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Soliton solutions to the Sawada–Kotera equation

Dedicated to the memory of Prof. Martin Klaus of Virginia Tech, a long-time friend and collaborator

Abstract: We consider the direct and inverse scattering problems for the third-order differential equation in the reflectionless case. We formulate a corresponding Riemann–Hilbert problem using input consisting of the bound-state poles of a transmission coefficient and the bound-state dependency constants. With the time-evolved dependency constants, using the solution to the Riemann–Hilbert problem, we construct soliton solutions to a certain integrable system of fifth-order nonlinear partial differential equations. By imposing some appropriate restrictions on the dependency constants, we show that those soliton solutions yield soliton solutions to the Sawada–Kotera equation.

Keywords: Inverse scattering for the third-order equation, bound-state dependency constants, soliton solutions, Sawada–Kotera equation

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1 Introduction

We are interested in soliton solutions to a certain integrable system of coupled fifth-order nonlinear partial differential equations and, in particular, one of its special cases, namely, the Sawada–Kotera equation [26]. We construct such solutions by solving the inverse scattering problem for a third-order linear differential equation in the reflectionless case. The solution to the inverse scattering problem is obtained by solving a related Riemann–Hilbert problem using input consisting of the bound-state poles of a transmission coefficient and the bound-state dependency constants. Using the time-evolved dependency constants in the input data set, we show that the solution to the inverse problem yields soliton solutions to the aforementioned integrable nonlinear system and also soliton solutions to the Sawada–Kotera equation. Our method also explains the physical origin of the $2N$ real parameters appearing in the relevant N -soliton solution formula obtained by the bilinear method [17] of Hirota, by relating N of the real parameters to

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the bound-state poles and relating the remaining N real parameters to the bound-state dependency constants.

Our work presents the construction of soliton solutions to the Sawada–Kotera equation via the inverse scattering transform method [14]. Such solutions to the Sawada–Kotera equation are usually obtained by using Hirota’s bilinear method [17] or a slight modification [16, 24] of that method. Hirota’s bilinear method is an effective algebraic method to obtain soliton solutions to various integrable nonlinear partial differential equations. However, it is an ad hoc procedure, and it does not provide any insight or physical motivation for the construction of those soliton solutions. In particular, it does not explain how the parameters appearing in those soliton solutions may be related to any physical quantities. On the other hand, the method we use to construct those soliton solutions is a fundamental procedure that can be applied to a wide variety of other integrable evolution equations, and it also relates the parameters appearing in soliton solutions to the bound-state poles and bound-state dependency constants for the relevant ordinary linear differential operator.

In the inverse scattering transform method [14], a time-evolved potential is constructed from the time-evolved scattering data. A soliton solution corresponds to a time-evolved potential corresponding to a reflectionless scattering data set. A reflectionless scattering data set comprises the bound-state information only. The bound-state information in turn consists of the bound-state poles of a transmission coefficient and a bound-state dependency constant for each bound-state pole. For example, an N -soliton solution to the Sawada–Kotera equation is a real-valued function of x and t and it contains $2N$ real parameters. As our method indicates, N of those real parameters identify the locations of the bound-state poles of a transmission coefficient and the remaining N real parameters identify the corresponding bound-state dependency constants.

Our chapter is organized as follows. In Section 2, we introduce the fifth-order integrable system (2.6) of coupled nonlinear partial differential equations associated with the third-order linear equation (2.8). We indicate how the Sawada–Kotera equation arises as a special case of (2.6) by uncoupling the integrable system (2.6). In Section 3, we provide a summary of the direct scattering problem for (2.8) in the reflectionless case. This is done by introducing three relevant solutions to (2.8) at each k -value in the complex k -plane. We indicate how the left and right transmission coefficients are related to the spacial asymptotics of two of the relevant solutions to (2.8). In Section 4, we introduce the bound-state poles of the left transmission coefficient $T_1(k)$, and we describe the bound-state dependency constant for each bound-state pole. For simplicity we assume that each bound-state pole is simple. In Section 5, we describe the solution to the inverse scattering problem for (2.8) in the reflectionless case. This is done by providing some explicit expressions for each of the potentials Q and P in terms of the input data set consisting of $T_1(k)$, the bound-state poles of $T_1(k)$, and the bound-state dependency constants. In the case of the recovery of the potentials Q and P that depend on the parameter t , the use of time-evolved dependency constants yields soliton solutions to the corresponding integrable system. In Section 6, by imposing the appropriate

restrictions on a certain solution to the nonlinear system (2.6), we obtain the N -soliton solution to the Sawada–Kotera equation (2.1). Finally, in Section 7, we illustrate the N -soliton solution to the Sawada–Kotera equation when N takes the values of 1, 2, and 3.

2 The integrable system and the Sawada–Kotera equation

The Sawada–Kotera equation is the fifth-order nonlinear partial differential equation given by

$$Q_t + Q_{xxxxx} + 5Q_x Q_{xx} + 5Q Q_{xxx} + 5Q^2 Q_x = 0, \quad x, t \in \mathbb{R}, \quad (2.1)$$

where $\mathbb{R}$ is the real axis and the subscripts denote the corresponding partial derivatives. It is used as a model to describe the propagation of surface water waves in long, narrow, shallow canals, similar to the modeling used by the KdV (Korteweg–de Vries) equation [19]. For both the KdV equation and the Sawada–Kotera equation, we assume that the solutions are real valued. Hence, the quantity Q appearing in (2.1) is assumed to be real valued. Compared to the third-order nonlinear KdV equation, the Sawada–Kotera equation takes into account higher-order nonlinear and dispersive effects. In Section 7, we remark on some similar behaviors of soliton solutions to the Sawada–Kotera equation and to the KdV equation.

The Sawada–Kotera equation is integrable in the sense of the inverse scattering transform method. This is due to the fact that it has a Lax pair, i.e. it is related to two linear differential operators L and A satisfying the Lax operator equation [20]

$$L_t + LA - AL = 0, \quad (2.2)$$

where L_t is obtained by applying the partial t -derivative on L . When we use the Lax pair (L, A) for (2.1) on the left-hand side of (2.2), that left-hand side becomes the zero multiplication operator. There are actually two distinct Lax pairs for the Sawada–Kotera equation (2.1). The first pair (L, A) is given by

$$\begin{cases} L = D^3 + QD, \\ A = 9D^5 + 15QD^3 + 15Q_x D^2 + (10Q_{xx} + 5Q^2)D, \end{cases} \quad (2.3)$$

where we have let $D := d/dx$ and $D^n := d^n/dx^n$ for $n \geq 2$. The second Lax pair (L, A) for (2.1) is given by

$$\begin{cases} L = D^3 + QD + Q_x, \\ A = 9D^5 + 15QD^3 + 30Q_x D^2 + (25Q_{xx} + 5Q^2)D + (10Q_{xxx} + 10Q Q_x). \end{cases} \quad (2.4)$$

The Lax pairs in (2.3) and (2.4) can be obtained as the two special cases of the general Lax pair (L, A) given by

$$\begin{cases} L = D^3 + QD + P, \\ A = 9D^5 + 15QD^3 + (15P + 15Q_x)D^2 + (15P_x + 10Q_{xx} + 5Q^2)D + (10P_{xx} + 10QP), \end{cases} \quad (2.5)$$

where the potentials Q and P are independent of each other and are allowed to take complex values. The Lax pair in (2.3) is obtained from (2.5) when Q is real valued and we have $P \equiv 0$, and the Lax pair in (2.4) is obtained when Q is real valued and we have $P = Q_x$. The general Lax pair (L, A) given in (2.5) satisfies the Lax operator equation (2.2) provided the potentials Q and P appearing in (2.5) satisfy the coupled system of two fifth-order nonlinear partial differential equations given by

$$\begin{cases} Q_t + Q_{xxxxx} + 5Q_x Q_{xx} + 5Q Q_{xxx} + 5Q^2 Q_x + 15Q_{xx} P + 15Q_x P_x - 30P P_x = 0, \\ P_t + P_{xxxxx} + 5Q P_{xxx} + 15Q_x P_{xx} + 20Q_{xx} P_x + 5Q^2 P_x + 10Q_{xxx} P - 15P P_{xx} \\ \quad + 10Q Q_x P - 15(P_x)^2 = 0. \end{cases} \quad (2.6)$$

We assume that Q and P belong to the Schwartz class in $x \in \mathbb{R}$ for each fixed t . In order to uncouple the nonlinear system (2.6), from the last three terms on the left-hand side of the first coupled equation in (2.6), we see that we must have

$$15Q_{xx} P + 15Q_x P_x - 30P P_x = 0,$$

which is equivalent to

$$[(Q_x - P)P]_x = 0. \quad (2.7)$$

Integrating both sides of (2.7) with respect to x and using the fact that Q and P must vanish as $x \rightarrow \pm\infty$ for each fixed t , we obtain

$$(Q_x - P)P = 0,$$

which shows that the uncoupling in (2.6) occurs when $P \equiv 0$ or when $P = Q_x$. For each of those two cases, one can directly verify that the left-hand side of the second coupled equation in (2.6) vanishes identically when the first equation in (2.6) holds. Hence, we confirm that the use of $P \equiv 0$ or $P = Q_x$ in (2.6) yields (2.1).

Associated with the linear operator L in (2.5), we have the third-order ordinary linear differential equation

$$\psi''' + Q(x)\psi' + P(x)\psi = k^3\psi, \quad x \in \mathbb{R}, \quad (2.8)$$

where the prime denotes the x -derivative. We suppress the dependence on t for the potentials Q and P , and we write $Q(x)$ and $P(x)$, respectively, instead of $Q(x, t)$ and $P(x, t)$. The adjoint equation [3] for (2.8) is given by

$$\bar{\psi}''' + \bar{Q}(x) \bar{\psi}' + \bar{P}(x) \bar{\psi} = k^3 \bar{\psi}, \quad x \in \mathbb{R}, \quad (2.9)$$

with the adjoint potentials $\bar{Q}$ and $\bar{P}$ related to Q and P as

$$\bar{Q}(x) = Q(x)^*, \quad \bar{P}(x) = Q'(x)^* - P(x)^*, \quad x \in \mathbb{R}, \quad (2.10)$$

where we use an asterisk for complex conjugation and use an overbar to denote the quantities associated with the adjoint equation (2.9). Comparing (2.8) and (2.9), with the help of (2.10) we observe that, when the potential Q is real valued and the potential P is zero, the adjoint potential $\bar{Q}$ is equal to Q and the adjoint potential $\bar{P}$ becomes equal to Q_x . In that case, the equation in (2.8) reduces to

$$\psi''' + Q(x) \psi' = k^3 \psi, \quad x \in \mathbb{R}, \quad (2.11)$$

and the adjoint equation in (2.9) reduces to

$$\bar{\psi}''' + Q(x) \bar{\psi}' + Q_x(x) \bar{\psi} = k^3 \bar{\psi}, \quad x \in \mathbb{R}. \quad (2.12)$$

The Lax pair in (2.3) is associated with the reduced equation (2.11), and the Lax pair in (2.4) is associated with the reduced adjoint equation (2.12). This explains why the Sawada–Kotera equation (2.1) has two distinct Lax pairs given in (2.3) and (2.4), respectively.

3 The direct scattering problem in the reflectionless case

In this section, we present the basic ingredients for the direct scattering problem for (2.8) in the reflectionless case. We assume that the two complex-valued potentials Q and P appearing in (2.8) each belong to the Schwartz class in $x \in \mathbb{R}$ for each fixed value of t . That particular direct problem consists of the determination of the solutions to (2.8) and the left and right transmission coefficients $T_l(k)$ and $T_r(k)$, respectively, for (2.8) when the potentials Q and P are known. For the description of the direct scattering problem when the reflection coefficients are not zero, we refer the reader to [3, 18, 27].

It is convenient to divide the complex k -plane into four open sectors $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ as indicated on the left plot of Figure 1.1 by using the directed half lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$, which are parametrized as

$$\mathcal{L}_1 := \{k \in \mathbb{C} : k = zs \text{ for } s \in [0, +\infty)\}, \quad (3.1)$$

$$\mathcal{L}_2 := \{k \in \mathbb{C} : k = z^2 s \text{ for } s \in [0, +\infty)\},$$

$$\mathcal{L}_3 := \{k \in \mathbb{C} : k = -zs \text{ for } s \in [0, +\infty)\}, \quad (3.2)$$

$$\mathcal{L}_4 := \{k \in \mathbb{C} : k = -z^2 s \text{ for } s \in [0, +\infty)\}.$$

We use z to denote the special complex number $e^{2\pi i/3}$, which is also expressed as

$$z := -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \quad (3.3)$$

The open sectors $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ are described by using the parametrizations given by

$$\begin{aligned} \Omega_1 &:= \left\{ k \in \mathbb{C} : \frac{2\pi}{3} < \arg[k] < \frac{4\pi}{3} \right\}, \\ \Omega_2 &:= \left\{ k \in \mathbb{C} : -\frac{2\pi}{3} < \arg[k] < -\frac{\pi}{3} \right\}, \\ \Omega_3 &:= \left\{ k \in \mathbb{C} : -\frac{\pi}{3} < \arg[k] < \frac{\pi}{3} \right\}, \\ \Omega_4 &:= \left\{ k \in \mathbb{C} : \frac{\pi}{3} < \arg[k] < \frac{2\pi}{3} \right\}, \end{aligned}$$

where $\arg[k]$ denotes the argument function taking values in the interval $(-2\pi/3, 4\pi/3)$. We use $\overline{\Omega}_1, \overline{\Omega}_2, \overline{\Omega}_3, \overline{\Omega}_4$ to denote the closures of the open sectors $\Omega_1, \Omega_2, \Omega_3, \Omega_4$, respectively, where we recall that the closures are obtained by adding the boundaries to the corresponding open sectors.

We define the left Jost solution $f(k, x)$ and the right Jost solution $g(k, x)$ as the solutions to (2.8) with the respective spacial asymptotics given by

$$\begin{cases} f(k, x) = e^{kx} [1 + o(1)], & x \rightarrow +\infty, k \in \overline{\Omega}_1, \\ f'(k, x) = k e^{kx} [1 + o(1)], & x \rightarrow +\infty, k \in \overline{\Omega}_1, \\ f''(k, x) = k^2 e^{kx} [1 + o(1)], & x \rightarrow +\infty, k \in \overline{\Omega}_1, \end{cases} \quad (3.4)$$

$$\begin{cases} g(k, x) = e^{kx} [1 + o(1)], & x \rightarrow -\infty, k \in \overline{\Omega}_3, \\ g'(k, x) = k e^{kx} [1 + o(1)], & x \rightarrow -\infty, k \in \overline{\Omega}_3, \\ g''(k, x) = k^2 e^{kx} [1 + o(1)], & x \rightarrow -\infty, k \in \overline{\Omega}_3. \end{cases} \quad (3.5)$$

When the reflection coefficients for (2.8) are all zero, the left transmission coefficient $T_1(k)$ and the right transmission coefficient $T_r(k)$ are obtained by using the appropriate respective spacial asymptotics of the Jost solutions $f(k, x)$ and $g(k, x)$. We have

$$\begin{aligned} f(k, x) &= e^{kx} T_1(k)^{-1} [1 + o(1)], & x \rightarrow -\infty, k \in \overline{\Omega}_1, \\ g(k, x) &= e^{kx} T_r(k)^{-1} [1 + o(1)], & x \rightarrow +\infty, k \in \overline{\Omega}_3. \end{aligned}$$

When the potentials Q and P in (2.8) belong to the Schwartz class in $x \in \mathbb{R}$, the transmission coefficients $T_1(k)$ and $T_r(k)$ have meromorphic extensions from their respective k -domains $\overline{\Omega}_1$ and $\overline{\Omega}_3$ to the entire complex k -plane. In fact, those extensions satisfy [3]

$$T_r(k) = \frac{1}{T_1(k)}, \quad k \in \mathbb{C}. \quad (3.6)$$

It is known [3] that (2.8) has the particular solutions $m(k, x)$ and $n(k, x)$ with the respective k -domains $\overline{\Omega_2}$ and $\overline{\Omega_4}$, and in the reflectionless case they satisfy the spacial asymptotics given by

$$\begin{aligned} m(k, x) &= e^{kx} [1 + o(1)], \quad x \rightarrow -\infty, k \in \overline{\Omega_2}, \\ n(k, x) &= e^{kx} [1 + o(1)], \quad x \rightarrow -\infty, k \in \overline{\Omega_4}. \end{aligned}$$

In the reflectionless case, we have [3] the spacial asymptotics

$$\begin{aligned} m(k, x) &= e^{kx} T_1(z^2 k)^{-1} T_r(zk) [1 + o(1)], \quad x \rightarrow +\infty, k \in \overline{\Omega_2}, \\ n(k, x) &= e^{kx} T_1(zk)^{-1} T_r(z^2 k) [1 + o(1)], \quad x \rightarrow +\infty, k \in \overline{\Omega_4}, \end{aligned}$$

where we recall that z is the cube root of unity appearing in (3.3).

Since k appears as k^3 in (2.8), it follows that $\psi(zk, x)$ and $\psi(z^2 k, x)$ are also solutions to (2.8) whenever $\psi(k, x)$ is a solution. The k -domain of $\psi(zk, x)$ is obtained from the k -domain of $\psi(k, x)$ by a clockwise rotation of $2\pi/3$ radians around the origin of the complex k -plane. Similarly, the k -domain of $\psi(z^2 k, x)$ is obtained from the k -domain of $\psi(k, x)$ by a clockwise rotation of $4\pi/3$ radians. This allows us to determine three linearly independent solutions to (2.8) at each k -value in the complex k -plane. Toward that goal, we divide Ω_1 into the open subsectors Ω_1^{up} and Ω_1^{down} by using the negative real axis $\mathbb{R}^-$, and we divide the open sector Ω_3 into the open subsectors Ω_3^{down} and Ω_3^{up} by using the positive real axis $\mathbb{R}^+$. On the middle plot of Figure 1.1, we show the partitioning of the complex k -plane into the six open sectors Ω_1^{up} , Ω_1^{down} , Ω_2 , Ω_3^{down} , Ω_3^{up} , and Ω_4 . We use $-\mathbb{R}^-$ and $\mathbb{R}^+$, respectively, to denote the directed half lines parametrized as

$$\begin{aligned} \mathbb{R}^+ &:= \{k = s : s \in [0, +\infty)\}, \\ -\mathbb{R}^- &:= \{k = -s : s \in [0, +\infty)\}. \end{aligned}$$

On the right plot of Figure 1.1 we indicate three linearly independent solutions to (2.8) in each of the six closed sectors $\overline{\Omega_1^{\text{up}}}$, $\overline{\Omega_1^{\text{down}}}$, $\overline{\Omega_2}$, $\overline{\Omega_3^{\text{down}}}$, $\overline{\Omega_3^{\text{up}}}$, and $\overline{\Omega_4}$, where we recall that the overbar denotes the closure of the corresponding open sector.

For fixed real values of x and t , the large k -asymptotics of the basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, and $n(k, x)$ in their respective k -domains are given by [3]

$$f(k, x) = e^{kx} \left[1 + \frac{u_1(x)}{k} + \frac{u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_1}, \quad (3.7)$$

$$g(k, x) = e^{kx} \left[1 + \frac{v_1(x)}{k} + \frac{v_2(x)}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_3}, \quad (3.8)$$

$$m(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_2}, \quad (3.9)$$

$$n(k, x) = e^{kx} \left[1 + O\left(\frac{1}{k}\right) \right], \quad k \rightarrow \infty \text{ in } \overline{\Omega_4}, \quad (3.10)$$

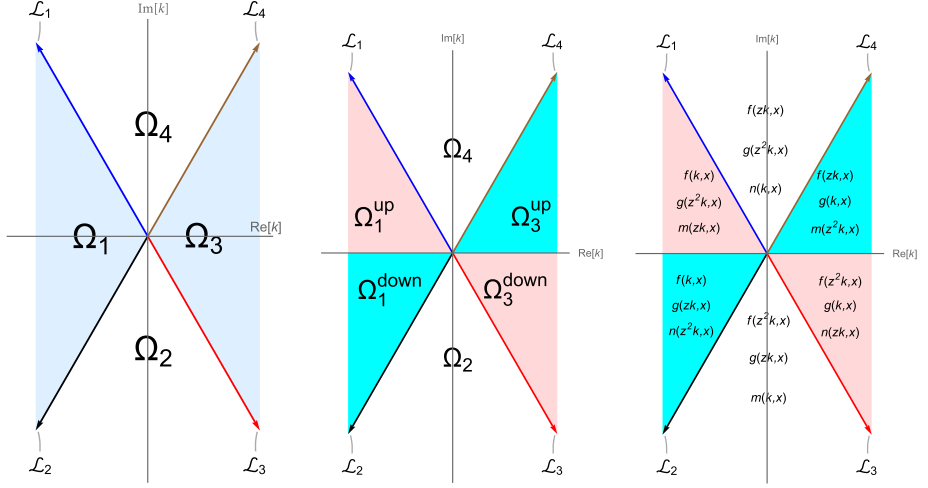


Figure 1.1: The directed half lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ and the open sectors $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in the complex k -plane are shown on the left plot. The complex k -plane is divided into the six sectors $\Omega_1^{\text{up}}, \Omega_1^{\text{down}}, \Omega_2, \Omega_3^{\text{down}}, \Omega_3^{\text{up}},$ and Ω_4 as shown on the middle plot, and the k -domains of three basic solutions to (2.8) in each of the six regions, respectively, are shown on the right plot.

where we have defined

$$u_1(x) := \frac{1}{3} \int_x^\infty dy Q(y), \quad x \in \mathbb{R}, \quad (3.11)$$

$$u_2(x) := -\frac{1}{3} \int_x^\infty dy [Q'(y) - P(y)] + \frac{1}{18} \left[\int_x^\infty dy Q(y) \right]^2, \quad x \in \mathbb{R}, \quad (3.12)$$

$$v_1(x) := -\frac{1}{3} \int_{-\infty}^x dy Q(y), \quad x \in \mathbb{R}, \quad (3.13)$$

$$v_2(x) := \frac{1}{3} \int_{-\infty}^x dy [Q'(y) - P(y)] + \frac{1}{18} \left[\int_{-\infty}^x dy Q(y) \right]^2, \quad x \in \mathbb{R}. \quad (3.14)$$

Using (3.11)–(3.14), we express the potentials Q and P in terms of $u_1(x)$ and $u_2(x)$ as

$$Q(x) = -3 \frac{du_1(x)}{dx}, \quad x \in \mathbb{R}, \quad (3.15)$$

$$P(x) = 3 \left[u_1(x) \frac{du_1(x)}{dx} - \frac{d^2 u_1(x)}{dx^2} - \frac{du_2(x)}{dx} \right], \quad x \in \mathbb{R}, \quad (3.16)$$

or in terms of $v_1(x)$ and $v_2(x)$ as

$$Q(x) = -3 \frac{dv_1(x)}{dx}, \quad x \in \mathbb{R}, \quad (3.17)$$

$$P(x) = 3 \left[v_1(x) \frac{dv_1(x)}{dx} - \frac{d^2 v_1(x)}{dx^2} - \frac{dv_2(x)}{dx} \right], \quad x \in \mathbb{R}. \quad (3.18)$$

4 The bound states and the bound-state dependency constants

A bound state corresponds to a nontrivial solution to (2.8) which is square integrable in $x \in \mathbb{R}$. If a bound state occurs at $k = k_j$ somewhere in the complex k -plane, then the number of linearly independent square-integrable solutions to (2.8) at $k = k_j$ determines the multiplicity of that bound state. In our work, we only consider simple bound states, where the multiplicity of each bound state is equal to 1. We refer the reader to [4, 5, 6] for the treatment of bound states with multiplicities for various differential and difference equations. The analysis of solutions to (2.8) at $k = 0$ is more challenging. When we analyze bound states for (2.8), we assume that a bound state does not occur at $k = 0$.

We consider the bound states occurring at the zeros of $T_1(k)^{-1}$ in Ω_1^{down} or Ω_1^{up} . Without loss of generality, it is enough to consider the bound states at the poles of $T_1(k)$ in Ω_1^{down} . It turns out [3] that the analysis of a bound state at $k = k_j$ with $k_j \in \Omega_1^{\text{down}}$ yields useful information about the bound state at $k = k_j^*$ with $k_j^* \in \Omega_1^{\text{up}}$. We recall that we use an asterisk to denote complex conjugation.

We define the 3-Wronskian of three functions $F(x)$, $G(x)$, and $H(x)$ as

$$[F(x); G(x); H(x)] := \begin{vmatrix} F(x) & G(x) & H(x) \\ F'(x) & G'(x) & H'(x) \\ F''(x) & G''(x) & H''(x) \end{vmatrix},$$

where we have the determinant of the relevant 3×3 matrix on the right-hand side. The 3-Wronskian of any three solutions to (2.8) at any particular k -value is zero if and only if those three solutions are linearly dependent. Furthermore, because of the absence of the term ψ'' in (2.8), the 3-Wronskian of any three solutions is independent of x and its value can be evaluated at any particular x -value. For example, using their asymptotics as $x \rightarrow \pm\infty$, we evaluate the 3-Wronskian of $f(k, x)$, $g(zk, x)$, and $n(z^2k, x)$ in Ω_1^{down} as

$$[f(k, x); g(zk, x); n(z^2k, x)] = -3z(1 - z)k^3 T_1(k)^{-1}, \quad k \in \overline{\Omega_1^{\text{down}}}. \quad (4.1)$$

Let us assume that the nonzero complex constant k_j is located in the open sector Ω_1^{down} and that it corresponds to a bound state. Then, we have $T_1(k_j)^{-1} = 0$. From (4.1) we see that the three solutions $f(k_j, x)$, $g(zk_j, x)$, and $n(z^2k_j, x)$ to (2.8) are linearly dependent. This allows us [3] to express $f(k_j, x)$ as a linear combination of $g(zk_j, x)$ and $n(z^2k_j, x)$ as

$$f(k_j, x) = D(k_j) g(zk_j, x) + W(k_j) n(z^2k_j, x), \quad x \in \mathbb{R}, \quad (4.2)$$

for some complex-valued constants $D(k_j)$ and $W(k_j)$.

Let us divide the open sector Ω_1^{down} , shown on the middle plot of Figure 1.1, into two parts, the first of which is the sector with $\arg[k] \in (\pi, 7\pi/6)$ and the second sector is described via $\arg[k] \in [7\pi/6, 4\pi/3)$. The analysis in Section 3 of [3] shows that $D(k_j) = 0$ when $\arg[k_j] \in (\pi, 7\pi/6)$, and hence (4.2) yields

$$f(k_j, x) = W(k_j) n(z^2 k_j, x), \quad \arg[k_j] \in \left(\pi, \frac{7\pi}{6}\right), \quad x \in \mathbb{R}, \quad (4.3)$$

where the nonzero complex constant $W(k_j)$ corresponds to the dependency constant at the bound state with $k = k_j$. On the other hand, again from the analysis in Section 3 of [3], it follows that $W(k_j) = 0$ if we have $\arg[k_j] \in [7\pi/6, 4\pi/3)$. In that case, from (4.2) with $W(k_j) = 0$ we get

$$f(k_j, x) = D(k_j) g(zk_j, x), \quad \arg[k_j] \in \left[\frac{7\pi}{6}, \frac{4\pi}{3}\right), \quad x \in \mathbb{R}, \quad (4.4)$$

with the nonzero complex constant $D(k_j)$ corresponding to the dependency constant at the bound state with $k = k_j$. If the potentials Q and P in (2.8) depend also on the parameter t , then the solutions to (2.8) also depend on t . Consequently, the dependency constants $W(k_j)$ and $D(k_j)$ appearing in (4.3) and (4.4), respectively, also depend on t . For further analysis related to the bound states of (2.8), we refer the reader to [3].

Let us consider the case where the potentials Q and P depend on the time variable t with the time evolution governed by the linear operator A given in the second line of (2.5). In that case, the bound-state dependency constants evolve in time as

$$\begin{aligned} D(k_j) &= E(k_j) e^{9(z^2-1)k_j^5 t}, \quad \arg[k_j] \in \left[\frac{7\pi}{6}, \frac{4\pi}{3}\right), \\ W(k_j) &= U(k_j) e^{9(z-1)k_j^5 t}, \quad \arg[k_j] \in \left(\pi, \frac{7\pi}{6}\right), \end{aligned} \quad (4.5)$$

where $E(k_j)$ and $U(k_j)$ are used to denote the values of the dependency constants $D(k_j)$ and $W(k_j)$, respectively, at $t = 0$. We refer the reader to [3] for further information on the time evolution of the bound-state dependency constants.

5 The inverse scattering problem in the reflectionless case

In this section, we present the basic results for the inverse problem for (2.8) in the reflectionless case. In that case, the inverse problem consists of the determination of the potentials Q and P when the transmission coefficients $T_1(k)$ and $T_r(k)$ are known. In

fact, as a result of (3.6), the right transmission coefficient $T_r(k)$ is known when $T_l(k)$ is specified. In general, the potentials Q and P cannot be determined uniquely when we only know the transmission coefficients. For the unique recovery of the potentials, we need some additional information. The specification of a dependency constant or a normalization constant at each bound state is a conventional way to recover the potentials uniquely. A bound-state dependency constant and a bound-state normalization constant are related [3] to each other via the residue of a transmission coefficient at the bound-state pole. The use of bound-state dependency constants is convenient when the potentials are recovered from the solution to a related Riemann–Hilbert problem. The use of bound-state normalization constants is appropriate when the potentials are recovered from the solution to a related linear integral equation, such as a Marchenko integral equation [7, 8, 10, 13, 21, 22] or a Gel’fand–Levitan integral equation [8, 21, 22].

In the reflectionless case, we formulate the inverse problem for (2.8) as a Riemann–Hilbert problem in the complex k -plane. Thus, the left transmission coefficient and the bound-state dependency constants can be used as input to the Riemann–Hilbert problem, and the potentials are uniquely recovered from the solution to the Riemann–Hilbert problem. We refer the reader to Section 4 of [3] for the details of the analysis of the formulation of the corresponding Riemann–Hilbert problem. In this section, we present the relevant Riemann–Hilbert problem when a specific left transmission coefficient is used as input. The particular choice of the left transmission coefficient is motivated by the goal of obtaining soliton solutions to the Sawada–Kotera equation (2.1).

We choose the left transmission coefficient $T_l(k)$ as

$$T_l(k) = \frac{\Gamma(k)}{\Gamma(-k)}, \quad (5.1)$$

where $\Gamma(k)$ is given by

$$\Gamma(k) := \prod_{j=1}^N (k + k_j)(k + k_j^*), \quad k \in \mathbb{C}. \quad (5.2)$$

From (5.1) and (5.2), we see that the left transmission coefficient $T_l(k)$ is defined on the whole complex k -plane and that it has $2N$ poles located at $k = k_j$ and $k = k_j^*$, respectively, for $1 \leq j \leq N$. The large k -asymptotics of $T_l(k)$ is given by

$$T_l(k) = 1 + \frac{2\Sigma_N}{k} + \frac{2\Sigma_N^2}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty \text{ in } \mathbb{C},$$

where we have defined

$$\Sigma_N := \sum_{j=1}^N (k_j + k_j^*). \quad (5.3)$$

The quantity Q appearing in the Sawada–Kotera equation (2.1) is real valued. Since we would like to relate (5.1) eventually to (2.1), we are interested in the consideration of the potentials Q and P appearing in (2.8) only when they are real valued. Since the original k -domain of $T_1(k)$ is the sector $\overline{\Omega}_1$, the restriction of Q to the real values requires [3] that each of the k_j -values in (5.2) occurs on the directed half line with $\arg[k] = 7\pi/6$. Thus, we have a restriction on the locations of the bound-state poles. Since we only consider simple bound states, the restriction can be presented as

$$k_j = iz\eta_j, \quad 1 \leq j \leq N, \quad (5.4)$$

where we recall that z is the special complex constant in (3.3). Without loss of generality, we assume that we have the ordering $0 < \eta_1 < \dots < \eta_N$.

We refer the reader to Section 4 of [3] for the formulation of the Riemann–Hilbert problem in the presence of reflection coefficients. Here, we present the basic steps of the formulation of the Riemann–Hilbert problem in the reflectionless case and we also describe the basic steps to solve that Riemann–Hilbert problem. For the details of the steps outlined here, we refer the reader to Section 4 of [3]. Using the basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, and $n(k, x)$ introduced in Section 3, we form the solutions $\Phi_+(k, x)$ and $\Phi_-(k, x)$ to (2.8) as

$$\Phi_+(k, x) := \begin{cases} T_1(k)f(k, x), & k \in \overline{\Omega}_1, \\ m(k, x), & k \in \overline{\Omega}_2, \end{cases} \quad (5.5)$$

$$\Phi_-(k, x) := \begin{cases} g(k, x), & k \in \overline{\Omega}_3, \\ n(k, x), & k \in \overline{\Omega}_4. \end{cases} \quad (5.6)$$

Using the directed half lines $\mathcal{L}_1$ and $\mathcal{L}_3$ defined in (3.1) and (3.2), respectively, we obtain the directed full line $\mathcal{L}$ via $\mathcal{L} := \mathcal{L}_1 \cup (-\mathcal{L}_3)$, where we recall that $-\mathcal{L}_3$ is obtained from $\mathcal{L}_3$ by changing its direction. The parametrization of $\mathcal{L}$ is given by

$$\mathcal{L} := \{k \in \mathbb{C} : k = zs \text{ for } s \in (-\infty, +\infty)\}.$$

The directed line $\mathcal{L}$ divides the complex k -plane into two half planes $\mathcal{P}^+$ and $\mathcal{P}^-$ as shown in Figure 1.2. The open left-half complex plane $\mathcal{P}^+$ and the open right-half complex plane $\mathcal{P}^-$ can be parameterized as

$$\begin{aligned} \mathcal{P}^+ &:= \{k \in \mathbb{C} : k = zs \text{ for } s \in \mathbb{C}^+\}, \\ \mathcal{P}^- &:= \{k \in \mathbb{C} : k = zs \text{ for } s \in \mathbb{C}^-\}, \end{aligned}$$

where we use $\mathbb{C}^+$ and $\mathbb{C}^-$ to denote the upper-half and lower-half complex planes. We refer to $\mathcal{P}^+$ as the plus region and refer to $\mathcal{P}^-$ as the minus region, as indicated in Figure 1.2. We use $\overline{\mathcal{P}^+}$ and $\overline{\mathcal{P}^-}$ to denote their closures. Hence, we have $\overline{\mathcal{P}^+} := \mathcal{P}^+ \cup \mathcal{L}$ and $\overline{\mathcal{P}^-} := \mathcal{P}^- \cup \mathcal{L}$.

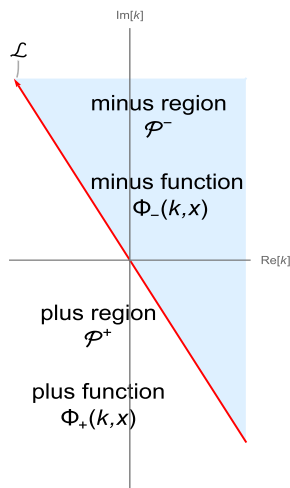


Figure 1.2: The plus and minus regions in the complex k -plane are separated by the directed full line $\mathcal{L}$, and the plus and minus functions have their respective k -domains as indicated.

We remark that the quantities $\Phi_+(k, x)$ and $\Phi_-(k, x)$ in (5.5) and (5.6), respectively, are certain specific solutions to (2.8). We proceed to set up our Riemann–Hilbert problem. It is known [3] in the reflectionless case that $\Phi_+(k, x)$ and $\Phi_-(k, x)$ satisfy

$$\Phi_+(k, x) = \Phi_-(k, x), \quad k \in \mathcal{L}. \quad (5.7)$$

When (5.1), (5.2), and (5.4) hold, it is known [3] that, for each fixed $x \in \mathbb{R}$, the quantity $\Phi_+(k, x)$ is meromorphic in $k \in \mathcal{P}^+$ with simple poles located at $k = k_j$ and $k = k_j^*$ for $1 \leq j \leq N$. On the other hand, for each fixed $x \in \mathbb{R}$, the quantity $\Phi_-(k, x)$ is analytic in $k \in \mathcal{P}^-$. From (5.7), we conclude that $\Phi_+(k, x)$ is the meromorphic continuation of $\Phi_-(k, x)$ from $k \in \mathcal{P}^-$ to $k \in \mathcal{P}^+$ and that $\Phi_-(k, x)$ is the analytic continuation of $\Phi_+(k, x)$ from $k \in \mathcal{P}^+$ to $k \in \mathcal{P}^-$. Multiplying both sides of (5.7) by $e^{-kx} \Gamma(-k)$, we get

$$e^{-kx} \Gamma(-k) \Phi_+(k, x) = e^{-kx} \Gamma(-k) \Phi_-(k, x), \quad k \in \mathcal{L}. \quad (5.8)$$

For each fixed $x \in \mathbb{R}$, the left-hand side of (5.8) is analytic in $k \in \mathcal{P}^+$ and the right-hand side is analytic in $k \in \mathcal{P}^-$. Hence, those two sides are analytic continuations of each other. Consequently, each side of (5.8) is entire with their respective analytic continuations in $k \in \mathbb{C}$. With the help of (3.7)–(3.14), by using the generalized Liouville theorem [25], we conclude that each side of (5.8) is equal to a monic polynomial in k of degree $2N$, where the coefficients may depend on x and t . By suppressing the t -dependence in our notation, we obtain the solution to our Riemann–Hilbert problem posed in (5.7) as

$$\Phi_+(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\mathcal{P}^+}, \quad x \in \mathbb{R}, \quad (5.9)$$

$$\Phi_-(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\mathcal{P}^-}, \quad x \in \mathbb{R}, \quad (5.10)$$

where $V(k)$ is the row vector with $2N$ components defined as

$$V(k) := [k^{2N-1} \quad k^{2N-2} \quad \dots \quad k \quad 1], \quad k \in \mathbb{C},$$

and $\mathbf{A}(x)$ is a column vector with $2N$ entries that are functions of x and t . Because of the aforementioned meromorphic extensions, each of (5.9) and (5.10) holds when $k \in \mathbb{C}$. We write $\mathbf{A}(x)$ in terms of its components as

$$\mathbf{A}(x) = \begin{bmatrix} A_{2N-1}(x) \\ A_{2N-2}(x) \\ \vdots \\ A_1(x) \\ A_0(x) \end{bmatrix}, \quad x \in \mathbb{R}.$$

We emphasize that we suppress the t -dependence in our notation for $\mathbf{A}(x)$ and its components $A_j(x)$ for $0 \leq j \leq 2N-1$.

We remark that the solutions $\Phi_+(k, x)$ and $\Phi_-(k, x)$ appearing in (5.9) and (5.10) comprise the general solution to the Riemann–Hilbert problem (5.7). In other words, those solutions contain $2N$ arbitrary constants not depending on k but depending on x and t through the scalar functions $A_j(x)$ for $0 \leq j \leq 2N-1$. From (5.9), with the help of (5.1) and the first line of (5.5), we get

$$f(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(k)}, \quad k \in \overline{\Omega_1}, \quad x \in \mathbb{R}. \quad (5.11)$$

From (5.9) and the second line of (5.5), we obtain

$$m(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\Omega_2}, \quad x \in \mathbb{R}. \quad (5.12)$$

Similarly, from (5.10) with the help of (5.6) we have

$$g(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\Omega_3}, \quad x \in \mathbb{R}, \quad (5.13)$$

$$n(k, x) = e^{kx} \frac{k^{2N} + V(k) \mathbf{A}(x)}{\Gamma(-k)}, \quad k \in \overline{\Omega_4}, \quad x \in \mathbb{R}. \quad (5.14)$$

With the help of (5.2), we observe that the right-hand sides in (5.12)–(5.14) are analytic in their respective k -domains and that the right-hand side in (5.11) is meromorphic in its k -domain. Thus, we have confirmed the aforementioned meromorphic and analytic properties stated below (5.7) for $\Phi_+(k, x)$ and $\Phi_-(k, x)$, respectively.

In order to get a unique solution to (5.7), we use the dependency constants $D(k_j)$ and $D(k_j^*)$ at the bound states at $k = k_j$ and $k = k_j^*$, respectively, as follows. We recall that $D(k_j)$ appears in (4.4) and (4.5). It is known [3] that when k_j has the form given in (5.4),

the bound-state dependency constants $D(k_j)$ and $D(k_j^*)$ can be written in simpler forms by introducing the modified bound-state dependency constant γ_j by letting

$$\gamma_j := -\frac{\Gamma(k_j) E(k_j)}{\Gamma(-zk_j)}, \quad 1 \leq j \leq N, \quad (5.15)$$

where we recall that $E(k_j)$ and $\Gamma(k)$ are the quantities appearing in (4.5) and (5.2), respectively. To use the bound-state dependency constants $D(k_j)$ and $D(k_j^*)$ in the solution to the Riemann–Hilbert problem (5.7), we proceed as follows. With the help of (4.5), (5.11), and (5.13), from (4.4) we obtain

$$e^{k_j x} \frac{k_j^{2N} + V(k_j) \mathbf{A}(x)}{\Gamma(k_j)} = E(k_j) e^{9(z^2-1)k_j^5 t} e^{zk_j x} \frac{(zk_j)^{2N} + V(zk_j) \mathbf{A}(x)}{\Gamma(-zk_j)}, \quad 1 \leq j \leq N. \quad (5.16)$$

The analog of (4.4) at $k = k_j^*$ is given by [3]

$$f(k_j^*, x) = D(k_j^*) g(z^2 k_j^*, x), \quad \arg[k_j^*] \in \left(\frac{2\pi}{3}, \frac{5\pi}{6} \right], \quad (5.17)$$

and the analog of (4.5) at $k = k_j^*$ is given by [3]

$$D(k_j^*) = E(k_j^*) e^{9(z-1)(k_j^*)^5 t}, \quad \arg[k_j^*] \in \left(\frac{2\pi}{3}, \frac{5\pi}{6} \right]. \quad (5.18)$$

With the help of (5.11), (5.13), and (5.18), from (5.17) for $1 \leq j \leq N$ we get

$$e^{k_j^* x} \frac{(k_j^*)^{2N} + V(k_j^*) \mathbf{A}(x)}{\Gamma(k_j^*)} = E(k_j^*) e^{9(z-1)(k_j^*)^5 t} e^{z^2 k_j^* x} \frac{(z^2 k_j^*)^{2N} + V(z^2 k_j^*) \mathbf{A}(x)}{\Gamma(-z^2 k_j^*)}. \quad (5.19)$$

Let us introduce the quantity χ_j , which is an exponential function of x and t , as

$$\chi_j := e^{(z-1)k_j x + 9(z^2-1)k_j^5 t}, \quad 1 \leq j \leq N. \quad (5.20)$$

When k_j has the form given in (5.4), from (5.20) we conclude that

$$\chi_j^* = \chi_j, \quad 1 \leq j \leq N,$$

which indicates that each χ_j is real valued. Similarly, when k_j is as in (5.4), with the help of (4.5) and (5.2) where the quantities $\Gamma(k)$ and $E(k_j)$ appear, respectively, we get

$$\frac{\Gamma(k_j^*) E(k_j^*)}{\Gamma(-z^2 k_j^*)} = \left(\frac{\Gamma(k_j) E(k_j)}{\Gamma(-zk_j)} \right)^*, \quad 1 \leq j \leq N. \quad (5.21)$$

Then, by comparing (5.21) with the right-hand side of (5.15), we conclude that the modified bound-state dependency constant at $k = k_j^*$ is equal to γ_j^* , which is the complex

conjugate of the modified bound-state dependency constant γ_j at $k = k_j$ appearing in (5.15). Consequently, we can rewrite (5.16) and (5.19), respectively, as

$$k_j^{2N} + V(k_j) \mathbf{A}(x) = -\gamma_j \chi_j [(zk_j)^{2N} + V(zk_j) \mathbf{A}(x)], \quad 1 \leq j \leq N, \quad (5.22)$$

$$(k_j^*)^{2N} + V(k_j^*) \mathbf{A}(x) = -\gamma_j^* \chi_j [(z^2 k_j^*)^{2N} + V(z^2 k_j^*) \mathbf{A}(x)], \quad 1 \leq j \leq N. \quad (5.23)$$

We remark that (5.22) and (5.23) comprise a linear algebraic system of $2N$ equations for the $2N$ unknowns $A_j(x)$ with $0 \leq j \leq 2N - 1$. We write that linear algebraic system in the matrix form as

$$\mathbf{M}(x) \mathbf{A}(x) = -\mathbf{B}(x), \quad (5.24)$$

where we have defined the $2N \times 2N$ matrix $\mathbf{M}(x)$ and the column vector $\mathbf{B}(x)$ with $2N$ entries as

$$\mathbf{M}(x) := \begin{bmatrix} m_{2N-1}(k_1) & m_{2N-2}(k_1) & \cdots & m_1(k_1) & m_0(k_1) \\ m_{2N-1}(k_1^*) & m_{2N-2}(k_1^*) & \cdots & m_1(k_1^*) & m_0(k_1^*) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{2N-1}(k_N) & m_{2N-2}(k_N) & \cdots & m_1(k_N) & m_0(k_N) \\ m_{2N-1}(k_N^*) & m_{2N-2}(k_N^*) & \cdots & m_1(k_N^*) & m_0(k_N^*) \end{bmatrix}, \quad (5.25)$$

$$\mathbf{B}(x) := \begin{bmatrix} m_{2N}(k_1) \\ m_{2N}(k_1^*) \\ \vdots \\ m_{2N}(k_N) \\ m_{2N}(k_N^*) \end{bmatrix}. \quad (5.26)$$

The entries $m_l(k_j)$ and $m_l(k_j^*)$ appearing in (5.25) and (5.26) are defined as

$$\begin{cases} m_l(k_j) := k_j^l + (zk_j)^l \gamma_j \chi_j, & 1 \leq j \leq N, \quad 0 \leq l \leq 2N, \\ m_l(k_j^*) := (k_j^*)^l + (z^2 k_j^*)^l \gamma_j^* \chi_j, & 1 \leq j \leq N, \quad 0 \leq l \leq 2N. \end{cases} \quad (5.27)$$

From (5.27), we see that we have

$$m_l(k_j^*) = m_l(k_j)^*, \quad 1 \leq j \leq N, \quad 0 \leq l \leq 2N.$$

We emphasize that we suppress the t -dependence in our notation for $\mathbf{M}(x)$, $\mathbf{A}(x)$, $\mathbf{B}(x)$, and $m_l(k_j)$. It turns out that the potential Q is determined by $A_{2N-1}(x)$ alone, and the potential P is determined by $A_{2N-1}(x)$ and $A_{2N-2}(x)$ only. The remaining entries $A_j(x)$ for $0 \leq j \leq 2N - 3$, along with $A_{2N-1}(x)$ and $A_{2N-2}(x)$, are used to construct the basic solutions $f(k, x)$, $g(k, x)$, $m(k, x)$, and $n(k, x)$ to (2.8) given in (5.11)–(5.14). It is understood that, when $N = 1$, we only have $A_1(x)$ and $A_0(x)$ and those two quantities determine both the potentials and the basic solutions.

In order to determine $A_j(x)$ for $0 \leq j \leq 2N - 1$, we solve the linear system in (5.24) and obtain $\mathbf{A}(x)$ as

$$\mathbf{A}(x) = -\mathbf{M}(x)^{-1}\mathbf{B}(x). \quad (5.28)$$

From (5.28), we recover $A_{2N-1}(x)$ and $A_{2N-2}(x)$, respectively, as

$$A_{2N-1}(x) = -[1 \ 0 \ 0 \ \cdots \ 0] \mathbf{M}(x)^{-1}\mathbf{B}(x), \quad (5.29)$$

$$A_{2N-2}(x) = -[0 \ 1 \ 0 \ \cdots \ 0] \mathbf{M}(x)^{-1}\mathbf{B}(x), \quad (5.30)$$

where the row vector in (5.29) has 1 in its first entry and the zeros in each of the remaining $2N - 1$ entries. Similarly, the row vector in (5.30) has 1 in its second entry and the zeros in all the remaining entries. By using (15) on p. 12 of [9], from (5.29) and (5.30) we obtain $A_{2N-1}(x)$ and $A_{2N-2}(x)$ as the ratio of two determinants as

$$A_{2N-1}(x) = \frac{\det \left[\begin{array}{c|cccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline \mathbf{B}(x) & & & & & \mathbf{M}(x) \end{array} \right]}{\det [\mathbf{M}(x)]},$$

$$A_{2N-2}(x) = \frac{\det \left[\begin{array}{c|cccc} 0 & 0 & 1 & 0 & \cdots & 0 \\ \hline \mathbf{B}(x) & & & & & \mathbf{M}(x) \end{array} \right]}{\det [\mathbf{M}(x)]}.$$

Alternatively, by using Cramer's rule [15], from (5.24) we get

$$A_{2N-1}(x) = -\frac{\det[\mathbf{M}_1(x)]}{\det[\mathbf{M}(x)]}, \quad (5.31)$$

$$A_{2N-2}(x) = -\frac{\det[\mathbf{M}_2(x)]}{\det[\mathbf{M}(x)]}. \quad (5.32)$$

Here, $\mathbf{M}_1(x)$ denotes the $2N \times 2N$ matrix obtained by replacing the first column of the matrix $\mathbf{M}(x)$ with the column vector $\mathbf{B}(x)$, and $\mathbf{M}_2(x)$ denotes the $2N \times 2N$ matrix obtained by replacing the second column of $\mathbf{M}(x)$ with $\mathbf{B}(x)$.

In order to solve the inverse scattering problem for (2.8) in the reflectionless case, we recall that we use the input data set $\{\eta_j, \gamma_j\}_{j=1}^N$, which allows a unique determination of the potentials Q and P . From (5.27), we see that all entries in the matrix $\mathbf{M}(x)$ and all components in the column vector $\mathbf{B}(x)$ are uniquely determined by the elements of the set $\{\eta_j, \gamma_j\}_{j=1}^N$. Hence, as shown in (5.28) we uniquely determine $\mathbf{A}(x)$ by using the input data set $\{\eta_j, \gamma_j\}_{j=1}^N$. In particular, we determine $A_{2N-1}(x)$ and $A_{2N-2}(x)$ uniquely in terms of the elements in the input data set $\{\eta_j, \gamma_j\}_{j=1}^N$.

Having determined $A_{2N-1}(x)$ and $A_{2N-2}(x)$ uniquely by using the input data set $\{\eta_j, \gamma_{jj=1}^N\}$, we proceed to determine the potentials Q and P appearing in (2.8). Using (5.2), (5.11), and (5.13), we obtain the asymptotics of $f(k, x)$ as $k \rightarrow \infty$ in $\overline{\Omega}_1$ as

$$f(k, x) = e^{kx} \left[1 + \frac{A_{2N-1}(x) - \Sigma_N}{k} + \frac{A_{2N-2}(x) - \Sigma_N A_{2N-1}(x) + \Pi_N}{k^2} + O\left(\frac{1}{k^3}\right) \right], \quad (5.33)$$

and the asymptotics of $g(k, x)$ as $k \rightarrow \infty$ in $\overline{\Omega}_3$ as

$$g(k, x) = e^{kx} \left[1 + \frac{A_{2N-1}(x) + \Sigma_N}{k} + \frac{A_{2N-2}(x) + \Sigma_N A_{2N-1}(x) + \Pi_N}{k^2} + O\left(\frac{1}{k^3}\right) \right]. \quad (5.34)$$

We note that Π_N appearing in (5.33) and (5.34) is the constant defined as

$$\begin{aligned} \Pi_N := & k_1(k_1 + k_1^* + \cdots + k_N + k_N^*) + k_1^*(k_1^* + k_2 + \cdots + k_N + k_N^*) \\ & + k_2(k_2 + k_2^* + \cdots + k_N + k_N^*) + \cdots + k_N(k_N + k_N^*) + k_N^*(k_N^*), \end{aligned}$$

and we recall that Σ_N is the constant defined in (5.3). Comparing (3.7) with (5.33) and comparing (3.8) with (5.34), we obtain

$$u_1(x) = A_{2N-1}(x) - \Sigma_N, \quad (5.35)$$

$$v_1(x) = A_{2N-1}(x) + \Sigma_N, \quad (5.36)$$

$$u_2(x) = A_{2N-2}(x) - \Sigma_N A_{2N-1}(x) + \Pi_N, \quad (5.37)$$

$$v_2(x) = A_{2N-2}(x) + \Sigma_N A_{2N-1}(x) + \Pi_N. \quad (5.38)$$

With the help of (3.15)–(3.18) and (5.35)–(5.38), we get

$$Q(x) = -3 \frac{dA_{2N-1}(x)}{dx}, \quad (5.39)$$

$$P(x) = 3 \left(A_{2N-1}(x) \frac{dA_{2N-1}(x)}{dx} - \frac{d^2 A_{2N-1}(x)}{dx^2} - \frac{dA_{2N-2}(x)}{dx} \right). \quad (5.40)$$

Since $A_{2N-1}(x)$ and $A_{2N-2}(x)$ are uniquely determined by the input data set $\{\eta_j, \gamma_{jj=1}^N\}$, we see from (5.39) and (5.40) that the same input data set uniquely determines the potentials Q and P appearing in (2.8). With the help of the first lines of (3.4) and (3.5), from (5.33) and (5.34), we obtain the spacial asymptotics given by

$$\lim_{x \rightarrow \pm\infty} A_{2N-1}(x) = \pm \Sigma_N, \quad (5.41)$$

$$\lim_{x \rightarrow \pm\infty} A_{2N-2}(x) = \Sigma_N^2 - \Pi_N. \quad (5.42)$$

We remark that the determinants of the matrices $\mathbf{M}(x)$, $\mathbf{M}_1(x)$, and $\mathbf{M}_2(x)$ appearing in (5.31) and (5.32) have various useful properties. Next, we elaborate on some of

those properties. For example, the determinant of $\mathbf{M}_1(x)$ is determined in terms of the determinant of $\mathbf{M}(x)$ as

$$\det[\mathbf{M}_1(x)] = \Sigma_N \det[\mathbf{M}(x)] + \frac{d \det[\mathbf{M}(x)]}{dx}, \quad (5.43)$$

which enables us to write (5.31) in the equivalent form as

$$A_{2N-1}(x) = -\Sigma_N - \frac{1}{\det[\mathbf{M}(x)]} \frac{d \det[\mathbf{M}(x)]}{dx}. \quad (5.44)$$

Consequently, using (5.44) on the right-hand side of (5.39), we express Q in terms of the determinant of $\mathbf{M}(x)$ as

$$Q(x) = 3 \frac{d}{dx} \left(\frac{1}{\det[\mathbf{M}(x)]} \frac{d \det[\mathbf{M}(x)]}{dx} \right). \quad (5.45)$$

As seen from (5.22) and (5.25)–(5.27), the appearance of the parameters x and t is restricted through the quantities χ_j appearing in (5.20). In fact, from (5.25)–(5.27) we observe that each of the determinants of the matrices $\mathbf{M}(x)$, $\mathbf{M}_1(x)$, and $\mathbf{M}_2(x)$ is a polynomial of degree $2N$ in the N real-valued variables $\chi_1, \dots, \chi_N$. Because of the specific appearance of k_j and k_j^* in (5.2) for $1 \leq j \leq N$, the variables $\chi_1, \dots, \chi_N$ appear on those three determinants in a symmetrical way. Each variable χ_j appears in the polynomial at most quadratically. The aforementioned symmetry allows us to determine the coefficients in those polynomials explicitly in terms of the parameters in the set $\{k_j, k_j^*, \gamma_j, \gamma_j^*\}_{j=1}^N$. For example, if we determine the coefficient of the term χ_1 in the polynomial, then we also get the coefficient of each χ_j by exploiting the aforementioned symmetry. By setting $\chi_j = 0$ for $2 \leq j \leq N$, from (5.25), we get

$$\det[\mathbf{M}(x)] = \det \begin{bmatrix} k_1^{2N-1} + (zk_1)^{2N-1} \gamma_1 \chi_1 & \cdots & k_1 + zk_1 \gamma_1 \chi_1 & 1 + \gamma_1 \chi_1 \\ (k_1^*)^{2N-1} + (z^2 k_1^*)^{2N-1} \gamma_1^* \chi_1 & \cdots & k_1^* + z^2 k_1^* \gamma_1^* \chi_1 & 1 + \gamma_1^* \chi_1 \\ \vdots & \ddots & \vdots & \vdots \\ k_N^{2N-1} & \cdots & k_N & 1 \\ (k_N^*)^{2N-1} & \cdots & k_N^* & 1 \end{bmatrix}, \quad (5.46)$$

which can be written as

$$\det[\mathbf{M}(x)] = a_0 + a_{11} \chi_1 + a_{12} \chi_1^2, \quad (5.47)$$

where the coefficients a_0 , a_{11} , and a_{12} are explicitly expressed in terms of the parameters in the set $\{k_j, k_j^*, \gamma_j, \gamma_j^*\}_{j=1}^N$. The coefficient a_0 is obtained by letting $\chi_1 = 0$ in (5.46), from which we have

$$\alpha_0 = \det \begin{bmatrix} k_1^{2N-1} & k_1^{2N-2} & \cdots & k_1 & 1 \\ (k_1^*)^{2N-1} & (k_1^*)^{2N-2} & \cdots & k_1^* & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_N^{2N-1} & k_N^{2N-2} & \cdots & k_N & 1 \\ (k_N^*)^{2N-1} & (k_N^*)^{2N-2} & \cdots & k_N^* & 1 \end{bmatrix}. \quad (5.48)$$

The right-hand side in (5.48) is related to a Vandermonde determinant, and we write (5.48) as

$$\alpha_0 = \mathcal{V}(k_1, k_1^*, k_2, k_2^*, \dots, k_N, k_N^*), \quad (5.49)$$

where the Vandermonde coefficient on the right-hand side of (5.49) is defined as

$$\mathcal{V}(a_1, a_2, a_3, a_4, \dots, a_{2p-1}, a_{2p}) := \prod_{1 \leq j < l \leq 2p} (a_j - a_l). \quad (5.50)$$

We write (5.49) as

$$\alpha_0 = \mathcal{U}_{00}, \quad (5.51)$$

we recover the coefficient α_{11} appearing in (5.47) from (5.46) as

$$\alpha_{11} = \gamma_1 \mathcal{U}_{10} + \gamma_1^* \mathcal{U}_{01}, \quad (5.52)$$

and we also obtain the coefficient α_{12} in (5.47) as

$$\alpha_{12} = \gamma_1 \gamma_1^* \mathcal{U}_{11}, \quad (5.53)$$

where we have defined

$$\begin{cases} \mathcal{U}_{00} := \mathcal{V}(k_1, k_1^*, k_2, k_2^*, \dots, k_N, k_N^*), \\ \mathcal{U}_{10} := \mathcal{V}(zk_1, k_1^*, k_2, k_2^*, \dots, k_N, k_N^*), \\ \mathcal{U}_{01} := \mathcal{V}(k_1, z^2 k_1^*, k_2, k_2^*, \dots, k_N, k_N^*), \\ \mathcal{U}_{11} := \mathcal{V}(zk_1, z^2 k_1^*, k_2, k_2^*, \dots, k_N, k_N^*). \end{cases} \quad (5.54)$$

In a similar manner, by setting $\chi_j = 0$ for $2 \leq j \leq N$ in the expression for $\det[\mathbf{M}_2(x)]$, we get

$$\det[\mathbf{M}_2(x)] = \beta_0 + \beta_{11} \chi_1 + \beta_{12} \chi_1^2. \quad (5.55)$$

During the evaluation of $\det[\mathbf{M}_2(x)]$, we can factor out certain common terms from the rows of the matrix $\mathbf{M}_2(x)$ and then evaluate the determinant of the resulting simplified matrix as a Vandermonde determinant. Hence, in order to express the coefficients β_0 ,

β_{11}, β_{12} in (5.55) explicitly in terms of the parameters in the set $\{k_j, k_j^*, \gamma_j, \gamma_j^*\}_{j=1}^N$, it is convenient to introduce the modified Vandermonde coefficient $\tilde{\mathcal{V}}(a_1, a_2, a_3, a_4, \dots, a_{2p-1}, a_{2p})$ as

$$\tilde{\mathcal{V}}(a_1, a_2, a_3, a_4, \dots, a_{2p-1}, a_{2p}) := \left(\sum_{1 \leq j < l \leq 2p} a_j a_l \right) \mathcal{V}(a_1, a_2, a_3, a_4, \dots, a_{2p-1}, a_{2p}).$$

With the help of the modified Vandermonde coefficient, we determine the coefficients $\beta_0, \beta_{11}, \beta_{12}$ appearing in (5.55) as

$$\begin{aligned} \beta_0 &= -\tilde{\mathcal{V}}(k_1, k_1^*, k_2, k_2^*, \dots, k_N, k_N^*), \\ \beta_{11} &= -\gamma_1 \tilde{\mathcal{V}}(zk_1, k_1^*, k_2, k_2^*, \dots, k_N, k_N^*) - \gamma_1^* \tilde{\mathcal{V}}(k_1, z^2 k_1^*, k_2, k_2^*, \dots, k_N, k_N^*), \\ \beta_{12} &= -\gamma_1 \gamma_1^* \tilde{\mathcal{V}}(zk_1, z^2 k_1^*, k_2, k_2^*, \dots, k_N, k_N^*). \end{aligned}$$

As mentioned already, the determinant of $\mathbf{M}(x)$ is a polynomial of degree $2N$ in the N real-valued variables $\chi_1, \dots, \chi_N$. The coefficient of the highest-degree term $\chi_1^2 \chi_2^2 \cdots \chi_N^2$ in $\det[\mathbf{M}(x)]$ is obtained in a manner similar to the evaluation of a_{12} given in (5.53), and that coefficient is equal to the quantity given by

$$\left(\prod_{j=1}^N \gamma_j \gamma_j^* \right) \mathcal{V}(zk_1, z^2 k_1^*, zk_2, z^2 k_2^*, \dots, zk_N, z^2 k_N^*). \quad (5.56)$$

Since $\det[\mathbf{M}_1(x)]$ is also a polynomial of degree $2N$ in the N real-valued variables $\chi_1, \dots, \chi_N$, as in (5.47) and (5.55), by setting $\chi_j = 0$ for $2 \leq j \leq N$ in the expression for $\det[\mathbf{M}_1(x)]$, we obtain

$$\det[\mathbf{M}_1(x)] = \epsilon_0 + \epsilon_{11} \chi_1 + \epsilon_{12} \chi_1^2. \quad (5.57)$$

With the help of (5.20), (5.43), and (5.47), we evaluate the coefficients in (5.57) as

$$\epsilon_0 = \Sigma_N \alpha_0, \quad \epsilon_{11} = [\Sigma_N + (z-1)k_1] \alpha_{11}, \quad \epsilon_{12} = [\Sigma_N + 2(z-1)k_1] \alpha_{12},$$

where we recall that Σ_N, α_{11} , and α_{12} are the quantities in (5.3), (5.52), and (5.53), respectively.

As already indicated, the symmetrical appearance of the quantities k_j in (5.2) for $1 \leq j \leq N$ allows us to determine the coefficients of χ_j and χ_j^2 in the expressions for $\det[\mathbf{M}(x)]$, $\det[\mathbf{M}_1(x)]$, and $\det[\mathbf{M}_2(x)]$. For example, with $\chi_j \neq 0$ if we choose the remaining quantities χ_l as zero for $l \neq j$, then we get the analog of (5.47) as

$$\det[\mathbf{M}(x)] = \alpha_0 + \alpha_{jj} \chi_j + \alpha_{j(j+1)} \chi_j^2, \quad (5.58)$$

where α_{jj} and $\alpha_{j(j+1)}$ are the analogs of α_{11} and α_{12} in (5.52) and (5.53), respectively, and they are expressed as

$$a_{jj} = \gamma_j \mathcal{U}_{j0} + \gamma_j^* \mathcal{U}_{0j}, \quad a_{j(j+1)} = \gamma_j \gamma_j^* \mathcal{U}_{jj}, \quad 1 \leq j \leq N. \quad (5.59)$$

We remark that $\mathcal{U}_{j0}$ for $2 \leq j \leq N$ is defined in a similar way $\mathcal{U}_{10}$ appearing in the second line of (5.54) is defined, i.e. by replacing k_j appearing on the right-hand side of (5.49) by zk_j . Similarly, $\mathcal{U}_{0j}$ is defined in an analogous manner $\mathcal{U}_{01}$ appearing in the third line of (5.54) is defined, i.e. by replacing k_j^* appearing on the right-hand side of (5.49) by $z^2 k_j^*$. Likewise, $\mathcal{U}_{jj}$ is defined in a similar way $\mathcal{U}_{11}$ appearing in the fourth line of (5.54) is defined, i.e. by replacing k_j and k_j^* appearing on the right-hand side of (5.49) by zk_j and $z^2 k_j^*$, respectively.

6 Soliton solutions to the Sawada–Kotera equation

In Section 5, we have constructed complex-valued solution pairs (Q, P) to the integrable system (2.6). This is done by solving the inverse scattering problem for (2.8) in the reflectionless case by using as input the bound-state poles of the left transmission coefficient $T_1(k)$ given in (5.1) and the time-evolved modified bound-state dependency constants. By using the input data set $\{k_j, \gamma_j\}_{j=1}^N$, with k_j being the quantity in (5.4) and γ_j being the complex-valued modified bound-state dependency constant in (5.15), we have constructed the potential pair (Q, P) as in (5.39) and (5.40). Since we have used the time evolution for the dependency constants as specified in (4.5) and (5.18), the resulting time-evolved potential pair (Q, P) constitutes a solution to the integrable coupled system (2.6).

In this section, by imposing the appropriate restriction on the solution pair (Q, P) appearing in (5.39) and (5.40), we construct the N -soliton solution to the Sawada–Kotera equation (2.1). It turns out [3] that relevant restriction is accomplished by specifying each of the ratios s_j/r_j for $1 \leq j \leq N$ in the input data set $\{k_j, \gamma_j\}_{j=1}^N$ appropriately in terms of the elements in the set $\{k_j\}_{j=1}^N$ or equivalently in the set $\{\eta_j\}_{j=1}^N$. Here, we use r_j and s_j to denote the real and imaginary parts of γ_j , respectively. Thus, we have

$$\gamma_j = r_j + is_j, \quad 1 \leq j \leq N. \quad (6.1)$$

Because of the symmetrical appearance of the quantities k_j in (5.2), for the determination of s_j/r_j for $1 \leq j \leq N$, it is sufficient to determine the explicit expression only for s_1/r_1 and use the appropriate symmetries to determine the remaining ratios s_j/r_j . The restrictions on the ratios s_j/r_j for $1 \leq j \leq N$ assure that Q is real valued and that we have either $P \equiv 0$ or $P = Q_x$. The resulting real-valued expression for Q contains the $2N$ real-valued parameters η_j and r_j for $1 \leq j \leq N$, where we recall that η_j is related to k_j as in (5.4).

In the following steps, we show how the real-valued N -soliton solution Q to (2.1) expressed in terms of the parameters in the set $\{\eta_j, r_j\}_{j=1}^N$ is obtained from the complex-valued solution pair (Q, P) in (5.39) and (5.40) expressed in terms of the parameters in the set $\{\eta_j, r_j, s_j\}_{j=1}^N$.

- (a) When $P \equiv 0$, by integrating both sides of (5.40) and using (5.41) and (5.42), we write (5.40) in the equivalent form as

$$\frac{1}{2}[A_{2N-1}(x)]^2 - A'_{2N-1}(x) - A_{2N-2}(x) = \Pi_N - \frac{1}{2}\Sigma_N^2, \quad (6.2)$$

where we recall that we use a prime to denote the x -derivative. Using (5.32) and (5.44) in (6.2), after some simplifications, we obtain the expression equivalent to (6.2), which is given by

$$\begin{aligned} (\Pi_N - \Sigma_N^2) \det[\mathbf{M}(x)] + \Sigma_N \frac{d \det[\mathbf{M}(x)]}{dx} + \frac{d^2 \det[\mathbf{M}(x)]}{dx^2} + \det[\mathbf{M}_2(x)] \\ = \frac{1}{2 \det[\mathbf{M}(x)]} \left(\frac{d \det[\mathbf{M}(x)]}{dx} \right)^2. \end{aligned} \quad (6.3)$$

- (b) On the other hand, if $P = Q_x$ then (5.39) and (5.40) yield

$$A_{2N-1}(x) A'_{2N-1}(x) - A'_{2N-2}(x) = 0. \quad (6.4)$$

Integrating both sides of (6.4) and using (5.41) and (5.42), we write (6.4) in the equivalent form as

$$\frac{1}{2}[A_{2N-1}(x)]^2 - A_{2N-2}(x) = \Pi_N - \frac{1}{2}\Sigma_N^2. \quad (6.5)$$

Using (5.32) and (5.44) in (6.5), after some simplifications, we write (6.5) in the equivalent form given by

$$\begin{aligned} (\Pi_N - \Sigma_N^2) \det[\mathbf{M}(x)] + \Sigma_N \frac{d \det[\mathbf{M}(x)]}{dx} + \frac{d^2 \det[\mathbf{M}(x)]}{dx^2} + \det[\mathbf{M}_2(x)] \\ = -\frac{1}{2 \det[\mathbf{M}(x)]} \left(\frac{d \det[\mathbf{M}(x)]}{dx} \right)^2. \end{aligned} \quad (6.6)$$

- (c) From Section 5, we know that each of $\det[\mathbf{M}(x)]$ and $\det[\mathbf{M}_2(x)]$ is a polynomial of degree $2N$ in the N real-valued variables $\chi_1, \dots, \chi_N$. Consequently, each of $d \det[\mathbf{M}(x)]/dx$ and $d^2 \det[\mathbf{M}(x)]/dx^2$ is also a polynomial of degree $2N$ in $\chi_1, \dots, \chi_N$. Thus, the left-hand sides of (6.3) and (6.6) are some polynomials of degree $2N$ in $\chi_1, \dots, \chi_N$. We remark that the left-hand sides of (6.3) and (6.6) coincide while their right-hand sides differ by a minus sign. Thus, those right-hand sides must also be polynomials of degree $2N$ in $\chi_1, \dots, \chi_N$.
- (d) In order to satisfy the restrictions that the right-hand sides of (6.3) and (6.6) must each be a polynomial of degree $2N$ in $\chi_1, \dots, \chi_N$, we let

$$\det[\mathbf{M}(x)] = \alpha_0 [\Delta(x)]^2, \quad (6.7)$$

where α_0 is the constant appearing in (5.47)–(5.49) and $\Delta(x)$ is a polynomial of degree N in $\chi_1, \dots, \chi_N$. The use of α_0 in (6.7) ensures that $\Delta(x)$ is normalized in the sense that $\Delta(x) = 1$ when we let $\chi_j = 0$ for $1 \leq j \leq N$. Comparing (5.47) with (6.7), we see that, if we set $\chi_j = 0$ for $2 \leq j \leq N$, then $\Delta(x)$ must satisfy

$$[\Delta(x)]^2 = 1 + \frac{\alpha_{11}}{\alpha_0} \chi_1 + \frac{\alpha_{12}}{\alpha_0} \chi_1^2, \quad (6.8)$$

and hence the coefficients $\alpha_0, \alpha_{11}, \alpha_{12}$ appearing in (5.47) must be related to each other as

$$\frac{\alpha_{12}}{\alpha_0} = \left(\frac{\alpha_{11}}{2\alpha_0} \right)^2, \quad (6.9)$$

or equivalently as

$$4\alpha_0\alpha_{12} = (\alpha_{11})^2. \quad (6.10)$$

- (e) From (5.51)–(5.54) we observe that the constraint (6.10) can be expressed in terms of the Vandermonde coefficients appearing in (5.54) as

$$4\mathcal{U}_{00}\mathcal{U}_{11}\gamma_1\gamma_1^* = (\mathcal{U}_{10}\gamma_1 + \mathcal{U}_{01}\gamma_1^*)^2. \quad (6.11)$$

Using (6.1) in (6.11), we write (6.11) in terms of r_1 and s_1 as

$$4\mathcal{U}_{00}\mathcal{U}_{11}(r_1^2 + s_1^2) = [\mathcal{U}_{10}(r_1 + is_1) + \mathcal{U}_{01}(r_1 - is_1)]^2. \quad (6.12)$$

We divide both sides of (6.12) by r_1^2 and write the resulting equality in the equivalent form expressed as a quadratic equation in s_1/r_1 as

$$[4\mathcal{U}_{00}\mathcal{U}_{11} + (\mathcal{U}_{10} - \mathcal{U}_{01})^2] \left(\frac{s_1}{r_1} \right)^2 - 2i(\mathcal{U}_{10}^2 - \mathcal{U}_{01}^2) \left(\frac{s_1}{r_1} \right) + [4\mathcal{U}_{00}\mathcal{U}_{11} - (\mathcal{U}_{10} + \mathcal{U}_{01})^2] = 0. \quad (6.13)$$

The two solutions to the quadratic equation (6.13) are given by

$$\frac{s_1}{r_1} = \frac{i(\mathcal{U}_{10}^2 - \mathcal{U}_{01}^2) \pm 4\sqrt{\mathcal{U}_{00}\mathcal{U}_{11}(\mathcal{U}_{10}\mathcal{U}_{01} - \mathcal{U}_{00}\mathcal{U}_{11})}}{4\mathcal{U}_{00}\mathcal{U}_{11} + (\mathcal{U}_{10} - \mathcal{U}_{01})^2}. \quad (6.14)$$

From (5.50) and (5.54), it follows that the denominator on the right-hand side of (6.14) is real and nonzero, the quantity inside the square root in the numerator is positive, and $(\mathcal{U}_{10}^2 - \mathcal{U}_{01}^2)$ is purely imaginary. Hence, the quadratic equation (6.13) has two distinct real roots specified in (6.14). In fact, one of them is related to the constraint regarding the right-hand side of (6.3), and the other is related to the constraint regarding the right-hand side of (6.6).

- (f) By exploiting the symmetries existing in (5.47) and (5.58), with the help of (5.59), from (6.14) we obtain the ratios s_j/r_j as

$$\frac{s_j}{r_j} = \frac{i(\mathcal{U}_{j0}^2 - \mathcal{U}_{0j}^2) \pm 4\sqrt{\mathcal{U}_{00}\mathcal{U}_{jj}(\mathcal{U}_{j0}\mathcal{U}_{0j} - \mathcal{U}_{00}\mathcal{U}_{jj})}}{4\mathcal{U}_{00}\mathcal{U}_{jj} + (\mathcal{U}_{j0} - \mathcal{U}_{0j})^2}, \quad 1 \leq j \leq N. \quad (6.15)$$

As indicated in (e), the two root values appearing on the right-hand side of (6.15) are real and distinct.

- (g) We already know that each of $\mathcal{U}_{00}, \mathcal{U}_{j0}, \mathcal{U}_{0j}, \mathcal{U}_{jj}$ is determined in terms of the elements in the set $\{k_j\}_{j=1}^N$ or equivalently in the set $\{\eta_j\}_{j=1}^N$ due to (5.4). From (6.15) we observe that each ratio s_j/r_j is uniquely expressed in terms of the elements in $\{\eta_j\}_{j=1}^N$ in each of the cases $P \equiv 0$ and $P = Q_x$. Thus, each of the quantities $(1 + is_j/r_j)$ is a constant uniquely expressed in terms of the elements in the set $\{\eta_j\}_{j=1}^N$. Let us write (6.1) as

$$\gamma_j = r_j \left(1 + i \frac{s_j}{r_j} \right), \quad 1 \leq j \leq N. \quad (6.16)$$

Using (6.15) on the right-hand side of (6.16), we express each γ_j as a product of r_j and a constant determined by the elements of the set $\{\eta_j\}_{j=1}^N$. By using the right-hand side of (6.16) on the right-hand side of (5.27), we express all entries of the matrix $\mathbf{M}(x)$ appearing on the right-hand side of (5.25) in terms of the elements in $\{\eta_j, r_j\}_{j=1}^N$. Using the resulting expression for $\mathbf{M}(x)$ on the right-hand side of (5.45), we obtain the solution Q to (2.1) in terms of the elements in $\{\eta_j, r_j\}_{j=1}^N$.

- (h) Alternatively, we obtain Q by proceeding as follows. Using either of the two roots specified in (6.15), we uniquely determine the quantity $\Delta(x)$ appearing in (6.7) in terms of the parameters in the set $\{k_j, r_j\}_{j=1}^N$ or equivalently in the set $\{\eta_j, r_j\}_{j=1}^N$. Using (6.7) in (5.45) we recover the solution Q to (2.1) as

$$Q(x) = 6 \frac{d}{dx} \left(\frac{\Delta'(x)}{\Delta(x)} \right), \quad (6.17)$$

where we recall that the prime is used for the x -derivative and we suppress the t -dependence in our notation for $\Delta(x)$.

- (i) We recall that $\Delta(x)$ is a real-valued polynomial of degree N in the N real variables $\chi_1, \dots, \chi_N$. The constant term not containing any of those N variables is equal to 1. In each term of $\Delta(x)$, a specific χ_j either does not appear or appears only to the first power. From (5.51), (5.52), (6.1), and (6.7)–(6.10), it follows that the coefficient of χ_1 in $\Delta(x)$ is equal to $\alpha_{11}/(2\alpha_0)$, where we have

$$\frac{\alpha_{11}}{2\alpha_0} = \frac{r_1}{2\mathcal{U}_{00}} \left[(\mathcal{U}_{10} + \mathcal{U}_{01}) + i(\mathcal{U}_{10} - \mathcal{U}_{01}) \frac{s_1}{r_1} \right], \quad (6.18)$$

with the understanding that (6.14) is used for the expression s_1/r_1 appearing on the right-hand side of (6.18). In a similar way, with the help of (5.58), (5.59), (6.1), (6.7)–(6.10) it follows that the coefficient of χ_j is equal to $\alpha_{jj}/(2\alpha_0)$, where we have

$$\frac{a_{jj}}{2a_0} = \frac{r_j}{2\mathcal{U}_{00}} \left[(\mathcal{U}_{j0} + \mathcal{U}_{0j}) + i(\mathcal{U}_{j0} - \mathcal{U}_{0j}) \frac{s_j}{r_j} \right], \quad 1 \leq j \leq N, \quad (6.19)$$

with the understanding that (6.15) is used for the expression s_j/r_j appearing on the right-hand side of (6.19). The sign of the free real parameter r_j is chosen so that the right-hand side of (6.19) is positive.

(j) We introduce the quantities y_j as

$$y_j := \frac{a_{jj}}{2a_0} \chi_j, \quad 1 \leq j \leq N, \quad (6.20)$$

where $a_{jj}/(2a_0)$ is given by the right-hand side of (6.19), and we recall that χ_j is the quantity in (5.20). Using (5.4) in (5.20) we get

$$\chi_j = e^{\sqrt{3}\eta_j(x-9\eta_j^4 t)}, \quad 1 \leq j \leq N, \quad (6.21)$$

and hence χ_j remains positive for all real values of x and t . By choosing the sign of each r_j appropriately so that the right-hand side of (6.19) is positive, we observe from (6.20) and (6.21) that the quantity y_j defined in (6.20) remains positive for $1 \leq j \leq N$ for all x and t . We can then write $\Delta(x)$ as

$$\begin{aligned} \Delta(x) = & 1 + \sum_{j=1}^N y_j + \sum_{1 \leq j_1 < j_2 \leq N} A_{j_1 j_2} y_{j_1} y_{j_2} \\ & + \sum_{1 \leq j_1 < j_2 < j_3 \leq N} A_{j_1 j_2} A_{j_1 j_3} A_{j_2 j_3} y_{j_1} y_{j_2} y_{j_3} \\ & + \cdots + [A_{12} A_{13} \cdots A_{(N-1)N}] y_1 y_2 \cdots y_N, \end{aligned} \quad (6.22)$$

where the right-hand side is a polynomial of degree N in $y_1, y_2, \dots, y_N$ with the last term containing the product $y_1 y_2 \cdots y_N$. We remark that (6.22) is the analog of (32) in [24]. With the help of (5.56) and (6.7), the double-indexed quantity A_{jl} is determined by the elements in the set $\{k_j\}_{j=1}^N$ and is given by

$$A_{jl} := \frac{(k_j - k_l)^3 (k_j^3 + k_l^3)}{(k_j + k_l)^3 (k_j^3 - k_l^3)}, \quad 1 \leq j < l \leq N. \quad (6.23)$$

Using (5.4) in (6.23), we can express A_{jl} in terms of the elements of $\{\eta_j\}_{j=1}^N$ as

$$A_{jl} = \frac{(\eta_j - \eta_l)^3 (\eta_j^3 + \eta_l^3)}{(\eta_j + \eta_l)^3 (\eta_j^3 - \eta_l^3)}, \quad 1 \leq j < l \leq N. \quad (6.24)$$

We remark that (6.24) is the analog of (33) in [24]. Because the η_j -values are distinct, from (6.24) it follows that A_{jl} is positive. Consequently, from (6.22) we observe that the quantity $\Delta(x)$ remains positive for all real values of x and t , and this assures that the N -soliton solution Q to (2.1) appearing in (6.17) and containing the $2N$ real-valued parameters η_j and r_j for $1 \leq j \leq N$ does not have any singularities.

7 Soliton behaviors

In Section 6, we have presented the construction of the N -soliton solution to the Sawada–Kotera equation (2.1) with the input consisting of the $2N$ real parameters in the set $\{\eta_j, r_j\}_{j=1}^N$. In this section, we elaborate on the method of Section 6, when N takes the values of 1, 2, and 3. We remark that in this section we no longer suppress the t -dependence in the relevant quantities.

When $N = 1$, we use the input data set $\{\eta_1, r_1\}$ to have the 1-soliton solution $Q(x, t)$ via (6.17) by constructing the key quantity $\Delta(x, t)$ appearing in (6.22). In this case, (6.22) reduces to

$$\Delta(x, t) = 1 + y_1, \quad (7.1)$$

where y_1 is obtained from (6.20) with χ_1 being the quantity we get from (6.21) when $j = 1$, i.e. we have

$$\chi_1 = e^{\sqrt{3}\eta_1(x-9\eta_1^4t)}. \quad (7.2)$$

The coefficient $\alpha_{11}/(2\alpha_0)$ in (6.20) is evaluated with the help of (5.54), (6.14), and (6.18). By using (5.54) in (6.14), we obtain

$$\frac{s_1}{r_1} = \begin{cases} \sqrt{3}, & P \equiv 0, \\ 0, & P = Q_x, \end{cases} \quad (7.3)$$

where the first line on the right-hand side refers to the case $P \equiv 0$ and the second line refers to the case $P = Q_x$. Next, using (5.54) and (7.3) in (6.20), we obtain the quantity y_1 as

$$y_1 = \begin{cases} -2r_1\chi_1, & P \equiv 0, \\ r_1\chi_1, & P = Q_x. \end{cases} \quad (7.4)$$

We remark that either of the two lines on the right-hand side of (7.4) yields two equivalent values for y_1 by restricting the parameter r_1 in the first line to negative values and by restricting the parameter r_1 in the second line to positive values. Using (7.1) and (7.4) in (6.17), we obtain the 1-soliton solution to (2.1) as

$$Q(x, t) = \begin{cases} -\frac{36r_1\eta_1^2\chi_1}{(1-2r_1\chi_1)^2}, & P \equiv 0, \\ \frac{18r_1\eta_1^2\chi_1}{(1+r_1\chi_1)^2}, & P = Q_x, \end{cases} \quad (7.5)$$

where it is understood that $r_1 < 0$ in the first line on the right-hand side and $r_1 > 0$ in the second line. We observe that the substitution $r_1 \mapsto -r_1/2$ in the first line yields the second line in (7.5). Since χ_1 appearing in (7.2) is a function of $(x - 9\eta_1^4t)$, from (7.5) we see that the 1-soliton $Q(x, t)$ moves from the left to the right with the speed $9\eta_1^4$.

In Figure 1.3 we show the four snapshots at $t = -2$, $t = -1$, $t = 0$, and $t = 1$, respectively, for the 1-soliton solution to (2.1) constructed with the input parameters given by

$$(\eta_1, r_1) = (1, 1), \quad (7.6)$$

where we use the second line on the right-hand side of (7.5) corresponding to $P = Q_x$. From (7.5), we see that the same soliton solution can be constructed by using the first line of the right-hand side of (7.5) with the input parameters η_1 and r_1 given by

$$(\eta_1, r_1) = \left(1, -\frac{1}{2}\right),$$

in which case the corresponding P has the value $P(x, t) \equiv 0$. The 1-soliton solution $Q(x, t)$ in (7.5) is a real-valued solitary wave moving from the left to the right with the constant speed $9\eta_1^4$, which is equal to 9 in this particular case. We remark that the 1-soliton behavior for the Sawada–Kotera equation (2.1) illustrated in Figure 1.3 is similar to the behavior [1, 2, 11, 12, 23] of the 1-soliton solution to the KdV equation in the sense that the soliton moves with a constant speed from the left to the right without changing its shape.

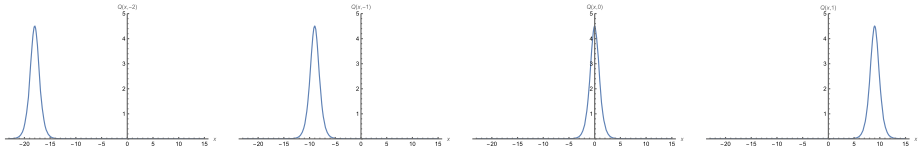


Figure 1.3: The snapshots for the 1-soliton solution $Q(x, t)$ to (2.1) with the parameter values in (7.6) at $t = -2$, $t = -1$, $t = 0$, and $t = 1$, respectively.

When $N = 2$, we use the input data set $\{\eta_1, \eta_2, r_1, r_2\}$ to construct the 2-soliton solution $Q(x, t)$ to (2.1) via (6.17) by forming the quantity $\Delta(x, t)$ appearing in (6.22). In this case, (6.22) yields

$$\Delta(x, t) = 1 + y_1 + y_2 + A_{12}y_1y_2, \quad (7.7)$$

where A_{12} is obtained from (6.24) and the quantities y_1 and y_2 are constructed with the help of (6.15) and (6.18)–(6.21). Using (5.54) in (6.14) and exploiting the symmetrical appearance of k_1 and k_2 in (5.2), we obtain

$$\frac{s_1}{r_1} = \begin{cases} \frac{\sqrt{3}\eta_2^2}{2\eta_1^2 + \eta_2^2}, & P \equiv 0, \\ -\frac{\sqrt{3}\eta_1^2}{\eta_1^2 + 2\eta_2^2}, & P = Q_x, \end{cases} \quad \frac{s_2}{r_2} = \begin{cases} \frac{\sqrt{3}\eta_1^2}{\eta_1^2 + 2\eta_2^2}, & P \equiv 0, \\ -\frac{\sqrt{3}\eta_2^2}{2\eta_1^2 + \eta_2^2}, & P = Q_x, \end{cases} \quad (7.8)$$

where the first lines on the right-hand sides refer to the case $P \equiv 0$ and the second lines refer to the case $P = Q_x$. Next, using (5.54) and (7.8) in (6.18), we obtain the quantity y_1 appearing in (6.20) and (7.7) as

$$y_1 = \begin{cases} \frac{2r_1(\eta_1^3 + 2\eta_1^2\eta_2 + 2\eta_1\eta_2^2 + \eta_2^3)\chi_1}{(\eta_1 - \eta_2)(2\eta_1^2 + \eta_2^2)}, & P \equiv 0, \\ -\frac{2r_1(\eta_1^3 + 2\eta_1^2\eta_2 + 2\eta_1\eta_2^2 + \eta_2^3)\chi_1}{(\eta_1 - \eta_2)(\eta_1^2 + 2\eta_2^2)}, & P = Q_x, \end{cases} \quad (7.9)$$

with χ_1 being the exponential function given in (7.2). Due to the ordering $0 < \eta_1 < \eta_2$, in the first line on the right-hand side of (7.9) we use $r_1 < 0$ and in the second line we use $r_1 > 0$ so that the quantity y_1 remains positive for all real values of x and t . The value of y_2 is obtained with the help of (7.9) by using the replacement $(\eta_1, \eta_2, r_1) \mapsto (\eta_2, \eta_1, r_2)$. Thus, we get

$$y_2 = \begin{cases} -\frac{2r_2(\eta_1^3 + 2\eta_1^2\eta_2 + 2\eta_1\eta_2^2 + \eta_2^3)\chi_2}{(\eta_1 - \eta_2)(\eta_1^2 + 2\eta_2^2)}, & P \equiv 0, \\ \frac{2r_2(\eta_1^3 + 2\eta_1^2\eta_2 + 2\eta_1\eta_2^2 + \eta_2^3)\chi_2}{(\eta_1 - \eta_2)(2\eta_1^2 + \eta_2^2)}, & P = Q_x, \end{cases} \quad (7.10)$$

with χ_2 being the exponential function obtained from (6.21) when $j = 2$, i.e. we have

$$\chi_2 = e^{\sqrt{3}\eta_2(x-9\eta_2^4t)}. \quad (7.11)$$

In order to have y_2 to remain positive for all real values of x and t , as a result of the ordering $0 < \eta_1 < \eta_2$, we see that we must use $r_2 > 0$ in the first line on the right-hand side of (7.10) and use $r_2 < 0$ in the second line.

In Figure 1.4, we show the behavior of the 2-soliton solution corresponding to the input data set $\{\eta_1, \eta_2, r_1, r_2\}$ with the specific values of the parameters given by

$$(\eta_1, \eta_2, r_1, r_2) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, 1, -1 \right), \quad (7.12)$$

where we use the second lines on the right-hand sides in (7.8)–(7.10) corresponding to the case $P = Q_x$. Using the method of Section 6, we explicitly construct the 2-soliton solution with the input data set related to (7.12). Using the values specified in (7.12) as input to the key quantity $\Delta(x, t)$ appearing in (7.7), we obtain

$$\Delta(x, t) = 1 + \frac{14}{3}\chi_1 + 7\chi_2 + \frac{14}{9}\chi_1\chi_2, \quad (7.13)$$

where the quantities χ_1 and χ_2 are constructed with the help of (7.2) and (7.11) as

$$\chi_1 = e^{x-t}, \quad \chi_2 = e^{2(x-16t)}.$$

Using (7.13) in (6.17), we obtain the solution $Q(x, t)$ to (2.1) as

$$Q(x, t) = \frac{28\chi_1 + 168\chi_2 + 280\chi_1\chi_2 + \frac{1568}{9}\chi_1^2\chi_2 + \frac{196}{3}\chi_1\chi_2^2}{(1 + \frac{14}{3}\chi_1 + 7\chi_2 + \frac{14}{9}\chi_1\chi_2)^2}, \quad (7.14)$$

with the understanding that we have $P(x, t) = Q_x(x, t)$. By exploiting the symmetrical appearance of k_1 and k_2 in (5.2), we determine that we obtain the same $Q(x, t)$ given in

(7.14) with $P(x, t) \equiv 0$ if we use not the input values in (7.12) but instead use the input values given by

$$(\eta_1, \eta_2, r_1, r_2) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{3}, \frac{3}{2} \right).$$

The left plot in Figure 1.4 shows the snapshot at $t = -0.6$ when the two solitons are apart from each other and not yet interacting with each other. The taller soliton has the speed 16 and is behind the shorter soliton having the speed 1. Both solitons move from the left to the right. The next plot shows the snapshot at $t = -0.2$ soon after the beginning of the nonlinear interactions between the two solitons. The third plot shows the snapshot at $t = 0.2$, when the nonlinear interactions are about to end. Finally, the right plot shows the snapshot at $t = 0.6$, when the two solitons are apart from each other and there are no longer any interactions. By comparing the four snapshots in Figure 1.4, we observe that, as a result of the nonlinear interactions, the shorter soliton has been pushed backward as the taller soliton overtakes that shorter soliton. During the nonlinear interactions the taller soliton overtakes the shorter soliton. We also remark that, even though we do not show the snapshot at $t = 0$ in Figure 1.4, the two solitons at $t = 0$ in this case overlap in such a way as if there were a single soliton at that moment. The 2-soliton behavior for the Sawada–Kotera equation is similar to the 2-soliton behavior for the KdV equation.

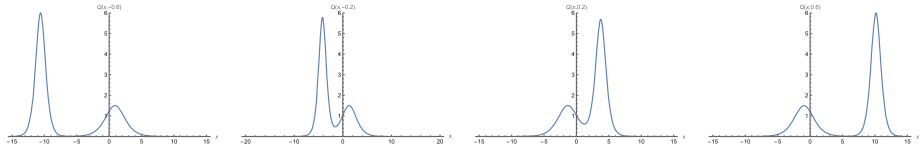


Figure 1.4: The snapshots for the 2-soliton solution $Q(x, t)$ to (2.1) with the parameter values in (7.12) at $t = -0.6$, $t = -0.2$, $t = 0.2$, and $t = 0.6$, respectively.

When $N = 3$, by using the input data set $\{\eta_1, \eta_2, \eta_3, r_1, r_2, r_3\}$, we construct the 3-soliton solution to (2.1). As explained in Section 6, we first construct the quantity $\Delta(x, t)$ given in (6.22) and then obtain the 3-soliton solution $Q(x, t)$ via (6.17). When $N = 3$, from (6.22) we get

$$\Delta(x, t) = 1 + y_1 + y_2 + y_3 + A_{12}y_1y_2 + A_{13}y_1y_3 + A_{23}y_2y_3 + A_{12}A_{13}A_{23}y_1y_2y_3, \quad (7.15)$$

where the coefficients A_{12}, A_{13}, A_{23} are obtained in terms of η_1, η_2, η_3 by using (6.24). We have

$$A_{12} = \frac{(\eta_1 - \eta_2)^3(\eta_1^3 + \eta_2^3)}{(\eta_1 + \eta_2)^3(\eta_1^3 - \eta_2^3)}, \quad A_{13} = \frac{(\eta_1 - \eta_3)^3(\eta_1^3 + \eta_3^3)}{(\eta_1 + \eta_3)^3(\eta_1^3 - \eta_3^3)}, \quad A_{23} = \frac{(\eta_2 - \eta_3)^3(\eta_2^3 + \eta_3^3)}{(\eta_2 + \eta_3)^3(\eta_2^3 - \eta_3^3)}.$$

We remark that the quantities A_{13} and A_{23} can readily be obtained from the quantity A_{12} by exploiting the symmetrical appearance of k_1, k_2 , and k_3 in (5.2). The quantities y_1 ,

y_2, y_3 are as in (6.20) with χ_1, χ_2, χ_3 being the exponential functions appearing in (6.21). Thus, we have

$$y_1 = \frac{\alpha_{11}}{2\alpha_0} \chi_1, \quad y_2 = \frac{\alpha_{22}}{2\alpha_0} \chi_2, \quad y_3 = \frac{\alpha_{33}}{2\alpha_0} \chi_3,$$

where the three coefficients $\alpha_{jj}/(2\alpha_0)$ for $j = 1, 2, 3$ are obtained with the help of (6.19). We then construct the 3-soliton solution $Q(x, t)$ by using (7.15) in (6.17).

In Figure 1.5, we illustrate the behavior of the 3-soliton solution corresponding to the input data set $\{\eta_1, \eta_2, \eta_3, r_1, r_2, r_3\}$ with the parameter values given by

$$(\eta_1, \eta_2, \eta_3, r_1, r_2, r_3) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, 1, -1, -1 \right), \quad (7.16)$$

where we use the case $P = Q_\chi$. Via the method of Section 6, we explicitly construct the 3-soliton solution with the input data set corresponding to (7.16). For this, we first use (7.16) in (7.15) and obtain the corresponding $\Delta(x, t)$ as

$$\Delta(x, t) = 1 + 13\chi_1 + 95\chi_2 + \frac{1235}{7}\chi_3 + \frac{1235}{21}\chi_1\chi_2 + \frac{1235}{4}\chi_1\chi_3 + 247\chi_2\chi_3 + \frac{247}{12}\chi_1\chi_2\chi_3, \quad (7.17)$$

where the quantities χ_1, χ_2, χ_3 are obtained with the help of (6.21) and (7.16) as

$$\chi_1 = e^{x-t}, \quad \chi_2 = e^{2(x-16t)}, \quad \chi_3 = e^{3(x-81t)}.$$

The 3-soliton solution $Q(x, t)$ to (2.1) is then constructed by using (7.17) in (6.17).

The left plot in Figure 1.5 shows the snapshot at $t = -0.6$, when the three solitons are apart from each other and not yet interacting with each other. The next plot shows the snapshot at $t = -0.1$ near the beginning of the nonlinear interactions among the three solitons. The third plot shows the snapshot at $t = 0.13$ near the end of the nonlinear interactions. Finally, the right plot shows the snapshot at $t = 0.4$, when the three solitons are apart from each other and they no longer interact. We also observe the following. All three solitons travel from the left to the right. Initially, before the nonlinear interactions start, they are aligned in such a way that the tallest soliton is behind the other two solitons, the shortest soliton is ahead of the other two, and the middle-height soliton is between the other two. Prior to any nonlinear interactions, the tallest soliton has the speed 81, the middle-height soliton has the speed 16, and the shortest soliton has the speed 1. During the nonlinear interactions, the tallest soliton overtakes the other two solitons and the middle-height soliton overtakes the shortest soliton. We further observe that the shortest soliton has been pushed backward as a result of the nonlinear interactions. In Figure 1.5, we do not show the snapshot at $t = 0$, but we remark that the three solitons at $t = 0$ in this case overlap in such a way as if there were a single soliton at that moment. The 3-soliton behavior for the Sawada–Kotera equation is similar to the 3-soliton behavior for the KdV equation.

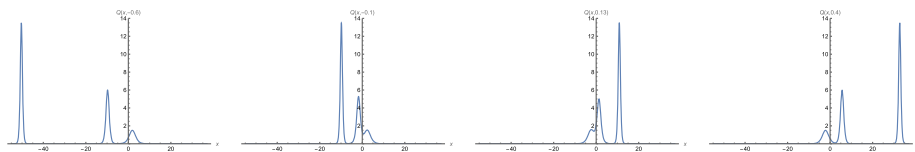


Figure 1.5: The snapshots for the 3-soliton solution $Q(x, t)$ to (2.1) with the parameter values in (7.16) at $t = -0.6$, $t = -0.1$, $t = 0.13$, and $t = 0.4$, respectively.

We recall that the input data set with the parameter values in (7.16) yields the quantity $\Delta(x, t)$ in (7.17) and the corresponding 3-soliton solution $Q(x, t)$ to (2.1) via (6.17), where the associated quantity $P(x, t)$ is related to $Q(x, t)$ as $P(x, t) = Q_x(x, t)$. By exploiting the symmetrical appearance of k_1 , k_2 , and k_3 in (5.2), we construct the same $\Delta(x, t)$ in (7.17) and hence the same $Q(x, t)$ with the associated quantity $P(x, t)$ satisfying $P(x, t) \equiv 0$ if we use not the input values in (7.16) but instead use the input values given by

$$(\eta_1, \eta_2, \eta_3, r_1, r_2, r_3) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, -\frac{3}{4}, \frac{5}{2}, -\frac{25}{4} \right).$$

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