7.

Example 9.8. In this example, we still use the potential V given in (9.6) and the boundary matrices A and B given in (9.5). Using the explicit expression for the regular solution obtained in Example 9.3, the bound-state k -values listed in (9.25) and (9.26), and the Gel'fand–Levitan normalization matrices C_1 and C_2 listed in (9.52), with the help of (3.11) we obtain the Gel'fand–Levitan normalized bound-state solutions $\Phi_1(x)$ and $\Phi_2(x)$ as

$$\Phi_1(x) = \frac{e^{-\kappa_1 x}}{2 + e^{2x/3}} \begin{bmatrix} 2.5273\bar{4} + 1.1587\bar{4} e^{2x/3} & -0.76182\bar{3} - 0.34928\bar{1} e^{2x/3} \\ -5.5256\bar{4} - 2.8677\bar{5} e^{2x/3} & 1.6656\bar{1} + 0.86443\bar{3} e^{2x/3} \end{bmatrix}, \quad (9.53)$$

$$\Phi_2(x) = \frac{e^{-\kappa_2 x}}{2 + e^{2x/3}} \begin{bmatrix} -0.44109\bar{7} + 0.52254\bar{8} e^{2x/3} & 0.0024816\bar{6} - 0.0029399\bar{1} e^{2x/3} \\ -0.49632\bar{5} + 0.49493\bar{4} e^{2x/3} & 0.0027923\bar{8} - 0.0027845\bar{6} e^{2x/3} \end{bmatrix}. \quad (9.54)$$

From (9.53) and (9.54), we obtain the large x -asymptotics

$$\begin{aligned} \Phi_1(x) &= e^{-\kappa_1 x} \left(\begin{bmatrix} 1.1587\bar{4} & -0.34928\bar{1} \\ -2.8677\bar{5} & 0.86443\bar{3} \end{bmatrix} + o(1) \right), \quad x \rightarrow +\infty, \\ \Phi_2(x) &= e^{-\kappa_2 x} \left(\begin{bmatrix} 0.52254\bar{8} & -0.0029399\bar{1} \\ 0.49493\bar{4} & -0.0027845\bar{6} \end{bmatrix} + o(1) \right), \quad x \rightarrow +\infty. \end{aligned}$$

Using $\Phi_1(x)$ given in (9.53), $\Phi_2(x)$ given in (9.54), and the orthogonal projection matrices Q_1 and Q_2 given in the first equalities of (9.32) and (9.33), respectively, we confirm that the orthonormalizations stated in (3.12) and (3.13) indeed hold for the bound states at $k = i\kappa_1$ and $k = i\kappa_2$. From (9.53) and (9.54), we obtain

$$\Phi_1(0) = \begin{bmatrix} 1.2286\bar{9} & -0.37036\bar{8} \\ -2.797\bar{8} & 0.84334\bar{6} \end{bmatrix}, \quad \Phi_1'(0) = \begin{bmatrix} -6.2764\bar{1} & 1.8919\bar{1} \\ 14.240\bar{8} & -4.2926\bar{3} \end{bmatrix}, \quad (9.55)$$

$$\Phi_2(0) = \begin{bmatrix} 0.027150\bar{1} & -0.0001527\bar{5} \\ -0.00046346\bar{4} & 2.607\bar{5} \times 10^{-6} \end{bmatrix}, \quad \Phi_2'(0) = \begin{bmatrix} 0.10674\bar{7} & -0.00060056\bar{9} \\ 0.11014\bar{5} & -0.00061969\bar{1} \end{bmatrix}. \quad (9.56)$$

Using the boundary matrices A and B appearing in (9.5) and also using the expressions (9.55) and (9.56), we verify that each of the Gel'fand–Levitan normalized bound-state solutions $\Phi_1(x)$ and $\Phi_2(x)$ satisfies the boundary condition (2.7). Having obtained the Marchenko normalized bound-state solutions $\Psi_1(x)$ and $\Psi_2(x)$ and the Gel'fand–Levitan normalized bound-state solutions $\Phi_1(x)$ and $\Phi_2(x)$, we next determine the dependency matrices D_1 and D_2 . Using $\Psi_1(0)$ and $\Psi_2(0)$ listed in the first equalities of (9.44) and (9.45), respectively, we obtain the Moore–Penrose inverses $\Psi_1(0)^+$ and $\Psi_2(0)^+$ as

$$\Psi_1(0)^+ = \begin{bmatrix} 0.047199\bar{7} & -0.10747\bar{6} \\ -0.11681\bar{4} & 0.26599\bar{2} \end{bmatrix}, \quad \Psi_2(0)^+ = \begin{bmatrix} 26.733\bar{1} & -0.45634\bar{5} \\ 25.320\bar{4} & -0.4322\bar{3} \end{bmatrix}. \quad (9.57)$$

Using $x = 0$ in (3.21), we get

$$D_1 = \Psi_1(0)^+ \Phi_1(0), \quad D_2 = \Psi_2(0)^+ \Phi_2(0). \quad (9.58)$$

Using $\Phi_1(0)$ from the first equality of (9.55), $\Phi_2(0)$ from the first equality of (9.56), and $\Psi_1(0)^+$ and $\Psi_2(0)^+$ from (9.57), with the help of (9.58) we obtain the dependency matrix D_1 at the bound state $k = i\kappa_1$ and the dependency matrix D_2 at the bound state $k = i\kappa_2$ as

$$D_1 = \begin{bmatrix} 0.3586\bar{9} & -0.10812\bar{1} \\ -0.88772\bar{1} & 0.26758\bar{8} \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.72601\bar{8} & -0.0040846\bar{6} \\ 0.68765\bar{2} & -0.0038688\bar{1} \end{bmatrix}. \quad (9.59)$$

Using D_1 and D_2 appearing in (9.59), we verify the properties listed in (3.19) and (3.23)–(3.26).

Consider the orthogonal projection matrix P satisfying the first set of equalities in (3.1) in the special case when $n = 2$. The rank of P can only be 2, 1, or 0. If the rank of P is equal to 2, then P must be the 2×2 identity matrix. This can be seen as follows. From the first set of equalities in (3.1), we have $P^2 = P$, which can be written as $P(P - I) = 0$. Since P is invertible, by multiplying $P(P - I) = 0$ on the left by P^{-1} we obtain $P - I = 0$ or equivalently $P = I$. If the rank of P is zero, then P must be the 2×2 zero matrix. In the next theorem, we characterize all 2×2 orthogonal projection matrices with rank 1. The result is useful in the determination of the projection matrices P used in (7.19) and (8.22) in the transformation of the Jost matrix in adding a new bound state or in increasing the multiplicity of an existing bound state. Even though this result is not necessarily new, the elementary proof we provide yields some insight on the computation of such projection matrices not only in the 2×2 matrix case but also when the matrix size is larger.

Theorem 9.9. *Any 2×2 rank-one projection matrix P is given by*

$$P = \begin{bmatrix} \frac{1}{2} \pm \sqrt{\frac{1}{4} - \beta^2 - \gamma^2} & \beta + i\gamma \\ \beta - i\gamma & \frac{1}{2} \mp \sqrt{\frac{1}{4} - \beta^2 - \gamma^2} \end{bmatrix}, \quad (9.60)$$

where β and γ are two real constants satisfying $0 \leq \beta^2 + \gamma^2 \leq 1/4$.

Proof. Consider the orthogonal projection matrix P satisfying the first set of equalities in (3.1) in the special case when $n = 2$. We recall that we consider the case where the rank of P is equal to 1. Since P is self-adjoint, it must have the form

$$P = \begin{bmatrix} \alpha & \beta + i\gamma \\ \beta - i\gamma & \epsilon \end{bmatrix}, \quad (9.61)$$

for some real constants α, β, γ , and ϵ . The idempotent property $P^2 = P$ yields

$$\begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 & (\alpha + \epsilon)(\beta + i\gamma) \\ (\alpha + \epsilon)(\beta - i\gamma) & \epsilon^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta + i\gamma \\ \beta - i\gamma & \epsilon \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} \alpha^2 - \alpha + \beta^2 + \gamma^2 & (\alpha + \epsilon - 1)(\beta + i\gamma) \\ (\alpha + \epsilon - 1)(\beta - i\gamma) & \epsilon^2 - \epsilon + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (9.62)$$

The inspection of the (1, 2)-entries in (9.62) shows that we must have either $\beta + i\gamma = 0$ or $\alpha + \epsilon = 1$. In the former case, we must have $\beta = 0$ and $\gamma = 0$, and the inspection of the diagonal entries in (9.62) shows that the real parameters α and ϵ can only take the values of 0 or 1. This case yields the two possibilities for P given by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (9.63)$$

In the latter case, i.e. when we have $\alpha + \epsilon = 1$, the four constraints in (9.62) can equivalently be expressed as the single constraint given by

$$\alpha^2 - \alpha + |p|^2 = 0, \quad (9.64)$$

where we have let $p := \beta + i\gamma$. Solving (9.64), we obtain the two possible values for α given by

$$\alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} - |p|^2}, \quad (9.65)$$

with the constraint $0 < 4|p|^2 \leq 1$. Using (9.65) and $\alpha + \epsilon = 1$ in (9.61), we observe that P has the two possible forms given by

$$P = \begin{bmatrix} \frac{1}{2} \pm \sqrt{\frac{1}{4} - |p|^2} & p \\ p^* & \frac{1}{2} \mp \sqrt{\frac{1}{4} - |p|^2} \end{bmatrix}. \quad (9.66)$$

The determinant of either of the two matrices appearing on the right-hand side of (9.66) is zero, and hence those two matrices each have rank equal to one. The four possibilities for P given in (9.63) and (9.66) cover all the possibilities for any 2×2 rank-one projection matrix P . Thus, the proof is complete. $\square$

In the next theorem, in the 2×2 matrix case we show how the projection matrices $\hat{P}$ and $\hat{\hat{P}}$ used to add a new bound state and to increase the multiplicity of that bound state, respectively, are related to each other. This result is useful in practical computations, and it provides some insight into the process of increasing the multiplicity of a bound state in general. The notation used in the theorem is similar to the notation used in Theorem 8.1.

Theorem 9.10. *Let $\hat{P}$ denote an orthogonal projection matrix to be used in (7.19) in the 2×2 case in order to add a new bound-state with the multiplicity 1 at $k = i\tilde{\kappa}$. Let $\hat{\hat{P}}$ denote an orthogonal projection matrix to be used in (8.22) in the 2×2 case in order to increase the multiplicity of the bound-state at $k = i\tilde{\kappa}$ from 1 to 2. Then, $\hat{P}$ must be of the form given on the right-hand side of (9.60), and we must have $\hat{\hat{P}} = I - \hat{P}$. Consequently, we also have $\hat{P}\hat{\hat{P}} = 0$ and $\hat{\hat{P}}\hat{P} = 0$.*

Proof. Let us suppose that the unperturbed operator does not have a bound state at $k = i\tilde{\kappa}$ and let $J(k)$ be the corresponding unperturbed 2×2 Jost matrix. Let us add a bound state at $k = i\tilde{\kappa}$ with multiplicity one, and let us use $\hat{J}(k)$ to denote the corresponding Jost matrix for the perturbed operator. From (7.19), we have

$$\hat{J}(k) = \left[I - \frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} \hat{P} \right] J(k), \quad (9.67)$$

where I is the 2×2 identity matrix. We remark that the matrix $\hat{P}$ appearing in (9.67) must be equal to one of the two matrices given in (9.60) for some appropriate β and γ . Let us also increase the multiplicity of the bound state at $k = i\tilde{\kappa}$ from 1 to 2 by using the projection matrix $\hat{\hat{P}}$. As seen from (8.22), we then end up with the further perturbed Jost matrix $\hat{\hat{J}}(k)$, which is related to $\hat{J}(k)$ as

$$\hat{\hat{J}}(k) = \left[I - \frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} \hat{\hat{P}} \right] \hat{J}(k), \quad (9.68)$$

where the matrix $\hat{\hat{P}}$ should have one of the two forms appearing on the right-hand side of (9.60) with some appropriate β and γ . From (9.67) and (9.68), it follows that

$$\hat{\hat{J}}(k) = \left[I - \frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} \hat{\hat{P}} \right] \left[I - \frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} \hat{P} \right] J(k), \quad (9.69)$$

which can be written as

$$\hat{\hat{J}}(k) = \left[I - \frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} (\hat{\hat{P}} + \hat{P}) + \left(\frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} \right)^2 \hat{\hat{P}} \hat{P} \right] J(k). \quad (9.70)$$

If the bound state at $k = i\tilde{\kappa}$ with multiplicity 2 is added to the unperturbed problem, from (7.19) with the projection matrix equal to the 2×2 identity matrix, we get

$$\hat{\hat{J}}(k) = \left[I - \frac{2i\tilde{\kappa}}{k + i\tilde{\kappa}} I \right] J(k), \quad (9.71)$$

where we have also used Theorem 8.1 indicating that the addition of the bound state in one step or in two steps yields the same result. Subtracting (9.71) from (9.70), we obtain

$$\left[\frac{2i\tilde{k}}{k+i\tilde{k}} I - \frac{2i\tilde{k}}{k+i\tilde{k}} (\hat{P} + \hat{P}) + \left(\frac{2i\tilde{k}}{k+i\tilde{k}} \right)^2 \hat{P} \hat{P} \right] J(k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (9.72)$$

The Jost matrix $J(k)$ is invertible for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$ because it corresponds to the case with no bound states. Multiplying (9.72) on the right by the inverse of $J(k)$, we obtain

$$\frac{2i\tilde{k}}{k+i\tilde{k}} I - \frac{2i\tilde{k}}{k+i\tilde{k}} (\hat{P} + \hat{P}) + \left(\frac{2i\tilde{k}}{k+i\tilde{k}} \right)^2 \hat{P} \hat{P} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}, \quad (9.73)$$

from which we get $\hat{P} + \hat{P} = I$ and $\hat{P} \hat{P} = 0$. Thus, we have $\hat{P} = I - \hat{P}$. Multiplying the matrix equality $\hat{P} = I - \hat{P}$ with the matrix $\hat{P}$ on the right and using $\hat{P}^2 = \hat{P}$, we also get $\hat{P} \hat{P} = 0$. $\square$

The matrix equalities stated in Theorem 9.10, i.e.

$$\hat{P}_j + \hat{P}_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{P}_j \hat{P}_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (9.74)$$

are useful in the addition of a bound state at $k = i\tilde{k}_j$ and increasing the multiplicity of a bound state in the 2×2 matrix case as follows. It is enough to construct $\hat{J}(k)$, and this amounts to the determination of the two real constants β and γ and picking the correct sign in the diagonal entries on the right-hand side of (9.60). That task is accomplished by requiring that the orthogonal projection matrix $\hat{Q}_j$ appearing in the second set of equalities in (3.1) is the projection onto the kernel of $\hat{J}(i\tilde{k}_j)$. Note that $\hat{Q}_j$ is the orthogonal projection matrix onto the orthogonal complement of the kernel of the Gel'fand–Levitan normalization matrix $\hat{C}_j$ appearing in (3.17). Hence, $\hat{Q}_j$ can be determined from the Gel'fand–Levitan normalization matrix $\hat{C}_j$. In order to determine the exact form of $\hat{P}_j$ in (9.67), we simply impose the condition that $\hat{J}(i\tilde{k}_j) \hat{Q}_j = 0$, which allows us to pick the correct complex constant and the correct sign on the right-hand side in (9.60). The reasoning and conclusion presented in (9.67)–(9.74) in the 2×2 matrix case can readily be generalized to the case of adding or removing a bound state with any multiplicities. For example, we have the following results. If $\hat{P}$ is the $n \times n$ orthogonal projection matrix used in (9.67) to add a new bound state at $k = i\tilde{k}$ with multiplicity $\tilde{m}$ and if $\hat{P}$ is the $n \times n$ orthogonal projection matrix used in (9.68) to increase the multiplicity of the bound state at $k = i\tilde{k}$ from $\tilde{m}$ to n , then we get

$$\hat{P} = I - \hat{P}, \quad \hat{P} \hat{P} = 0, \quad \hat{P} \hat{P} = 0,$$

where we assume that $1 \leq \tilde{m} < n$.

In the next example, we deal with a self-adjoint potential that belongs to $L^1(\mathbb{R}^+)$ but not $L^1_1(\mathbb{R}^+)$. For such a potential, we show that the Jost solution $f(k, x)$, the physical solution $\Psi(k, x)$, the Jost matrix $J(k)$, and the scattering matrix $S(k)$ may not have continuous extensions to $k = 0$.

Example 9.11. The Whittaker equation [1]

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2} \right) w = 0, \quad (9.75)$$

has the two linearly independent solutions $M_{\kappa, \mu}(z)$ and $W_{\kappa, \mu}(z)$, which are known as the Whittaker functions. Using $\kappa = 0$ and $z = -2ik(1+x)$ in (9.75), we obtain the Schrödinger equation

$$-\frac{d^2 w}{dx^2} + V_\mu(x) w = k^2 w, \quad x \in \mathbb{R}^+, \quad (9.76)$$

where the potential $V_\mu(x)$ is given by

$$V_\mu(x) := \frac{\mu^2 - 1/4}{(1+x)^2}, \quad x \in \mathbb{R}^+. \quad (9.77)$$

We can express $M_{0, \mu}(-2ik(1+x))$ and $W_{0, \mu}(-2ik(1+x))$ in terms of Kummer's confluent hypergeometric function $M(a, b, z)$ and Tricomi's confluent hypergeometric function $U(a, b, z)$, respectively, as

$$M_{0, \mu}(-2ik(1+x)) = e^{ik(1+x)} [-2ik(1+x)]^{\mu+1/2} M(\mu+1/2, 1+2\mu, -2ik(1+x)), \quad (9.78)$$

$$W_{0, \mu}(-2ik(1+x)) = e^{ik(1+x)} [-2ik(1+x)]^{\mu+1/2} U(\mu+1/2, 1+2\mu, -2ik(1+x)). \quad (9.79)$$

The general solution to (9.76) is a linear combination of the two functions appearing on the right-hand sides of (9.78) and (9.79), respectively. The Jost solution $f_\mu(k, x)$ satisfying (2.10) and (9.76) is then obtained as

$$f_\mu(k, x) = e^{-ik} W_{0, \mu}(-2ik(1+x)), \quad \overline{\mathbb{C}^+} \setminus \{0\},$$

or equivalently as

$$f_\mu(k, x) = e^{ikx} [-2ik(1+x)]^{\mu+1/2} U(\mu+1/2, 1+2\mu, -2ik(1+x)), \quad \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.80)$$

The right-hand side of (9.80) can be expressed in terms of the modified Bessel function of the second kind $K_\mu(z)$, and we have

$$f_\mu(k, x) = \sqrt{\frac{2}{\pi}} e^{-ik} \sqrt{-ik(1+x)} K_\mu(-ik(1+x)), \quad \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.81)$$

Let us use the Dirichlet boundary condition by choosing the matrices A and B appearing in (2.7) as $A = 0$ and $B = 1$, where in the scalar case we use a scalar to represent a 1×1 matrix. In this case, from (2.11) and (9.80), we see that the Jost matrix $J_\mu(k)$ is equal to $f_\mu(k, 0)$, i.e. we have

$$J_\mu(k) = (-2ik)^{\mu+1/2} U(\mu + 1/2, 1 + 2\mu, -2ik). \quad (9.82)$$

By using (9.81) in (9.82), we get

$$J_\mu(k) = \sqrt{\frac{2}{\pi}} e^{-ik} \sqrt{-ik} K_\mu(-ik), \quad \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.83)$$

Using (9.83) in (2.12) and (2.13), we obtain the corresponding scattering matrix $S_\mu(k)$ and the physical solution $\Psi_\mu(k, x)$, respectively, as

$$S_\mu(k) = -\frac{i k e^{2ik} K_\mu(ik)}{\sqrt{k^2} K_\mu(-ik)}, \quad \mathbb{R} \setminus \{0\}, \quad (9.84)$$

$$\Psi_\mu(k, x) = f_\mu(-k, x) + f_\mu(k, x) S_\mu(k), \quad \mathbb{R} \setminus \{0\}. \quad (9.85)$$

The small k -asymptotics of the right-hand sides of (9.81)–(9.85) are obtained by using the small z -asymptotics of $K_\mu(z)$. For example, when we have

$$\mu = \sqrt{\ell(\ell + 1) + \frac{1}{4}},$$

where ℓ is a positive integer, the corresponding potential $V_\mu(x)$ appearing in (9.77) is given by

$$V_\mu(x) = \frac{\ell(\ell + 1)}{(1 + x)^2}, \quad x \in \mathbb{R}^+. \quad (9.86)$$

The Jost solution $f_\mu(k, x)$ corresponding to the potential $V_\mu(x)$ in (9.86) can be expressed in terms of the Hankel function of the first kind $H_\nu^{(1)}(z)$ of order ν , and we have

$$f_\mu(k, x) = i^{\ell+1} \sqrt{\frac{\pi k}{2}} (1 + x) H_{\ell+1/2}^{(1)}(k(1 + x)), \quad \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.87)$$

The corresponding Jost matrix $J_\mu(k)$ is obtained from (9.87) as $f_\mu(k, 0)$, and hence we get

$$J_\mu(k) = i^{\ell+1} \sqrt{\frac{\pi k}{2}} H_{\ell+1/2}^{(1)}(k), \quad \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.88)$$

Using (9.88) in (2.12), we obtain the corresponding scattering matrix $S_\mu(k)$ as

$$S_\mu(k) = -i \frac{H_{\ell+1/2}^{(1)}(-k)}{H_{\ell+1/2}^{(1)}(k)}, \quad \mathbb{R} \setminus \{0\}. \quad (9.89)$$

The corresponding physical solution $\Psi_\mu(k, x)$ is obtained by using (9.87) and (9.89) on the right-hand side of (9.85). Using the small k -asymptotics of the Hankel function of the first kind on the right-hand sides of (9.87) and (9.88), we obtain the respective asymptotics

$$f_\mu(k, x) = \frac{1}{[k(1+x)]^\ell} \left[\left(\frac{i}{2} \right)^\ell \frac{(2\ell)!}{\ell!} + O(k^2) \right], \quad k \rightarrow 0 \text{ in } \overline{\mathbb{C}^+} \setminus \{0\}, \quad (9.90)$$

$$J_\mu(k) = \frac{1}{k^\ell} \left[\left(\frac{i}{2} \right)^\ell \frac{(2\ell)!}{\ell!} + O(k^2) \right], \quad k \rightarrow 0 \text{ in } \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.91)$$

From (9.90) and (9.91), we observe that the right-hand sides of (9.90) and (9.91) each have a pole of order ℓ at $k = 0$. Thus, neither the zero-energy Jost solution $f_\mu(0, x)$ nor the zero-energy Jost matrix $J_\mu(0)$ exist. On the other hand, using (9.91) on the right-hand side of (2.12), we observe that the scattering matrix $S_\mu(k)$ given in (9.89) has a continuous extension to $k = 0$, and we have $S_\mu(0) = (-1)^{\ell+1}$. The zero-energy Schrödinger equation with the potential V_μ in (9.86) can be written as

$$-\psi'' + \frac{\ell(\ell+1)}{(1+x)^2} \psi = 0, \quad x \in \mathbb{R}^+, \quad (9.92)$$

and it has the two linearly independent solutions given by

$$\psi(x) = (1+x)^{1/2+\sqrt{\ell(\ell+1)+1/4}}, \quad \psi(x) = (1+x)^{1/2-\sqrt{\ell(\ell+1)+1/4}}, \quad x \in \mathbb{R}^+. \quad (9.93)$$

Any other solution to (9.92) must be a linear combination of the two solutions listed in (9.93). In this case, the zero-energy Jost solution $f_0(0, x)$ does not exist because we cannot construct a solution to (9.92) with the asymptotic behavior $1 + o(1)$ as $x \rightarrow +\infty$. This is because the first solution in (9.93) blows up and the second solution there decays as $x \rightarrow +\infty$ if ℓ is a positive integer. The zero-energy regular solution $\varphi_\mu(0, x)$ can be expressed as a linear combination of the two solutions in (9.93) as

$$\varphi_\mu(0, x) = \frac{(1+x)^{1/2+\sqrt{\ell(\ell+1)+1/4}} - (1+x)^{1/2-\sqrt{\ell(\ell+1)+1/4}}}{2\sqrt{\ell(\ell+1)+\frac{1}{4}}}, \quad x \in \mathbb{R}^+.$$

The corresponding zero-energy physical solution $\Psi_\mu(0, x)$ does not exist in this case as the following argument shows. Using (2.14), we express the physical solution $\Psi_\mu(k, x)$ in terms of the regular solution $\varphi_\mu(k, x)$ and the inverse of the Jost matrix $J_\mu(k)$ as

$$\Psi_\mu(k, x) = -\varphi_\mu(k, x)[2ikJ_\mu(k)^{-1}], \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.94)$$

From (9.91), we see that $J_\mu(k)^{-1} = O(k^\ell)$ as $k \rightarrow 0$. Since $\varphi_\mu(k, x)$ exists for $k \in \mathbb{C}$, we observe that the right-hand side of (9.94) has a limit as $k \rightarrow 0$, and that limit is equal to the function that is identically zero for all $x \geq 0$. However, we cannot use that limit as the zero-energy physical solution $\Psi_\mu(0, x)$ because the latter must be a nontrivial bounded

solution to (9.92). Since we cannot construct a nontrivial bounded solution to (9.92) as a linear combination of the two solutions listed in (9.93), we conclude that the zero-energy physical solution $\Psi_\mu(0, x)$ in this case does not exist. This is also compatible with the limit as $k \rightarrow 0$ for (9.85) and the fact that the zero-energy Jost solution $f_\mu(0, x)$ does not exist. In general, for other μ -values, the quantity $S_\mu(k)$ does not have a continuous extension to $k = 0$. For example, when $\mu = 0$ the potential $V_\mu(x)$ appearing in (9.77) is given by

$$V_0(x) = -\frac{1}{4(1+x)^2}, \quad x \in \mathbb{R}^+. \quad (9.95)$$

In that case, using $\mu = 0$ in (9.81)–(9.84), we obtain the Jost solution $f_0(k, x)$, the Jost matrix $J_0(k)$, and the scattering matrix $S_0(k)$, respectively, as

$$f_0(k, x) = \sqrt{\frac{2}{\pi}} e^{-ik} \sqrt{-ik(1+x)} K_0(-ik(1+x)), \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}, \quad (9.96)$$

$$J_0(k) = \sqrt{\frac{2}{\pi}} e^{-ik} \sqrt{-ik} K_0(-ik), \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}, \quad (9.97)$$

$$S_0(k) = -\frac{ik e^{2ik} K_0(ik)}{\sqrt{k^2} K_0(-ik)}, \quad k \in \mathbb{R} \setminus \{0\}. \quad (9.98)$$

The physical solution $\Psi_0(k, x)$ is obtained by using (9.96) and (9.98) in (2.13), and we get

$$\Psi_0(k, x) = f_0(-k, x) + f_0(k, x) S_0(k), \quad k \in \overline{\mathbb{C}^+} \setminus \{0\}. \quad (9.99)$$

Similarly, we obtain the regular solution $\varphi_0(k, x)$ by using (9.96) and (9.97) in (2.15), and we have

$$\varphi_0(k, x) = \frac{1}{2ik} [f_0(k, x) J_0(-k) - f_0(-k, x) J_0(k)], \quad k \in \mathbb{R} \setminus \{0\}. \quad (9.100)$$

The small k -asymptotics of $f_0(k, x)$ is obtained by letting $k \rightarrow 0$ on the right-hand side of (9.96). As $k \rightarrow 0$ in $\overline{\mathbb{C}^+}$, we get

$$f_0(k, x) = -\sqrt{\frac{2}{\pi}} \sqrt{-ik(1+x)} (\gamma + \log(-ik(1+x)/2)) + O(|k|^{3/2}), \quad (9.101)$$

where we use $\log$ to denote the complex-valued principal logarithm and use γ for the Euler–Mascheroni constant with the approximate value of 0.577. Similarly, from (9.97) we obtain the small k -asymptotics of $J_0(k)$ as

$$J_0(k) = -\sqrt{\frac{2}{\pi}} \sqrt{-ik} (\gamma + \log(-ik/2)) + O(|k|^{3/2}), \quad k \rightarrow 0 \text{ in } \overline{\mathbb{C}^+}. \quad (9.102)$$

From (9.102), we conclude that the Jost matrix $J_0(k)$, which is defined when $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, extends continuously to $k = 0$ and satisfies $J_0(0) = 0$. On the other hand, using (9.102)

in (2.12), we see that the scattering matrix $S_0(k)$, which is defined when $k \in \mathbb{R} \setminus \{0\}$, has the right and left limits, obtained as $k \rightarrow 0^+$ and $k \rightarrow 0^-$, respectively, and we have

$$S_0(0^+) = -i, \quad S_0(0^-) = i. \quad (9.103)$$

From (9.103), we conclude that $S_0(k)$ cannot be extended to $k = 0$ in a continuous manner. Hence, $S_0(0)$ does not exist. From (9.101), we observe that the limit as $k \rightarrow 0$ for the Jost solution $f_0(k, x)$ is identically zero. Then, since $S_0(k)$ has modulus 1, from (9.99) we conclude that the limit as $k \rightarrow 0$ for the physical solution $\Psi_0(k, x)$ is also identically zero. In this case, we conclude that neither the Jost solution $f_0(k, x)$ nor the physical solution $\Psi_0(k, x)$ can be defined at $k = 0$. If $f_0(0, x)$ existed at $k = 0$, it would have to satisfy the asymptotics $f_0(0, x) = 1 + o(1)$ as $x \rightarrow +\infty$. Similarly, if $\Psi_0(0, x)$ existed, it would have to be a nontrivial bounded solution to the zero-energy Schrödinger equation. The zero-energy Schrödinger equation with the potential $V_0(x)$ appearing in (9.95) is given by

$$-\psi'' - \frac{1}{4(1+x)^2} \psi = 0, \quad x \in \mathbb{R}^+, \quad (9.104)$$

and it has two linearly independent solutions given by

$$\psi(x) = \sqrt{1+x}, \quad \psi(x) = \sqrt{1+x} \ln(1+x), \quad x \in \mathbb{R}^+. \quad (9.105)$$

where $\ln$ denotes the real-valued natural logarithm function. By using a linear combination of the two solutions listed in (9.105), it is impossible to construct a solution to (9.104) with the required asymptotic behavior as $x \rightarrow +\infty$. Thus, the zero-energy Jost solution $f_0(0, x)$ does not exist, and the question regarding the continuity at $k = 0$ for the Jost solution $f_0(k, x)$ becomes irrelevant. Similarly, the zero-energy physical solution $\Psi_0(0, x)$ does not exist, and the question regarding the continuity at $k = 0$ for the physical solution $\Psi_0(k, x)$ becomes irrelevant. The fact that the limit as $k \rightarrow 0$ for the right-hand side of (9.101) is identically zero does not mean that the zero-energy Jost solution $f_0(0, x)$ is given by the function that is identically zero. Similarly, the fact that the limit as $k \rightarrow 0$ for the right-hand side of (9.99) is identically zero does not mean that the zero-energy physical solution $\Psi_0(0, x)$ is given by the function that is identically zero. As we have already observed, the Jost matrix $J_0(k)$, defined when $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, extends continuously to $k = 0$ and satisfies $J_0(0) = 0$. Since the Jost solution $f_0(k, x)$ does not exist at $k = 0$, the zero-energy Jost matrix $J_0(0)$ cannot be defined by using (2.11) at $k = 0$. The regular solution given in (9.100) has a limit as $k \rightarrow 0$. That limit can be evaluated by using (9.101) and (9.102) on the right-hand side of (9.100), and it is equal to $\sqrt{1+x} \ln(1+x)$. In fact, with the help of (9.105), we can construct the zero-energy regular solution satisfying the initial conditions (2.6) with $A = 0$ and $B = 1$. We then get

$$\varphi_0(0, x) = \sqrt{1+x} \ln(1+x), \quad x \in \mathbb{R}^+. \quad (9.106)$$

This confirms that $\varphi_0(k, x)$ is continuous at $k = 0$. We note that a regular solution to the Schrödinger equation exists for $k \in \mathbb{C}$ whenever the potential V is locally integrable for $x \in [0, +\infty)$.

10 Examples of changing the discrete spectrum

In this section, we provide some illustrative examples to demonstrate some modifications in the spectrum of the matrix-valued Schrödinger operator. In the explicit examples provided, we start with an unperturbed Schrödinger operator for which we know the potential V and the boundary matrices A and B . We then evaluate the corresponding perturbed Schrödinger operator and the relevant quantities when a discrete eigenvalue is added to or removed from the spectrum. We supplement our examples with various remarks for better understanding of the methods used. Such remarks include a correction to the result presented in (IV.1.30) of [6] concerning asymptotic behavior of the potential increment when a discrete eigenvalue is added to the spectrum of the scalar-valued Schrödinger operator with the Dirichlet boundary condition.

In our first example, we illustrate the addition of a discrete eigenvalue in the scalar case with a non-Dirichlet boundary condition. This example demonstrates that the addition of a bound state in the non-Dirichlet case does not necessarily change the zero unperturbed potential, but it changes the unperturbed boundary condition so that the perturbed problem has a bound state as a result of the perturbed boundary condition. This example was treated in [2] by a different method.

Example 10.1. In the scalar case, i.e. when $n = 1$, let us choose the unperturbed potential V and the boundary matrices A and B as

$$V(x) \equiv 0, \quad (A, B) = (1, \kappa_1), \quad (10.1)$$

where κ_1 is a positive parameter and we omit the matrix brackets in the 1×1 matrix case. Using the input data set in (10.1), we evaluate the Jost solution $f(k, x)$, the regular solution $\varphi(k, x)$, the Jost matrix $J(k)$, and the scattering matrix $S(k)$, respectively, as

$$f(k, x) = e^{ikx}, \quad \varphi(k, x) = \cos(kx) + \frac{\kappa_1}{k} \sin(kx), \quad (10.2)$$

$$J(k) = -i(k + i\kappa_1), \quad S(k) = \frac{k - i\kappa_1}{k + i\kappa_1}. \quad (10.3)$$

From the first equality of (10.3), we see that $J(k)$ has no zeros on the positive imaginary axis in the complex k -plane, and hence the unperturbed problem has no bound states. To obtain our perturbed problem, we add one bound state at $k = i\tilde{\kappa}_1$ with the Gel'fand–Levitan normalization matrix $\tilde{C}_1$ by using

$$\tilde{\kappa}_1 = \kappa_1, \quad \tilde{C}_1 = \tilde{c}_1, \quad (10.4)$$

where $\tilde{c}_1$ is a positive constant. To evaluate the relevant quantities for the perturbed problem, we use the method of Section 7. From the second equality of (10.2), we obtain

$$\varphi(i\tilde{\kappa}_1, x) = e^{\kappa_1 x}. \quad (10.5)$$

Using (10.5) and the second equality of (10.2) in (7.6), we get

$$\xi_1(x) = \tilde{c}_1 e^{\kappa_1 x}. \quad (10.6)$$

Since we are in the scalar case, the orthogonal projection matrices $\tilde{P}_1$ and $\tilde{Q}_1$ are both equal to 1, i.e. we have

$$\tilde{P}_1 = 1, \quad \tilde{Q}_1 = 1. \quad (10.7)$$

Using (10.6) and (10.7) in (7.9), we obtain

$$\Omega_1(x) = 1 + \frac{\tilde{c}_1^2}{2\kappa_1} [e^{2\kappa_1 x} - 1]. \quad (10.8)$$

With the help of (10.6), (10.8), and the first equality of (10.1), from (7.10) we obtain the perturbed potential $\tilde{V}$ as

$$\tilde{V}(x) = \frac{8\tilde{c}_1^2 \kappa_1^2 (\tilde{c}_1^2 - 2\kappa_1) e^{2\kappa_1 x}}{(2\kappa_1 - \tilde{c}_1^2 + \tilde{c}_1^2 e^{2\kappa_1 x})^2}, \quad x \in \mathbb{R}^+. \quad (10.9)$$

Using the second equalities of (10.1) and (10.4) in (7.13), we obtain the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = 1, \quad \tilde{B} = \kappa_1 - \tilde{c}_1^2.$$

With the help of (10.6), (10.8), and the second equality of (10.2), from (7.12) we obtain the perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \cos(kx) + \frac{\kappa_1 \sin(kx)}{k} \frac{2\kappa_1 - \tilde{c}_1^2 - \tilde{c}_1^2 e^{2\kappa_1 x}}{2\kappa_1 - \tilde{c}_1^2 + \tilde{c}_1^2 e^{2\kappa_1 x}}.$$

The perturbed Jost solution $\tilde{f}(k, x)$ is obtained from (7.26), by using (10.6), (10.8), and the first equalities of (10.2), (10.4), and (10.7) as

$$\tilde{f}(k, x) = e^{ikx} \left[1 + \frac{2i\kappa_1(\tilde{c}_1^2 - 2\kappa_1)}{(k + i\kappa_1)(2\kappa_1 - \tilde{c}_1^2 + \tilde{c}_1^2 e^{2\kappa_1 x})} \right]. \quad (10.10)$$

The perturbed Jost matrix $\tilde{J}(k)$ is obtained by using the first equalities of (10.3), (10.4), and (10.7) in (7.19) as

$$\tilde{J}(k) = -i(k - i\kappa_1), \quad (10.11)$$

which has a zero on the positive imaginary axis at $k = i\kappa_1$, confirming that the perturbed problem has a bound state at $k = i\kappa_1$. Using (10.11) in (7.24), we obtain the perturbed scattering matrix $\tilde{S}(k)$ as

$$\tilde{S}(k) = \frac{k + i\kappa_1}{k - i\kappa_1}. \quad (10.12)$$

From (10.9), we see that the perturbed potential becomes $\tilde{V}(x) \equiv 0$ if the positive parameter $\tilde{c}_1$ in (10.4) is chosen as $\tilde{c}_1 = \sqrt{2\kappa_1}$. In that case, (10.10) yields $\tilde{f}(k, x) = e^{ikx}$. For any other choices for $\tilde{c}_1$, the support of $\tilde{V}$ is the interval $[0, +\infty)$, and from (10.9) we get

$$\tilde{V}(x) = O(e^{-2\kappa_1 x}), \quad x \rightarrow +\infty. \quad (10.13)$$

The asymptotic decay estimate in (10.13) is consistent with our general estimate (7.18).

We remark that any example where the perturbed problem is obtained by adding a bound state to the unperturbed problem also provides an example where a bound state is removed, and this is done by interchanging the roles of the perturbed and unperturbed problems. In the next example, we provide such an example by using the quantities in Example 10.1 to illustrate the removal of a bound state.

Example 10.2. In the scalar case, i.e. when $n = 1$, as the unperturbed potential V let us use

$$V(x) = \frac{8\kappa_1^2(c_1^2 - 2\kappa_1)e^{2\kappa_1 x}}{(2\kappa_1 - c_1^2 + c_1^2 e^{2\kappa_1 x})^2}, \quad x \in \mathbb{R}^+, \quad (10.14)$$

and as the unperturbed boundary matrices A and B let us use

$$A = 1, \quad B = \kappa_1 - c_1^2, \quad (10.15)$$

where κ_1 and c_1 are two positive parameters. Corresponding to the input data set specified in (10.14) and (10.15), with the help of (9.4) we obtain the Jost solution $f(k, x)$. We then use (2.11), (2.12), and (2.15) to construct the Jost matrix $J(k)$, the scattering matrix $S(k)$, and the regular solution $\phi(k, x)$. We get

$$f(k, x) = e^{ikx} \left[1 + \frac{2i\kappa_1(c_1^2 - 2\kappa_1)}{(k + i\kappa_1)(2\kappa_1 - c_1^2 + c_1^2 e^{2\kappa_1 x})} \right], \quad (10.16)$$

$$J(k) = -i(k - i\kappa_1), \quad S(k) = \frac{k + i\kappa_1}{k - i\kappa_1}, \quad (10.17)$$

$$\phi(k, x) = \cos(kx) + \frac{\kappa_1 \sin(kx)}{k} \frac{2\kappa_1 - c_1^2 - c_1^2 e^{2\kappa_1 x}}{2\kappa_1 - c_1^2 + c_1^2 e^{2\kappa_1 x}}. \quad (10.18)$$

From the first equality of (10.17), we observe that $J(k)$ has a zero on the positive imaginary axis, i.e. at $k = i\kappa_1$. Thus, the unperturbed problem has a bound state at $k = i\kappa_1$. In

the scalar case, the orthogonal projections P_1 and Q_1 associated with the bound state at $k = i\kappa_1$ are given by

$$P_1 = 1, \quad Q_1 = 1. \quad (10.19)$$

From (10.18) we have

$$\varphi(i\kappa_1, x) = \frac{2\kappa_1 e^{\kappa_1 x}}{2\kappa_1 - c_1^2 + c_1^2 e^{2\kappa_1 x}}. \quad (10.20)$$

Using (10.20) and the second equality of (10.19), with the help of (3.14), (3.15), and (3.17) we obtain the Gel'fand–Levitan normalization matrix, which is a scalar in this case, as

$$C_1 = c_1, \quad (10.21)$$

where c_1 is the positive constant appearing in (10.14)–(10.18) and (10.20). Using (10.20) and (10.21) in (3.11), we get

$$\Phi_1(x) = \frac{2c_1\kappa_1 e^{\kappa_1 x}}{2\kappa_1 - c_1^2 + c_1^2 e^{2\kappa_1 x}}. \quad (10.22)$$

Using (10.22) in (5.5), we obtain

$$W_1(x) = \frac{2\kappa_1}{2\kappa_1 - c_1^2 + c_1^2 e^{2\kappa_1 x}}. \quad (10.23)$$

With the help of (10.14), (10.22), and (10.23), from (5.8) we get the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) \equiv 0.$$

Next, using (10.18), (10.22), and (10.23) in (5.7), we obtain the perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \cos(kx) + \frac{\kappa_1}{k} \sin(kx).$$

Similarly, using the first equalities of (10.17) and (10.19) in (5.13), we obtain the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = -i(k + i\kappa_1),$$

from the zero of which at $k = -i\kappa_1$ we conclude that the perturbed operator does not have any bound states. The perturbed scattering matrix $\tilde{S}(k)$ is obtained by using the second equality of (10.17) and the first equality of (10.19) in (5.15) as

$$\tilde{S}(k) = \frac{k - i\kappa_1}{k + i\kappa_1}.$$

The perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ are obtained from (5.9) by using (10.15) and (10.21), and we have

$$\tilde{A} = 1, \quad \tilde{B} = \kappa_1.$$

Finally, using (10.16), (10.22), (10.23), and the first equality of (10.19) in (5.18), we obtain the perturbed Jost solution $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) \equiv e^{ikx}.$$

Thus, we confirm that the method of Section 5 used in this example to remove a bound state and the method of Section 7 used in Example 10.2 to add a bound state are complementary in the sense that the removal undoes the addition of the bound state.

In (IV.1.30) on p. 63 of [6], it is claimed that, in the scalar case with the Dirichlet boundary condition, when we add a bound state at $k = i\kappa$, the potential increment $\tilde{V}(x) - V(x)$ decays as $Fe^{-2\kappa x}$ as $x \rightarrow +\infty$, where F is a nonzero constant. The following example shows that the result (IV.1.30) of [6] is incorrect. In order to demonstrate that the aforementioned incorrectness is independent of the asymptotic decay of the unperturbed potential V , we include here the result in Example 8.9 of [4] but by also providing a full complement to that result.

Example 10.3. Consider the scalar case, i.e. when $n = 1$, where the unperturbed potential is given by

$$V(x) = -\frac{8e^{2x}}{(1 + e^{2x})^2}, \quad x \in \mathbb{R}^+,$$

and the boundary matrices A and B , which are scalars in this case, are given by

$$A = 0, \quad B = -1. \tag{10.24}$$

Using the boundary matrices of (10.24) in (2.7), we observe that the corresponding boundary condition is the Dirichlet boundary condition $\psi(0) = 0$. We apply the method of Section 7 as detailed in Example 10.1 in order to add a bound state to the unperturbed operator. The corresponding Jost solution $f(k, x)$, regular solution $\phi(k, x)$, Jost matrix $J(k)$, and the scattering matrix $S(k)$ are obtained as

$$\begin{aligned} f(k, x) &= e^{ikx} \left[1 - \frac{2i}{(k + i)(1 + e^{2x})} \right], \\ \phi(k, x) &= -\frac{k \sin(kx) + \cos(kx) \tanh x}{k^2 + 1}, \end{aligned}$$

$$\begin{aligned} J(k) &= -\frac{k}{k+i}, \\ S(k) &= -\frac{k+i}{k-i}. \end{aligned} \quad (10.25)$$

From the zero of the Jost matrix in (10.25), we see that the unperturbed operator does not have any bound states. Let us add a bound state at $k = i\kappa_1$ with the Gel'fand–Levitan normalization matrix C_1 , where we have

$$\kappa_1 = 1, \quad C_1 = 4. \quad (10.26)$$

Although this particular case is already covered in Example 8.9 of [4], we include it here to consider all the possibilities in order to demonstrate that our result is independent of the asymptotic decay of the unperturbed potential $V(x)$ as $x \rightarrow +\infty$. The corresponding perturbed Jost matrix $\tilde{J}(k)$, perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$, and perturbed potential $\tilde{V}(x)$ are given by

$$\begin{aligned} \tilde{J}(k) &= -\frac{k(k-i)}{(k+i)^2}, \\ \tilde{A} &= 0, \quad \tilde{B} = -1. \\ \tilde{V}(x) &= \frac{q_{13}(x) + q_{14}(x)}{[(1-2x)\cosh x + (1+4x^2 + \cosh(2x))\sinh x]^2}, \quad x \in \mathbb{R}^+, \end{aligned}$$

where we have defined

$$\begin{aligned} q_{13}(x) &:= 7 - 24x + 32x^4 + 64x^2 \cosh(2x) - (16 + 32x) \sinh(2x), \\ q_{14}(x) &:= -(9 + 8x^2) \cosh(4x) + (-2 + 20x) \sinh(4x). \end{aligned}$$

With the choices in (10.26), the unperturbed potential $V(x)$ behaves as $O(e^{-2x})$ as $x \rightarrow +\infty$. The perturbed potential $\tilde{V}(x)$ behaves as $O(x^2 e^{-2x})$ as $x \rightarrow +\infty$, and hence the potential difference $\tilde{V}(x) - V(x)$ also behaves as $O(x^2 e^{-2x})$ as $x \rightarrow +\infty$. This is in contradiction to the result stated in (IV.1.30) of [6], which incorrectly predicts $\tilde{V}(x) - V(x) = O(e^{-2x})$ as $x \rightarrow +\infty$. We remark that our decay estimate of $O(x^2 e^{-2x})$ for $\tilde{V}(x) - V(x)$ is consistent with our general estimate given in (7.14). In this case, our estimate in (7.14) is not sharp. Instead of using the values in (10.26), let us add a bound state at $k = i\kappa_1$ with the Gel'fand–Levitan normalization matrix C_1 , where we have

$$\kappa_1 = 2, \quad C_1 = 3. \quad (10.27)$$

The corresponding perturbed Jost matrix $\tilde{J}(k)$, the perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$, and the perturbed potential $\tilde{V}(x)$ are then given by

$$\tilde{J}(k) = -\frac{k(k-2i)}{(k+i)(k+2i)},$$

$$\tilde{A} = 0, \quad \tilde{B} = -1,$$

$$\tilde{V}(x) = \frac{q_{15}(x) + q_{16}(x) + q_{17}(x)}{[(16 - 24x) \cosh x - 8 \sinh x + 9 \sinh(3x) + \sinh(5x)]^2}, \quad x \in \mathbb{R}^+,$$

where we have let

$$\begin{aligned} q_{15}(x) &:= 2468 + 1536x - 1152x^2 - 1728 \cosh(2x) - 1152 \cosh(4x) - 64 \cosh(6x), \\ q_{16}(x) &:= -36 \cosh(8x) - 2304 \sinh(2x) + 3456x \sinh(2x) - 1152 \sinh(4x), \\ q_{17}(x) &:= 1728x \sinh(4x) - 256 \sinh(6x) + 384x \sinh(6x). \end{aligned}$$

With the choices in (10.27), the unperturbed potential $V(x)$, the perturbed potential $\tilde{V}(x)$, and the potential difference $\tilde{V}(x) - V(x)$ each behave as $O(e^{-2x})$ as $x \rightarrow +\infty$. The result stated in (IV.1.30) of [6] incorrectly predicts $\tilde{V}(x) - V(x) = O(e^{-4x})$ as $x \rightarrow +\infty$. Our decay estimate of $O(e^{-2x})$ for $\tilde{V}(x) - V(x)$ in this case agrees with our general asymptotic decay estimate stated in (7.14). In this case, our estimate in (7.14) is sharp. Instead of using the values in (10.26) or in (10.27), let us add a bound state at $k = i\kappa_1$ with the Gel'fand–Levitan normalization matrix C_1 , where we have

$$\kappa_1 = 3, \quad C_1 = 1. \quad (10.28)$$

The corresponding perturbed Jost matrix $\tilde{J}(k)$, perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$, and perturbed potential $\tilde{V}(x)$ are given by

$$\begin{aligned} \tilde{J}(k) &= -\frac{k(k - 3i)}{(k + i)(k + 3i)}, \\ \tilde{A} &= 0, \quad \tilde{B} = -1, \\ \tilde{V}(x) &= \frac{q_{18}(x) + q_{19}(x) + q_{20}(x) + q_{21}(x)}{[192 - 12x - 3 \sinh(2x) + 3 \sinh(4x) + \sinh(6x)]^2}, \quad x \in \mathbb{R}^+, \end{aligned}$$

where we have defined

$$\begin{aligned} q_{18}(x) &:= -2 \operatorname{sech}^2 x + 73728 \cosh^3 x \sinh x - 4608x \cosh^3 x \sinh x, \\ q_{19}(x) &:= -221184 \cosh^3 x \cosh(2x) \sinh x + 13824x \cosh^3 x \cosh(2x) \sinh x, \\ q_{20}(x) &:= -6336 \cosh^3 x \sinh x \sinh(2x) + 1152 \cosh^3 x \sinh x \sinh(4x), \\ q_{21}(x) &:= -192 \cosh^3 x \sinh x \sinh(6x). \end{aligned}$$

With the choices in (10.28), the unperturbed potential $V(x)$, the perturbed potential $\tilde{V}(x)$, and the potential difference $\tilde{V}(x) - V(x)$ each behave as $O(e^{-2x})$ as $x \rightarrow +\infty$. This is also in contradiction to the result stated in (IV.1.30) of [6], which incorrectly predicts the asymptotic decay $\tilde{V}(x) - V(x) = O(e^{-6x})$ as $x \rightarrow +\infty$. Our asymptotic decay estimate of $O(e^{-2x})$ for $\tilde{V}(x) - V(x)$ agrees with the asymptotic estimate stated in (7.14). In this case, our estimate in (7.14) is sharp.

In the next example, we consider the addition of a bound state of multiplicity 1 in the 2×2 matrix case with a non-Dirichlet boundary condition. In the example, we illustrate the construction of the orthogonal projection matrix $\tilde{P}_{N+1}$ appearing in (7.19) by using two methods. The first method uses the construction specified (7.20), and the second method uses the procedure described in Theorem 9.9.

Example 10.4. In this example we use the unperturbed potential $V(x)$, and the boundary matrices A and B given by

$$V(x) = -\frac{8e^{2x/3}}{9(1+2e^{2x/3})^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad x \in \mathbb{R}^+, \quad (10.29)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \frac{1}{18} \begin{bmatrix} 17 & -1 \\ -1 & 17 \end{bmatrix}. \quad (10.30)$$

As described in Example 9.3, we construct the corresponding Jost solution $f(k, x)$, Jost matrix $J(k)$, and regular solution as

$$f(k, x) = e^{ikx} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{i}{(3k+i)(1+2e^{2x/3})} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right), \quad (10.31)$$

$$J(k) = \frac{-i(k+i)}{2(3k+i)} \begin{bmatrix} 6k+i & -i \\ -i & 6k+i \end{bmatrix}, \quad (10.32)$$

$$\varphi(k, x) = \frac{1}{q_{22}(k, x)} \begin{bmatrix} q_{23}(k, x) - q_{23}(-k, x) & q_{24}(k, x) - q_{24}(-k, x) \\ q_{24}(k, x) - q_{24}(-k, x) & q_{23}(k, x) - q_{23}(-k, x) \end{bmatrix}, \quad (10.33)$$

where we have defined

$$q_{22}(k, x) := 4k(3k+i)(3k-i)(1+2e^{2x/3}), \quad (10.34)$$

$$q_{23}(k, x) := e^{ikx}(k-i)[18k^2 - 3ik + 1 + e^{2x/3}(36k^2 + 6ik + 2)], \quad (10.35)$$

$$q_{24}(k, x) := e^{ikx}(k-i)[-3ik - 1 + e^{2x/3}(6ik - 2)]. \quad (10.36)$$

Even though $q_{22}(k, x)$ appearing in the denominator on the right-hand side of (10.33) has simple zeros at $k = 0$, $k = i/3$, and $k = -i/3$, that right-hand side in (10.33) is entire in k . Hence, those three singularities of the right-hand side in (10.33) are removable singularities. From (10.32), we evaluate the determinant of $J(k)$ as

$$\det[J(k)] = -\frac{3k(k+i)^2}{3k+i}.$$

From (10.32) we obtain

$$J(i) = \frac{1}{4} \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}. \quad (10.37)$$

Since $\det[J(k)]$ does not have a zero on the positive imaginary axis, the unperturbed operator does not have any bound states. Using the method of Section 7, let us add a simple bound state at $k = i\tilde{\kappa}_1$ with the Gel'fand–Levitan normalization matrix $\tilde{C}_1$, where we use

$$\tilde{\kappa}_1 = 1, \quad \tilde{C}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \quad (10.38)$$

From (3.17) and the second equality of (10.38), we see that the corresponding orthogonal projection $\tilde{Q}_1$ associated with the bound-state data in (10.38) is given by

$$\tilde{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (10.39)$$

Let us remark that $\tilde{C}_1$ and $\tilde{Q}_1$ are compatible. In fact, if we start with the matrix $\tilde{Q}_1$ given in (10.39) and try to construct all nonzero nonnegative matrices $\tilde{C}_1$ satisfying

$$\tilde{C}_1 \tilde{Q}_1 = \tilde{Q}_1 \tilde{C}_1 = \tilde{C}_1, \quad (10.40)$$

we observe that $\tilde{C}_1$ must have the form

$$\tilde{C}_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},$$

where a is a positive constant. In our example, we simply let $a = 2$. Since the rank of $\tilde{Q}_1$ is equal to one, the corresponding orthogonal projection $\tilde{P}_1$ must also have rank one. Hence, $\tilde{P}_1$ must be equal to one of the two matrices appearing on the right-hand side of (9.66). In order to determine the exact matrix $\tilde{P}_1$, we proceed as in Theorem 9.9 and impose the condition that $\tilde{J}(i) \tilde{Q}_1 = 0$, which is equivalent to

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2i}{2i} \begin{bmatrix} \frac{1}{2} \pm \sqrt{\frac{1}{4} - |\tilde{p}_1|^2} & \tilde{p}_1 \\ \tilde{p}_1^* & \frac{1}{2} \mp \sqrt{\frac{1}{4} - |\tilde{p}_1|^2} \end{bmatrix} \right) \frac{1}{4} \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (10.41)$$

We remark that, in order to establish (10.41), we have used the value of $\tilde{J}(i)$ from (7.19) and the value of $\tilde{Q}_1$ from (10.39). In the evaluation of $\tilde{J}(i)$, we have in turn used the value of the projection matrix in (9.66) and the value of $J(i)$ in (10.32). We note that (10.41) is equivalent to a system of equations containing the unknown $\tilde{p}_1$, and we have

$$\begin{cases} 7(\frac{1}{2} \mp \sqrt{\frac{1}{4} - |\tilde{p}_1|^2}) + \tilde{p}_1 = 0, \\ -7\tilde{p}_1^* - 7(\frac{1}{2} \pm \sqrt{\frac{1}{4} - |\tilde{p}_1|^2}) = 0, \end{cases} \quad (10.42)$$

where we recall that the asterisk is used to denote complex conjugation. We write the system in (10.42) as

$$\begin{cases} 7(\frac{1}{2} \mp \sqrt{\frac{1}{4} - |\tilde{p}_1|^2}) + \tilde{p}_1 = 0, \\ 49\tilde{p}_1^* + 7(\frac{1}{2} \pm \sqrt{\frac{1}{4} - |\tilde{p}_1|^2}) = 0, \end{cases} \quad (10.43)$$

which is obtained from (10.42) by multiplying the second equality by -7 on both sides. By adding the two equalities in (10.43) we obtain

$$7 + \tilde{p}_1 + 49\tilde{p}_1^* = 0,$$

and hence $\tilde{p}_1$ must be real and we must have $\tilde{p}_1 = -7/50$. One can directly verify that any one of the three equivalent systems in (10.41)–(10.43) is satisfied when $\tilde{p}_1 = -7/50$ provided we use only the upper signs there. Thus, the orthogonal projection matrix $\tilde{P}_1$ is given by

$$\tilde{P}_1 = \frac{1}{50} \begin{bmatrix} 49 & -7 \\ -7 & 1 \end{bmatrix}. \quad (10.44)$$

Let us construct the same matrix $\tilde{P}_1$ appearing in (10.44) by using (7.20). For this, we proceed as follows. We first need to construct the invertible matrix L_1 appearing in (7.20) when $N = 1$. From (7.21), we observe that L_1 is the constant 2×2 matrix obtained as the coefficient of the term e^x in the expression for $\varphi(i, x)$. With the help of (10.33)–(10.36), we see that L_1 is given by

$$L_1 = \lim_{x \rightarrow +\infty} \frac{-e^x}{q_{22}(i, x)} \begin{bmatrix} q_{23}(-i, x) & q_{24}(-i, x) \\ q_{24}(-i, x) & q_{23}(-i, x) \end{bmatrix},$$

which yields

$$L_1 = \frac{1}{8} \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}, \quad (10.45)$$

Using (10.38) and (10.45), we obtain

$$L_1 \tilde{C}_1 = \frac{1}{4} \begin{bmatrix} 7 & 0 \\ -1 & 0 \end{bmatrix}. \quad (10.46)$$

Finally, using (10.46) in (7.20), we get $\tilde{P}_1$ and observe that it is identical to the matrix appearing on the right-hand side of (10.44). The perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ are obtained by using (10.30) and the second equality of (10.38), and we have

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \frac{1}{18} \begin{bmatrix} -55 & -1 \\ -1 & 17 \end{bmatrix}.$$

We obtain the perturbed Jost matrix $\tilde{J}(k)$ by using (10.32), the first equality of (10.38), and (10.44), and we get

$$\tilde{J}(k) = \frac{1}{50(3k+i)} \begin{bmatrix} (k-i)(-150ik+31) & 17k+31i \\ 17(k-i) & -150ik^2+169k+17i \end{bmatrix}. \quad (10.47)$$

Using (10.47) in (2.12), we obtain the corresponding scattering matrix $\tilde{S}(k)$ as

$$\tilde{S}(k) = \frac{1}{625(k^2 + 1)(3k - i)} \begin{bmatrix} q_{25}(k) & 17i(31 - 25k^2) \\ 17i(31 - 25k^2) & q_{25}(k)^* \end{bmatrix}, \quad (10.48)$$

where we have defined

$$q_{25}(k) := 336i - 1525k + 3600ik^2 + 1875k^3.$$

From (10.47), we see that the determinant of the perturbed Jost matrix is given by

$$\det[\tilde{J}(k)] = \frac{-3k(k + i)(k - i)}{3k + i},$$

from which we see that there is a simple bound state at $k = i$. The corresponding perturbed potential $\tilde{V}(x)$ is obtained as in (7.10) with the help of (7.6) and (7.9). We obtain

$$\tilde{V}(x) = -\frac{8e^{2x/3}}{9(-9 - 18e^{2x/3} + 26e^{2x} + 25e^{8x/3})^2} \begin{bmatrix} q_{26}(x) & q_{27}(x) \\ q_{27}(x) & q_{28}(x) \end{bmatrix}, \quad x \in \mathbb{R}^+,$$

where we have defined

$$\begin{aligned} q_{26}(x) &:= 81 - 2025e^{4x/3} - 5328e^{2x} - 3969e^{8x/3} + 361e^{4x}, \\ q_{27}(x) &:= -81 + 405e^{4x/3} + 144e^{2x} - 567e^{8x/3} + 323e^{4x}, \\ q_{28}(x) &:= 81 - 81e^{4x/3} - 144e^{2x} - 81e^{8x/3} + 289e^{4x}. \end{aligned}$$

The corresponding perturbed Jost solution $\tilde{f}(k, x)$ is obtained from (7.26) with the help of (7.6), (7.9), (10.31), the first equality of (10.38), and (10.44). We get

$$\tilde{f}(k, x) = e^{ikx} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{q_{29}(k, x)} \begin{bmatrix} q_{30}(k, x) & q_{31}(k, x) \\ q_{32}(k, x) & q_{33}(k, x) \end{bmatrix} \right), \quad (10.49)$$

where we have let

$$\begin{aligned} q_{29}(k, x) &:= 25(3k + i)(k + i)(-9 - 18e^{2x/3} + 26e^{2x} + 25e^{8x/3}), \\ q_{30}(k, x) &:= 36(-8 + 43ik) + 882e^{2x/3}(-1 + 3ik) + 722e^{2x}(1 - ik), \\ q_{31}(k, x) &:= 36(-6 + ik) + 126e^{2x/3}(1 - 3ik) + 646e^{2x}(-1 + ik), \\ q_{32}(k, x) &:= 36(6 + ik) + 126e^{2x/3}(1 - 3ik) + 646e^{2x}(-1 + ik), \\ q_{33}(k, x) &:= 36(-8 + 7ik) + 18e^{2x/3}(-1 + 3ik) + 578e^{2x}(1 - ik). \end{aligned}$$

In order to obtain $\tilde{f}(k, x)$ given in (10.49), we have constructed the Moore–Penrose inverse of the matrix $\Omega_1(x)$ defined in (7.9) when $N = 0$. For the evaluation of $\Omega_1(x)^+$, we

have used the symbolic software Mathematica, but we have also independently verified that the matrices $\Omega_1(x)$ and $\Omega_1(x)^+$ satisfy the four matrix equalities given in (3.2) of [4] used in the definition [5] of the Moore–Penrose inverse of a matrix. With the help of the perturbed scattering matrix $\tilde{S}(k)$ given in (10.48) and the perturbed Jost solution $\tilde{f}(k, x)$ given in (10.49), from (2.15) we construct the perturbed regular solution $\tilde{\varphi}(k, x)$ explicitly. Since the expression is too lengthy, we do not display it here. We obtain $\tilde{\varphi}(i, x)$ as

$$\tilde{\varphi}(i, x) = \frac{1}{-9 - 18e^{2x/3} + 26e^{2x} + 25e^{8x/3}} \begin{bmatrix} q_{34}(x) & q_{35}(x) \\ q_{36}(x) & q_{37}(x) \end{bmatrix}, \quad (10.50)$$

where we have let

$$\begin{aligned} q_{34}(x) &:= 10e^x + 14e^{5x/3}, & q_{35}(x) &:= -e^x + 4e^{5x/3} - 6e^{3x} + 3e^{11x/3}, \\ q_{36}(x) &:= 2e^x - 2e^{5x/3}, & q_{37}(x) &:= -11e^x - 16e^{5x/3} + 30e^{3x} + 21e^{11x/3}. \end{aligned}$$

Using $\tilde{Q}_1$ given in (10.39) and $\tilde{\varphi}(i, x)$ given in (10.50), with the help of (3.14)–(3.17) we construct the corresponding Gel'fand–Levitan normalization matrix $\tilde{C}_1$, and we observe that the constructed 2×2 matrix $\tilde{C}_1$ coincides with the matrix appearing in the second equality of (10.38). In this example, the unperturbed potential $V(x)$, the perturbed potential $\tilde{V}(x)$, and the potential difference $\tilde{V}(x) - V(x)$ each behave as $O(e^{-2x/3})$ as $x \rightarrow +\infty$. The asymptotic decay for $\tilde{V}(x) - V(x)$ agrees with the asymptotic estimate stated in (7.14). In this case, our estimate in (7.14) is sharp.

In the next example, we demonstrate the addition of a bound state of multiplicity 2 by using the method of Section 7. The unperturbed operator in this example coincides with the unperturbed operator of Example 10.4.

Example 10.5. In this example, let us assume that our unperturbed operator corresponds to the potential $V(x)$ given in (10.29) and to the boundary matrices A and B given in (10.30). We know from Example 10.4 that the unperturbed operator does not have any bound states. Instead of adding a simple bound state at $k = i$ as in Example 10.4, let us add a bound state at $k = i$ with multiplicity two. We choose our Gel'fand–Levitan normalization matrix $\tilde{C}_1$ as

$$\tilde{C}_1 = \frac{1}{6} \begin{bmatrix} 4 + 3\sqrt{2} & 4 - 3\sqrt{2} \\ 4 - 3\sqrt{2} & 4 + 3\sqrt{2} \end{bmatrix}, \quad (10.51)$$

The matrix $\tilde{C}_1$ is a positive matrix because its eigenvalues are $4/3$ and $\sqrt{2}$. Since the multiplicity of the bound state to be added is 2 in this 2×2 matrix case, we know that the orthogonal projection matrices $\tilde{Q}_1$ and $\tilde{P}_1$ associated with the bound state at $k = i$ are both equal to the 2×2 identity matrix. Using $\tilde{\kappa}_1 = 1$ and the unperturbed Jost matrix $J(k)$ given in (10.32), from (7.19) we obtain the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \frac{-i(k-i)}{2(3k+i)} \begin{bmatrix} 6k+i & -i \\ -i & 6k+i \end{bmatrix}. \quad (10.52)$$

In fact, in this case we have

$$\tilde{J}(k) = \frac{k-i}{k+i} J(k).$$

The determinant of the perturbed Jost matrix $\tilde{J}(k)$ is given by

$$\det[\tilde{J}(k)] = -\frac{3k(k-i)^2}{3k+i},$$

and hence there is a bound state at $k = i$ with the multiplicity 2. The corresponding perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$ are obtained from (7.13) with the help of (10.30) and (10.51). We have

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \frac{1}{18} \begin{bmatrix} -17 & 1 \\ 1 & -17 \end{bmatrix}. \quad (10.53)$$

The perturbed potential $\tilde{V}(x)$ is obtained from (7.10) with the help of (7.6), (7.9), and (10.51). We get

$$\tilde{V}(x) = -\frac{8e^{2x/3}}{9(2+e^{2x/3})^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad x \in \mathbb{R}^+. \quad (10.54)$$

The corresponding perturbed Jost solution $\tilde{f}(k, x)$ is obtained from (7.26) with the help of (7.6), (7.9), and (10.31). We have

$$\tilde{f}(k, x) = e^{ikx} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2i}{(3k+i)(2+e^{2x/3})} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right). \quad (10.55)$$

As done in Example 10.4, we verify the compatibility of the Gel'fand–Levitan normalization matrix $\tilde{C}_1$ and the orthogonal projection matrix $\tilde{Q}_1$, where we have $\tilde{Q}_1 = I$. We observe that the two equalities in (10.40) are trivially satisfied. As mentioned already the matrix $\tilde{C}_1$ is chosen as a positive matrix. We construct the corresponding perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \frac{1}{4k(9k^2+1)(2+e^{2x/3})} \begin{bmatrix} q_{38}(k, x) - q_{38}(-k, x) & q_{39}(k, x) - q_{39}(-k, x) \\ q_{39}(k, x) - q_{39}(-k, x) & q_{38}(k, x) - q_{38}(-k, x) \end{bmatrix}, \quad (10.56)$$

where we have defined

$$q_{38}(k, x) := e^{ikx} (k+i) [36k^2 - 6ik + 2 + e^{2x/3} (18k^2 + 3ik + 1)], \quad (10.57)$$

$$q_{39}(k, x) := e^{ikx} (k+i) [-6ik - 2 + e^{2x/3} (3ik - 1)]. \quad (10.58)$$

Using $\tilde{Q}_1 = I$ and $\tilde{\varphi}(i, x)$, with the help of (3.14)–(3.17), we construct the corresponding Gel'fand–Levitan normalization matrix $\tilde{C}_1$, and we observe that the constructed 2×2 matrix $\tilde{C}_1$ coincides with the matrix appearing in (10.51). In this example, the unperturbed potential $V(x)$, the perturbed potential $\tilde{V}(x)$, and the potential difference $\tilde{V}(x) - V(x)$ each behave as $O(e^{-2x/3})$ as $x \rightarrow +\infty$. The asymptotic decay of $\tilde{V}(x) - V(x)$ is the same as the decay estimate predicted in (7.14). In this case, our estimate given in (7.14) is sharp.

In the next example we illustrate the method of Section 6 used to reduce the multiplicity of a bound state. As our unperturbed problem, we use the perturbed problem of Example 10.5.

Example 10.6. We choose our unperturbed operator in such a way that the unperturbed potential $V(x)$ is

$$V(x) = -\frac{8e^{2x/3}}{9(2 + e^{2x/3})^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad x \in \mathbb{R}^+, \quad (10.59)$$

which coincides with the right-hand side of (10.54), and we choose the unperturbed boundary matrices A and B as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \frac{1}{18} \begin{bmatrix} -17 & 1 \\ 1 & -17 \end{bmatrix}, \quad (10.60)$$

which coincide with the boundary matrices listed in (10.53). The unperturbed Jost solution $f(k, x)$, the unperturbed Jost matrix $J(k)$, and the unperturbed regular solutions $\varphi(k, x)$ are respectively given by

$$f(k, x) = e^{ikx} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2i}{(3k + i)(2 + e^{2x/3})} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right), \quad (10.61)$$

$$J(k) = \frac{-i(k - i)}{2(3k + i)} \begin{bmatrix} 6k + i & -i \\ -i & 6k + i \end{bmatrix}, \quad (10.62)$$

$$\varphi(k, x) = \frac{1}{4k(9k^2 + 1)(2 + e^{2x/3})} \begin{bmatrix} q_{38}(k, x) - q_{38}(-k, x) & q_{39}(k, x) - q_{39}(-k, x) \\ q_{39}(k, x) - q_{39}(-k, x) & q_{38}(k, x) - q_{38}(-k, x) \end{bmatrix}, \quad (10.63)$$

where $q_{38}(k, x)$ and $q_{39}(k, x)$ are the quantities defined in (10.57) and (10.58), respectively. We remark that the Jost solution $f(k, x)$ given in (10.61) coincides with $\tilde{f}(k, x)$ appearing in (10.55), the Jost matrix given in (10.62) coincides with $\tilde{J}(k)$ appearing in (10.52), and the regular solution $\varphi(k, x)$ given in (10.63) coincides with $\tilde{\varphi}(k, x)$ appearing in (10.56). Even though the denominator in (10.63) appears to have simple poles at $k = 0$, $k = i/3$, and $k = -i/3$, the left-hand side in (10.63) is entire in k . Hence, those three singularities of the right-hand side in (10.63) are removable singularities. From (10.62), we get the determinant of the Jost matrix $J(k)$ as

$$\det[J(k)] = \frac{-3k(k - i)^2}{3k + i}. \quad (10.64)$$

Since the right-hand side of (10.64) has a double zero at $k = i$ occurring on the positive imaginary axis in the complex k -plane, we see that the unperturbed operator has a bound state at $k = i$ with the multiplicity 2. We are interested in reducing that multiplicity from 2 to 1 by using the method of Section 6. To simplify our notation, we use $Q_r, \mathbf{G}_r, \mathbf{H}_r, C_r, \Phi_r, W_r, P_r$, to denote the quantities $Q_{Nr}, \mathbf{G}_{Nr}, \mathbf{H}_{Nr}, C_{Nr}, \Phi_{Nr}, W_{Nr}, P_{Nr}$, respectively, appearing in Section 6. From (10.62), we observe that $J(i)$ is the 2×2 zero matrix, and hence the orthogonal projection Q onto the kernel of $J(i)$ is equal to the 2×2 identity matrix. This is not surprising because the multiplicity of the bound state is 2 and we are in the 2×2 matrix case. In our example, let us choose the matrix Q_r as

$$Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (10.65)$$

and we observe that (6.2) is satisfied. Using (10.63) at $k = i$, we get

$$\varphi(i, x) = \frac{e^{-x}}{4(2 + e^{2x/3})} \begin{bmatrix} 7 + 5e^{2x/3} & -1 + e^{2x/3} \\ -1 + e^{2x/3} & 7 + 5e^{2x/3} \end{bmatrix}. \quad (10.66)$$

Using (10.65) and (10.66) on the right-hand side of (6.3), we have

$$\mathbf{G}_r = \begin{bmatrix} \frac{17}{32} & 0 \\ 0 & 0 \end{bmatrix}, \quad (10.67)$$

Next, using (10.65) and (10.67) in (6.4), we obtain

$$\mathbf{H}_r = \begin{bmatrix} \frac{17}{32} & 0 \\ 0 & 1 \end{bmatrix}, \quad (10.68)$$

which is a positive matrix. Then, with the help of (6.5), (10.65), and (10.68), we construct the nonnegative matrix C_r as

$$C_r = \begin{bmatrix} \frac{4\sqrt{2}}{\sqrt{17}} & 0 \\ 0 & 0 \end{bmatrix}. \quad (10.69)$$

Using (10.60) and (10.69) in (6.13), we get

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \frac{287}{306} & \frac{1}{18} \\ \frac{1}{18} & -\frac{17}{18} \end{bmatrix}. \quad (10.70)$$

Using (10.66) and (10.69) in (6.7), we have

$$\Phi_r(x) = \frac{\sqrt{2}e^{-x}}{\sqrt{17}(2 + e^{2x/3})} \begin{bmatrix} 7 + 5e^{2x/3} & 0 \\ -1 + e^{2x/3} & 0 \end{bmatrix}. \quad (10.71)$$

Then, using (10.71) in (6.10), we obtain

$$W_r(x) = \frac{e^{-2x} (25 + 26 e^{2x/3})}{17 (2 + e^{2x/3})} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (10.72)$$

The Moore–Penrose inverse $W_r(x)^+$ of the matrix $W_r(x)$ is obtained as

$$W_r(x)^+ = \frac{17 (2 + e^{2x/3})}{e^{-2x} (25 + 26 e^{2x/3})} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (10.73)$$

where we observe that the (1, 1)-entries in (10.72) and (10.73) are reciprocals of each other. Using (10.59), (10.71), and (10.73) in (6.11), we obtain the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) = \frac{e^{2x/3}}{9 (25 + 26 e^{2x/3})^2} \begin{bmatrix} -2888 & 2584 \\ 2584 & -2312 \end{bmatrix}, \quad x \in \mathbb{R}^+. \quad (10.74)$$

Using (10.63), (10.71), and (10.73) in (6.12), we obtain the perturbed regular solution $\tilde{\varphi}(k, x)$ as

$$\tilde{\varphi}(k, x) = \frac{1}{q_{40}(k, x)} \begin{bmatrix} q_{41}(k, x) - q_{41}(-k, x) & q_{42}(k, x) - q_{42}(-k, x) \\ q_{43}(k, x) - q_{43}(-k, x) & q_{44}(k, x) - q_{44}(-k, x) \end{bmatrix}, \quad (10.75)$$

where we have defined

$$q_{40}(k, x) := 4k(9k^2 + 1)(25 + 26 e^{2x/3}), \quad (10.76)$$

$$q_{41}(k, x) := e^{ikx}(-31i - 26k - 507ik^2 + 450k^3) + 2 e^{ikx+2x/3}(-7i + 64k - 177ik^2 + 234k^3), \quad (10.77)$$

$$q_{42}(k, x) := e^{ikx}(31i - 76k + 51ik^2) + 2 e^{ikx+2x/3}(7i + 38k - 51ik^2), \quad (10.78)$$

$$q_{43}(k, x) := 17e^{ikx}(-i + 4k + 3ik^2) + 34 e^{ikx+2x/3}(-i - 2k - 3ik^2), \quad (10.79)$$

$$q_{44}(k, x) := e^{ikx}(17i + 118k + 357ik^2 + 450k^3) + 2 e^{ikx+2x/3}(17i - 8k + 255ik^2 + 234k^3). \quad (10.80)$$

From (10.65), we see that a unit vector generating Q_r is given by $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. In our example, the equality (6.18) is given by

$$\varphi(i, x) w = f(i, x) \beta. \quad (10.81)$$

Using (10.61) and (10.66) in (10.81), we obtain $\beta = \begin{bmatrix} 5/4 \\ 1/4 \end{bmatrix}$. Hence, a unit vector generating the orthogonal projection matrix P_r is given by $\begin{bmatrix} 5 \\ 1 \end{bmatrix} / \sqrt{26}$. Thus, we obtain P_r as

$$P_r = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \frac{1}{\sqrt{26}} \begin{bmatrix} 5 & 1 \end{bmatrix},$$

which yields

$$P_r = \frac{1}{26} \begin{bmatrix} 25 & 5 \\ 5 & 1 \end{bmatrix}. \quad (10.82)$$

Using (10.62) and (10.82) in (6.17), we obtain the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \frac{1}{26(3k+i)} \begin{bmatrix} (k+i)(-78ik+7) & 17k-7i \\ 17(k+i) & 78k^2-59ik+17 \end{bmatrix}, \quad (10.83)$$

which yields the determinant $\det[\tilde{J}(k)]$ as

$$\det[\tilde{J}(k)] = \frac{-3k(k-i)(k+i)}{3k+i}. \quad (10.84)$$

The only zero on the positive imaginary axis of the right-hand side of (10.84) is a simple zero occurring at $k = i$. Thus, the transformation has reduced the multiplicity of the bound state at $k = i$ from 2 to 1. Using (10.61), (10.71), (10.73), and (10.82) in (6.19), we obtain the perturbed Jost solution $\tilde{f}(k, x)$ as

$$\tilde{f}(k, x) = \frac{e^{ikx}}{13(3k+i)(25+26e^{2x/3})} \begin{bmatrix} q_{45}(k, x) & 323i \\ 323i & q_{46}(k, x) \end{bmatrix}, \quad (10.85)$$

where we have let

$$q_{45}(k, x) := -36i + 975k + 338e^{2x/3}(3k+i), \quad (10.86)$$

$$q_{46}(k, x) := 36i + 975k + 338e^{2x/3}(3k+i). \quad (10.87)$$

In this example, we have $V(x)$, $\tilde{V}(x)$, and $\tilde{V}(x) - V(x)$, all having the behavior $O(e^{-2x/3})$ as $x \rightarrow +\infty$. This asymptotic behavior agrees with the predicted asymptotic behavior stated in (6.14). In this case, our estimate given in (6.14) is sharp.

In the next example, we illustrate the method of Sect.8 used to increase the multiplicity of an existing bound state. As our unperturbed problem, we use the perturbed problem of Example 10.6.

Example 10.7. In this example we choose our unperturbed half-line matrix Schrödinger operator so that the unperturbed potential $V(x)$ is given by

$$V(x) = \frac{e^{2x/3}}{9(25+26e^{2x/3})^2} \begin{bmatrix} -2888 & 2584 \\ 2584 & -2312 \end{bmatrix}, \quad x \in \mathbb{R}^+, \quad (10.88)$$

which coincides with the right-hand side of (10.74), and as our unperturbed boundary matrices, we use the 2×2 constant matrices A and B given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{287}{306} & \frac{1}{18} \\ \frac{1}{18} & -\frac{17}{18} \end{bmatrix}, \quad (10.89)$$

which coincide with the matrices on the right-hand side of (10.70). The unperturbed Jost solution $f(k, x)$ is obtained as the 2×2 matrix appearing on the right-hand side of (10.85), i.e. we have

$$f(k, x) = \frac{e^{ikx}}{13(3k+i)(25+26e^{2x/3})} \begin{bmatrix} q_{45}(k, x) & 323i \\ 323i & q_{46}(k, x) \end{bmatrix}, \quad (10.90)$$

with $q_{45}(k, x)$ and $q_{46}(k, x)$ defined as in (10.86) and (10.87), respectively. From Example 10.6, we know that the unperturbed operator corresponding to (10.88) and (10.89) has a simple bound state at $k = i$. We are interested in increasing the multiplicity of that bound state from 1 to 2. For this, we proceed as follows. To simplify our notation, we use $\tilde{Q}_i, \tilde{\mathbf{G}}_i, \mathbf{H}_i, \tilde{C}_i, \xi_i(x), \Omega_i(x), L_i, P_i$ to denote the quantities $\tilde{Q}_{Ni}, \tilde{\mathbf{G}}_{Ni}, \mathbf{H}_{Ni}, \tilde{C}_{Ni}, \xi_{Ni}(x), \Omega_{Ni}(x), L_{Ni}, P_{Ni}$, respectively, appearing in Section 8. From (10.83), we know that the unperturbed Jost matrix $J(k)$ is given by

$$J(k) = \frac{1}{26(3k+i)} \begin{bmatrix} (k+i)(-78ik+7) & 17k-7i \\ 17(k+i) & -78ik^2-59k-17i \end{bmatrix}. \quad (10.91)$$

Using (10.91), we obtain

$$J(i) = \frac{1}{52} \begin{bmatrix} 85 & 5 \\ 17 & 1 \end{bmatrix},$$

and hence we see that the kernel of $J(i)$ is spanned by a unit vector along $\begin{bmatrix} 1 \\ -17 \end{bmatrix}$. Thus, the orthogonal projection matrix Q onto $\text{Ker}[J(i)]$ is given by

$$Q = \frac{1}{1+17^2} \begin{bmatrix} 1 & \\ & -17 \end{bmatrix} \begin{bmatrix} 1 & -17 \end{bmatrix}, \quad (10.92)$$

or equivalently we have

$$Q = \frac{1}{290} \begin{bmatrix} 1 & -17 \\ -17 & 289 \end{bmatrix}.$$

In order to increase the multiplicity of the bound state at $k = i$ from 1 to 2, we need to specify the matrix $\tilde{C}_i$ appearing in (8.7). We remark that we cannot specify $\tilde{C}_i$ arbitrarily because it has to be compatible with the construction described in (8.2)–(8.7). Using (8.2) we first need determine the rank-one orthogonal projection matrix $\tilde{Q}_i$ onto a subspace of the orthogonal complement in $\mathbb{C}^n$ of $Q\mathbb{C}^n$, where in this case we have $n = 2$. From (10.92), we already know that the orthogonal complement in $\mathbb{C}^2$ of $Q\mathbb{C}^2$ is spanned by a unit vector along $\begin{bmatrix} 17 \\ 1 \end{bmatrix}$. Hence, the orthogonal projection matrix $\tilde{Q}_i$ is given by

$$\tilde{Q}_i = \frac{1}{290} \begin{bmatrix} 289 & 17 \\ 17 & 1 \end{bmatrix}. \quad (10.93)$$

We let

$$\tilde{\mathbf{G}}_i = \begin{bmatrix} a & b + ic \\ b - ic & d \end{bmatrix}, \quad (10.94)$$

where a, b, c , and d are real constants. The two equalities in (8.3) impose the restrictions on (10.94) so that we have

$$a = 289d, \quad b = 17d, \quad c = 0. \quad (10.95)$$

Using (10.95) in (10.94), we get

$$\tilde{\mathbf{G}}_i = d \begin{bmatrix} 289 & 17 \\ 17 & 1 \end{bmatrix}. \quad (10.96)$$

From (10.96), we see that the eigenvalues of $\tilde{\mathbf{G}}_i$ are 0 and $290d$. Since $\tilde{\mathbf{G}}_i$ must be nonnegative and must have its rank equal to 1, it follows that we must have $d > 0$. We further require that the restriction of $\tilde{\mathbf{G}}_i$ to the subspace $\tilde{Q}_i \mathbb{C}^2$ be invertible. From (10.93), we see that the subspace $\tilde{Q}_i \mathbb{C}^2$ is spanned by the vector $\begin{bmatrix} 17 \\ 1 \end{bmatrix}$. Hence, we must have

$$\tilde{\mathbf{G}}_i \begin{bmatrix} 17 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 17 \\ 1 \end{bmatrix}, \quad (10.97)$$

for some nonzero constant α . We observe that (10.97) holds because the column vector $\begin{bmatrix} 17 \\ 1 \end{bmatrix}$ is an eigenvector of $\tilde{\mathbf{G}}_i$ with the eigenvalue α being equal to $290d$. Consequently, $\tilde{\mathbf{G}}_i$ is given by the matrix on the right-hand side of (10.96) with any positive constant d . Using (10.93) and (10.96) in (8.4), we get

$$\tilde{\mathbf{H}}_i = \begin{bmatrix} \frac{1}{290} + 289d & -\frac{17}{290} + 17d \\ -\frac{17}{290} + 17d & -\frac{289}{290} + d \end{bmatrix}. \quad (10.98)$$

Using (8.6) and (10.98), we obtain the positive matrix $\tilde{\mathbf{H}}_i^{1/2}$ and its inverse $\tilde{\mathbf{H}}_i^{-1/2}$ as

$$\begin{aligned} \tilde{\mathbf{H}}_i^{1/2} &= \begin{bmatrix} \frac{1}{290} + \frac{289\sqrt{d}}{\sqrt{290}} & -\frac{17}{290} + \frac{17\sqrt{d}}{\sqrt{290}} \\ -\frac{17}{290} + \frac{17\sqrt{d}}{\sqrt{290}} & \frac{289}{290} + \frac{\sqrt{d}}{\sqrt{290}} \end{bmatrix}, \\ \tilde{\mathbf{H}}_i^{-1/2} &= \begin{bmatrix} \frac{1}{290} + \frac{289}{290\sqrt{290d}} & -\frac{17}{290} + \frac{17}{290\sqrt{290d}} \\ -\frac{17}{290} + \frac{17}{290\sqrt{290d}} & \frac{289}{290} + \frac{1}{290\sqrt{290d}} \end{bmatrix}. \end{aligned} \quad (10.99)$$

Using (10.93) and (10.99) in (8.7), we get the matrix $\tilde{\mathbf{C}}_i$ as

$$\tilde{\mathbf{C}}_i = \frac{1}{290\sqrt{290d}} \begin{bmatrix} 289 & 17 \\ 17 & 11 \end{bmatrix}. \quad (10.100)$$

Thus, the only freedom we have in the specification of $\tilde{C}_i$ is the choice of the positive constant d . We will see that each specific choice of d yields a particular perturbed potential $\tilde{V}(x)$, a particular perturbed regular solution $\tilde{\varphi}(k, x)$, a particular perturbed Jost solution $\tilde{f}(k, x)$, and a particular perturbed boundary matrix $\tilde{B}$. We will also see that the constructed perturbed boundary matrix $\tilde{A}$, perturbed orthogonal projection $\tilde{P}_i$, and perturbed Jost matrix $\tilde{J}(k)$ are not affected by any particular choice of d . Using (10.89) and (10.100) in (8.15), we get

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \frac{287}{306} - \frac{289}{84100d} & \frac{1}{18} - \frac{17}{84100d} \\ \frac{1}{18} - \frac{17}{84100d} & -\frac{17}{18} - \frac{1}{84100d} \end{bmatrix}.$$

In order to construct the orthogonal projection matrix $\tilde{P}_i$, we first use (8.16) and (8.23) and construct the matrix product $L_i Q_i$. With the help of (10.75)–(10.80), for the unperturbed regular solution $\varphi(k, x)$ at $k = i$, we obtain

$$\varphi(i, x) = \frac{1}{4(25 + 26 e^{2x/3})} \begin{bmatrix} q_{47}(x) & q_{48}(x) \\ q_{49}(x) & q_{50}(x) \end{bmatrix}, \quad (10.101)$$

where we have defined

$$\begin{aligned} q_{47}(x) &:= 17e^x(7 + 5e^{2x/3}), & q_{48}(x) &:= 2(12e^{-x} - 24e^{-x/3} + 7e^x + 5e^{5x/3}), \\ q_{49}(x) &:= 17e^x(-1 + e^{2x/3}), & q_{50}(x) &:= 2(84e^{-x} + 120e^{-x/3} - e^x + e^{5x/3}). \end{aligned}$$

From (10.93) and (10.101), we get

$$\varphi(i, x) Q_i = \frac{1}{1160(25 + 26 e^{2x/3})} \begin{bmatrix} q_{47}(x) & q_{48}(x) \\ q_{49}(x) & q_{50}(x) \end{bmatrix} \begin{bmatrix} 289 & 17 \\ 17 & 1 \end{bmatrix}. \quad (10.102)$$

Using the first asymptotics as $x \rightarrow +\infty$ in each of (2.10) and (7.22), from (8.16) with $\kappa_1 = 1$, we obtain the matrix product $L_i Q_i$ as

$$L_i Q_i = \frac{1}{104} \begin{bmatrix} 85 & 5 \\ 17 & 1 \end{bmatrix}. \quad (10.103)$$

Hence, (10.100) and (10.103) yield the matrix product $L_i Q_i \tilde{C}_i$ as

$$L_i Q_i \tilde{C}_i = \frac{1}{104 \sqrt{290d}} \begin{bmatrix} 85 & 5 \\ 17 & 1 \end{bmatrix}, \quad (10.104)$$

where we remark that the right-hand side of (10.104) contains the positive parameter d even though the right-hand side of (10.103) does not contain d . Using (10.104) in (8.24), we obtain the orthogonal projection matrix P_i into the kernel of $\tilde{f}(i)^\dagger$ as

$$P_i = \frac{1}{26} \begin{bmatrix} 25 & 5 \\ 5 & 1 \end{bmatrix}. \quad (10.105)$$

From (10.105), we observe that P_i is independent of the positive parameter d . Using (10.91) and (10.105) in (8.22), we get the perturbed Jost matrix $\tilde{J}(k)$ as

$$\tilde{J}(k) = \frac{1}{2(3k+i)} \begin{bmatrix} -i(k-i)(6k+i) & -k+i \\ -k+i & -i(k-i)(6k+i) \end{bmatrix}. \quad (10.106)$$

From (10.106), we see that $\tilde{J}(k)$ is independent of the positive parameter d . The determinant of the perturbed Jost matrix $\tilde{J}(k)$ appearing in (10.106) is given by

$$\det[\tilde{J}(k)] = \frac{-3k(k-i)^2}{3k+i}. \quad (10.107)$$

From the factor $(k-i)^2$ in the numerator on the right-hand side of (10.107), we see that the perturbed operator has a bound state at $k=i$ with the multiplicity 2. Using (10.100) and (10.102) in (8.9), we obtain $\xi_i(x)$ as

$$\xi_i(x) = \frac{1}{580 \sqrt{290} d (25 + 26 e^{2x/3})} \begin{bmatrix} 17 q_{51}(x) & q_{51}(x) \\ 17 q_{52}(x) & q_{52}(x) \end{bmatrix}, \quad (10.108)$$

where we have defined

$$\begin{aligned} q_{51}(x) &:= 6 e^{-x} - 12 e^{-x/3} + 1015 e^x + 725 e^{5x/3}, \\ q_{52}(x) &:= 42 e^{-x} + 60 e^{-x/3} - 145 e^x + 145 e^{5x/3}. \end{aligned}$$

We remark that $\xi_i(x)$ contains the positive parameter d in its denominator. Using (10.93) and (10.108) in (8.12), we obtain $\Omega_i(x)$ as

$$\Omega_i(x) = \begin{bmatrix} \frac{289}{290} - 289 q_{53}(x, d) & \frac{17}{290} - 17 q_{53}(x, d) \\ \frac{17}{290} - 17 q_{53}(x, d) & \frac{1}{290} - q_{53}(x, d) \end{bmatrix}, \quad (10.109)$$

where we have defined

$$q_{53}(x, d) := \frac{29585 + 72 e^{-2x} + 144 e^{-4x/3} + 33274 e^{2x/3} - 42050 e^{2x} - 21025 e^{8x/3}}{195112000 d (25 + 26 e^{2x/3})}.$$

We note that $\Omega_i(x)$ contains the positive parameter d . From (10.109), as also done in Example 10.4, with the help of the symbolic software Mathematica, we obtain the Moore–Penrose inverse of $\Omega_i(x)$ as

$$\Omega_i(x)^+ = \frac{d e^{2x} (25 + 26 e^{2x/3})}{q_{54}(x, d)} \begin{bmatrix} 670480 & 39440 \\ 39440 & 2320 \end{bmatrix}, \quad (10.110)$$

where we have let

$$q_{54}(x, d) := -72 - 144 e^{2x/3} + 5(-5917 + 3364000 d) e^{2x} \\ + 2(-16637 + 8746400 d) e^{8x/3} + 42050 e^{4x} + 21025 e^{14x/3}.$$

We verify that the matrix $\Omega_i(x)^+$ appearing in (10.110) is indeed the Moore–Penrose inverse of the matrix $\Omega_i(x)$ by verifying that those two matrices satisfy the four equalities given in (3.2) of [4] used in the definition of the Moore–Penrose inverse [5] of a matrix. With the help of (10.88), (10.108), and (10.110), from (8.13) we recover the perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) = -\frac{8 e^{2x/3}}{9 [q_{54}(x, d)]^2} \begin{bmatrix} q_{55}(x, d) & q_{56}(x, d) \\ q_{56}(x, d) & q_{57}(x, d) \end{bmatrix}, \quad x \in \mathbb{R}^+,$$

where we have defined

$$q_{55}(x, d) := 5184 - 324(-209 + 672800 d) e^{4x/3} - 1440(-1601 + 672800 d) e^{2x} \\ - 1296(-1949 + 672800 d) e^{8x/3} - 54496800 e^{10x/3} \\ + (541979641 - 657593374400 d + 163410202240000 d^2) e^{4x} \\ - 54496800 e^{14x/3} + 1324575(-8423 + 4709600 d) e^{16x/3} \\ + 336400(-45143 + 24893600 d) e^{6x} \\ + 4730625(-1253 + 672800 d) e^{20x/3} + 442050625 e^{8x}, \\ q_{56}(x, d) := 5184 - 324(-4943 + 4709600 d) e^{4x/3} + 1152(-3689 + 672800 d) e^{2x} \\ + 6480(-1601 + 672800 d) e^{8x/3} \\ + (-455148503 + 528668747200 d - 146209128320000 d^2) e^{4x} \\ - 189225(-4943 + 4709600 d) e^{16x/3} + 336400(-3689 + 672800 d) e^{6x} \\ + 946125(-1601 + 672800 d) e^{20x/3} + 442050625 e^{8x}, \\ q_{57}(x, d) := 5184 - 2268(-8423 + 4709600 d) e^{4x/3} \\ - 288(-184261 + 100247200 d) e^{2x} \\ - 32400(-1253 + 672800 d) e^{8x/3} - 54496800 e^{10x/3} \\ + (216875449 - 419599793600 d + 130818693760000 d^2) e^{4x} \\ - 54496800 e^{14x/3} + 189225(-209 + 672800 d) e^{16x/3} \\ + 336400(-1079 + 672800 d) e^{6x} \\ + 189225(-1949 + 672800 d) e^{20x/3} + 442050625 e^{8x}.$$

We note that the potential $\tilde{V}(x)$ contains the parameter d . In this example, we have $V(x)$, $\tilde{V}(x)$, and $\tilde{V}(x) - V(x)$, all having the behavior $O(e^{-2x/3})$ as $x \rightarrow +\infty$. That asymptotic

behavior agrees with the predicted asymptotic behavior described in (8.17). In this case, our estimate given in (8.17) is sharp. Finally, using (10.90), (10.105), (10.108), and (10.110) in (8.25), we obtain the perturbed Jost solution $\tilde{f}(k, x)$ explicitly. Since that expression is too lengthy, we do not display it here.

In the next example, we illustrate the removal of a discrete eigenvalue in the scalar case where the unperturbed potential decays as $O(1/x^2)$ when $x \rightarrow +\infty$. We remark that the potential in this example is integrable but does not belong to the class $L_1^1(\mathbb{R}^+)$. Hence, it is outside the class of potentials for which the theory of removing bound states is developed of Section 5. Nevertheless, we follow the procedure of Section 5 to remove a bound state in this example.

Example 10.8. Consider the scalar case, i.e. $n = 1$, where the unperturbed potential is given by

$$V(x) = \frac{2}{(1+x)^2}, \quad x \in \mathbb{R}^+, \quad (10.111)$$

and the boundary matrices A and B , which are scalars in this case, are given by

$$A = 1, \quad B = -2. \quad (10.112)$$

The corresponding Jost solution $f(k, x)$, Jost matrix $J(k)$, scattering matrix $S(k)$, and regular solution $\varphi(k, x)$ are evaluated by proceeding as in Example 9.2, and we get

$$f(k, x) = e^{ikx} \left[1 + \frac{i}{k(1+x)} \right], \quad (10.113)$$

$$J(k) = \frac{[k + i(\sqrt{5} + 1)/2][k - i(\sqrt{5} + 1)/2]}{ik}, \quad (10.114)$$

$$S(k) = \frac{k^2 + ik + 1}{k^2 - ik + 1},$$

$$\varphi(k, x) = \frac{q_{58}(k, x)}{2k^3(1+x)},$$

where we have defined

$$q_{58}(k, x) := e^{-ikx}(k^2 - ik + 1)(kx + k - i) + e^{ikx}(k^2 + ik + 1)(kx + k + i).$$

From the zeros of the Jost matrix $J(k)$ given in (10.114), we observe that there is a simple bound state at $k = i\kappa_1$, where the value of κ_1 is given by

$$\kappa_1 = \frac{1 + \sqrt{5}}{2}. \quad (10.115)$$

The Gel'fand–Levitan normalization matrix C_1 at the bound state at $k = i\kappa_1$ is evaluated as described in Example 10.2, and we have

$$C_1 = \sqrt{2 + \frac{4}{\sqrt{5}}}. \quad (10.116)$$

By proceeding as in Example 10.2, we remove the bound state and obtain the corresponding perturbed Jost matrix $\tilde{J}(k)$, perturbed scattering matrix $\tilde{S}(k)$, perturbed boundary matrices $\tilde{A}$ and $\tilde{B}$, and perturbed potential $\tilde{V}(x)$ as

$$\begin{aligned} \tilde{J}(k) &= \frac{[k + i(\sqrt{5} + 1)/2][k + i(\sqrt{5} - 1)/2]}{ik}, \\ \tilde{S}(k) &= \frac{(k^2 + ik + 1)[k - i(\sqrt{5} + 1)/2]^2}{(k^2 - ik + 1)[k + i(\sqrt{5} + 1)/2]^2}, \\ \tilde{A} &= 1, \quad \tilde{B} = \frac{4}{\sqrt{5}}, \\ \tilde{V}(x) &= \frac{2}{(\sqrt{5} + x)^2}, \quad x \in \mathbb{R}^+, \\ \tilde{f}(k, x) &= \frac{e^{ikx}(2ik - 1 - \sqrt{5})q_{59}(k, x)}{ik(2ik + 1 + \sqrt{5})(5 + \sqrt{5}x)(2k^2 + 3 + \sqrt{5})}, \end{aligned} \quad (10.117)$$

where we have defined

$$q_{59}(k, x) := 5 + 3\sqrt{5} - ik(5 + 3\sqrt{5})(x + 1) + k^2[10 + 8\sqrt{5} + (10 + 2\sqrt{5})x] + ik^3(10 + 2\sqrt{5}x).$$

From (10.117), we see that $\tilde{J}(k)$ does not have any zeros on the positive imaginary axis, and hence the perturbed operator does not have any bound states. In this example, the unperturbed potential $V(x)$ and the perturbed potential $\tilde{V}(x)$ each behave as $O(1/x^2)$ as $x \rightarrow +\infty$, and the potential difference $\tilde{V}(x) - V(x)$ behaves as $O(1/x^3)$ as $x \rightarrow +\infty$.

In the next example, we revisit Example 10.8, but we add a bound state to the unperturbed problem instead of removing the existing bound state in the unperturbed problem.

Example 10.9. We consider the scalar case, i.e. $n = 1$, where the unperturbed quantities are the same as in Example 10.8. Therefore, the unperturbed potential $V(x)$, boundary matrices A and B , Jost solution $f(k, x)$, Jost matrix $J(k)$, scattering matrix $S(k)$, and regular solution $\varphi(k, x)$ are as in (10.111)–(10.114), respectively. The unperturbed operator has one bound state at $k = i\kappa_1$ with the Gel'fand–Levitan normalization matrix C_1 , where the values of κ_1 and C_1 are as in (10.115) and (10.116), respectively. We add one bound state to the unperturbed problem at $k = i\tilde{\kappa}_2$ with the Gel'fand–Levitan normalization matrix $\tilde{C}_2$, where we use

$$\tilde{\kappa}_2 = 1, \quad \tilde{C}_2 = 2.$$

By proceeding as in Example 10.1, we determine the perturbed quantities. We obtain the perturbed potential $\tilde{V}(x)$, the boundary matrices $\tilde{A}$ and $\tilde{B}$, Jost matrix $\tilde{J}(k)$, scattering matrix $\tilde{S}(k)$, and Jost solution $\tilde{f}(k, x)$ as

$$\tilde{V}(x) = \frac{q_{60}(x) + q_{61}(x)}{[3 + x + e^{2x}(-6 - 6x + 4x^2) + e^{4x}(1 - x)]^2}, \quad x \in \mathbb{R}^+, \quad (10.118)$$

where we have defined

$$\begin{aligned} q_{60}(x) &:= 2 + 2e^{2x}(84 + 8x - 40x^2 - 16x^3), \\ q_{61}(x) &:= 4e^{4x}(33 - 8x + 24x^2) + 8e^{6x}(-5 + 6x - 14x^2 + 4x^3) + 2e^{8x}, \\ \tilde{A} &= 1, \quad \tilde{B} = -6, \\ \tilde{J}(k) &= \frac{(k - i)(k - i(\sqrt{5} + 1)/2)}{ik(k + i)(k + i(\sqrt{5} + 1)/2)}, \\ \tilde{S}(k) &= \frac{(k + i)^2(k^2 + ik + 1)}{(k - i)^2(k^2 - ik + 1)}, \\ \tilde{f}(k, x) &= \frac{e^{ikx}[-(k - i)^2[i + k(3 + x)] + e^{2x}q_{62}(k, x) + e^{4x}q_{63}(k, x)]}{k(k + i)^2[-3 - x + e^{2x}(6 + 6x - 4x^2) + e^{4x}(-1 + x)], \\ q_{62}(k, x) &:= -2i(-5 + 2x) + k(10 + 6x - 4x^2) - 2ik^2(-3 + 4x) + k^3(6 + 6x - 4x^2), \\ q_{63}(k, x) &:= (i + k)^2[i + k(-1 + x)]. \end{aligned} \quad (10.119)$$

From (10.119), we observe that $\tilde{J}(k)$ has two zeros on the positive imaginary axis at $k = i$ and $k = i(\sqrt{5} + 1)/2$, confirming that the perturbed operator has two bound states. We remark that, from (10.111) and (10.118), we obtain $V(x) = O(1/x^2)$, $\tilde{V}(x) = O(1/x^2)$, and $\tilde{V}(x) - V(x) = O(1/x^3)$ as $x \rightarrow +\infty$.

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