

Inverse problems for selfadjoint Schrödinger operators on the half line with compactly supported potentials

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For a selfadjoint Schrödinger operator on the half line with a real-valued, integrable, and compactly supported potential, it is investigated whether the boundary parameter at the origin and the potential can uniquely be determined by the scattering matrix or by the absolute value of the Jost function known at positive energies, without having the bound-state information. It is proved that, except in one special case where the scattering matrix has no bound states and its value is +1 at zero energy, the determination by the scattering matrix is unique. In the special case, it is shown that there are exactly two distinct sets consisting of a potential and a boundary parameter yielding the same scattering matrix, and a characterization of the nonuniqueness is provided. A reconstruction from the scattering matrix is outlined yielding all the corresponding potentials and boundary parameters. The concept of “eligible resonances” is introduced, and such resonances correspond to real-energy resonances that can be converted into bound states via a Darboux transformation without changing the compact support of the potential. It is proved that the determination of the boundary parameter and the potential by the absolute value of the Jost function is unique up to the inclusion of eligible resonances. Several equivalent characterizations are provided to determine whether a resonance is eligible or ineligible. A reconstruction from the absolute value of the Jost function is given, yielding all the corresponding potentials and boundary parameters. The results obtained are illustrated with various explicit examples. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4907558>]

I. INTRODUCTION

In this paper, we consider the half-line Schrödinger operator with the general selfadjoint boundary condition at the origin when the potential is real valued, integrable, and compactly supported. We examine the inverse problem of recovery of the potential and boundary condition from two distinct types of input data, investigate whether the determination from each input data set is unique, present the characterization of the nonuniqueness if the unique determination is not possible, and provide a procedure to reconstruct all the corresponding potentials and boundary conditions from each input data set.

The first set of input data we use is the scattering matrix known at all positive energies, but without any explicit information on the bound states. The second input data set we use is the absolute value of the so-called Jost function given at all positive energies, but again without any explicit information on the bound states. Assuming that the existence problem is solved, i.e., by assuming that there exists at least one set consisting of a potential and a boundary condition corresponding to our input data, we investigate whether we have two or more distinct sets containing a potential and a boundary condition corresponding to our input data, and we provide a reconstruction of all such sets.

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Recall that the parameter k^2 appearing in Schrödinger equation (2.1) corresponds to the energy and the parameter k corresponds to the momentum variable. As stated in Theorem 2.1(k), the knowledge of the scattering matrix at all positive energies is equivalent to knowing the scattering matrix for all real momentum values. Hence, as our first input data set, we equivalently use the scattering matrix known for all real k -values. Similarly, as stated in Theorem 2.1(k), the knowledge of the absolute value of the Jost function at positive energies is equivalent to knowing that absolute value for all real k -values, and hence we use the latter quantity as our second input data set.

Our inverse scattering problem can be paraphrased as follows: To what extent, can the lack of bound-state information in our input data set be compensated for by the knowledge that the potential is compactly supported? We certainly need to restrict our study to a specific class of potentials so that the problem under study is mathematically well stated. Real-valued, integrable potentials naturally arise^{1,7,8,15-17} in the theory of inverse problems for Schrödinger operators on the half line. The potentials of compact support appear in our analysis because for such potentials, the corresponding Jost function has an analytic extension from the real k -axis to the entire complex k -plane. Such an analytic extension is crucial in our analysis in order to compensate for the lack of bound-state information in our data.

A motivation to study our inverse problems comes from the inverse problem of determining the radius of the human vocal tract from sound pressure measurements at the lips.⁴ The vocal tract radius as a function of the distance from the glottis is related to the potential of the Schrödinger equation, the length of the vocal tract corresponds to the length of the support interval of the potential, the behavior of the vocal tract at the glottis is accounted for by the selfadjoint boundary condition for the Schrödinger operator, and the sound pressure at the lips as a function of the sound frequency is related to the absolute value of the Jost function as a function of k . The human speech consists of phonemes, and during the utterance of a phoneme, if the upper lip opens downward (i.e., when the slope of the radius of the vocal tract at the upper lip is negative) as in the utterance of the vowel /o/, then the corresponding Schrödinger operator has one bound state, and the Schrödinger operator has no bound states if the slope of the radius function at the upper lip is positive or zero as in the utterance of /a/ or /u/, respectively.

There are two main methods to solve the inverse problem for a selfadjoint Schrödinger operator on the half line. The first is the Marchenko method,^{1,7-10,15,16} and it uses the input data set consisting of the scattering matrix and the bound-state information. In the Marchenko method, the bound-state information consists of the bound-state energies and the so-called bound-state norming constants. The second method is the Gel'fand-Levitan method,^{7,8,12,15,16} and that method uses the input data set consisting of the absolute value of the Jost function and the bound-state information. In the Gel'fand-Levitan method, the bound-state information consists of the bound-state energies (such energies are the same as the bound-state energies used in the Marchenko method) and the bound-state norming constants (the Marchenko norming constants and the Gel'fand-Levitan norming constants differ from each other even though they are related to each other). In this paper, we consider the Marchenko recovery method when the bound-state information is absent from the standard Marchenko input data but instead we know that the corresponding potential is compactly supported. Similarly, we consider the Gel'fand-Levitan method when the Gel'fand-Levitan input data set does not contain the bound-state information but instead we know that the corresponding potential is compactly supported.

The results proved in our paper are analogous to some results related to the full-line Schrödinger equation where the bound-state information is missing from the input data. For example, a real-valued, integrable potential with a finite first moment is uniquely determined^{2,18} from the corresponding left (right) reflection coefficient alone if the support of the potential is confined to the right (left) half line, or such a potential is uniquely determined^{3,14} by the data consisting of the left (right) reflection coefficient and knowledge of the potential on the left (right) half line.

The analysis of the two inverse problems under study in our paper turns out to have an impact on other related problems. One contribution of our study is in the area of resonances for selfadjoint Schrödinger operators on the half line. The nonzero zeros of the analytic extension of the Jost function to the complex k -plane correspond to either bound states or resonances. If such zeros are located in the open upper-half complex plane, they correspond to bound states. It is known^{1,7,8,15,16}

that each such bound-state zero is simple and that the number of such zeros is either zero or a positive integer. It is also known^{8,15,16} that the bound states can be added or removed by using the mathematical procedure known as the Darboux transformation.

If the zeros of the Jost function are located in the open lower-half complex k -plane, then those zeros correspond to resonances. Equivalently, the poles of the meromorphic extension of the scattering matrix correspond to bound states if such poles occur in the open upper-half complex plane, and those poles of the scattering matrix occurring in the open lower-half complex plane correspond to resonances. The number of resonances can be zero, one, or countably infinite. A zero of the Jost function corresponding to a resonance may or may not be simple. The only real zero of the Jost function can occur at zero, and such a zero must be simple.

In our paper, we specifically deal with resonances corresponding to the zeros of the Jost function on the negative imaginary axis in the complex k -plane, i.e., with real-energy resonances. In our analysis, in a natural way, we are prompted to classify such resonances into two mutually exclusive groups. The first group consists of “eligible” resonances because such resonances can be converted into bound states through a Darboux transformation^{8,9,16} without changing the compact support of the potential. The remaining resonances occurring on the negative imaginary axis consist of “ineligible” resonances because such resonances cannot be converted into bound states under a Darboux transformation without changing the compact support of the potential. It is remarkable that ineligible resonances still remain ineligible if we add or remove any number of bound states via a Darboux transformation without changing the compact support of the potential. On the other hand, an eligible resonance either remains eligible or is converted into a bound state if we add any number of bound states via a Darboux transformation without changing the compact support of the potential. Similarly, a bound state removed via a Darboux transformation is converted into an eligible resonance.

Consider the sequence where each element in the sequence consists of a potential and a boundary parameter in such a way that one element in the sequence is connected to another element through a number of Darboux transformations related to removing or adding bound states without changing the compact support of the potentials. For such a sequence, we define the “maximal number of eligible resonances” as the number of eligible resonances corresponding to a pair with no bound states. Without causing any ambiguity, for any term in the sequence, we can define the maximal number of eligible resonances as the maximal number of eligible resonances associated with the sequence itself. Hence, for any term in the sequence, the sum of the number of eligible resonances and the number of bound states must be equal to the maximal number of eligible resonances. It turns out that each eligible resonance is simple in the sense that the corresponding zero of the related Jost function is a simple zero. Hence, we do not need to be concerned about the multiplicity of an eligible resonance. On the other hand, an ineligible resonance does not need be simple, i.e., the corresponding zero of the related Jost function may not necessarily be a simple zero.

It is remarkable that the identification of each resonance on the negative imaginary axis either as eligible or ineligible arises in a natural way and is motivated by physics, and the identification can be unambiguously given mathematically. One could certainly insist on converting an ineligible resonance into a bound state, but in that case the resulting potential would no longer be in the original class; either the compact support property would be lost or the resulting potential would no longer be integrable. We illustrate the concepts of eligible and ineligible resonances with some explicit examples in Sec. VI.

We use θ appearing in (2.3) to describe the selfadjoint boundary condition at the origin. Let $S_\theta(k)$ be the scattering matrix corresponding to the boundary parameter θ and the momentum variable k when the potential V is real valued, integrable, and compactly supported. In the recovery of the potential and the selfadjoint boundary condition from the scattering matrix $S_\theta(k)$, we summarize our main findings as follows. We have the unique recovery, except in one special case. That special case occurs when there are precisely two simultaneous constraints on $S_\theta(k)$, namely, $S_\theta(0) = +1$ and at the same time there are no bound-state poles associated with $S_\theta(k)$. The latter restriction is equivalent to the statement that $S_\theta(k)$ has no poles on the positive imaginary axis in the complex k -plane. In the special case, it turns out that the scattering matrix corresponds to exactly two distinct sets, each consisting of a potential and a selfadjoint boundary condition. Interestingly, when such

a nonuniqueness occurs, the boundary condition in one set must be the Dirichlet boundary condition and the boundary condition in the other set must be a Neumann boundary condition. In Sec. IV, we further explore the nonuniqueness in the special case and provide an interpretation of the nonuniqueness by viewing the compactly supported potential in the context of the corresponding full-line Schrödinger operator. We then find that one of the nonunique potentials corresponds to the reflection coefficient $R(k)$ and the other corresponds to $-R(k)$, and this occurs when the corresponding full-line Schrödinger operator has no bound states and is exceptional, i.e., $R(0) \neq -1$. In Sec. VI, we illustrate the nonuniqueness with an explicit example.

Concerning the recovery of the potential and the selfadjoint boundary condition from the absolute value of the Jost function, we have the unique recovery up to the inclusion of eligible resonances. From our input data set, we are able to uniquely determine all eligible resonances. Let us use M to denote the maximal number of eligible resonances corresponding to our input data set. We find that there are precisely 2^M distinct sets, each consisting of a potential and a selfadjoint boundary condition, corresponding to the same input data. We note^{19,20} that M can be infinite for our selfadjoint Schrödinger operator on the half line when the potential is real valued, integrable, and compactly supported. A further minimal assumption²⁰ on the potential guarantees that M is finite. In Sec. V, we present the details of the recovery from the absolute value of the Jost function and elaborate on the 2^M -fold nonuniqueness.

Our paper is organized as follows. In Sec. II, we provide the preliminary mathematical tools needed to analyze the two inverse problems under study. This is done by introducing the half-line Schrödinger operator, the selfadjoint boundary condition at the origin, the Jost solution and the regular solution to the half-line Schrödinger equation, the associated Jost function, the scattering matrix, the bound states, the norming constants, the resonances, and the relevant properties of all such quantities. In Sec. III, we introduce the Darboux transformations to add or remove bound states, obtain a few results related to the Darboux transformations for potentials of compact support, and provide several equivalent characterizations of eligible resonances. In Sec. IV, we analyze the recovery of the potential and the boundary condition from the scattering matrix alone. We show that the recovery of the corresponding potential and the boundary parameter is unique except in one special case, and we characterize the double nonuniqueness in that special case. In Sec. V, we study the recovery problem from the absolute value of the Jost function. We show that the recovery is unique up to the inclusion of eligible resonances, which is equivalent to having a 2^M -fold nonuniqueness. Finally, in Sec. VI, we provide some explicit examples to illustrate the theoretical results presented in Secs. III-V.

II. PRELIMINARIES

In this section, we present the preliminaries needed to prove the main results given in Secs. III-V. We use \mathbf{R} to denote the real axis, let $\mathbf{R}^+ := (0, +\infty)$, use \mathbf{C} for the complex plane, \mathbf{C}^+ for the open upper-half complex plane, \mathbf{C}^- for the open lower-half complex plane, $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$, and $\overline{\mathbf{C}^-} := \mathbf{C}^- \cup \mathbf{R}$.

Consider the half-line Schrödinger equation

$$-\psi'' + V(x)\psi = k^2\psi, \quad x \in \mathbf{R}^+, \quad (2.1)$$

where the prime denotes the x -derivative and the potential V is assumed to belong to class \mathcal{A} defined as

$$\mathcal{A} := \left\{ V : V(x) \in \mathbf{R}, V(x) = 0 \text{ for } x > b, \int_0^b dx |V(x)| < +\infty \right\}, \quad (2.2)$$

i.e., V is real valued and integrable, and it vanishes when $x > b$ for some nonnegative b . We obtain a selfadjoint Schrödinger operator on the half line by supplementing (2.1) and (2.2) with the general selfadjoint boundary condition at $x = 0$ given by^{7,12,15,16}

$$(\sin \theta)\psi'(0) + (\cos \theta)\psi(0) = 0, \quad (2.3)$$

where the boundary parameter θ is a fixed real constant in the interval $(0, \pi]$. The case $\theta = \pi$ in (2.3) corresponds to the Dirichlet boundary condition $\psi(0) = 0$ and a case with $\theta \in (0, \pi)$ corresponds to a non-Dirichlet boundary condition $\psi(0) \neq 0$. The non-Dirichlet case with $\theta = \pi/2$ in (2.3), i.e., $\psi'(0) = 0$, is known as the Neumann boundary condition. The Dirichlet case arises especially when (2.1) is related to the three-dimensional Schrödinger equation with a spherically symmetric potential. On the other hand, there are various vibration problems¹³ where a non-Dirichlet boundary condition is more appropriate to use. The non-Dirichlet case also arises in the inverse problem of determining the shape of a human vocal tract from sound pressure measurements at the lips.⁴

The so-called Jost solution associated with (2.1) and (2.2) is usually denoted by $f(k, x)$, and it satisfies

$$f(k, x) = e^{ikx}, \quad x \geq b. \quad (2.4)$$

For each fixed $x \in \mathbf{R}^+ \cup \{0\}$, the quantities $f(k, x)$ and $f'(k, x)$ have analytic extensions^{7-9,15,16} from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ as a consequence of V belonging to class \mathcal{A} . Thus, for each fixed x , the Jost function $f(k, x)$ has a Taylor series expansion around any k -value in \mathbf{C} .

The so-called regular solution associated with (2.1)-(2.3), denoted by $\varphi_\theta(k, x)$, satisfies the initial conditions

$$\begin{cases} \varphi_\theta(k, 0) = 1, & \varphi'_\theta(k, 0) = -\cot \theta, & \theta \in (0, \pi), \\ \varphi_\theta(k, 0) = 0, & \varphi'_\theta(k, 0) = 1, & \theta = \pi. \end{cases} \quad (2.5)$$

The subscript θ in $\varphi_\theta(k, x)$ indicates the dependence on the particular value of θ used in (2.3). We also use the subscript θ with certain other quantities to emphasize their dependence on θ .

We recall^{7,11,15-17} that the bound states for the Schrödinger operator associated with (2.1)-(2.3) correspond to square-integrable solutions to (2.1) satisfying the boundary condition (2.3). Therefore, the bound-state energies, i.e., the k^2 -values at which bound states occur, depend on the boundary parameter θ . When V belongs to class \mathcal{A} given in (2.2), it is known^{7,12,15-17} that there can be at most a finite number of bound states and that the number of bound states is also affected by the parameter θ . Because of the selfadjointness of the corresponding Schrödinger operator, each bound-state energy must be real. It is already known^{7,12,15-17} that for each positive k^2 -value in (2.1), there correspond two linearly independent solutions, e.g., $f(k, x)$ and $f(-k, x)$, neither of which is square integrable in $x \in \mathbf{R}^+$ as a result of (2.4). Each bound state is known^{7,12,15-17} to be simple in the sense that there exists only one linearly independent square-integrable solution to (2.1) satisfying (2.3) at a bound-state energy. The bound states, if there are any, can only occur at certain negative values of k^2 , and we will assume that they occur at $k = i\gamma_s$ for $s = 1, \dots, N$ for some nonnegative integer N and distinct positive values γ_s . Note that the γ_s -values are not assumed to be in an increasing or decreasing order. Note also that even though the value of N and the values of γ_s all depend on the choice of θ , for notational simplicity we usually suppress the dependence on θ for those quantities.

The so-called Jost function associated with (2.1)-(2.3), usually denoted by $F_\theta(k)$, is defined^{7,12,15,16} as

$$F_\theta(k) := \begin{cases} -i[f'(k, 0) + \cot \theta f(k, 0)], & \theta \in (0, \pi), \\ f(k, 0), & \theta = \pi, \end{cases} \quad (2.6)$$

and it helps us to identify the bound states and to define the scattering matrix. It is known^{8-10,15-17} that $f(k, x)$ and $f(-k, x)$ are linearly independent for each fixed $k \in \mathbf{C} \setminus \{0\}$. Thus, we can express the regular solution $\varphi_\theta(k, x)$ appearing in (2.5) as a linear combination of $f(k, x)$ and $f(-k, x)$. In fact, with the help of (2.5) and (2.6) we get

$$\varphi_\theta(k, x) = \begin{cases} \frac{1}{2k} [F_\theta(k) f(-k, x) - F_\theta(-k) f(k, x)], & \theta \in (0, \pi), \\ \frac{i}{2k} [F_\theta(k) f(-k, x) - F_\theta(-k) f(k, x)], & \theta = \pi. \end{cases} \quad (2.7)$$

From (2.3), (2.4), and (2.7), we see that a bound state can only occur at a zero of $F_\theta(k)$, which is equivalent to the linear dependence of the two solutions $\varphi_\theta(k, x)$ and $f(k, x)$ at that particular

k -value. This is because the linear dependence on $\varphi_\theta(k, x)$ assures the satisfaction of the boundary condition (2.3), and the linear dependence on $f(k, x)$ guarantees an exponential decay as $x \rightarrow +\infty$ and in turn the square integrability in $x \in \mathbf{R}^+$.

We have seen that there are at most a finite number of zeros of the Jost function $F_\theta(k)$ in \mathbf{C}^+ and such zeros can only occur on the positive imaginary axis, and those zeros correspond to bound states of the Schrödinger operator given in (2.1)-(2.3). Let us now consider the zeros of $F_\theta(k)$ in \mathbf{C}^- , which are called resonances. When $V(x) \equiv 0$, from (2.4) and (2.6) it follows that

$$F_\theta(k) = \begin{cases} k - i \cot \theta, & \theta \in (0, \pi), \\ 1, & \theta = \pi. \end{cases} \quad (2.8)$$

Thus, the number of resonances is at most one when $V(x) \equiv 0$. As stated in Theorem 2.1(g) later, if $V(x) \not\equiv 0$ then there must be a countably infinite number of resonances, and each resonance occurs either on the negative imaginary axis or a pair of resonances are symmetrically located with respect to the negative imaginary axis.

In our paper, we are primarily interested in imaginary resonances (real-energy resonances), i.e., those resonances with the corresponding k -values located on the negative imaginary axis. Through a pathological example,²⁰ it is known that the number of imaginary resonances can be countably infinite even when the potential V is in class \mathcal{A} . On the other hand, the number of imaginary resonances is guaranteed to be finite under some minimal further assumptions, e.g., see Proposition 7 of Ref. 20, such as $V(x) \geq 0$, or $V(x) \leq 0$, in some neighborhood of $x = b$, where b is the parameter appearing in (2.2) and related to the compact support of V . In Sec. III, we develop various equivalent criteria to identify each imaginary resonance either as an eligible resonance or an ineligible resonance and explore the connection between bound states and eligible resonances. In other words, in Sec. III, we analyze whether or not an imaginary resonance can be converted to a bound state without changing the compact support of the potential.

Having seen that the zeros of $F_\theta(k)$ in \mathbf{C}^+ correspond to bound states and the zeros in \mathbf{C}^- correspond to resonances, let us now consider zeros of $F_\theta(k)$ occurring on the real axis. It is known^{7,15,16} that the only real zero of $F_\theta(k)$ can occur at $k = 0$ and such a zero, if it exists, must be a simple zero. The case $F_\theta(0) = 0$ corresponds to the exceptional case, and the case $F_\theta(0) \neq 0$ corresponds to the generic case. In the exceptional case, the number of bound states may change by one under a small perturbation of the potential. Let us also consider the Jost solution $f(k, x)$ and the regular solution $\varphi_\theta(k, x)$ appearing in (2.4) and (2.5), respectively, at $k = 0$. Generically $\varphi_\theta(0, x)$ becomes unbounded as $x \rightarrow +\infty$, whereas in the exceptional case it remains bounded as $x \rightarrow +\infty$. The behavior of $\varphi_\theta(0, x)$ as $x \rightarrow +\infty$ is obtained by letting $k \rightarrow 0$ in (2.7), using (2.4) and exploiting the known behaviors of $f(0, x)$ and $\dot{f}(0, x)$ as $x \rightarrow +\infty$, where we use an overdot to indicate the k -derivative. As seen from (2.4), we have $f(0, x) = 1$ and $\dot{f}(0, x) = ix$ for $x \geq b$. From (2.7) at $k = 0$ we get

$$\varphi_\theta(0, x) = \begin{cases} \dot{F}_\theta(0) f(0, x) - F_\theta(0) \dot{f}(0, x), & \theta \in (0, \pi), \\ i [\dot{F}_\theta(0) f(0, x) - F_\theta(0) \dot{f}(0, x)], & \theta = \pi, \end{cases}$$

which shows that $\varphi_\theta(0, x)$ is proportional to $f(0, x)$ and hence remains bounded in the exceptional case and that $\varphi_\theta(0, x)$ contains $\dot{f}(0, x)$ and hence becomes unbounded in the generic case.

Recall that we assume the bound states occur at the zeros $k = i\gamma_s$ of $F_\theta(k)$ appearing in (2.6) for $s = 1, \dots, N$. It is known^{6,15,16} that $\varphi_\theta(i\gamma_s, x)$ is real valued and square integrable. The positive quantity g_s , defined as

$$g_s := \frac{1}{\sqrt{\int_0^\infty dx \varphi_\theta(i\gamma_s, x)^2}}, \quad s = 1, \dots, N, \quad (2.9)$$

is known as the Gel'fand-Levitan norming constant for the bound state at $k = i\gamma_s$. Let us use \mathcal{G} to denote the Gel'fand-Levitan spectral data set^{7,8,15,16} given by

$$\mathcal{G} := \{|F_\theta(k)| : k \in \mathbf{R}; \{\gamma_s, g_s\}_{s=1}^N\}. \quad (2.10)$$

We refer to the information consisting of $|F_\theta(k)|$ for $k \in \mathbf{R}$ as the continuous part of the Gel'fand-Levitan spectral data and refer to the portion $\{\gamma_s, g_s\}_{s=1}^N$ as the discrete part of the Gel'fand-Levitan spectral data. For the construction of V and θ from \mathcal{G} via the Gel'fand-Levitan method, we outline the recovery procedure below and refer the reader to Refs. 7, 12, 15, and 16 for the details.

(a) From the large- k asymptotics⁷

$$|F_\theta(k)| = \begin{cases} |k| + O(1), & k \rightarrow \pm\infty, \quad \theta \in (0, \pi), \\ 1 + O\left(\frac{1}{k}\right), & k \rightarrow \pm\infty, \quad \theta = \pi, \end{cases} \tag{2.11}$$

we can tell whether $\theta \in (0, \pi)$ or $\theta = \pi$.

(b) We form^{7,12,15,16} the Gel'fand-Levitan kernel $G_\theta(x, y)$, where for $\theta \in (0, \pi)$ we have

$$G_\theta(x, y) := \frac{1}{\pi} \int_{-\infty}^{\infty} dk \left[\frac{k^2}{|F_\theta(k)|^2} - 1 \right] (\cos kx)(\cos ky) + \sum_{s=1}^N g_s^2 (\cosh \gamma_s x)(\cosh \gamma_s y), \tag{2.12}$$

and for $\theta = \pi$ we have

$$G_\theta(x, y) := \frac{1}{\pi} \int_{-\infty}^{\infty} dk \left[\frac{1}{|F_\theta(k)|^2} - 1 \right] (\sin kx)(\sin ky) + \sum_{s=1}^N \frac{g_s^2}{\gamma_s^2} (\sinh \gamma_s x)(\sinh \gamma_s y). \tag{2.13}$$

(c) Using $G_\theta(x, y)$ as input to the Gel'fand-Levitan integral equation

$$A_\theta(x, y) + G_\theta(x, y) + \int_0^x dz A_\theta(x, z) G_\theta(z, y), \quad 0 < y < x,$$

we obtain $A_\theta(x, y)$. The unique solvability of (2.14) is known^{12,15,16} for the spectral data set corresponding to a potential in class \mathcal{A} and a boundary condition as in (2.3).

(d) We obtain the potential $V(x)$ and the boundary parameter θ via^{7,12,15,16}

$$\begin{aligned} V(x) &= 2 \frac{d}{dx} A_\theta(x, x), & \theta \in (0, \pi], \\ \cot \theta &= -A_\theta(0, 0), & \theta \in (0, \pi). \end{aligned} \tag{2.14}$$

(e) The regular solution $\varphi_\theta(k, x)$ is recovered from $A_\theta(x, y)$ via^{7,12,15,16}

$$\varphi_\theta(k, x) = \begin{cases} \cos kx + \int_0^x dy A_\theta(x, y) \cos ky, & \theta \in (0, \pi), \\ \frac{\sin kx}{k} + \int_0^x dy A_\theta(x, y) \frac{\sin ky}{k}, & \theta = \pi. \end{cases}$$

An alternative to the Gel'fand-Levitan procedure is the Marchenko method,^{7,15,16} which uses the input data set \mathcal{M} given by

$$\mathcal{M} := \{S_\theta(k) : k \in \mathbf{R}; \{\gamma_s, m_s\}_{s=1}^N\}, \tag{2.15}$$

where the scattering matrix $S_\theta(k)$ is defined in terms of the Jost function $F_\theta(k)$ as^{7,15,16}

$$S_\theta(k) := \begin{cases} -\frac{F_\theta(-k)}{F_\theta(k)}, & \theta \in (0, \pi), \\ \frac{F_\theta(-k)}{F_\theta(k)}, & \theta = \pi, \end{cases} \tag{2.16}$$

and the Marchenko bound-state norming constants m_s are given by^{7,15,16}

$$m_s := \frac{1}{\sqrt{\int_0^\infty dx f(i\gamma_s, x)^2}}, \quad s = 1, \dots, N. \tag{2.17}$$

We refer to the information consisting of $S_\theta(k)$ for $k \in \mathbf{R}$ as the continuous part of the Marchenko scattering data and the portion $\{\gamma_s, m_s\}_{s=1}^N$ as the discrete part of the Marchenko scattering data.

For the construction of V and θ from \mathcal{M} given in (2.15), we outline the steps of the Marchenko recovery method below and refer the reader to Refs. 7, 15, and 16 for further details.

- (a) Using the data \mathcal{M} , we construct the Marchenko kernel $M_\theta(y)$ for $y > 0$ as

$$M_\theta(y) := \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk [S_\theta(k) - 1] e^{iky} + \sum_{s=1}^N m_s^2 e^{-\gamma_s y}, & \theta \in (0, \pi), \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk [1 - S_\theta(k)] e^{iky} + \sum_{s=1}^N m_s^2 e^{-\gamma_s y}, & \theta = \pi. \end{cases} \tag{2.18}$$

- (b) Using $M_\theta(y)$ given in (2.18) as input to the Marchenko integral equation

$$K(x, y) + M_\theta(x + y) + \int_x^\infty dz K(x, z) M_\theta(z + y) = 0, \quad y > x, \tag{2.19}$$

we obtain $K(x, y)$. The unique solvability of (2.19) is guaranteed^{8-10,15,16} if the scattering data set corresponds to a potential in class \mathcal{A} given in (2.2).

- (c) The potential $V(x)$ and the Jost solution $f(k, x)$ are obtained from $K(x, y)$ via

$$V(x) = -2 \frac{dK(x, x)}{dx}, \quad f(k, x) = e^{ikx} + \int_x^\infty dy K(x, y) e^{iky}. \tag{2.20}$$

- (d) Having $K(x, y)$ and $S_\theta(k)$ at hand, we can recover $\cot \theta$ as well. For this purpose, we can proceed as follows. From the second equation in (2.20), we get

$$f(k, 0) = 1 + \int_0^\infty dy K(0, y) e^{iky}, \tag{2.21}$$

$$f'(k, 0) = ik - K(0, 0) + \int_0^\infty dy K_x(0, y) e^{iky}, \tag{2.22}$$

where $K_x(0, y)$ denotes the x -derivative of $K(x, y)$ evaluated at $x = 0$. In light of the second line of (2.16) we then check if we have

$$S_\theta(k) = \frac{1 + \int_0^\infty dy K(0, y) e^{-iky}}{1 + \int_0^\infty dy K(0, y) e^{iky}}, \tag{2.23}$$

which is obtained by using (2.21) and (2.22) in the second line of (2.16). We conclude that $\theta = \pi$ if (2.23) is satisfied. If (2.23) is not satisfied, we conclude that $\theta \in (0, \pi)$ and uniquely determine $\cot \theta$ as

$$\cot \theta = \frac{-f'(-k, 0) - S_\theta(k) f'(k, 0)}{f(-k, 0) + S_\theta(k) f(k, 0)}, \tag{2.24}$$

which is obtained with the help of (2.6), (2.16), (2.21), and (2.22).

For easy citation later on, we summarize the results presented above and several additional known facts^{1,7-10,15-17} in the following theorem.

Theorem 2.1. Consider the Schrödinger operator given in (2.1)-(2.3) with the potential V in class \mathcal{A} , a fixed boundary parameter $\theta \in (0, \pi]$ and b being the constant appearing in (2.2) related to the compact support of the potential. Let $F_\theta(k)$ be the corresponding Jost function given in (2.6) and $S_\theta(k)$ be the corresponding scattering matrix appearing in (2.16). Then:

- (a) The Jost function $F_\theta(k)$ has an analytic extension from $k \in \mathbf{R}$ to the entire complex plane $k \in \mathbf{C}$. There are at most a finite number of zeros of $F_\theta(k)$ in \mathbf{C}^+ , such zeros occur on the positive imaginary axis, say at $k = i\gamma_s$ for $s = 1, \dots, N$, they are all simple, and they correspond to the bound states of (2.1) with the selfadjoint boundary condition (2.3). A real zero of $F_\theta(k)$ can only occur at $k = 0$, and such a zero, if it exists, must be simple.

(b) As $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$ we have

$$F_\theta(k) = \begin{cases} k - i \cot \theta + \frac{i}{2} \int_0^b dx V(x) + o(1), & \theta \in (0, \pi), \\ 1 - \frac{1}{2ik} \int_0^b dx V(x) + o\left(\frac{1}{k}\right), & \theta = \pi. \end{cases} \quad (2.25)$$

(c) As $k \rightarrow \infty$ in $\overline{\mathbf{C}^-}$ we have

$$F_\theta(k) = \begin{cases} k - i \cot \theta + \frac{i}{2} \int_0^b dx V(x) + e^{2ikb} o(1), & \theta \in (0, \pi), \\ 1 - \frac{1}{2ik} \int_0^b dx V(x) + e^{2ikb} o\left(\frac{1}{k}\right), & \theta = \pi. \end{cases}$$

(d) As $k \rightarrow \pm\infty$ in \mathbf{R} , the large- $|k|$ asymptotics of the scattering matrix $S_\theta(k)$ is given by

$$S_\theta(k) = \begin{cases} 1 - \frac{i}{k} \int_0^b dx V(x) + \frac{2i}{k} \cot \theta + o\left(\frac{1}{k}\right), & \theta \in (0, \pi), \\ 1 - \frac{i}{k} \int_0^b dx V(x) + o\left(\frac{1}{k}\right), & \theta = \pi. \end{cases}$$

(e) The scattering matrix $S_\theta(k)$ defined in (2.16) has a meromorphic extension from $k \in \mathbf{R}$ to $k \in \mathbf{C}$. The poles of $S_\theta(k)$ in \mathbf{C}^+ are all simple and occur at $k = i\gamma_s$ for $s = 1, \dots, N$. The Marchenko norming constants m_s defined in (2.17) are related to the residues of the scattering matrix at those poles as

$$\text{Res}(S_\theta, i\gamma_s) = \begin{cases} i m_s^2, & \theta \in (0, \pi), \\ -i m_s^2, & \theta = \pi, \end{cases} \quad (2.26)$$

where $\text{Res}(S_\theta, i\gamma_s)$ denotes the residue of $S_\theta(k)$ at $k = i\gamma_s$.

(f) For each $\theta \in (0, \pi]$, the scattering matrix $S_\theta(k)$ is analytic at $k = 0$ in \mathbf{C} . The value of $S_\theta(0)$ is either $+1$ or -1 . Specifically, for $\theta = \pi$, we have

$$S_\pi(0) = \begin{cases} +1, & f(0, 0) \neq 0, \\ -1, & f(0, 0) = 0, \end{cases} \quad (2.27)$$

and for any $\theta \in (0, \pi)$, we have

$$S_\theta(0) = \begin{cases} -1, & F_\theta(0) \neq 0, \\ +1, & F_\theta(0) = 0. \end{cases} \quad (2.28)$$

(g) Unless $V(x) \equiv 0$, there are infinitely many zeros of $F_\theta(k)$ in \mathbf{C}^- , and such zeros are known as resonances. The resonances need not be simple, and they are located either on the negative imaginary axis or occur as pairs located symmetrically with respect to the negative imaginary axis.

(h) The Gel'fand-Levitan norming constants g_s appearing in (2.9) and the Marchenko norming constants m_s appearing in (2.17) are related to each other as

$$g_s = \frac{2\gamma_s m_s}{|F_\theta(-i\gamma_s)|}, \quad \theta \in (0, \pi]. \quad (2.29)$$

(i) The potential V and the boundary parameter θ are uniquely determined from the Gel'fand-Levitan spectral data \mathcal{G} given in (2.10).

(j) The potential V and the boundary parameter θ are uniquely determined from the Marchenko scattering data \mathcal{M} given in (2.15).

(k) The knowledge of $S_\theta(k)$ for $k > 0$ is equivalent to the knowledge of $S_\theta(k)$ for $k \in \mathbf{R}$, and we have for $k \in \mathbf{R}$

$$S_\theta(-k) = S_\theta(k)^*, \quad \theta \in (0, \pi],$$

where the asterisk denotes complex conjugation. Similarly, the knowledge of $|F_\theta(k)|$ for $k > 0$ is equivalent to the knowledge of $|F_\theta(k)|$ for $k \in \mathbf{R}$, and we have for $k \in \mathbf{R}$

$$F_\theta(-k) = \begin{cases} -F_\theta(k)^*, & \theta \in (0, \pi), \\ F_\theta(k)^*, & \theta = \pi. \end{cases}$$

Proof. For (a), (i), (j), we refer the reader to Refs. 7, 15, and 16. For (b), (c), (d), (e), the reader is referred to Refs. 6–8, 11, and 16. The result in (f) is obtained by using (2.6) and (2.16) with the help of a series expansion around $k = 0$. The proof of (g) is as follows. From (2.8), we already know that the number of resonances corresponding to $V(x) \equiv 0$ is either zero or one. For $V(x) \not\equiv 0$ with $b > 0$ in (2.2), we conclude, from (a)–(c), that $e^{2ikb}F_\theta(k)$ is entire in k and behaves as $O(k)$ as $k \rightarrow \infty$ in \mathbf{C} . If $F_\theta(k)$ had no zeros or had only a finite number of zeros in \mathbf{C} , then the Hadamard factorization of $e^{2ikb}F_\theta(k)$ and the use of Liouville’s theorem would force $F_\theta(k)$ to be equal to e^{-2ikb} multiplied with either a constant or a polynomial in k . However, such a behavior would contradict (2.25). Thus, the number of resonances must be countably infinite. Since k appears as ik in $f(k, 0)$ and $f'(k, 0)$, it follows from (2.6), that the zeros of $F_\theta(k)$ in \mathbf{C}^- either occur on the negative imaginary axis or a pair of resonances are symmetrically located with respect to the negative imaginary axis. From Example 6.2(c) later in our paper, we know that a resonance need not be simple. Thus, the proof of (g) is complete. Note that (2.29) holds for $\theta = \pi$ as well as for $\theta \in (0, \pi)$. The result in (2.29) is obtained by evaluating (2.7) at the bound state $k = i\gamma_s$, using $F_\theta(i\gamma_s) = 0$ in that equation, taking the square of both sides of the resulting equation, followed by an integration on $x \in \mathbf{R}^+$, and by using (2.9) and (2.17) in the resulting equation. Finally, the results in (j) follow from the use of the identities^{1,7,8,15–17} for $k \in \mathbf{R}$

$$f(-k, 0) = f(k, 0)^*, \quad f'(-k, 0) = f'(k, 0)^*,$$

in (2.8) and (2.16) and the fact that $f(k, 0)$ and $f'(k, 0)$ are continuous at $k = 0$. ■

Next, we elaborate on the exceptional case for the half-line Schrödinger operator and present the behavior of the corresponding scattering coefficients for the full-line Schrödinger operator at $k = 0$. Such results are needed in Secs. III and IV in the elaboration of the nonuniqueness arising in the special case, i.e., case (iii) of Sec. IV.

Recall that the exceptional case for the half-line Schrödinger operator occurs when $F_\theta(0) = 0$, where $F_\theta(k)$ is the Jost function defined in (2.6). Since we can view the potential V appearing in (2.1) as the potential on the full line with $V(x) = 0$ for $x < 0$, we can uniquely^{7,8} associate with V the scattering coefficients T, L , and R , where T is the transmission coefficient, L is the reflection coefficient from the left, and R is the reflection coefficient from the right. This is done via^{7,8}

$$f(k, 0) = \frac{1 + L(k)}{T(k)}, \quad f'(k, 0) = ik \frac{1 - L(k)}{T(k)}, \quad R(k) = -\frac{L(-k)T(k)}{T(-k)}. \tag{2.30}$$

The exceptional case for the full-line Schrödinger operator occurs when $T(0) \neq 0$, and the full-line generic case occurs when $T(0) = 0$.

Theorem 2.2. *Consider the half-line Schrödinger operator given in (2.1)–(2.3) with the potential V in class \mathcal{A} and with a fixed boundary parameter $\theta \in (0, \pi]$. Let $f(k, x)$ and $F_\theta(k)$ be the corresponding Jost solution and the Jost function appearing in (2.4) and (2.6), respectively. Further, let $T(k), L(k)$, and $R(k)$ be the corresponding scattering coefficients appearing in (2.30). Then:*

- (a) *The half-line exceptional case with the Dirichlet boundary condition, i.e., $f(0, 0) = 0$, corresponds to the following zero-energy behavior of the scattering coefficients:*

$$T(0) = 0, \quad \dot{T}(0) \neq 0, \quad L(0) = -1, \quad \dot{L}(0) = 0, \quad \ddot{L}(0) \neq 0, \tag{2.31}$$

$$R(0) = -1, \quad \dot{R}(0) = -\frac{\ddot{T}(0)}{\dot{T}(0)}, \quad \ddot{R}(0) = -\dot{T}(0)^2 - \frac{\dot{T}(0)^2}{\dot{T}(0)^2}, \tag{2.32}$$

where we recall that an overdot denotes the k -derivative.

(b) *The half-line exceptional case with the Neumann boundary condition, i.e., $f'(0,0) = 0$, corresponds to the following zero-energy behavior of the scattering coefficients:*

$$T(0) \neq 0, \quad L(0) \neq -1, \quad R(0) \neq -1. \tag{2.33}$$

(c) *The half-line exceptional case with the non-Dirichlet and non-Neumann boundary conditions, i.e., the case $F_\theta(0) = 0$ with $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$, corresponds to the following zero-energy behavior of the scattering coefficients:*

$$\begin{aligned} T(0) = 0, \quad \dot{T}(0) \neq 0, \quad L(0) = -1, \quad R(0) = -1, \\ \dot{L}(0) = -\frac{2i}{\cot \theta}, \quad \dot{R}(0) = \frac{2i}{\cot \theta} - \frac{\dot{T}(0)}{\dot{T}(0)}. \end{aligned} \tag{2.34}$$

Proof. The behavior of the scattering coefficients around $k = 0$ is already known.^{5,8} In the full-line generic case, we have

$$T(0) = 0, \quad \dot{T}(0) \neq 0, \quad L(0) = -1, \quad R(0) = -1, \tag{2.35}$$

and in the full-line exceptional case, we have

$$T(0) \neq 0, \quad L(0) \in (-1, 1), \quad R(0) \in (-1, 1). \tag{2.36}$$

From Theorem 2.1(a), when $V \in \mathcal{A}$ we know that $f(k,0)$ and $f'(k,0)$ are entire, and hence with the help of (2.30) we see that $T(k), R(k)$, and $L(k)$ are analytic at $k = 0$. Expanding around $k = 0$ the first identity in (2.30), we see that (2.36) is incompatible with $f(0,0) = 0$ and hence in case of (a) in our theorem, we must have (2.35). Then, the expansion of the first identity in (2.30) yields

$$f(0,0) + k \dot{f}(0,0) + O(k^2) = \frac{\dot{L}(0)}{\dot{T}(0)} + \frac{k}{2} \left[\frac{\ddot{L}(0)}{\dot{T}(0)} - \frac{\dot{L}(0)\ddot{T}(0)}{\dot{T}(0)^2} \right] + O(k^2), \quad k \rightarrow 0 \text{ in } \mathbf{C}. \tag{2.37}$$

From Theorem 2.1(a) we already know that $k = 0$ must be a simple zero of $f(k,0)$ and hence $\dot{f}(0,0) \neq 0$. Thus from (2.37), we get $\dot{L}(0) = 0$ and $\ddot{L}(0) \neq 0$. Hence, we have proved (2.31). In fact, the expansion around $k = 0$ of the identity⁸⁻¹⁰

$$L(k)L(-k) + T(k)T(-k) = 1, \quad k \in \mathbf{C},$$

indicates that in the full-line generic case we have

$$\ddot{L}(0) + \dot{L}(0)^2 + \dot{T}(0)^2 = 0, \tag{2.38}$$

and hence (2.38) shows that in case of (a) we have

$$\ddot{L}(0) = -\dot{T}(0)^2, \tag{2.39}$$

which also confirms that $\dot{L}(0) \neq 0$ in (2.31). We establish (2.32) by expanding around $k = 0$ the third identity in (2.30) and using (2.31) and (2.39). Let us now turn to the proof of (b). Expanding around $k = 0$ the second identity in (2.30), we see that (2.35) is incompatible with $f'(0,0) = 0$. Thus, we must have (2.36) in case of (b), which establishes (2.33). Finally, let us prove (c). Using the first two identities in (2.6), we get

$$F_\theta(k) = k \frac{1 - L(k)}{T(k)} - i \cot \theta \frac{1 + L(k)}{T(k)}. \tag{2.40}$$

Note that (2.36) is not compatible with $\cot \theta \neq 0$ and $F_\theta(0) = 0$. Thus, we must have (2.35) in case of (c). Then, expanding around $k = 0$ both sides of (2.40) we get

$$\begin{aligned} F_\theta(0) &= \frac{2 - i \cot \theta \dot{L}(0)}{\dot{T}(0)}, \\ \dot{F}_\theta(0) &= -\frac{\dot{T}(0)}{2\dot{T}(0)} \frac{2 - i \cot \theta \dot{L}(0)}{\dot{T}(0)} - \frac{2\dot{L}(0) + i \cot \theta \ddot{L}(0)}{2\dot{T}(0)}. \end{aligned} \tag{2.41}$$

Since $F_\theta(0) = 0$, from (2.41) we get $\dot{L}(0) = -2i/\cot \theta$. Finally, with the help of (2.30), we obtain $\dot{R}(0)$ given in (2.34). ■

The next theorem shows that if the half-line Schrödinger operator with the Neumann boundary condition and with a potential V belonging to class \mathcal{A} has no bound states, then the full-line Schrödinger operator with the same potential V cannot have any bound states either. The result is needed for the proof of Theorem 2.4 and later in the analysis in Sec. IV.

Theorem 2.3. Consider the half-line Schrödinger operator given in (2.1)-(2.3) with the potential V in class \mathcal{A} and with a fixed boundary parameter $\theta \in (0, \pi]$, and let $f(k, x)$ and $F_\theta(k)$ be the corresponding Jost solution and the Jost function appearing in (2.4) and (2.6), respectively. Let N_θ denote the number of bound states, i.e., the number of zeros of $F_\theta(i\beta)$ when $\beta \in (0, +\infty)$. Let $T(k), L(k),$ and $R(k)$ be the corresponding scattering coefficients appearing in (2.30). Let \tilde{N} denote the number of bound states for the corresponding full-line Schrödinger operator, i.e., let \tilde{N} denote the number of zeros of $1/T(i\beta)$ in the interval $\beta \in (0, +\infty)$. If $N_{\pi/2} = 0$, then we must have $\tilde{N} = 0$.

Proof. It is already known⁷ that $N_{\theta_1} \leq N_{\theta_2}$ if $\theta_1 \geq \theta_2$. Thus, in particular, we have $N_\pi \leq N_{\pi/2}$. Since we assume $N_{\pi/2} = 0$, we then also have $N_\pi = 0$. Thus, neither $f'(i\beta, 0)$ nor $f(i\beta, 0)$ vanishes for $\beta > 0$. From (2.25), we then conclude that $-f'(i\beta, 0) > 0$ and $f(i\beta, 0) > 0$ for all $\beta > 0$. The first two identities in (2.30) yield

$$\frac{2ik}{T(k)} = f'(k, 0) + ik f(k, 0), \quad k \in \mathbf{C}. \tag{2.42}$$

From (2.42), using $k = i\beta$, we obtain

$$\frac{2\beta}{T(i\beta)} = -f'(i\beta, 0) + \beta f(i\beta, 0). \tag{2.43}$$

Since the right-hand side of (2.43) is positive for all $\beta > 0$, we conclude that $T(i\beta)$ does not have any poles for $\beta > 0$ and hence $\tilde{N} = 0$. ■

The following theorem shows that in the absence of any bound states, the Marchenko equation given in (2.19) is equivalent to the full-line Marchenko equation given by

$$K(x, y) + \hat{R}(x + y) + \int_x^\infty dz K(x, z) \hat{R}(z + y) = 0, \quad y > x, \tag{2.44}$$

where $\hat{R}(y)$ denotes the Fourier transform of the reflection coefficient $R(k)$ appearing in (2.30), namely,

$$\hat{R}(y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk R(k) e^{iky}. \tag{2.45}$$

The result in Theorem 2.4 is needed in the characterization of the double nonuniqueness in the special case in Sec. IV, i.e., case (iii) there.

Theorem 2.4. Consider the half-line Schrödinger operator given in (2.1)-(2.3) with the potential V in class \mathcal{A} and with a fixed boundary parameter $\theta \in (0, \pi]$. Let $F_\theta(k), S_\theta(k),$ and $M_\theta(y)$ be the corresponding Jost function, the scattering matrix, and the Marchenko kernel defined in (2.6), (2.16), and (2.18), respectively. Let $T(k), L(k),$ and $R(k)$ be the corresponding scattering coefficients appearing in (2.30). Assume that neither the half-line Schrödinger operator nor the full-line Schrödinger operator has any bound states, i.e., $F_\theta(k)$ has no zeros on the positive imaginary axis and $T(k)$ has no poles on the positive imaginary axis. Then, we have

$$M_\theta(y) = \hat{R}(y), \quad y > 0, \quad \theta \in (0, \pi], \tag{2.46}$$

where $\hat{R}(y)$ is the quantity given in (2.45).

Proof. From (2.30), we get

$$\frac{2ik L(k)}{T(k)} = ik f(k, 0) - f'(k, 0), \tag{2.47}$$

and hence from (2.42) and (2.47) we have

$$T(k) = \frac{2ik}{f'(k,0) + ik f(k,0)}, \quad \frac{L(-k)}{T(-k)} = \frac{f'(-k,0) + ik f(-k,0)}{2ik}. \tag{2.48}$$

Using (2.48) in the third equation in (2.30) we obtain

$$R(k) = -\frac{f'(-k,0) + ik f(-k,0)}{f'(k,0) + ik f(k,0)}. \tag{2.49}$$

Thus, from (2.49) and the second line of (2.16), we get

$$1 - S_\pi(k) - R(k) = 1 - \frac{f(-k,0)}{f(k,0)} + \frac{f'(-k,0) + ik f(-k,0)}{f'(k,0) + ik f(k,0)}. \tag{2.50}$$

Using the Wronskian relation⁸⁻¹⁰

$$f(-k,0)f'(k,0) - f'(-k,0)f(k,0) = 2ik,$$

and the first equality in (2.48), we can rewrite (2.50) as

$$1 - S_\pi(k) - R(k) = 1 - \frac{T(k)}{f(k,0)}. \tag{2.51}$$

In the absence of bound states for the full-line Schrödinger equation, it is known⁸⁻¹⁰ that $T(k)$ is analytic in \mathbf{C}^+ and continuous in $\overline{\mathbf{C}^+}$ and $T(k) = 1 + O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. In the absence of bound states for the half-line Schrödinger equation, $f(k,0)$ and hence also $1/f(k,0)$ are analytic in \mathbf{C}^+ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$ and behave as $1 + O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Furthermore, from Theorem 2.2(a), the continuity of $T(k)/f(k,0)$ at $k = 0$ is assured. Thus, the right-hand side of (2.51) is analytic in \mathbf{C}^+ and continuous in $\overline{\mathbf{C}^+}$ and behaves as $O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Hence, its Fourier transform vanishes for $y > 0$, i.e.,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk [1 - S_\pi(k) - R(k)] e^{iky} = 0, \quad y > 0. \tag{2.52}$$

Comparing (2.52) with (2.45) and the second line of (2.18) without the summation term there, we see that $M_\pi(y) = \hat{R}(y)$ for $y > 0$, establishing (2.46) for $\theta = \pi$. In a similar way, we can show that, for $\theta \in (0, \pi)$, we have

$$R(k) - S_\theta(k) + 1 = 1 - \frac{(k + i \cot \theta)T(k)}{F_\theta(k)}. \tag{2.53}$$

In the absence of bound states for the half-line Schrödinger operator, from Theorem 2.1 we know that $1/F_\theta(k)$ is analytic in \mathbf{C}^+ , continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$, and behaves like $O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. In the absence of bound states for the full-line Schrödinger operator, we already know that $T(k)$ is analytic in \mathbf{C}^+ , continuous in $\overline{\mathbf{C}^+}$, and behaves as $1 + O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Furthermore, from (b) and (c) of Theorem 2.2 it follows that the second term on the right-hand side in (2.53) is continuous at $k = 0$. Thus, the right-hand side in (2.53) is analytic in $k \in \mathbf{C}^+$ and continuous in $k \in \overline{\mathbf{C}^+}$ and behaves as $O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Hence, its Fourier transform for $y > 0$ vanishes, i.e., we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk [R(k) - S_\theta(k) + 1] e^{iky} = 0, \quad y > 0,$$

yielding $M_\theta(y) = \hat{R}(y)$ for $y > 0$. Therefore, (2.46) holds also when $\theta \in (0, \pi)$. ■

III. DARBOUX TRANSFORMATIONS AND ELIGIBLE RESONANCES

Recall that a Darboux transformation^{8,9,16} allows us to change the discrete spectrum of a differential operator by adding or removing a finite number of discrete eigenvalues without changing the continuous spectrum and without affecting the remainder of the discrete spectrum. In preparation for the analysis in Sec. V, in this section we provide the Darboux transformation formulas when a bound state is added or removed from the spectrum of the Schrödinger operator on the

half line. We also provide various results related to the Darboux transformation with compactly supported potentials. In particular, we provide the necessary and sufficient conditions for retaining the compact-support property of the potential when we add a bound state. We show that such a bound state can only come from an eligible resonance, which is a zero of the Jost function $F_\theta(k)$ occurring on the negative imaginary axis and can be converted to a bound state via a Darboux transformation without changing the compact support of the potential. We provide various equivalent characterizations of eligible resonances, such as (3.19), (3.38), and (3.53).

For clarity, we use the notation $\theta_j, V(x; j), \varphi(k, x; j)$, and $F(k; j)$ to denote the relevant quantities corresponding to the Schrödinger operator with bound states at $k = i\gamma_1, \dots, i\gamma_j$, where the case $j = 0$ refers to the quantities without bound states. Note that θ_j is the boundary parameter appearing in (2.3), $\varphi(k, x; j)$ is the regular solution in (2.5), $F(k; j)$ is the Jost function in (2.6), and g_j is the Gel'fand-Levitan bound-state norming constant in (2.9).

We recall that the γ_s -values are not necessarily in an increasing or decreasing order, and the ordering only refers to the order in which the bound states are added. We suppose that the bound states are added in succession by starting with the potential $V(x; 0)$ containing no bound states and by first adding the bound state at $k = i\gamma_1$ with the Gel'fand-Levitan norming constant g_1 , then by adding the bound state at $k = i\gamma_2$ with the norming constant g_2 , and so on. In the presence of N bound states, when the bound states are removed in succession, we start with the potential $V(x; N)$ and first remove the bound state at $k = i\gamma_N$ with the norming constant g_N , then remove the bound state at $k = i\gamma_{N-1}$ with the norming constant g_{N-1} , and so on.

The following theorem summarizes the Darboux transformation when a bound state at $k = i\gamma_{j+1}$ with the Gel'fand-Levitan norming constant g_{j+1} is added to the half-line Schrödinger operator with the potential $V(\cdot; j)$ and the boundary parameter θ_j . In the Dirichlet case, i.e., when $\theta_j = \pi$, we refer the reader to Ref. 8 for the Darboux transformation formulas provided in the theorem. In the non-Dirichlet case, i.e., when $\theta_j \in (0, \pi)$, we refer the reader to (2.3.23) of Ref. 16 for the Darboux transformation formulas when a bound state is added. The formulas in the non-Dirichlet case look similar to those in the Dirichlet case except that the boundary parameter θ_j has to be allowed to change so that the two conditions given in the first line of (2.5) are satisfied. We invite the interested reader to directly verify the results by showing that (2.1) and (2.5) are satisfied after the bound state is added.

Theorem 3.1. *Let $V(\cdot; j)$ for some fixed $j \geq 0$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$, where we assume that there are no bound states in case $j = 0$. Assume that one bound state at $k = i\gamma_{j+1}$ is added to the spectrum with the Gel'fand-Levitan norming constant g_{j+1} , but otherwise the relevant spectral data set is unchanged. The resulting boundary parameter θ_{j+1} , potential $V(x; j + 1)$, regular solution $\varphi(k, x; j + 1)$, and Jost function $F(k; j + 1)$ are related to the original quantities $\theta_j, V(x; j), \varphi(k, x; j)$, and $F(k; j)$ as*

$$\begin{cases} \cot \theta_{j+1} = \cot \theta_j + g_{j+1}^2, & \theta_j \in (0, \pi), \\ \theta_{j+1} = \theta_j, & \theta_j = \pi, \end{cases} \tag{3.1}$$

$$V(x; j + 1) = V(x; j) - \frac{d}{dx} \left[\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j)^2}{1 + g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j)^2} \right], \tag{3.2}$$

$$F(k; j + 1) = \frac{k - i\gamma_{j+1}}{k + i\gamma_{j+1}} F(k; j), \tag{3.3}$$

$$\varphi(k, x; j + 1) = \varphi(k, x; j) - \frac{g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j) \int_0^x dy \varphi(k, y; j) \varphi(i\gamma_{j+1}, y; j)}{1 + g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j)^2}. \tag{3.4}$$

The following theorem summarizes the Darboux transformation when the bound state at $k = i\gamma_j$ with the Gel'fand-Levitan norming constant g_j is removed from the half-line Schrödinger operator with the potential $V(\cdot; j)$ and the boundary parameter θ_j . The formulas in the non-Dirichlet case resemble the corresponding formulas in the Dirichlet case except that the boundary parameter changes in a way compatible with the first line of (3.1). We omit the proof of the theorem and invite the interested reader to directly verify the formulas by showing that (2.1) and (2.5) are satisfied after the bound state is removed.

Theorem 3.2. *Let $V(\cdot; j)$ for some fixed $j \geq 1$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$. Assume that the bound state at $k = i\gamma_j$ is removed from the spectrum with the Gel'fand-Levitan norming constant g_j , but otherwise the relevant spectral data set is unchanged. The resulting boundary parameter θ_{j-1} , potential $V(x; j - 1)$, regular solution $\varphi(k, x; j - 1)$, and Jost function $F(k; j - 1)$ are related to $\theta_j, V(x; j), \varphi(k, x; j)$, and $F(k; j)$ as*

$$\begin{cases} \cot \theta_{j-1} = \cot \theta_j - g_j^2, & \theta_j \in (0, \pi), \\ \theta_{j-1} = \theta_j, & \theta_j = \pi, \end{cases}$$

$$V(x; j - 1) = V(x; j) + \frac{d}{dx} \left[\frac{2g_j^2 \varphi(i\gamma_j, x; j)^2}{1 - g_j^2 \int_0^x dy \varphi(i\gamma_j, y; j)^2} \right], \tag{3.5}$$

$$F(k; j - 1) = \frac{k + i\gamma_j}{k - i\gamma_j} F(k; j), \tag{3.6}$$

$$\varphi(k, x; j - 1) = \varphi(k, x; j) + \frac{g_j^2 \varphi(i\gamma_j, x; j) \int_0^x dy \varphi(k, y; j) \varphi(i\gamma_j, y; j)}{1 - g_j^2 \int_0^x dy \varphi(i\gamma_j, y; j)^2}, \tag{3.7}$$

where $\theta_0, V(x; 0), F(k; 0)$, and $\varphi(k, x; 0)$ correspond to the relevant quantities with no bound states.

Let us remark that (2.11), (3.3), and (3.6) imply that the boundary conditions cannot switch from a Dirichlet condition to a non-Dirichlet condition or vice versa when bound states are added or removed via a Darboux transformation. This is because (3.3) and (3.6) show that the leading term in (2.11) for the large- k asymptotics of the Jost function $F_\theta(k)$ cannot change from 1 to k or vice versa as $k \rightarrow +\infty$.

The next theorem indicates that the compact-support property of the potential is retained if a bound state is removed.

Theorem 3.3. *Let $V(\cdot; j) \in \mathcal{A}$ for some fixed $j \geq 1$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j , the constant b in (2.2) related to the compact support of $V(\cdot; j)$, and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$. Assume that the bound state at $k = i\gamma_j$ is removed from the spectrum with the Gel'fand-Levitan norming constant g_j , but otherwise the relevant spectral data set is unchanged. If the compact support of $V(\cdot; j)$ is confined to the interval $[0, b]$, then the support of $V(\cdot; j - 1)$ is also confined to $[0, b]$ and we have $V(\cdot; j - 1) \in \mathcal{A}$.*

Proof. We know that (3.5) holds because $V(\cdot; j)$ has a bound state at $k = i\gamma_j$ with the norming constant g_j given in (2.9). It is enough to show that the quantity inside the brackets in (3.5) is a constant for $x \geq b$ and hence its x -derivative vanishes. Because $\varphi(i\gamma_j, x; j)$ is a bound state, it decays exponentially as $x \rightarrow +\infty$. Thus, from (2.7), by using (2.4) and $F(i\gamma_j; j) = 0$ we get

$$\varphi(i\gamma_j, x; j)^2 = \mp \frac{1}{4\gamma_j^2} F(-i\gamma_j; j)^2 e^{-2\gamma_j x}, \quad x \geq b, \tag{3.8}$$

where the upper sign refers to the non-Dirichlet case $\theta_j \in (0, \pi)$ and the lower sign to the Dirichlet case $\theta_j = \pi$. For $x \geq b$, we can evaluate the denominator inside the brackets in (3.5) by using

$\int_0^x = \int_0^\infty - \int_x^\infty$ there. Because of (2.9), we have

$$g_j^2 \int_0^\infty dy \varphi(i\gamma_j, y; j)^2 = 1, \tag{3.9}$$

and with the help of (3.8), we get

$$g_j^2 \int_x^\infty dy \varphi(i\gamma_j, y; j)^2 = \mp \frac{1}{8\gamma_j^3} F(-i\gamma_j; j)^2 e^{-2\gamma_j x}, \quad x \geq b. \tag{3.10}$$

Using (3.8)-(3.10) in the quantity inside the brackets in (3.5), we get

$$\frac{2g_j^2 \varphi(i\gamma_j, x; j)^2}{1 - g_j^2 \int_0^x dy \varphi(i\gamma_j, y; j)^2} = 4\gamma_j, \quad x \geq b, \tag{3.11}$$

and hence from (3.5) we see that $V(x; j) = V(x; j - 1)$ for $x > b$ and thus the support of $V(\cdot; j - 1)$ is contained within the support as $V(\cdot; j)$. The property $V(\cdot; j - 1) \in \mathcal{A}$ then follows from the fact that the quantity inside the brackets in the second term on the right-hand side of (3.5) is real valued and continuous in x when $x \in [0, b]$. ■

In the notation used in this section, we can express the definition of the Gel'fand-Levitan norming constant g_s given in (2.9) as

$$g_s := \frac{1}{\sqrt{\int_0^\infty dx \varphi(i\gamma_s, x; N)^2}}, \quad s = 1, \dots, N, \tag{3.12}$$

where $\varphi(k, x; N)$ is the regular solution appearing in (2.7). The following result shows that we can obtain g_j by normalizing not only $\varphi(i\gamma_j, x; N)$ but any one of $\varphi(i\gamma_j, x; s)$ for $s = j, j + 1, \dots, N$.

Theorem 3.4. *Let $V(\cdot; N) \in \mathcal{A}$ for some fixed $N \geq 2$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_N , the bound states at $k = i\gamma_s$ for $s = 1, \dots, N$, and the corresponding Gel'fand-Levitan norming constants g_s defined as in (2.9). For any j with $1 \leq j < N$, we then have*

$$\int_0^\infty dx \varphi(i\gamma_j, x; j)^2 = \int_0^\infty dx \varphi(i\gamma_j, x; j + 1)^2 = \dots = \int_0^\infty dx \varphi(i\gamma_j, x; N)^2. \tag{3.13}$$

Proof. From (3.4), for any positive integer s with $j + 1 \leq s \leq N$, we obtain

$$\varphi(i\gamma_j, x; s) = \varphi(i\gamma_j, x; j) - \frac{g_s^2 \varphi(i\gamma_s, x; j) \int_0^x dy \varphi(i\gamma_j, y; j) \varphi(i\gamma_s, y; j)}{1 + g_s^2 \int_0^x dy \varphi(i\gamma_s, y; j)^2}. \tag{3.14}$$

Squaring both sides of (3.14) and with some simplification, we observe that

$$\varphi(i\gamma_j, x; s)^2 = \varphi(i\gamma_j, x; j)^2 - \frac{d}{dx} \left[\frac{g_s^2 \left[\int_0^x dy \varphi(i\gamma_j, y; j) \varphi(i\gamma_s, y; j) \right]^2}{1 + g_s^2 \int_0^x dy \varphi(i\gamma_s, y; j)^2} \right]. \tag{3.15}$$

Integrating both sides of (3.15) over $x \in (0, +\infty)$, we see that the equalities in (3.13) all hold provided the quantity inside the brackets in (3.15) vanishes as $x \rightarrow +\infty$ because that quantity already vanishes at $x = 0$. Let us use $\int_0^x = \int_0^b + \int_b^x$ when $x \geq b$ and estimate the integrals in the numerator and in the denominator in (3.15). By Theorem 3.3, we know that $V(\cdot; j) \in \mathcal{A}$ because $V(\cdot; N) \in \mathcal{A}$. Thus, $V(x; j) = 0$ for $x > b$ and $f(k, x; j) = e^{ikx}$ for $x \geq b$ as a result of (2.4). We also have

$F(i\gamma_j; j) = 0$ and thus via (3.3) we have $F(i\gamma_s; j) \neq 0$ for $j + 1 \leq s \leq N$. Therefore, from (2.7) we obtain

$$\varphi(i\gamma_j, x; j)^2 = \mp \frac{1}{4\gamma_j^2} F(-i\gamma_j; j)^2 e^{-2\gamma_j x}, \quad x \geq b, \tag{3.16}$$

and for $j + 1 \leq s \leq N$, we have

$$\varphi(i\gamma_s, x; j)^2 = \mp \frac{1}{4\gamma_s^2} [F(i\gamma_s; j) e^{\gamma_s x} - F(-i\gamma_s; j) e^{-\gamma_s x}]^2, \quad x \geq b, \tag{3.17}$$

where the upper sign refers to the non-Dirichlet case $\theta_j \in (0, \pi)$ and the lower sign refers to the Dirichlet case $\theta_j = \pi$. With the help of (3.16) and (3.17) we get

$$\int_0^x dy \varphi(i\gamma_j, y; j) \varphi(i\gamma_s, y; j) = O(e^{(\gamma_s - \gamma_j)x}), \quad x \rightarrow +\infty,$$

$$\int_0^x dy \varphi(i\gamma_s, y; j)^2 = O(e^{2\gamma_s x}), \quad x \rightarrow +\infty.$$

Thus, the quantity inside the brackets in (3.15) has the behavior $O(e^{-2\gamma_j x})$ as $x \rightarrow +\infty$. Hence, our proof is complete. ■

Using the result in Theorem 3.4, we can comment on the denominator in (3.5). As seen from (3.12) and (3.13), the Gel'fand-Levitan norming constant g_j can be obtained by normalizing $\varphi(i\gamma_j, x; s)$ for any integer s with $j \leq s \leq N$, i.e., via

$$g_j = \frac{1}{\sqrt{\int_0^\infty dx \varphi(i\gamma_j, x; s)^2}}, \quad s = j, j + 1, \dots, N. \tag{3.18}$$

Using (3.5) and the positivity of $\varphi(i\gamma_j, y; j)^2$, we conclude that function defined by the integral $\int_0^x dy \varphi(i\gamma_j, y; j)^2$ is an increasing function of x . With the help of (3.18), we see that it increases from the value of zero at $x = 0$ to the value of $1/g_j^2$ as x increases from $x = 0$ to $x = +\infty$. Thus, the denominator in (3.5) remains positive for $x \in \mathbf{R}^+$.

The following theorem is one of the key results needed for the characterization of eligible and ineligible resonances. Recall that an eligible resonance corresponds to a zero of the Jost function defined in (2.6) in such a way that such a zero occurs on the negative imaginary axis and can be converted into a bound state through a Darboux transformation without changing the compact support of the potential. If a zero of the Jost function occurring on the negative imaginary axis cannot be converted into a bound state under a Darboux transformation without changing the compact support of the potential, then we refer to such an imaginary resonance as an ineligible resonance.

Theorem 3.5. *Let $V(\cdot; j) \in \mathcal{A}$ for some fixed $j \geq 0$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$, where $j = 0$ corresponds to the absence of bound states. Assume that a bound state at $k = i\gamma_{j+1}$ is added to the spectrum with the Gel'fand-Levitan norming constant g_{j+1} , but otherwise the relevant spectral data set is unchanged. Let b be the constant appearing in (2.2) related to the compact support of $V(\cdot; j)$. The support of $V(\cdot; j + 1)$ is also confined to $[0, b]$ if and only if*

$$F(-i\gamma_{j+1}; j) = 0, \quad g_{j+1}^2 = \frac{2\gamma_{j+1}}{\varphi(i\gamma_{j+1}, b; j)^2 - 2\gamma_{j+1} \int_0^b dy \varphi(i\gamma_{j+1}, y; j)^2}. \tag{3.19}$$

Note that the second condition in (3.19) implies that we must have

$$\frac{2\gamma_{j+1}}{\varphi(i\gamma_{j+1}, b; j)^2 - 2\gamma_{j+1} \int_0^b dy \varphi(i\gamma_{j+1}, y; j)^2} > 0. \tag{3.20}$$

When (3.19) is satisfied, the resulting potential $V(\cdot; j + 1)$ belongs to class \mathcal{A} .

Proof. In order to prove our theorem, from (3.2) we see that it is enough to prove that (3.19) is equivalent to $F(i\gamma_{j+1}; j + 1) = 0$ and that

$$\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j)^2}{1 + g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j)^2} = c_1, \quad x \geq b, \tag{3.21}$$

for some constant c_1 . In fact, from (3.11) we know that the value of c_1 must be $4\gamma_{j+1}$. We first show that (3.19) holds with $c_1 = 4\gamma_{j+1}$ there. For this, we proceed as follows. Because $k = i\gamma_{j+1}$ corresponds to a bound state, we have $F(i\gamma_{j+1}; j + 1) = 0$. By Theorem 2.1, we know that $F(k; j)$ is entire in k , and hence from (3.3) we see that we must have $F(-i\gamma_{j+1}; j) = 0$. Since $V(x; j) = 0$ for $x > b$, by (2.4) the corresponding Jost solution is given by $f(k, x; j) = e^{ikx}$ for $x \geq b$. Using $F(-i\gamma_{j+1}; j) = 0$ in (2.7) we get

$$\varphi(i\gamma_{j+1}, x; j)^2 = \mp \frac{1}{4\gamma_{j+1}^2} F(i\gamma_{j+1}; j)^2 e^{-2\gamma_{j+1}x}, \quad x \geq b,$$

where the upper sign refers to the non-Dirichlet case $\theta_j \in (0, \pi)$ and the lower sign to the Dirichlet case $\theta_j = \pi$. Thus, (3.21) is satisfied provided we have

$$4\gamma_{j+1} = \frac{\mp \frac{g_{j+1}^2}{2\gamma_{j+1}^2} F(i\gamma_{j+1}; j)^2 e^{-2\gamma_{j+1}x}}{1 + g_{j+1}^2 \int_0^b dy \varphi(i\gamma_{j+1}, y; j)^2 - \frac{g_{j+1}^2}{2\gamma_{j+1}} \varphi(i\gamma_{j+1}, b; j)^2 \mp \frac{g_{j+1}^2}{8\gamma_{j+1}^3} F(i\gamma_{j+1}; j)^2 e^{-2\gamma_{j+1}x}}. \tag{3.22}$$

After cross multiplying and simplifying, we see that (3.22) is equivalent to

$$1 + g_{j+1}^2 \int_0^b dy \varphi(i\gamma_{j+1}, y; j)^2 - \frac{g_{j+1}^2}{2\gamma_{j+1}} \varphi(i\gamma_{j+1}, b; j)^2 = 0,$$

which is satisfied because of the second equality in (3.19). Let us now prove the converse, namely, prove that $V(x; j + 1) = 0$ for $x > b$ implies (3.19). From (3.2) and the fact that $V(\cdot; j) \in \mathcal{A}$ we know that $V(x; j + 1) = 0$ for $x > b$ if and only if (3.21) holds with $c_1 = 4\gamma_{j+1}$ there, i.e.,

$$\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j)^2}{1 + g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j)^2} = 4\gamma_{j+1}, \quad x \geq b. \tag{3.23}$$

Evaluating (3.23) at $x = b$, we get the second equality in (3.19). Let us cross multiply in (3.23) and then take the x -derivative of both sides of the resulting equation. We get

$$4g_{j+1}^2 \varphi'(i\gamma_{j+1}, x; j) \varphi(i\gamma_{j+1}, x; j) = 4g_{j+1}^2 \gamma_{j+1} \varphi(i\gamma_{j+1}, x; j)^2, \quad x \geq b,$$

or equivalently

$$\varphi'(i\gamma_{j+1}, x; j) = \gamma_{j+1} \varphi(i\gamma_{j+1}, x; j), \quad x \geq b. \tag{3.24}$$

From (3.24), we see that

$$\varphi(i\gamma_{j+1}, x; j) = c_2 e^{\gamma_{j+1}x}, \quad x \geq b, \tag{3.25}$$

for some constant c_2 . On the other hand, with the help of (2.4) and (2.7) we get for $x \geq b$

$$\varphi(i\gamma_{j+1}, x; j) = \begin{cases} \frac{1}{2i\gamma_{j+1}} [F(i\gamma_{j+1}; j) e^{\gamma_{j+1}x} - F(-i\gamma_{j+1}; j) e^{-\gamma_{j+1}x}], & \theta \in (0, \pi), \\ \frac{1}{2\gamma_{j+1}} [F(i\gamma_{j+1}; j) e^{\gamma_{j+1}x} - F(-i\gamma_{j+1}; j) e^{-\gamma_{j+1}x}], & \theta = \pi. \end{cases} \tag{3.26}$$

Comparing (3.25) and (3.26), we see that we must have $F(-i\gamma_{j+1}; j) = 0$. When (3.22) is satisfied, the potential $V(\cdot; j + 1)$ belongs to \mathcal{A} because the quantity inside the brackets in the second term on the right-hand side of (3.2) is real valued and continuous in x when $x \in [0, b]$. ■

The result in the preceding theorem is fascinating in the sense that if we add a bound state to the compactly supported potential $V(\cdot; j)$ in class \mathcal{A} at $k = i\gamma_{j+1}$ with some arbitrary Gel'fand-Levitan norming constant g_{j+1} , in general the resulting potential $V(\cdot; j + 1)$ cannot be compactly supported. Theorem 3.5 states that the potential $V(\cdot; j + 1)$ is compactly supported if and only if $k = -i\gamma_{j+1}$ happens to be a zero of $F(k; j)$ and the norming constant g_{j+1} happens to be equal to the square root of the quantity on the right-hand side of the second equality in (3.19). Thus, if the left-hand side in (3.20) does not yield a positive number, then it is impossible for $V(\cdot; j + 1)$ to have the support in $[0, b]$ because there cannot be a corresponding positive norming constant g_{j+1} guaranteeing the compact support for the potential. Let us clarify that, if the left-hand side in (3.20) is not positive, one can find a potential with support in $[0, b]$, but such a potential must have a singularity and it cannot belong to class \mathcal{A} .

The result of Theorem 3.5 is analogous to the result⁹ from the full-line Schrödinger equation when a bound state is added to a compactly supported potential: Start with a compactly supported potential V associated with the transmission coefficient T and add a bound state to it at $k = i\kappa$ to obtain the potential \tilde{V} with the transmission coefficient \tilde{T} given by

$$\tilde{T}(k) = \frac{k + i\kappa}{k - i\kappa} T(k).$$

Then, \tilde{V} is also compactly supported if and only if the transmission coefficient $T(k)$ has a pole at $k = -i\kappa$. The analysis in the full-line case is less complicated due to the fact that in the full-line case there is no boundary condition at $x = 0$ such as (2.3).

In Theorem 3.5, in terms of $F(k; j)$ and $\varphi(k, x; j)$, we have expressed the necessary and sufficient conditions for the support of the potential $V(\cdot; j + 1)$ to be confined within the compact support of $V(\cdot; j)$. In the next theorem, the two conditions stated in (3.19) are expressed in terms of $F(k; j + 1)$ and $\varphi(k, x; j + 1)$.

Theorem 3.6. *Let $V(\cdot; j) \in \mathcal{A}$ for some fixed $j \geq 0$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$, where $j = 0$ corresponds to the absence of bound states. Assume that a bound state at $k = i\gamma_{j+1}$ is added to the spectrum with the Gel'fand-Levitan norming constant g_{j+1} , but otherwise the relevant spectral data set is unchanged. Let b be the constant appearing in (2.2) related to the compact support of $V(\cdot; j)$. The support of $V(\cdot; j + 1)$ is also confined to $[0, b]$ if and only if*

$$F(i\gamma_{j+1}; j + 1) = 0, \quad g_{j+1}^2 = \frac{2\gamma_{j+1}}{\varphi(i\gamma_{j+1}, b; j + 1)^2 + 2\gamma_{j+1} \int_0^b dt \varphi(i\gamma_{j+1}, t; j + 1)^2}. \quad (3.27)$$

Proof. The equivalence of $F(i\gamma_{j+1}; j + 1) = 0$ and $F(-i\gamma_{j+1}; j) = 0$ is already shown in the proof of Theorem 3.5. Let us now prove that the second equality in (3.19) is equivalent to the second equality in (3.27). From (3.5), we see that

$$V(x; j + 1) = V(x; j) - \frac{d}{dx} \left[\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j + 1)^2}{1 - g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j + 1)^2} \right]. \quad (3.28)$$

A comparison with (3.2) shows that the right-hand sides of (3.2) and of (3.28) are equal to each other for $x > b$, and we have

$$\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j + 1)^2}{1 - g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j + 1)^2} = \frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j)^2}{1 + g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j)^2} + c_3, \quad x \geq b, \quad (3.29)$$

for some constant c_3 . Using (3.23) on the right-hand side of (3.29), we get

$$\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j + 1)^2}{1 - g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j + 1)^2} = 4\gamma_{j+1} + c_3, \quad x \geq b. \tag{3.30}$$

Cross multiplying in (3.30) and then taking the x -derivative of the resulting equation, for $x \geq b$ we obtain

$$4g_{j+1}^2 \varphi'(i\gamma_{j+1}, x; j + 1) \varphi(i\gamma_{j+1}, x; j + 1) = -(4\gamma_{j+1} + c_3) g_{j+1}^2 \varphi(i\gamma_{j+1}, x; j + 1)^2,$$

which simplifies to

$$\varphi'(i\gamma_{j+1}, x; j + 1) = -\left(\gamma_{j+1} + \frac{c_3}{4}\right) \varphi(i\gamma_{j+1}, x; j + 1), \quad x \geq b. \tag{3.31}$$

On the other hand, since $V(x; j + 1) = 0$ for $x > b$, we have the analog of (3.16) given by

$$\varphi(i\gamma_{j+1}, x; j + 1)^2 = \mp \frac{1}{4\gamma_{j+1}^2} F(-i\gamma_{j+1}; j)^2 e^{-2\gamma_{j+1}x}, \quad x \geq b, \tag{3.32}$$

where the upper sign refers to the non-Dirichlet case $\theta_{j+1} \in (0, \pi)$ and the lower sign to the Dirichlet case $\theta_{j+1} = \pi$. Comparing (3.31) and (3.32), we get $c_3 = 0$, and hence (3.30) yields

$$\frac{2g_{j+1}^2 \varphi(i\gamma_{j+1}, b; j + 1)^2}{1 - g_{j+1}^2 \int_0^b dy \varphi(i\gamma_{j+1}, y; j + 1)^2} = 4\gamma_{j+1}. \tag{3.33}$$

By isolating g_{j+1}^2 to one side of the equation in (3.33), we observe from (3.29) and (3.33) that the second equality in (3.19) is equivalent to the second equality in (3.27). ■

We can ask whether we can predict if (3.20) is satisfied without actually evaluating the left-hand side in (3.20). For this purpose, we will exploit the signs of $\varphi(i\gamma_{j+1}, x; j)$ and $\varphi(i\gamma_{j+1}, x; j + 1)$ as $x \rightarrow +\infty$. It is convenient to define

$$H(\beta; j) := \begin{cases} -i F(i\beta; j), & \theta_j \in (0, \pi), \\ F(i\beta; j), & \theta_j = \pi, \end{cases} \tag{3.34}$$

where $F(k; j)$ is the Jost function corresponding to the potential $V(\cdot; j)$ and the boundary parameter θ_j . The advantage of using $H(\beta; j)$ rather than $F(i\beta; j)$ is that the former is real valued and hence its sign can be examined graphically. Note that

$$H'(\beta; j) := \frac{dH(\beta; j)}{d\beta} = \begin{cases} \left. \frac{dF(k; j)}{dk} \right|_{k=i\beta}, & \theta_j \in (0, \pi), \\ i \left. \frac{dF(k; j)}{dk} \right|_{k=i\beta}, & \theta_j = \pi. \end{cases} \tag{3.35}$$

Note also that, as seen from (2.25), as $\beta \rightarrow +\infty$ we have

$$H(\beta; j) = \begin{cases} \beta + O(1), & \theta_j \in (0, \pi), \\ 1 + O\left(\frac{1}{\beta}\right), & \theta_j = \pi, \end{cases} \tag{3.36}$$

and hence $H(\beta; j)$ is positive for large positive β -values.

The result in the following theorem can be used as a test to determine whether the inequality in (3.20) is satisfied or not.

Theorem 3.7. *Let $V(\cdot; j) \in \mathcal{A}$ for some fixed $j \geq 0$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$, where $j = 0$ corresponds to the absence of bound states. Let $F(k; j)$ be the corresponding Jost function defined in (2.6), $H(\beta; j)$ be the quantity defined in (3.34), and b be the*

constant appearing in (2.2). Assume that a bound state at $k = i\gamma_{j+1}$ is added to the spectrum, but otherwise the relevant spectral data set is unchanged. The support of $V(\cdot; j + 1)$ is also confined to the interval $[0, b]$ if and only if

$$F(-i\gamma_{j+1}; j) = 0, \quad \frac{i \dot{F}(-i\gamma_{j+1}; j)}{F(i\gamma_{j+1}; j)} > 0, \tag{3.37}$$

or equivalently, if and only if

$$H(-\gamma_{j+1}; j) = 0, \quad \frac{H'(-\gamma_{j+1}; j)}{H(\gamma_{j+1}; j)} > 0. \tag{3.38}$$

Proof. The equivalence of (3.37) and (3.38) is obtained directly by using (3.34) and (3.35). Thus, we only need to show that (3.37) is equivalent to the first condition given in (3.19) and the condition in (3.20). In other words, we need to prove that (3.37) is equivalent to

$$F(-i\gamma_{j+1}; j) = 0, \tag{3.39}$$

and to the positivity of the right-hand side in the equality involving g_{j+1}^2 in (3.19). Note that (3.39) appears also in (3.19) and hence we only need to show the equivalence of the inequality in (3.37) and the positivity of the relevant quantity. Using (3.39) in (3.26) we see that, for $x \geq b$, we have

$$\varphi(i\gamma_{j+1}, x; j) = \begin{cases} \frac{1}{2i\gamma_{j+1}} F(i\gamma_{j+1}; j) e^{\gamma_{j+1}x}, & \theta_j \in (0, \pi), \\ \frac{1}{2\gamma_{j+1}} F(i\gamma_{j+1}; j) e^{\gamma_{j+1}x}, & \theta_j = \pi. \end{cases} \tag{3.40}$$

Using (3.39) we can write (3.3) as

$$F(k, j + 1) = (k - i\gamma_{j+1}) \frac{F(k; j) - F(-i\gamma_{j+1}; j)}{k + i\gamma_{j+1}}. \tag{3.41}$$

Letting $k \rightarrow -i\gamma_{j+1}$, from (3.41), as a result of the analyticity of $F(k; j)$ in \mathbb{C} we obtain

$$F(-i\gamma_{j+1}, j + 1) = -2i\gamma_{j+1} \dot{F}(-i\gamma_{j+1}; j), \tag{3.42}$$

where we recall that an overdot indicates the k -derivative. With the help of (2.7), let us now evaluate $\varphi(i\gamma_{j+1}, x; j + 1)$. Using (2.4) in (2.7), for $x \geq b$ we obtain

$$\varphi(i\gamma_{j+1}, x; j + 1) = \begin{cases} \frac{1}{2i\gamma_{j+1}} [F(i\gamma_{j+1}; j + 1) e^{\gamma_{j+1}x} - F(-i\gamma_{j+1}; j + 1) e^{-\gamma_{j+1}x}], \\ \frac{1}{2\gamma_{j+1}} [F(i\gamma_{j+1}; j) e^{\gamma_{j+1}x} - F(-i\gamma_{j+1}; j + 1) e^{-\gamma_{j+1}x}], \end{cases} \tag{3.43}$$

where the first line holds if $\theta_{j+1} \in (0, \pi)$ and the second line holds if $\theta_{j+1} = \pi$. From Theorem 3.6, we know that $F(i\gamma_{j+1}, j + 1) = 0$ and hence (3.43), for $x \geq b$, is equivalent to

$$\varphi(i\gamma_{j+1}, x; j + 1) = \begin{cases} -\frac{1}{2i\gamma_{j+1}} F(-i\gamma_{j+1}; j + 1) e^{-\gamma_{j+1}x}, & \theta_j \in (0, \pi), \\ -\frac{1}{2\gamma_{j+1}} F(-i\gamma_{j+1}; j + 1) e^{-\gamma_{j+1}x}, & \theta_j = \pi. \end{cases} \tag{3.44}$$

Using (3.42) in (3.44) we see that, for $x \geq b$, we have

$$\varphi(i\gamma_{j+1}, x; j + 1) = \begin{cases} \dot{F}(-i\gamma_{j+1}; j) e^{-\gamma_{j+1}x}, & \theta_j \in (0, \pi), \\ i \dot{F}(-i\gamma_{j+1}; j) e^{-\gamma_{j+1}x}, & \theta_j = \pi. \end{cases} \tag{3.45}$$

With the help of (3.1), we observe that $\theta_{j+1} \in (0, \pi)$ if and only if $\theta_j \in (0, \pi)$. Hence, from (3.40) and (3.45) we obtain

$$\frac{\varphi(i\gamma_{j+1}, x; j + 1)}{\varphi(i\gamma_{j+1}, x; j)} = 2i\gamma_{j+1} \frac{\dot{F}(-i\gamma_{j+1}; j)}{F(i\gamma_{j+1}; j)} e^{-\gamma_{j+1}x}, \quad \theta_j \in (0, \pi], \quad x \geq b. \tag{3.46}$$

From (3.46), we see that the inequality in (3.37) is satisfied if and only if the quantity on the left-hand side of (3.46) is positive for any $x \geq b$. Let us now evaluate that quantity. From (3.7), using $j + 1$ instead of j there and letting $k = i\gamma_{j+1}$ there, we obtain

$$\varphi(i\gamma_{j+1}, x; j) = \frac{\varphi(i\gamma_{j+1}, x; j + 1)}{1 - g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j + 1)^2}, \quad x \geq 0,$$

or equivalently

$$\frac{\varphi(i\gamma_{j+1}, x; j + 1)}{\varphi(i\gamma_{j+1}, x; j)} = 1 - g_{j+1}^2 \int_0^x dy \varphi(i\gamma_{j+1}, y; j + 1)^2, \quad x \geq 0. \tag{3.47}$$

From (3.18) it follows that

$$\frac{1}{g_{j+1}^2} = \int_0^\infty dy \varphi(i\gamma_{j+1}, y; j + 1)^2, \tag{3.48}$$

and hence using $\int_0^x = \int_0^\infty - \int_x^\infty$ in (3.47), with the help of (3.48) we get

$$\frac{\varphi(i\gamma_{j+1}, x; j + 1)}{\varphi(i\gamma_{j+1}, x; j)} = g_{j+1}^2 \int_x^\infty dy \varphi(i\gamma_{j+1}, y; j + 1)^2, \quad x \geq 0. \tag{3.49}$$

Comparing (3.49) with (3.46), we observe that the inequality in (3.37) is satisfied if and only if g_{j+1}^2 appearing in (3.49) is positive. From (3.19) and (3.20), we already know that (3.39) and the positivity of g_{j+1}^2 are equivalent for $V(\cdot; j + 1)$ to have support in $[0, b]$. Thus, we have proved that (3.19) is equivalent to

$$F(-i\gamma_{j+1}; j) = 0; \quad \frac{\varphi(i\gamma_{j+1}, x; j + 1)}{\varphi(i\gamma_{j+1}, x; j)} > 0, \quad x \geq b. \tag{3.50}$$

With the help of (3.46), we see that (3.50) is equivalent to (3.37). Hence, the proof is complete. \blacksquare

One consequence of Theorem 3.7 is that the scattering matrix corresponding to a half-line Schrödinger operator has a meromorphic extension with simple poles at the bound states.

Proposition 3.8. Let $V(\cdot; j) \in \mathcal{A}$ for some fixed $j \geq 0$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$, where $j = 0$ corresponds to the absence of bound states. Let $F(k; j)$ and $S(k; j)$ be the corresponding Jost function and the scattering matrix defined in (2.6) and (2.16), respectively. Assume that a bound state at $k = i\gamma_{j+1}$ is added to the spectrum without changing the support of the potential and without changing the remaining part of the spectral data set. Under the corresponding Darboux transformation, the scattering matrix is transformed as

$$S(k; j + 1) = \left(\frac{k + i\gamma_{j+1}}{k - i\gamma_{j+1}} \right)^2 S(k; j). \tag{3.51}$$

The scattering matrix $S(k; j)$ has a meromorphic extension from $k \in \mathbf{R}$ to the entire complex plane. The only poles of $S(k; j)$ in \mathbf{C}^+ occur at the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$ and such poles are all simple. Furthermore, $S(k; j)$ has simple zeros at $k = -i\gamma_s$ for $s = 1, \dots, j$.

Proof. The meromorphic extension of $S(k; j)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ has already been established in Theorem 2.1(e). We get (3.51) by using (3.3) in (2.16). Using induction, from (3.51) it is seen that it is enough to prove that $S(k; 0)$ has no poles in \mathbf{C}^+ and that $S(k; j + 1)$ has a simple pole at $k = i\gamma_{j+1}$ and has a simple zero at $k = -i\gamma_{j+1}$. Note that $S(k; 0)$ has no poles in \mathbf{C}^+ , which follows from (2.16) and the fact that $F(k; 0)$ has no zeros in \mathbf{C}^+ . At first sight, (3.51) gives the wrong impression that $S(k; j + 1)$ has a double pole at $k = i\gamma_{j+1}$ and a double zero at $k = -i\gamma_{j+1}$. However, the pole at $k = i\gamma_{j+1}$ is a simple one and the zero at $k = -i\gamma_{j+1}$ is a simple one, as the following argument shows. Using (2.16), let us write (3.51) as

$$S(k; j + 1) = \mp \left(\frac{k + i\gamma_{j+1}}{k - i\gamma_{j+1}} \right) \left(\frac{F(-k; j)}{k - i\gamma_{j+1}} \right) \left(\frac{k + i\gamma_{j+1}}{F(k; j)} \right), \tag{3.52}$$

where the upper sign refers to the non-Dirichlet boundary condition $\theta_j \in (0, \pi)$ and the lower sign to the Dirichlet boundary condition $\theta_j = \pi$. From (3.37), we know that $F(-k; j)$ has a simple zero at $k = i\gamma_{j+1}$. Thus, the second factor on the right-hand side of (3.52) has a removable singularity at $k = i\gamma_{j+1}$ and no zero at $k = i\gamma_{j+1}$. Similarly, the third factor on the right-hand side of (3.52) has a removable singularity at $k = -i\gamma_{j+1}$ and no zero at $k = -i\gamma_{j+1}$. We also know that $F(k; j)$ in the third factor cannot vanish at $k = i\gamma_{j+1}$ because we already have $F(-i\gamma_{j+1}; j) = 0$ as a result of the fact that $F(-k; j)$ and $F(k; j)$ cannot vanish at the same k -value. Thus, the product of the second and third factors on the right-hand side of (3.52) does not have a pole at $k = i\gamma_{j+1}$ and that product does not have a zero at $k = -i\gamma_{j+1}$. Hence, the simple pole at $k = i\gamma_{j+1}$ in the first factor on the right-hand side of (3.52) is the only pole of $S(k; j + 1)$ at $k = i\gamma_{j+1}$ and that the simple zero at $k = -i\gamma_{j+1}$ in the first factor is the only zero of $S(k; j + 1)$ at $k = -i\gamma_{j+1}$. ■

It is useful to state the result of Theorem 3.7 in terms of the quantities associated with no bound states. Thus, we present the following result.

Theorem 3.9. *Let $V(\cdot; j) \in \mathcal{A}$ for some fixed $j \geq 0$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ_j and the bound states at $k = i\gamma_s$ for $s = 1, \dots, j$, where $j = 0$ corresponds to the absence of bound states. Let $H(\beta; j)$ be the quantity defined in (3.34) and b be the constant appearing in (2.2). Assume that a bound state at $k = i\gamma_{j+1}$ is added to the spectrum, but otherwise the spectral data set is unchanged. The support of $V(\cdot; j + 1)$ is also confined to $[0, b]$ if and only if*

$$H(-\gamma_{j+1}; 0) = 0, \quad H'(-\gamma_{j+1}; 0) > 0, \tag{3.53}$$

where we recall that $H(\beta; 0)$ refers to the quantity in (3.34) when there are no bound states.

Proof. From (3.3) and (3.34), we obtain

$$H(\beta; j) = H(\beta; 0) \prod_{s=1}^j \left(\frac{\beta - \gamma_s}{\beta + \gamma_s} \right). \tag{3.54}$$

Thus, through differentiation with respect to β , (3.54) yields

$$H'(\beta; j) = H(\beta; j) \sum_{s=1}^j \left(\frac{2\gamma_s}{\beta + \gamma_s} \right) + H'(\beta; 0) \prod_{s=1}^j \left(\frac{\beta - \gamma_s}{\beta + \gamma_s} \right). \tag{3.55}$$

From (3.54) and (3.55), we obtain

$$H(\gamma_{j+1}; j) = H(\gamma_{j+1}; 0) \prod_{s=1}^j \left(\frac{\gamma_{j+1} - \gamma_s}{\gamma_{j+1} + \gamma_s} \right), \tag{3.56}$$

$$H'(-\gamma_{j+1}; j) = H'(-\gamma_{j+1}; 0) \prod_{s=1}^j \left(\frac{\gamma_{j+1} + \gamma_s}{\gamma_{j+1} - \gamma_s} \right), \tag{3.57}$$

where we have used $H(-\gamma_{j+1}; j) = 0$ to get (3.57) from (3.55). From (3.56) and (3.57), we obtain

$$\frac{H'(-\gamma_{j+1}; j)}{H(\gamma_{j+1}; j)} = \frac{H'(-\gamma_{j+1}; 0)}{H(\gamma_{j+1}; 0)} \prod_{s=1}^j \left(\frac{\gamma_{j+1} + \gamma_s}{\gamma_{j+1} - \gamma_s} \right)^2. \tag{3.58}$$

Furthermore, from (3.36) and the fact that $F(k; 0)$ has no zeros on the positive imaginary axis, we know that $H(\beta; 0) > 0$ for $\beta > 0$. Thus, we see that (3.54) and (3.58) imply that (3.38) and (3.53) are equivalent. ■

One important consequence of Theorem 3.9 is that an ineligible resonance remains ineligible if a number of bound states are removed or added via Darboux transformations without changing the compact support of the potential. An examination of the graph of $H(\beta; j)$ or $H(\beta; 0)$ and the use of (3.38) or (3.53) reveal various facts about eligible and ineligible resonances. The following

proposition lists several such facts. We remind the reader that the meaning of the maximal number of eligible resonances is given in Sec. I.

Proposition 3.10. Let $V(\cdot; N) \in \mathcal{A}$ be the potential of the Schrödinger operator specified in (2.1)-(2.3) with the boundary parameter θ in (2.3) and N bound states at $k = i\gamma_j$ for $j = 1, \dots, N$, where we have $N = 0$ if there are no bound states. Let M and N_{inel} denote the maximal number of eligible resonances and the number of ineligible resonances, respectively, corresponding to the set $\{V(\cdot; N), \theta\}$. Let $H(\beta; N)$ be the quantity corresponding to the set $\{V(\cdot; N), \theta\}$, as defined in (3.34). We have the following:

- (a) The maximal number of eligible resonances corresponding to the set $\{V(\cdot; N), \theta\}$ is equal to the sum of the number eligible resonances and the number of bound states for $\{V(\cdot; N), \theta\}$.
- (b) The number of ineligible resonances for $\{V(\cdot; N), \theta\}$, i.e., the value of N_{inel} , remains unchanged if any number of bound states are removed or added via Darboux transformations without changing the compact support of the potential.
- (c) Between any two consecutive eligible resonances corresponding to $\{V(\cdot; N), \theta\}$, there must at least be one ineligible resonance.
- (d) We must have $M \leq 1 + N_{\text{inel}}$, and hence for $\{V(\cdot; N), \theta\}$ we must also have $N \leq 1 + N_{\text{inel}}$.
- (e) If there are at least two bound states associated with the set $\{V(\cdot; N), \theta\}$, then there must at least be one ineligible resonance.
- (f) If $k = -i\gamma$ corresponds to an imaginary resonance and if $H(\beta; N)$ has no zeros in the interval $\beta \in (-\gamma, \gamma)$, then $k = -i\gamma$ must correspond to an eligible resonance for the set $\{V(\cdot; N), \theta\}$.

Proof. The proof of (a) intuitively follows from the definition of the maximal number of eligible resonances, which is given in Sec. I. Here, we provide the technical details. Because $V(\cdot; N) \in \mathcal{A}$, by Theorem 2.1(a) the corresponding Jost function $F(k; N)$ is entire in $k \in \mathbf{C}$ and hence $H(\beta; N)$ appearing in (3.34) is a real-valued analytic function of $\beta \in \mathbf{R}$. By Theorems 2.1 and 3.3, it then follows that $H(\beta; s)$ is also a real-valued analytic function of $\beta \in \mathbf{R}$ for any $s = 0, 1, \dots, N$. By definition, $H(\beta; s)$ has exactly s zeros in the interval $\beta \in (0, +\infty)$, and by (3.53) we conclude that M is the number of zeros of $H(\beta; 0)$ in the interval $\beta \in (-\infty, 0)$ satisfying $H'(\beta; 0) > 0$. Thus, $H(\beta; N)$ is obtained from $H(\beta; 0)$ by converting N eligible resonances into bound states. Hence, $H(\beta; N)$ has exactly N bound states and $M - N$ eligible resonances, proving (a). From (3.53), it follows that an ineligible resonance for $\{V(\cdot; N), \theta\}$ corresponds to a zero of the associated $H(\beta; 0)$ in the interval $\beta \in (-\infty, 0)$ satisfying $H'(\beta; 0) \leq 0$. As bound states are added, no such zeros of $H(\beta; 0)$ are moved from the interval $(-\infty, 0)$ to the interval $(0, +\infty)$. Hence, (b) holds. Let us now consider (c) when there are no bound states so that we can use the eligibility criteria (3.53) of Theorem 3.9. In that case, $H(\beta; 0)$ is a real-valued analytic function of β in the interval $(-\infty, 0)$, and hence it is impossible to have two consecutive zeros of $H(\beta; 0)$ in the interval $(-\infty, 0)$ at which $H'(\beta; 0) > 0$. Thus, in the absence of bound states there has to be at least one ineligible resonance between two eligible resonances. As stated in the proof of (b), the ineligible resonances are unaffected if some eligible resonances are converted into bound states. Therefore, the process of adding bound states does not change the location of the ineligible resonances but only moves a number of eligible resonances into bound states. Hence, even in the presence of bound states, we must have at least one ineligible resonance between two consecutive eligible resonances, proving (c). Note that the first inequality in (d) directly follows from (c). By (a), we have $N \leq M$ and hence the second inequality in (d) is a consequence of the first inequality in (d). Note that (e) directly follows from the second inequality in (d) if we have $N \geq 2$. We prove (f) as follows. If $H(-\gamma; N) = 0$, we cannot have $H(\gamma; N) = 0$ because otherwise the corresponding regular solution $\varphi_\theta(k, x)$ given in (2.7) would have to be identically zero at $k = i\gamma$, contradicting (2.5). Furthermore, if $H(-\gamma; N) = 0$ and $H(\beta; N)$ has no zeros in the interval $\beta \in (-\gamma, \gamma]$, then $H'(-\gamma; N)$ and $H(\gamma; N)$ must have the same sign. Hence, (3.38) implies that $k = -i\gamma$ is an eligible resonance. ■

IV. RECOVERY FROM THE SCATTERING MATRIX

In this section, we assume that we are given a scattering matrix $S_\theta(k)$ for $k \in \mathbf{R}$ and we know that S_θ comes from a potential V in class \mathcal{A} and from a boundary parameter θ for some $\theta \in (0, \pi]$, where θ appears in (2.3). However, we do not know what V is and we do not know what the value of θ is. In fact, we do not even know whether $\theta = \pi$ or $\theta \in (0, \pi)$. In other words, we are only given the continuous part of the Marchenko data specified in (2.15) and we only know the existence of V in \mathcal{A} and the existence of $\theta \in (0, \pi]$. In this section, we have two main goals. Our first main goal is to determine whether S_θ uniquely determines both V and θ . Our second main goal is to reconstruct V and θ in the case of uniqueness, or to reconstruct all possible sets $\{V, \theta\}$ corresponding the same scattering matrix S_θ in the case of nonuniqueness.

To help the reader to understand better the theory developed in this section, we first summarize our findings:

- (i) If the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ has at least one pole on the positive imaginary axis, then S_θ uniquely determines V and θ . We present an explicit algorithm to reconstruct the corresponding V and θ from S_θ .
- (ii) If the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ has no poles on the positive imaginary axis and we have $S_\theta(0) = -1$, then S_θ uniquely determines V and θ . We present an explicit algorithm to reconstruct the corresponding V and θ from S_θ .
- (iii) If the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ has no poles on the positive imaginary axis and we have $S_\theta(0) = +1$, then there are precisely two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ corresponding to the same S_θ . We have $\theta_1 = \pi$ and $\theta_2 = \pi/2$, and the potentials V_1 and V_2 correspond to some full-line reflection coefficients $R(k)$ and $-R(k)$, respectively. Neither of the two corresponding full-line Schrödinger operators have any bound states, and they are both exceptional in the sense that $R(0) \neq -1$. We present an algorithm to reconstruct the sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$.

We already know from Theorem 2.1(f) that $S_\theta(0)$ must be either -1 or $+1$. Thus, the three cases listed above cover all possible scenarios. Having summarized our findings, we now present the theory yielding the results in (i), (ii), and (iii), starting with case (i).

Case (i) Given $S_\theta(k)$ for $k \in \mathbf{R}$, by the uniqueness of the meromorphic extension, the poles of $S_\theta(k)$ on the positive imaginary axis are uniquely determined. We already know from Theorem 2.1(e) that such poles must be simple. Let us assume that there are N such poles and they occur at $k = i\gamma_s$ for $s = 1, \dots, N$. For the unique reconstruction of V and θ , we proceed as follows:

- (a) We record the set $\{\gamma_1, \dots, \gamma_N\}$ as input to the Marchenko method in (2.18)-(2.20) toward the identification of the bound states.
- (b) Next, we evaluate the residues $\text{Res}(S_\theta, i\gamma_s)$ for $s = 1, \dots, N$; i.e., we uniquely determine the residue of $S_\theta(k)$ at each bound-state pole at $k = i\gamma_s$. We then look at the sign of $i \text{Res}(S_\theta, i\gamma_s)$ for any one value of s . With the help of (2.26), if that sign is positive then we conclude that $\theta = \pi$, and if that sign is negative then we conclude that $\theta \in (0, \pi)$.
- (c) From the previous step, we know whether we have $\theta = \pi$ or $\theta \in (0, \pi)$. Then, we use the appropriate line in (2.18) and the corresponding set $\{S_\theta, \{\gamma_s, m_s\}_{j=s}^N\}$ in the Marchenko procedure outlined in Sec. II and we uniquely determine V as in (2.20). In case $\theta \in (0, \pi)$, we use (2.24) to determine the value of θ .

Case (ii) Given $S_\theta(k)$ for $k \in \mathbf{R}$ with $S_\theta(0) = -1$ and with the further knowledge that the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ does not have any poles on the positive imaginary axis, we proceed as follows. From the Marchenko theory outlined in (2.18)-(2.24), we see that we only need to know whether we have $\theta = \pi$ or $\theta \in (0, \pi)$. This is because we will use either the first line or the second line of (2.18), but without the summation terms in those lines, as input to the corresponding Marchenko equation. Thus, in the Marchenko equation (2.19), we have the Marchenko kernel and the nonhomogeneous term are determined up to a sign, depending on whether we have $\theta = \pi$ or $\theta \in (0, \pi)$. Let us assume that corresponding to S_θ , we have two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$. We cannot have both θ_1 and θ_2 equal to π because then the second line of (2.18) would yield

$V_1(x) \equiv V_2(x)$ via the Marchenko method. Similarly, we cannot have both θ_1 and θ_2 different from π because then the first line of (2.18) would yield $V_1(x) \equiv V_2(x)$. Thus, one of θ_1 and θ_2 must be equal to π and the other must be different from π . Without loss of any generality, we can assume that $\theta_1 = \pi$ and $\theta_2 \in (0, \pi)$. Let us use $f_1(k, 0)$ to denote the Jost function corresponding to $\{V_1, \theta_1\}$ and use $F_2(k)$ to denote the Jost function corresponding to $\{V_2, \theta_2\}$. Because $S_\theta(0) = -1$, from (2.27) and (2.28) it follows that we must have $f_1(0, 0) = 0$ and $F_2(0) \neq 0$. From Theorem 2.1, we know that $k = 0$ must be a simple zero of $f_1(k, 0)$ and hence we have $f_1(k, 0) = k h_1(k)$ for some function $h_1(k)$ in such a way that $h_1(k)$ is analytic and nonzero in $k \in \mathbf{C}^+$ and $h_1(k) = 1/k + O(1/k^2)$ as $k \rightarrow \infty$ in $k \in \mathbf{C}^+$. Similarly, from Theorem 2.1 we know that $F_2(k)$ is analytic and nonzero in $k \in \mathbf{C}^+$ and $F_2(k) = k + O(1)$ as $k \rightarrow \infty$ in $k \in \mathbf{C}^+$. Since $f_1(k, 0)$ and $F_2(k)$ correspond to the same scattering matrix $S_\theta(k)$, because of (2.16) we must have

$$S_\theta(k) = \frac{f_1(-k, 0)}{f_1(k, 0)} = \frac{-F_2(-k)}{F_2(k)}, \quad k \in \mathbf{R}, \tag{4.1}$$

which implies

$$\frac{f_1(k, 0)}{F_2(k)} = \frac{-f_1(-k, 0)}{F_2(-k)}, \quad k \in \mathbf{R}. \tag{4.2}$$

Since $f_1(k, 0) = k h_1(k)$, we can write (4.2) also as

$$\frac{h_1(k)}{F_2(k)} = \frac{h_1(-k)}{F_2(-k)}, \quad k \in \mathbf{R}. \tag{4.3}$$

Note that the left-hand side of (4.3) has an analytic extension from $k \in \mathbf{R}$ to $k \in \mathbf{C}^+$, and that analytic extension is continuous in \mathbf{C}^+ and behaves as $O(1/k^2)$ as $k \rightarrow \infty$ in \mathbf{C}^+ . Similarly, the right-hand side of (4.3) has an analytic extension from $k \in \mathbf{R}$ to $k \in \mathbf{C}^-$, and that analytic extension is continuous in \mathbf{C}^- and behaves as $O(1/k^2)$ as $k \rightarrow \infty$ in \mathbf{C}^- . Thus, $h_1(k)/F_2(k)$ must be an entire function of k and behaving like $O(1/k^2)$ as $k \rightarrow \infty$ in \mathbf{C} . By Liouville’s theorem, we must then have $h_1(k) \equiv 0$. However, that would imply $f_1(k, 0) \equiv 0$, contradicting the second line of (2.27). Thus, we cannot have both $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ corresponding to the same $S_\theta(k)$ and we must have a unique set $\{V, \theta\}$ corresponding to the input $S_\theta(k)$. Having established the uniqueness, let us now consider the reconstruction problem. As explained in Sec. II, we can use the Marchenko method for the reconstruction. We can first try the second line of (2.18) as input to the Marchenko equation with $\theta = \pi$ without the summation term there. We can construct the corresponding potential and Jost solution via (2.20) and can check if the right-hand side of (2.21) is zero at $k = 0$, which is required by the second line of (2.27). Alternatively, we can check if the right-hand side of (2.23) is equal to our scattering matrix $S_\theta(k)$. If there is no agreement, we then know that $\theta \in (0, \pi)$, and hence use the first line of (2.18) without the summation term there as input to the Marchenko equation and uniquely construct the corresponding V and θ via the first equality in (2.20) and by using (2.24), respectively.

Case (iii) Given $S_\theta(k)$ for $k \in \mathbf{R}$ with $S_\theta(0) = +1$ and with the further knowledge that the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}$ does not have any poles on the positive imaginary axis, we proceed as follows. As in case (ii), from the Marchenko theory it follows that it is enough to check the nonuniqueness by assuming that, corresponding to S_θ , we have two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ with $\theta_1 = \pi$ and $\theta_2 \in (0, \pi)$. Contrary to case (ii), we will now prove that there are precisely two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ corresponding to the same $S_\theta(k)$. We again use $f_1(k, 0)$ to denote the Jost function corresponding to $\{V_1, \theta_1\}$ and use $F_2(k)$ to denote the Jost function corresponding to $\{V_2, \theta_2\}$. Let us use $f_2(k, x)$ to denote the Jost solution corresponding to V_2 . From (2.6), we have

$$F_2(k) = -i [f_2'(k, 0) + (\cot \theta_2) f_2(k, 0)]. \tag{4.4}$$

This time, from (2.27) and (2.28) it follows that $f_1(0, 0) \neq 0$ and $F_2(0) = 0$. From Theorem 2.1(a), we know that $k = 0$ must be a simple zero of $F_2(k)$, and hence we have $F_2(k) = k g_2(k)$ for some function $g_2(k)$ in such a way that $g_2(k)$ is analytic and nonzero in $k \in \mathbf{C}^+$ and $g_2(k) = 1 + O(1/k)$ as $k \rightarrow \infty$ in $k \in \mathbf{C}^+$. Similarly, from Theorem 2.1 we know that $f_1(k, 0)$ is analytic and nonzero in $k \in \mathbf{C}^+$ and $f_1(k, 0) = 1 + O(1/k)$ as $k \rightarrow \infty$ in $k \in \mathbf{C}^+$. Since $f_1(k, 0)$ and $F_2(k)$ correspond to the

same scattering matrix $S_\theta(k)$, we must have (4.1) and (4.2) satisfied. Since $F_2(k) = k g_2(k)$, we can write (4.2) also as

$$\frac{f_1(k,0)}{g_2(k)} = \frac{f_1(-k,0)}{g_2(-k)}, \quad k \in \mathbf{R}. \quad (4.5)$$

Note that the left-hand side of (4.5) has an analytic extension from $k \in \mathbf{R}$ to $k \in \mathbf{C}^+$, and that analytic extension is continuous in $\overline{\mathbf{C}^+}$ and behaves as $1 + O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Similarly, the right-hand side of (4.5) has an analytic extension from $k \in \mathbf{R}$ to $k \in \mathbf{C}^-$, and that analytic extension is continuous in $\overline{\mathbf{C}^-}$ and behaves like $1 + O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^-}$. Thus, we must have $f_1(k,0)/g_2(k)$ entire and behaving like $1 + O(1/k)$ as $k \rightarrow \infty$ in \mathbf{C} . By Liouville's theorem, we must then have $g_2(k) \equiv f_1(k,0)$, or equivalently we must have

$$F_2(k) \equiv k f_1(k,0). \quad (4.6)$$

Since there are no poles of $S_\theta(k)$ on the positive imaginary axis in \mathbf{C}^+ , it follows that the Marchenko kernel, which we call $M_1(y)$, corresponding to the first set $\{V_1, \theta_1\}$ with $\theta_1 = \pi$ is given by the second line of (2.18) but without the summation term there. Then, the Marchenko kernel, which we call $M_2(y)$, corresponding to the second set $\{V_2, \theta_2\}$ with $\theta_2 \in (0, \pi)$ is given by the first line of (2.18) but without the summation term there. From (2.18), it is clear that $M_2(y) = -M_1(y)$ for $y > 0$. Let us now view V_1 and V_2 as compactly supported potentials in the full-line Schrödinger equation with $V_1(x) = 0$ for $x < 0$ and $V_2(x) = 0$ for $x < 0$. As in (2.30), let us associate the scattering coefficients T_1, L_1, R_1 with V_1 and associate the scattering coefficients T_2, L_2, R_2 with V_2 . Since $S_\theta(k)$ has no poles on the positive imaginary axis, we know that $N_{\theta_2} = 0$ and $N_\pi = 0$, where N_{θ_2} and N_π denote the number of bound states corresponding to $\{V_2, \theta_2\}$ and $\{V_1, \theta_1\}$, respectively. From (4.6), we know that $F_2(0) = 0$, and hence Theorem 2.2(f) indicates that we cannot have $\theta_2 \in (0, \pi/2) \cup (\pi/2, \pi]$ and thus we must have $\theta_2 = \pi/2$, which yields $\cot \theta_2 = 0$. By Theorem 2.3, we then know that $\tilde{N} = 0$, i.e., neither $T_1(k)$ nor $T_2(k)$ has any poles on the positive imaginary axis. From Theorem 2.4, we then get $M_1(y) = \hat{R}_1(y)$ and $M_2(y) = \hat{R}_2(y)$ for $y > 0$ with $\hat{R}_1(y)$ and $\hat{R}_2(y)$ denoting the Fourier transforms as in (2.45). Since we already know that $M_2(y) \equiv -M_1(y)$, we then get $\hat{R}_2(y) \equiv -\hat{R}_1(y)$, and hence yielding $R_2(k) \equiv -R_1(k)$. Because $\tilde{N} = 0$, it then also follows that $T_1(k) \equiv T_2(k)$. From the characterization conditions^{8-10,15,16} for the full-line Schrödinger operators, we already know that if there exists $V_1 \in \mathcal{A}$ corresponding to R_1 and T_1 , we are assured the existence of $V_2 \in \mathcal{A}$ corresponding to $-R_1$ and T_1 by recalling that $R_1(0) \neq -1$ and that T_1 does not have any bound-state poles on the positive imaginary axis. Thus, we have established the existence of two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ with $\theta_1 = \pi$ and $\theta_2 = \pi/2$. Note that using $\cot \theta_2 = 0$ in (4.4), we get $F_2(k) = -i f_2'(k,0)$ and hence (4.6) indicates that

$$f_2'(k,0) = ik f_1(k,0).$$

One consequence of (4.6) is that we must have

$$\int_0^b dx V_1(x) = \int_0^b dx V_2(x). \quad (4.7)$$

We obtain (4.7) by expanding $F_2(k)$ with $\cot \theta_2 = 0$ with the help of the first line of (2.25) and by comparing it with the expansion of the right-hand side of (4.6) via the second line of (2.25). The potential V_1 can be reconstructed with the help of the second line of (2.18) without the summation term there. The potential V_1 is then obtained by solving (2.19) and using the first equality in (2.20). Similarly, V_2 can be reconstructed by using the first line of (2.18) without the summation term there. Thus, V_2 can be obtained by using (2.19) and (2.20).

We summarize our findings in this section in the following theorem.

Theorem 4.1. *Assume that we are given $S_\theta(k)$ for $k \in \mathbf{R}$ and we know that it comes from a potential V in class \mathcal{A} and from a boundary parameter θ for some $\theta \in (0, \pi]$, where θ appears in (2.3). We then have the following:*

- (a) *If $S_\theta(0) = +1$ and the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}^+$ has no poles on the positive imaginary axis, then there are precisely two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ with $\theta_1 =$*

$\pi, \theta_2 = \pi/2, V_1 \in \mathcal{A}$, and $V_2 \in \mathcal{A}$. The set $\{V_1, \theta_1\}$ corresponds to the Jost solution $f_1(k, x)$, and the corresponding Jost function $f_1(k, 0)$ satisfies $f_1(0, 0) \neq 0$. The Jost function $F_2(k)$ for the second set $\{V_2, \theta_2\}$ is equal to $k f_1(k, 0)$. Both sets can be uniquely reconstructed by the Marchenko procedure. The set $\{V_1, \theta_1\}$ is associated with some scattering coefficients R_1, L_1 , and T_1 in such a way that $T_1(0) \neq 0$ and that $T_1(k)$ does not have poles on the positive imaginary axis. The scattering coefficients R_1, L_1 , and T_1 are related to $f_1(k, 0)$ and $f_1'(k, 0)$ as in (2.30). The set $\{V_2, \theta_2\}$ is associated with the scattering coefficients R_2, L_2 , and T_2 where $R_2(k) \equiv -R_1(k)$, $L_2(k) \equiv -L_1(k)$, and $T_2(k) \equiv T_1(k)$. Although, in general the potentials V_1 and V_2 are distinct, their integrals have the same value, as seen from (4.7). The very special case $V_1(x) \equiv V_2(x)$ occurs when $R_1(k) \equiv 0, L_1(k) \equiv 0$, and $T_1(k) \equiv 1$, which yields $V_1(x) \equiv 0$ and $V_2(x) \equiv 0$.

- (b) If $S_\theta(0) \neq +1$ or the extension of $S_\theta(k)$ from $k \in \mathbf{R}$ to $k \in \mathbf{C}^+$ has at least one pole on the positive imaginary axis, then there is a unique potential $V \in \mathcal{A}$ and a unique boundary parameter θ in the interval $(0, \pi]$ corresponding to $S_\theta(k)$. The corresponding potential V and boundary parameter θ can be uniquely reconstructed by the Marchenko procedure outlined in Sec. II.

V. RECOVERY FROM ABSOLUTE VALUE OF THE JOST FUNCTION

Our goal in this section is to investigate the determination of a real-valued, integrable, compactly supported potential and a selfadjoint boundary condition from the input data consisting of the absolute value of the corresponding Jost function known for $k \in \mathbf{R}$. In other words, we assume that we only know the continuous part of the Gel'fand-Levitan spectral data given in (2.10) without having any explicit knowledge of its discrete part. Furthermore, we know that our input data set corresponds to a selfadjoint Schrödinger operator on the half line with a selfadjoint boundary condition at $x = 0$. However, we do not know if the boundary condition is Dirichlet or non-Dirichlet, and we do not know if there are any bound states and we do not know the number of bound states if there are any. In fact, we would like to determine all such characteristics from our input data set alone, if possible.

In this section, we use the notation introduced in Sec. III, namely, we use $\theta_j, V(x; j), \varphi(k, x; j)$, and $F(k; j)$ to denote the relevant quantities corresponding to the half-line Schrödinger operator with bound states at $k = i\gamma_1, \dots, i\gamma_j$, where the case $j = 0$ refers to the quantities with no bound states. Note that θ_j is the boundary parameter appearing in (2.3), $\varphi(k, x; j)$ is the regular solution in (2.5), $F(k; j)$ is the Jost function in (2.6), g_j is the Gel'fand-Levitan norming constant in (2.9), $G(x, y; j)$ is the Gel'fand-Levitan kernel appearing in (2.12) and (2.13), $A(x, y; j)$ is the solution in (2.14) to the Gel'fand-Levitan equation, and $H(\beta; j)$ is the quantity in (3.34).

Mathematically speaking, we consider the selfadjoint Schrödinger operator on the half line with the potential $V(\cdot; N)$ in class \mathcal{A} , the boundary parameter θ_N , the Jost function $F(k; N)$, the bound states at $k = i\gamma_s$ with the corresponding Gel'fand-Levitan norming constants g_s for $s = 1, \dots, N$, where N is a nonnegative integer. We assume that our input data set solely consists of $|F(k; N)|$ for $k \in \mathbf{R}$. We do not know the value of N , and we do not know anything about the set $\{\gamma_s, g_s\}_{s=1}^N$. We would like to investigate to what extent our input data set determines $N, \theta_N, \{\gamma_s, g_s\}_{s=1}^N$, and $V(\cdot; N)$. In other words, we know the existence of at least one potential V in class \mathcal{A} and the existence of one selfadjoint boundary parameter $\theta \in (0, \pi]$ corresponding to our input data, and we would like to investigate the uniqueness or nonuniqueness of the set $\{V, \theta\}$ by determining all potentials V in class \mathcal{A} and all boundary parameters θ in the interval $(0, \pi]$ corresponding to our input data set.

Our findings are summarized as follows: We can determine whether the boundary condition is Dirichlet or non-Dirichlet. We can determine all the corresponding potentials and boundary conditions, but the uniqueness is only up to the inclusion of the eligible resonances. Thus, if the maximal number of eligible resonances is zero, then we have the unique determination of the potential $V(\cdot; 0)$ and the boundary parameter θ_0 corresponding to our data. If the maximal number of eligible resonances is one, then we determine the two distinct sets $\{V(\cdot; 0), \theta_0\}$ and $\{V(\cdot; 1), \theta_1, \{\gamma_1, g_1\}\}$ corresponding to our input data. If the maximal number of eligible resonances is M , then we determine

that there is a 2^M -fold nonuniqueness and that any one of those 2^M sets corresponds to our input data. We remind the reader that the definition of the maximal number of eligible resonances is given in Sec. I.

As mentioned earlier, the number of imaginary resonances may be infinite, but under some mild additional assumptions²⁰ such as $V(x) \geq 0$, or $V(x) \leq 0$, in the vicinity of $x = b$, that number is guaranteed to be finite. We recall that b refers to the constant in (2.2) and related to the compact support of the potential V . Thus, under a mild additional assumption we are guaranteed that M , the maximal number of eligible resonances, is finite.

Having summarized our findings, let us now outline the method of determining all potentials and boundary conditions corresponding to our input data:

- (a) From our input data $|F(k; N)|$ for $k \in \mathbf{R}$, by using the asymptotic behavior in (2.11) we can tell whether the corresponding boundary parameter θ_N satisfies $\theta_N \in (0, \pi)$ or $\theta_N = \pi$.
- (b) From (3.3) and (3.6), it is clear that we have

$$|F(k; 0)| = |F(k; N)|, \quad k \in \mathbf{R}, \tag{5.1}$$

where $F(k; 0)$ is the Jost function corresponding to no bound states. Using the Gel'fand-Levitan procedure outlined in Sec. II, from $|F(k; 0)|$, which is equivalent to $|F(k; N)|$ as seen from (5.1), we uniquely construct $V(x; 0)$, θ_0 , and the regular solution $\varphi(k, x; 0)$. This is done, by first forming the Gel'fand-Levitan kernel as in (2.12) and (2.13) without the summation terms there, namely,

$$G(x, y; 0) := \begin{cases} \frac{1}{\pi} \int_{-\infty}^{\infty} dk \left[\frac{k^2}{|F(k; N)|^2} - 1 \right] (\cos kx)(\cos ky), & \theta_0 \in (0, \pi), \\ \frac{1}{\pi} \int_{-\infty}^{\infty} dk \left[\frac{1}{|F(k; N)|^2} - 1 \right] (\sin kx)(\sin ky), & \theta_0 = \pi. \end{cases}$$

Using $G(x, y; 0)$ in the corresponding Gel'fand-Levitan equation in (2.14), namely, in

$$A(x, y; 0) + G(x, y; 0) + \int_0^x dz A(x, z; 0) G(z, y; 0) = 0, \quad 0 < y < x,$$

we uniquely recover $A(x, y; 0)$, from which we get $V(x; 0)$ and θ_0 via

$$\begin{aligned} \cot \theta_0 &= -A(0, 0; 0), & \theta_0 \in (0, \pi), \\ V(x; 0) &= 2 \frac{dA(x, x; 0)}{dx}, & \theta_0 \in (0, \pi]. \end{aligned}$$

- (c) As a consequence of (5.1), we uniquely determine $F(k; 0)$ from our input data via^{7,8}

$$F(k; 0) = \begin{cases} k \exp \left(\frac{-1}{\pi i} \int_{-\infty}^{\infty} dt \frac{\log |t/F(t; N)|}{t - k - i0^+} \right), & \theta_N \in (0, \pi), \\ \exp \left(\frac{1}{\pi i} \int_{-\infty}^{\infty} dt \frac{\log |F(t; N)|}{t - k - i0^+} \right), & \theta_N = \pi, \end{cases} \tag{5.2}$$

where $i0^+$ indicates that the value for $k \in \mathbf{R}$ must be obtained as a limit from within \mathbf{C}^+ . Since $F(k; 0)$ has an analytic extension to the entire complex plane, we are assured that (5.2) holds for all $k \in \mathbf{C}$.

- (d) Having $F(k; 0)$ at hand for $k \in \mathbf{C}$, we construct the real-valued function $H(\beta; 0)$ defined in (3.34). We already know that $H(\beta; 0)$ does not have any zeros when $\beta > 0$. We can have $H(0; 0) \neq 0$ (generic case) or we can have $H(0; 0) = 0$ (exceptional case) with a simple zero of $H(\beta; 0)$ at $\beta = 0$. We then go ahead and determine all imaginary resonances, i.e., the zeros of $H(\beta; 0)$ when $\beta < 0$.
- (e) We then identify each imaginary resonance either as eligible or ineligible by using the eligibility criteria given in (3.53), namely, by finding all negative β -values satisfying

$$H(\beta; 0) = 0, \quad H'(\beta; 0) > 0. \tag{5.3}$$

Assuming that (5.3) is satisfied when $\beta = -\beta_s$ for $s = 1, \dots, M$, we uniquely determine the set $\{\beta_s\}_{s=1}^M$. Note that M is the maximal number of eligible resonances. We know that M may be

zero, a positive integer, or infinity. As mentioned previously, a mild additional assumption²⁰ guarantees the finiteness of M .

- (f) Each eligible resonance $k = -i\beta_s$ can be converted into a bound state by using the Darboux transformation formulas given in Theorem 3.1. Thus, it is possible to add N bound states, where N is an integer between 0 and M . We can choose N bound states at $k = i\gamma_s$ among the M possible choices $k = i\beta_s$ in $\binom{M}{N}$ ways, where $\binom{M}{N}$ denotes the binomial coefficient, which is equal to $M!/((N!)(M-N)!)$. Thus, as N takes all values between 0 and M , we find that we have 2^M distinct sets consisting of a potential and a boundary parameter, each corresponding to the same absolute value of the Jost function.

VI. EXPLICIT EXAMPLES

In this section, we illustrate our main results presented in Sections III-V with some explicit examples. The first example is provided to remind the reader that the boundary parameter θ appearing in (2.3) indeed affects the bound states and resonances, and in fact even the trivial potential can have a bound state or a resonance depending on the value of the boundary parameter θ appearing in (2.3).

Example 6.1. Assume that $V(x) \equiv 0$ in (2.1). The corresponding Jost function $F_\theta(k)$ is given by (2.8). Since $F_\pi(k)$ has no zeros in \mathbf{C} , there are no bound states and there are no resonances in the Dirichlet case $\theta = \pi$. Let us now consider the non-Dirichlet case with some fixed boundary parameter $\theta \in (0, \pi)$. Recall that the zeros of $F_\theta(k)$ in \mathbf{C}^+ correspond to the bound states and the zeros in \mathbf{C}^- correspond to the resonances. If $\cot \theta > 0$, then there is one bound state and there are no resonances. If $\cot \theta = 0$, then there are no bound states and there are no resonances. If $\cot \theta < 0$, then there are no bound states and there is one imaginary resonance. In fact, as a result of Proposition 3.10(f), $k = i \cot \theta$ is an eligible resonance when $\cot \theta < 0$. Thus, if $\cot \theta < 0$ we can add a bound state to $V(x) \equiv 0$ at $k = -i \cot \theta$, and if we choose the Gel'fand-Levitan bound-state norming constant g as in (3.19), i.e., with $g^2 = -2 \cot \theta$, then the transformed potential still vanishes everywhere, and hence the transformed potential and the original potential have the same (trivial) compact support. Note that such a choice is compatible with (3.1). Let us see what happens if we do not use $g^2 = -2 \cot \theta$ as our norming constant. With $f(k, x) = e^{ikx}$ and $F_\theta(k) = k - i \cot \theta$, using the first line in (2.7) we evaluate $\varphi_\theta(k, x)$ as

$$\varphi_\theta(k, x) = \frac{1}{2k} [(k - i \cot \theta) e^{-ikx} + (k + i \cot \theta) e^{ikx}].$$

If we add a bound state at $k = -i \cot \theta$ with the Gel'fand-Levitan norming constant g , then the quantity inside the brackets in (3.2) is given by the right-hand side in the following equation:

$$\frac{2g^2 \varphi_\theta(-i \cot \theta, x)^2}{1 + g^2 \int_0^x dy \varphi_\theta(-i \cot \theta, y)^2} = \frac{4g^2 \cot \theta}{-g^2 + (2 \cot \theta + g^2) e^{2x \cot \theta}}. \quad (6.1)$$

Thus, the choice $g^2 = -2 \cot \theta$ makes the right-hand side in (6.1) equal to the constant $-4 \cot \theta$, and hence the support of the potential is unchanged when we add the bound state at $k = -i \cot \theta$ with the norming constant $g = \sqrt{-2 \cot \theta}$. Any other choice for the norming constant g results in a potential with support on the entire half line.

Next, we provide some examples of eligible resonances when the potential and the boundary parameter are known.

Example 6.2. Let us assume that we are given the boundary parameter $\theta \in (0, \pi)$ and that V is the piecewise constant potential (potential barrier or potential well) given by

$$V(x) = \begin{cases} v, & 0 < x < 1, \\ 0, & x > 1, \end{cases} \quad (6.2)$$

where v is a constant parameter. With the help of (2.4)-(2.7) and (6.2) we can explicitly evaluate the regular solution $\varphi_\theta(k, x)$, the Jost solution $f(k, x)$, and the Jost function $F_\theta(k)$ and get

$$\varphi_\theta(k, x) = \begin{cases} \cosh \eta x - \cot \theta \frac{\sinh \eta x}{\eta}, & 0 \leq x \leq 1, \\ b_1 \cos k(x-1) + \frac{b_2}{k} \sin k(x-1), & 1 \leq x < +\infty, \end{cases} \quad (6.3)$$

$$f(k, x) = \begin{cases} e^{ik} \cosh \eta(x-1) + ik e^{ik} \frac{\sinh \eta(x-1)}{\eta}, & 0 \leq x \leq 1, \\ e^{ikx}, & 1 \leq x < +\infty, \end{cases}$$

$$F_\theta(k) = e^{ik}(k - i \cot \theta) \cosh \eta - e^{ik}(k \cot \theta - i\eta^2) \frac{\sinh \eta}{\eta}, \quad (6.4)$$

where we have defined

$$\eta := \sqrt{v - k^2}, \quad b_1 := \cosh \eta - \cot \theta \frac{\sinh \eta}{\eta}, \quad b_2 := \eta \sinh \eta - \cot \theta \cosh \eta.$$

Let us now analyze (6.2) for various values of v and $\cot \theta$. We use an overline on a digit to indicate a round off.

- (a) When $(v, \cot \theta) = (-10, 1)$, using (3.34) and (6.4) we obtain $H_\theta(\beta)$, plotted in the first graph of Figure 1. We observe from the graph of $H_\theta(\beta)$ that it has two positive zeros and one negative zero. Thus, there are two bound states occurring at $k = 0.76040\bar{9}i$ and $k = 3.2527\bar{3}i$ and that $F_\theta(k)$ has a simple zero at $k = -\gamma i$, where $\gamma = 2.8208\bar{4}$. From the graph of $H_\theta(\beta)$, we easily see that $H_\theta(\gamma) < 0$ and $H'_\theta(-\gamma) > 0$, and hence by (3.38) we conclude that $k = -\gamma i$ is an ineligible resonance and that it is impossible to add a bound state to V without changing the compact support property. Equivalently, using $b = 1$ for the constant b appearing in (2.2), with the help of (6.3) we evaluate the right-hand side of the second equality in (3.19) and hence obtain $g^2 = -4.2376\bar{1}$. Thus, we confirm that $k = -\gamma i$ is an ineligible resonance because (3.20) is not satisfied. The same conclusion can also be reached via Proposition 3.10(e) because we have precisely two bound states and one imaginary resonance and hence that imaginary resonance must be ineligible.
- (b) When $(v, \cot \theta) = (-0.2, 6)$, the plot of $H_\theta(\beta)$, given as the second graph in Figure 1, reveals that $H_\theta(\beta)$ has one positive zero and two negative zeros. Thus, there is a bound state at $k = 6.0166\bar{4}i$ and that $F_\theta(k)$ has simple zeros at $k = -\gamma_1 i$ and $k = -\gamma_2 i$, where $\gamma_1 = 3.3618\bar{2}$ and $\gamma_2 = 5.9584\bar{2}$. From the graph of $H_\theta(\beta)$, we easily see that $H_\theta(\gamma_2) < 0$ and $H'_\theta(-\gamma_2) > 0$, and hence $k = -\gamma_2 i$ is an ineligible resonance, as indicated by the criteria in (3.38). On the other hand, $H_\theta(\gamma_1) < 0$ and $H'_\theta(-\gamma_1) < 0$, so that $k = -\gamma_1 i$ is an eligible resonance because of the criteria in (3.38). In fact, from the second equality in (3.19), using $b = 1$ and $\gamma = \gamma_1$ we get $g^2 = g_1^2 > 0$ with $g_1^2 = 1.9320\bar{9}$. Thus, we can add a bound state to V at $k = i\gamma_1$ with the Gel'fand-Levitan norming constant $g_1 = 1.39$ and the resulting potential has also support in the interval $[0, 1]$.
- (c) When $(v, \cot \theta) = (0.003521, -3)$, from the plot of $H_\theta(\beta)$ given as the third graph in Figure 1 we observe that $H_\theta(\beta)$ has no positive zeros and has a double zero at a negative β -value. Thus, there are no bound states and $F_\theta(k)$ has a double zero at $k = -\gamma i$, where $\gamma = 3.620\bar{5}$. We have

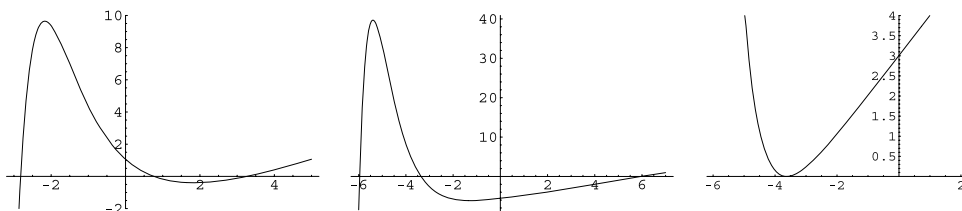


FIG. 1. The plots of $H_\theta(\beta)$ versus β in Examples 6.2(a), 6.2(b), and 6.2(c), respectively.

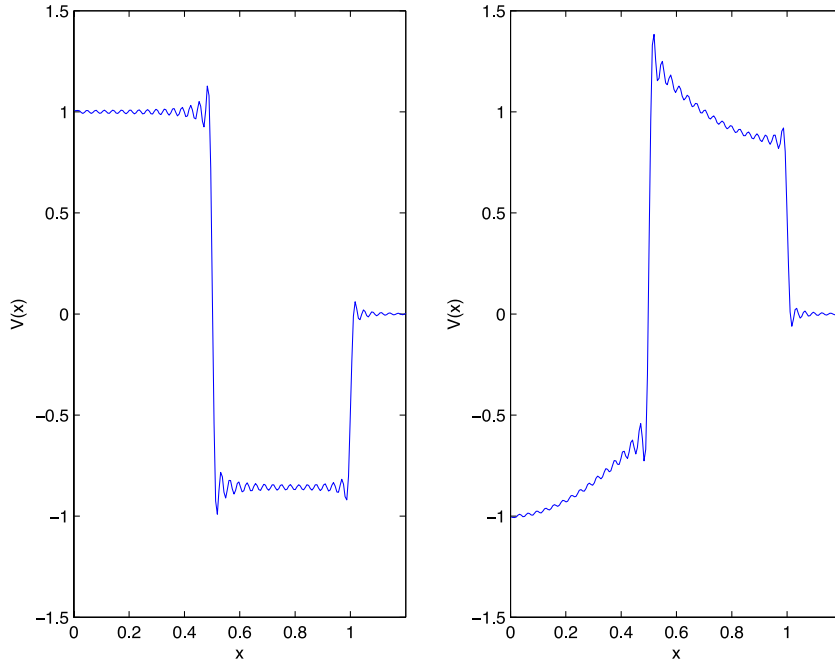


FIG. 2. The numerically reconstructed potentials V_1 and V_2 in Example 6.3 corresponding to L_1 and $-L_1$, respectively.

$H_\theta(\gamma) > 0$ and $H'_\theta(-\gamma) = 0$. Thus, the incompatibility with (3.38) shows that we cannot add any bound states to V without changing the compact support property.

In our final example, we elaborate on the nonuniqueness in the special case, case (iii) of Sec. IV, and present two distinct sets $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ corresponding to the same scattering matrix S .

Example 6.3. As stated in Theorem 4.1(a), we note that $\{V_1, \theta_1\}$ and $\{V_2, \theta_2\}$ with $V_1(x) \equiv 0$, $\theta_1 = \pi$, $V_2(x) \equiv 0$, $\theta_2 = \pi/2$ yield the same scattering matrix $S_\theta(k) \equiv 1$, as seen from (2.8) and (2.16), illustrating the double nonuniqueness indicated in Sec. IV. We now present a less trivial example of nonuniqueness by using the potential

$$V_1(x) = \begin{cases} 1, & 0 < x < 1, \\ -a, & \frac{1}{2} < x < 1, \\ 0, & x > 1, \end{cases} \tag{6.5}$$

where a is a positive parameter. We can evaluate the Jost solution $f_1(k, x)$ explicitly by using (6.5) in (2.1) and the asymptotic condition given in (2.4) and by satisfying the continuity of $f_1(k, x)$ and $f'_1(k, x)$ at $x = 1$ and at $x = 1/2$. We then evaluate $f_1(k, 0)$ and $f'_1(k, 0)$ explicitly as a function of k in the presence of the parameter a . Then, from (2.30) we obtain the corresponding scattering coefficients T_1, L_1 , and R_1 explicitly via

$$T_1(k) = \frac{2ik}{ik f_1(k, 0) + f'_1(k, 0)}, \quad L_1(k) = \frac{ik f_1(k, 0) - f'_1(k, 0)}{ik f_1(k, 0) + f'_1(k, 0)},$$

$$R_1(k) = -\frac{ik f_1(-k, 0) + f'_1(-k, 0)}{ik f_1(k, 0) + f'_1(k, 0)}.$$

We then choose the value of a so that $T_1(k)$ has no poles on the positive imaginary axis and that $T_1(0) \neq 0$. From the small- k limits of $T_1(k)$, we find that those two conditions are satisfied provided

a is obtained by solving near $a = 1$ the equation

$$\sqrt{a} \tan\left(\frac{\sqrt{a}}{2}\right) = \tanh\left(\frac{1}{2}\right),$$

which yields $a = 0.857\,24\bar{7}$. With this choice of a , we get $T_1(0) = 0.973\,82\bar{7}$, $L_1(0) = -0.227\bar{3}$, and $R_1(0) = 0.227\bar{3}$. Note that with $a = 0.857\,24\bar{7}$ in (6.5), the half-line scattering matrix $S_1(k)$ corresponding to the Dirichlet boundary condition $\theta_1 = \pi$ is obtained by using the second line of (2.16). With the same specific a -value, we then evaluate the potential $V_2(x)$ corresponding to the scattering coefficients T_2, L_2 , and R_2 , where

$$T_2(k) \equiv T_1(k), \quad L_2(k) \equiv -L_1(k), \quad R_2(k) \equiv -R_1(k).$$

Since $T_1(k)$ has no poles on the positive imaginary axis, one can uniquely reconstruct $V_2(x)$ from $R_2(k)$, or equivalently from $-R_1(k)$, with the help of (2.45), (2.44), and the first equation in (2.20). Note that V_1 and V_2 can also uniquely be reconstructed from L_1 and $-L_1$, respectively. In fact, the corresponding numerical approximations of V_1 and V_2 have been computed in MATLAB via the method of Ref. 19, using $L_1(k)$ and $-L_1(k)$ in the interval $k \in [0, 100]$ with a discretization length of $\Delta k = 0.01$. The resulting potentials are shown in Figure 2.

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