Wave focusing on the line

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Focusing of waves in one dimension is analyzed for the plasma-wave equation and the wave equation with variable speed. The existence of focusing causal solutions to these equations is established, and such wave solutions are constructed explicitly by deriving an orthogonality relation for the time-independent Schrödinger equation. The connection between wave focusing and inverse scattering is studied. The potential at any point is recovered from the incident wave that leads to focusing to that point. It is shown that focusing waves satisfy certain temporal-antisymmetry and support properties. Discontinuities in the spatial and temporal derivatives of the focusing waves are examined and related to the discontinuities in the potential of the Schrödinger equation. The theory is illustrated with some explicit examples. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1483894]

I. INTRODUCTION

Consider a (Dirac-delta) plane wave incident onto an inhomogeneous medium. As time progresses the plane wave is scattered by the inhomogeneity and consequently develops a ‘‘tail’’ that trails the wavefront. One of the questions considered in this article concerns the opposite process. Namely, ‘‘Can one prepare an incident wave (consisting of a plane wave plus a tail) such that the tail vanishes at a specified instant due to the interaction with the inhomogeneity, i.e., the wave reduces to the plane wave at that instant?’’ If this happens, we say that the wave focuses to the point being crossed by the wavefront at the specified instant. We are also interested in determining remotely the value of the inhomogeneity at any specified point in space from the incident wave that is going to focus to that point; this will be done by performing a measurement on the incident wave at some arbitrarily chosen moment in time before the wavefront reaches that point.

Mathematically speaking, our aim is to analyze focusing of causal solutions to the plasma-wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = V(x)u(x,t), \quad x,t \in \mathbf{R}, \tag{1.1}$$

where V is real valued and belongs to $L_1^1(\mathbf{R})$, the class of measurable potentials such that $\int_{-\infty}^{\infty} dx(1+|x|)|V(x)|$ is finite. In order to do this, we derive the orthogonality relation (3.6) for the associated Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2} + k^2\psi(k,x) = V(x)\psi(k,x), \quad x \in \mathbf{R}, \tag{1.2}$$

and exploit the connection between (1.1) and (1.2) through the Fourier transformation

$$u(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \psi(k,x)e^{-ikt}. \tag{1.3}$$

We use the subscript ''l'' to indicate waves incident from the left (i.e., coming from $x=-\infty$) and use ''r'' for incidence from the right. Our focusing waves consist of a (Dirac-delta distribution) wavefront and a tail lying either to the left or right of the wavefront in such a way that the tail completely disappears at a certain moment in time and thus the whole wave reduces to the wavefront at the focusing point. There is no loss of generality in choosing the focusing moment as $t=0$, and we denote the focusing point by x_0. Hence, we are interested in constructing causal solutions to (1.1) incident either from the left or right such that $u(x,0)=\delta(x-x_0)$, where $\delta(x)$ denotes the Dirac delta distribution. In Sec. VII we display $\partial u(x,0)/\partial t$ explicitly for our focusing waves and hence show that it is also possible to view them as some specific solutions to (1.1) satisfying certain initial conditions. Clearly, unless $\partial u(x,0)/\partial t\equiv 0$ when $x\in\mathbf{R}\setminus\{x_0\}$ for our focusing waves, their energy is not concentrated at x_0 when $t=0$; hence, in general, focusing of waves is not the same as focusing of the wave energy.

Our analysis helps us to understand better the connection between (1.1) and (1.2). In our treatment we include bound states of V, whereas such states are usually excluded in the analysis of (1.1) by imposing further restrictions on V such as positivity. Throughout our article, unless otherwise stated, V is only assumed to be real valued and belonging to $L_1^1(\mathbf{R})$; any other assumptions on V will be explicitly stated.

In our article we also investigate the connection between wave focusing and inverse scattering. The inverse scattering problem for (1.1) and (1.2) consists of the recovery of V from an appropriate set of scattering data. The recovery in the time domain is usually achieved by using some layer-stripping methods, see, e.g., Burridge (1980), Bube and Burridge (1983), Morawetz and Kriegsmann (1983), Bayliss *et al.* (1989), and Sacks (1993), in terms of the impulse response to a plane wave sent onto $V(x)$ either from $x=-\infty$ or from $x=+\infty$. In these techniques one considers the solution to (1.1) satisfying $u(x,t)=\delta(x-t)+o(1)$ and $\partial u(x,t)/\partial t=\delta'(x-t)+o(1)$ when $t\to-\infty$ as the wave incident from the left, or $u(x,t)=\delta(x+t)+o(1)$ and $\partial u(x,t)/\partial t=\delta'(x+t)+o(1)$ when $t\to-\infty$ as the wave incident from the right. The contrast with our focusing waves can be visualized by considering waves incident from the left especially when $V\equiv 0$ for $x<0$: our focusing wave for $x<0$ and $t<-x_0$ consists of the wavefront $\delta(x-x_0-t)$ followed by [cf. (5.19)] the nontrivial tail $K_r(x_0,x-t)$, whereas the wave in the aforementioned references is $\delta(x-t)$ for $x<0$ and $t<0$. We show in Sec. VII that the value of $V(x_0)$ for any fixed $x_0>0$ is recovered from the incident wave that is going to focus to x_0 with a measurement performed at an arbitrary moment $t<-x_0$ (i.e., before the wavefront reaches the inhomogeneity); in contrast, in the layer-stripping methods one lets the incident wave penetrate the inhomogeneity during the time interval $0<t<2x_0$ in order to recover $V(x)$ for $0<x<x_0$. A heuristic discussion of the physics connecting focusing and inverse scattering appears in Rose (2002), which is a strictly time-domain analysis that avoids reference to scattering solutions to (1.2).

This article is organized as follows. In Sec. II we introduce the Jost solutions, scattering coefficients, and normalized bound-state solutions of (1.2). In Sec. III we derive the orthogonality relation (3.6), a key result for obtaining the causal focusing wave solutions to (1.1) (incident either from the left or right) explicitly in terms of the Jost solutions, transmission coefficient, and normalized bound-state solutions of (1.2). In Sec. IV, we construct such causal waves that focus at $t=0$, namely U_{l} incident from the left and U_{r} incident from the right, and we study some of their properties; we also indicate how the value of $V(x_0)$ can be recovered by using waves focusing to x_0 and its vicinity. In Sec. V we examine the connection between wave focusing and inverse scattering problem; in particular, we analyze the relationship between wave focusing and the Marchenko inversion method, construct our focusing waves in terms of the solutions to the Marchenko integral equations, and show that wave focusing can be viewed as a consequence of the Marchenko method. In Sec. VI we explore certain temporal antisymmetries satisfied by the tails of our focusing waves; in this section we also show that, for potentials vanishing on a half line, the tail of a focusing wave may vanish in some regions at certain times and that a gap may develop between the wavefront and the tail. In Sec. VII, under more restrictive conditions on V, we analyze the discontinuities in the spatial and temporal derivatives of our focusing waves and relate such discontinuities to jump discontinuities of V; we also show that $V(x_0)$ can be recovered solely from the incident wave leading to focusing to x_0, where the measurement can be performed at one arbitrarily chosen moment before the wavefront reaches x_0; as a corollary we obtain the interesting identities (7.22) and (7.25) for the solutions to the Marchenko equations when the corresponding potential vanishes on a half line. In Sec. VIII we present some explicit examples to illustrate various aspects of wave focusing and the recovery of $V(x_0)$ via focusing, and we also provide some snapshots of focusing waves as their tails disappear and reappear. Finally, in Sec. IX we analyze wave focusing for the variable-speed wave equation (9.14).

II. PRELIMINARIES

Let $\mathbf{C}^+$ denote the upper-half complex plane and $\overline{\mathbf{C}^+}:=\mathbf{C}^+\cup\mathbf{R}$. There are two types of solutions to (1.2). The scattering solutions consist of linear combinations of e^{ikx} and e^{-ikx} as $x\to\pm\infty$, and they occur for $k\in\mathbf{R}\setminus\{0\}$; on the other hand, the bound-state solutions decay exponentially as $x\to\pm\infty$, and they can occur only at certain k-values on the imaginary axis in $\mathbf{C}^+$. Let us use N to denote the number of bound states, which is known to be finite, and suppose that the bound states occur at $k=i\kappa_j$ with $0<\kappa_1<\cdots<\kappa_N$.

Among the scattering solutions to (1.2) are the Jost solution from the left, $f_{\rm l}$, and the Jost solution from the right, $f_{\rm r}$, satisfying the respective boundary conditions

$$e^{-ikx}f_{\rm l}(k,x)=1+o(1),\quad e^{-ikx}f_{\rm l}'(k,x)=ik+o(1),\quad x\to+\infty, \tag{2.1}$$

$$e^{ikx}f_{\rm r}(k,x)=1+o(1),\quad e^{ikx}f_{\rm r}'(k,x)=-ik+o(1),\quad x\to-\infty, \tag{2.2}$$

where the prime is used for the derivative with respect to the spatial coordinate x. From the spatial asymptotics

$$f_{\rm l}(k,x)=\frac{e^{ikx}}{T(k)}+\frac{L(k)e^{-ikx}}{T(k)}+o(1),\quad x\to-\infty, \tag{2.3}$$

$$f_{\rm r}(k,x)=\frac{e^{-ikx}}{T(k)}+\frac{R(k)e^{ikx}}{T(k)}+o(1),\quad x\to+\infty, \tag{2.4}$$

we obtain the scattering coefficients, namely, the transmission coefficient T, and the reflection coefficients L and R from the left and right, respectively.

Each bound state corresponds to a pole of T in $\mathbf{C}^+$ and vice versa. It is known that the bound states are simple and there exists only one linearly independent solution to (1.2) at each $k=i\kappa_j$ belonging to $L^2(\mathbf{R})$. The bound-state norming constants $c_{{\rm l}j}$ and $c_{{\rm r}j}$ are defined as

$$c_{{\rm l}j}:=\left[\int_{-\infty}^{\infty}dx\, f_{\rm l}(i\kappa_j,x)^2\right]^{-1/2},\quad c_{{\rm r}j}:=\left[\int_{-\infty}^{\infty}dx\, f_{\rm r}(i\kappa_j,x)^2\right]^{-1/2}, \tag{2.5}$$

and they are related to each other via the residues of T as

$$\operatorname{Res}(T,i\kappa_j)=ic_{{\rm l}j}^2\gamma_j=i\frac{c_{{\rm r}j}^2}{\gamma_j}, \tag{2.6}$$

where γ_j is the dependency constant given by $\gamma_j:=f_{\rm l}(i\kappa_j,x)/f_{\rm r}(i\kappa_j,x)$. The sign of γ_j is the same as that of $(-1)^{N-j}$ and hence $c_{{\rm r}j}=(-1)^{N-j}\gamma_j c_{{\rm l}j}$. The normalized bound-state solution $\varphi_j(x)$ at $k=i\kappa_j$ is defined as

$$\varphi_j(x):=c_{{\rm l}j}f_{\rm l}(i\kappa_j,x)=(-1)^{N-j}c_{{\rm r}j}f_{\rm r}(i\kappa_j,x). \tag{2.7}$$

III. AN ORTHOGONALITY IDENTITY

The scattering and bound-state solutions to (1.2) satisfy the completeness relation, see, e.g., Newton (1983) and Chadan amd and Sabatier (1989),

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} dk[\psi_{\mathrm{l}}(k,x)\psi_{\mathrm{l}}(-k,x_0)+\psi_{\mathrm{r}}(k,x)\psi_{\mathrm{r}}(-k,x_0)]+\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)=\delta(x-x_0), \quad (3.1)$$

where ψ_{l} and ψ_{r} are the physical solutions to (1.2) related to the Jost solutions as

$$\psi_{\mathrm{l}}(k,x):=T(k)f_{\mathrm{l}}(k,x), \quad \psi_{\mathrm{r}}(k,x):=T(k)f_{\mathrm{r}}(k,x). \quad (3.2)$$

In the Jost solutions, physical solutions, and scattering coefficients, for real k, replacing k by $-k$ has the same effect as taking complex conjugation. Moreover, we have

$$f_{\mathrm{l}}(-k,x)=T(k)f_{\mathrm{r}}(k,x)-R(k)f_{\mathrm{l}}(k,x), \quad k\in\mathbf{R}, \quad (3.3)$$

$$f_{\mathrm{r}}(-k,x)=-L(k)f_{\mathrm{r}}(k,x)+T(k)f_{\mathrm{l}}(k,x), \quad k\in\mathbf{R}, \quad (3.4)$$

which are consequences of the fact that either $\{f_{\mathrm{l}}(k,\cdot),f_{\mathrm{r}}(k,\cdot)\}$ or $\{f_{\mathrm{l}}(-k,\cdot),f_{\mathrm{r}}(-k,\cdot)\}$ is a linearly independent set of solutions to (1.2) when $k\in\mathbf{R}\setminus\{0\}$ and that the functions in one set can be expressed as a linear combination of those in the other. It is known that

$$R(k)T(-k)=-L(-k)T(k), \quad k\in\mathbf{R}. \quad (3.5)$$

Next we prove an orthogonality identity for (1.2) that will be useful in the analysis of wave focusing for (1.1).

Theorem 3.1: *Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dkT(k)f_{\mathrm{l}}(k,x)f_{\mathrm{r}}(k,x_0)+\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)=\delta(x-x_0). \quad (3.6)$$

Proof: The proof will be given by showing that the integral term in (3.6) is identical to the integral in (3.1). From (3.2)–(3.4) we get

$$\psi_{\mathrm{r}}(k,x)=f_{\mathrm{l}}(-k,x)+R(k)f_{\mathrm{l}}(k,x), \quad k\in\mathbf{R},$$

$$\psi_{\mathrm{l}}(-k,x_0)=f_{\mathrm{r}}(k,x_0)+L(-k)f_{\mathrm{r}}(-k,x_0), \quad k\in\mathbf{R}.$$

Thus, for $k\in\mathbf{R}$ we have

$$\begin{aligned}&\psi_{\mathrm{l}}(k,x)\psi_{\mathrm{l}}(-k,x_0)+\psi_{\mathrm{r}}(k,x)\psi_{\mathrm{r}}(-k,x_0)\\ &\quad=T(k)f_{\mathrm{l}}(k,x)[f_{\mathrm{r}}(k,x_0)+L(-k)f_{\mathrm{r}}(-k,x_0)]+[f_{\mathrm{l}}(-k,x)+R(k)f_{\mathrm{l}}(k,x)]T(-k)f_{\mathrm{r}}(-k,x_0)\\ &\quad=T(k)f_{\mathrm{l}}(k,x)f_{\mathrm{r}}(k,x_0)+T(-k)f_{\mathrm{l}}(-k,x)f_{\mathrm{r}}(-k,x_0), \quad (3.7)\end{aligned}$$

where we have used (3.5) in the last step for simplification. Replacing the dummy integration variable k by $-k$, we get

$$\int_{-\infty}^{\infty} dkT(-k)f_{\mathrm{l}}(-k,x)f_{\mathrm{r}}(-k,x_0)=\int_{-\infty}^{\infty} dkT(k)f_{\mathrm{l}}(k,x)f_{\mathrm{r}}(k,x_0),$$

and hence from (3.7) we obtain

$$\int_{-\infty}^{\infty} dk[\psi_l(k,x)\psi_l(-k,x_0)+\psi_r(k,x)\psi_r(-k,x_0)]=2\int_{-\infty}^{\infty} dkT(k)f_l(k,x)f_r(k,x_0).$$

Thus, the integral on the left-hand side of (3.6) is the same as that on the left-hand side of (3.1).■

IV. WAVE FOCUSING FOR THE PLASMA-WAVE EQUATION

In this section we construct focusing waves of (1.1) incident either from the left or right in terms of the Jost solutions, transmission coefficient, and bound states for (1.2). We also relate the discontinuities at the wavefront of such focusing waves to an integral of V and show how the value of $V(x)$ at any specific point can be extracted from waves focusing to that point and its vicinity.

In terms of the Jost solutions of (1.2), let us define

$$K_l(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[f_l(k,x)-e^{ikx}]e^{-ikt}, \tag{4.1}$$

$$K_r(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[f_r(k,x)-e^{-ikx}]e^{ikt}. \tag{4.2}$$

Using (2.7) and the inverse Fourier transforms on (4.1) and (4.2), we obtain

$$\varphi_j(x)=(-1)^{N-j}c_{rj}\left[e^{\kappa_j x}+\int_{-\infty}^{x} dsK_r(x,s)e^{\kappa_j s}\right]=c_{lj}\left[e^{-\kappa_j x}+\int_{x}^{\infty} dsK_l(x,s)e^{-\kappa_j s}\right]. \tag{4.3}$$

The properties of K_l and K_r stated in the following theorem are already known, see, e.g., Faddeev (1967), Marchenko (1986), Chadan and Sabatier (1989), Deift and Trubowitz (1979), and they are used later in our analysis.

Theorem 4.1: *Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then*

(i) For each fixed $x\in\mathbf{R}$, $K_l(x,\cdot)$ and $K_r(x,\cdot)$ belong to $L^2(\mathbf{R})\cap L^1(\mathbf{R})$.

(ii) For any $a\in\mathbf{R}$, $K_l(x,t)$ is uniformly bounded in (x,t) for $x\geqslant a$; similarly, $K_r(x,t)$ is uniformly bounded in (x,t) for $x\leqslant a$. Moreover, we have

$$K_l(x,t)=0, \quad t<x; \quad K_r(x,t)=0, \quad t>x. \tag{4.4}$$

(iii) K_l and K_r are continuous in (x,t) except when $t=x$, and the jumps there are related to V as

$$K_l(x,x^+)=\frac{1}{2}\int_{x}^{\infty} dzV(z), \quad K_r(x,x^-)=\frac{1}{2}\int_{-\infty}^{x} dzV(z). \tag{4.5}$$

Define

$$\hat{L}(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dkL(k)e^{ikt}, \quad \hat{R}(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dkR(k)e^{ikt}. \tag{4.6}$$

When V is real valued and belongs to $L_1^1(\mathbf{R})$, each of $\hat{L}$ and $\hat{R}$ is continuous and belongs to $L^2(\mathbf{R})$. In fact, they are absolutely continuous and differentiable, and for each fixed $a\in\mathbf{R}$ their derivatives satisfy $\hat{L}'\in L_1^1(-\infty,a)$ and $\hat{R}'\in L_1^1(a,+\infty)$.

Let us define

$$P_l(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[T(k)-1]f_l(k,x)e^{-ikt}, \tag{4.7}$$

$$P_{\mathrm{r}}(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[T(k)-1]f_{\mathrm{r}}(k,x)e^{ikt}, \tag{4.8}$$

$$\Phi_{\mathrm{l}}(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[\psi_{\mathrm{l}}(k,x)-e^{ikx}]e^{-ikt}, \tag{4.9}$$

$$\Phi_{\mathrm{r}}(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[\psi_{\mathrm{r}}(k,x)-e^{-ikx}]e^{ikt}. \tag{4.10}$$

Proposition 4.2: Assume V is real valued and belongs to $L_1^1(\mathbf{R})$*. Then, for any fixed* $x\in\mathbf{R}$*, each of* $P_{\mathrm{l}}(x,\cdot)$*,* $P_{\mathrm{r}}(x,\cdot)$*,* $\Phi_{\mathrm{l}}(x,\cdot)$*, and* $\Phi_{\mathrm{r}}(x,\cdot)$ *belongs to* $L^2(\mathbf{R})$*. Moreover, we have*

$$\Phi_{\mathrm{l}}(x,t) = P_{\mathrm{l}}(x,t) + K_{\mathrm{l}}(x,t), \quad \Phi_{\mathrm{r}}(x,t) = P_{\mathrm{r}}(x,t) + K_{\mathrm{r}}(x,t), \tag{4.11}$$

$$\Phi_{\mathrm{l}}(x,t) = P_{\mathrm{l}}(x,t) = -\sum_{j=1}^{N} (-1)^{N-j} c_{\mathrm{r}j}\varphi_j(x)e^{\kappa_j t}, \quad t<x, \tag{4.12}$$

$$\Phi_{\mathrm{r}}(x,t) = P_{\mathrm{r}}(x,t) = -\sum_{j=1}^{N} c_{\mathrm{l}j}\varphi_j(x)e^{-\kappa_j t}, \quad t>x, \tag{4.13}$$

$$\Phi_{\mathrm{l}}(x,t) = K_{\mathrm{r}}(x,t) + \hat{L}(-x-t) + \int_{-\infty}^{x} ds\hat{L}(-t-s)K_{\mathrm{r}}(x,s), \quad t\neq x, \tag{4.14}$$

$$\Phi_{\mathrm{r}}(x,t) = K_{\mathrm{l}}(x,t) + \hat{R}(x+t) + \int_{x}^{\infty} ds\hat{R}(t+s)K_{\mathrm{l}}(x,s), \quad t\neq x, \tag{4.15}$$

$$P_{\mathrm{l}}(x,x_0+t) + \int_{-\infty}^{x_0} dsP_{\mathrm{l}}(x,s+t)K_{\mathrm{r}}(x_0,s) + \sum_{j=1}^{N} \varphi_j(x)\varphi_j(x_0)e^{\kappa_j t} = 0, \quad x>x_0+t, \tag{4.16}$$

$$P_{\mathrm{r}}(x,x_0-t) + \int_{x_0}^{\infty} dsP_{\mathrm{r}}(x,s-t)K_{\mathrm{l}}(x_0,s) + \sum_{j=1}^{N} \varphi_j(x)\varphi_j(x_0)e^{\kappa_j t} = 0, \quad x<x_0-t. \tag{4.17}$$

Proof: We obtain (4.11) by using (3.2), (4.1), (4.2), and (4.7)–(4.10). With the help of (2.6), (2.7), (4.4), (4.7), (4.8), and (4.11), by using a contour integration along the infinite semicircle enclosing $\mathbf{C}^+$, we obtain (4.12) and (4.13). Using (4.2), (4.6), and a Fourier transform on (3.4) we get (4.14). Similarly, by using (3.3), (4.1), and (4.6) we get (4.15). With the help of (4.3) and (4.12) we establish (4.16). In the same manner, using (4.3) and (4.13) we get (4.17). ■

Define

$$U_{\mathrm{l}}(x,t;x_0) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\psi_{\mathrm{l}}(k,x)f_{\mathrm{r}}(k,x_0)e^{-ikt} + \sum_{j=1}^{N} \varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}, \tag{4.18}$$

$$U_{\mathrm{r}}(x,t;x_0) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\psi_{\mathrm{r}}(k,x)f_{\mathrm{l}}(k,x_0)e^{-ikt} + \sum_{j=1}^{N} \varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}, \tag{4.19}$$

$$\Upsilon_{\mathrm{l}}(x,t;x_0) := U_{\mathrm{l}}(x,t;x_0) - \delta(x-x_0-t), \tag{4.20}$$

$$\Upsilon_{\mathrm{r}}(x,t;x_0) := U_{\mathrm{r}}(x,t;x_0) - \delta(x-x_0+t). \tag{4.21}$$

Theorem 4.3: *Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then $U_{\rm l}$ is a causal solution to (1.1) incident from the left and focusing to $x=x_0$ when $t=0$. Similarly, $U_{\rm r}$ is a causal solution to (1.1) that is incident from the right and that focuses to $x=x_0$ when $t=0$.*

Proof: First, $\psi_{\rm l}$, $\psi_{\rm r}$, $f_{\rm l}$, and $f_{\rm r}$ are solutions to (1.2) and they are transformed from the k-domain to the t-domain as in (1.3). Thus, with the help of (2.7), we see that each of $U_{\rm l}$ and $U_{\rm r}$ is a solution to (1.1). Using (4.2), (4.9), (4.11), (4.18), and (4.20) we get

$$\Upsilon_{\rm l}(x,t;x_0)=K_{\rm r}(x_0,x-t)+K_{\rm l}(x,x_0+t)+\int_{x-t}^{x_0}ds K_{\rm l}(x,s+t)K_{\rm r}(x_0,s)+P_{\rm l}(x,x_0+t)$$
$$+\int_{-\infty}^{x_0}ds P_{\rm l}(x,s+t)K_{\rm r}(x_0,s)+\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}. \tag{4.22}$$

Similarly, using (4.1), (4.10), (4.11), (4.19), and (4.21) we obtain

$$\Upsilon_{\rm r}(x,t;x_0)=K_{\rm l}(x_0,x+t)+K_{\rm r}(x,x_0-t)+\int_{x_0}^{x+t}ds K_{\rm r}(x,s-t)K_{\rm l}(x_0,s)+P_{\rm r}(x,x_0-t)$$
$$+\int_{x_0}^{\infty}ds P_{\rm r}(x,s-t)K_{\rm l}(x_0,s)+\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}. \tag{4.23}$$

With the help of (4.4) it follows that at any fixed moment t each of the first three terms on the right-hand side of (4.22) vanishes when $x>x_0+t$; moreover, using (4.16) it follows that the last three terms add to zero when $x>x_0+t$. Thus, $U_{\rm l}$ consists of the wavefront $\delta(x-x_0-t)$ followed by the tail $\Upsilon_{\rm l}$ on the left, and it is incident from the left. Similarly, at any fixed moment t each of the first three terms on the right-hand side of (4.23) vanishes when $x<x_0-t$; moreover, from (4.17) it follows that the last three terms add to zero when $x<x_0-t$. Hence, $U_{\rm r}$ is a wave consisting of the wavefront $\delta(x-x_0+t)$ followed by the tail $\Upsilon_{\rm r}$ on the right and the wave is incident from the right. Each of the waves $U_{\rm l}$ and $U_{\rm r}$ focuses to $x=x_0$ at $t=0$ because $U_{\rm l}(x,0;x_0)=\delta(x-x_0)$ and $U_{\rm r}(x,0;x_0)=\delta(x-x_0)$, as readily seen by comparing (4.18) and (4.19) with (3.6). ∎

Since (1.1) is linear and homogeneous, any linear combination of $U_{\rm l}$ and $U_{\rm r}$ also focuses at $t=0$. In fact, for the special choice $U_{\rm l}-U_{\rm r}$ even the wavefronts cancel each other and the wave vanishes on the entire x-axis at $t=0$. On the other hand, for the special choice $U_{\rm l}+U_{\rm r}$ the wavefronts superimpose on top of each other at $t=0$.

Proposition 4.4: Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then the only discontinuities of $P_{\rm l}$ and $P_{\rm r}$ can occur when $x=t$. Such discontinuities are given by

$$P_{\rm l}(x,x^-)=\Phi_{\rm l}(x,x^-)=-\sum_{j=1}^{N}(-1)^{N-j}c_{{\rm r}j}\varphi_j(x)e^{\kappa_j x}, \tag{4.24}$$

$$P_{\rm r}(x,x^+)=\Phi_{\rm r}(x,x^+)=-\sum_{j=1}^{N}c_{{\rm l}j}\varphi_j(x)e^{-\kappa_j x}, \tag{4.25}$$

$$P_{\rm l}(x,x^+)=-\frac{1}{2}\int_{-\infty}^{\infty}ds V(s)-\sum_{j=1}^{N}(-1)^{N-j}c_{{\rm r}j}\varphi_j(x)e^{\kappa_j x}, \tag{4.26}$$

$$P_{\rm r}(x,x^-)=-\frac{1}{2}\int_{-\infty}^{\infty}ds V(s)-\sum_{j=1}^{N}c_{{\rm l}j}\varphi_j(x)e^{-\kappa_j x}. \tag{4.27}$$

Proof: Note that (4.24) and (4.25) are equivalent to (4.12) and (4.13), respectively. Since $\hat{L}$ and $K_{\rm r}(x,\cdot)$ are square integrable, their product is integrable; hence the integral term in (4.14) is continuous in (x,t) as a result of the Lebesgue dominated convergence theorem. Similarly, from the square-integrability of $\hat{R}$ and of $K_{\mathrm{l}}(x,\cdot)$ it follows that the integral term in (4.15) is continuous in (x,t). Using the fact that $\hat{L}$ and $\hat{R}$ are continuous, from (4.11), (4.14), and (4.15) we see that the discontinuities of P_{l} and P_{r} coincide with those of K_{l} and K_{r}; hence, such discontinuities can occur only when $x=t$. In fact, with the help of (4.4) we get

$$P_{\mathrm{l}}(x,x^{+})-P_{\mathrm{l}}(x,x^{-})=-K_{\mathrm{r}}(x,x^{-})-K_{\mathrm{l}}(x,x^{+}), \tag{4.28}$$

$$P_{\mathrm{r}}(x,x^{+})-P_{\mathrm{r}}(x,x^{-})=K_{\mathrm{l}}(x,x^{+})+K_{\mathrm{r}}(x,x^{-}). \tag{4.29}$$

Thus, using (4.5), (4.24), (4.25), (4.28), and (4.29), we get (4.26) and (4.27). ■

Theorem 4.5: *Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then the only discontinuities of U_{l} and U_{r} occur at the wavefront, and the jumps in the tails at the wavefront are related to V as*

$$\Upsilon_{\mathrm{l}}(x_0^{-}+t,t;x_0)=-\frac{1}{2}\int_{x_0}^{x_0+t} dz\, V(z), \tag{4.30}$$

$$\Upsilon_{\mathrm{r}}(x_0^{+}-t,t;x_0)=-\frac{1}{2}\int_{x_0-t}^{x_0} dz\, V(z). \tag{4.31}$$

Proof: From (4.22) and (4.23) we see that the discontinuities in Υ_{l} and Υ_{r} can come only from the first, second, and fourth terms on the right-hand sides of (4.22) and (4.23), respectively; the third and fifth terms are continuous in (x,t) because the integrands there, being products of L^2-functions, are integrable in s. Thus, with the help of Proposition 4.4 and the fact that K_{r} and K_{l} can have discontinuities only when $x=t$, we conclude that the discontinuities in Υ_{l} and Υ_{r} can only occur at the wavefront, and we have

$$\begin{aligned}\Upsilon_{\mathrm{l}}(x_0^{-}+t,t;x_0)-\Upsilon_{\mathrm{l}}(x_0^{+}+t,t;x_0)={}&K_{\mathrm{r}}(x_0,x_0^{-})-K_{\mathrm{r}}(x_0,x_0^{+})+K_{\mathrm{l}}(x_0+t,x_0^{+}+t)-K_{\mathrm{l}}(x_0+t,x_0^{-}+t)\\&+P_{\mathrm{l}}(x_0+t,x_0^{+}+t)-P_{\mathrm{l}}(x_0+t,x_0^{-}+t),\end{aligned} \tag{4.32}$$

$$\begin{aligned}\Upsilon_{\mathrm{r}}(x_0^{+}-t,t;x_0)-\Upsilon_{\mathrm{r}}(x_0^{-}-t,t;x_0)={}&K_{\mathrm{l}}(x_0,x_0^{+})-K_{\mathrm{l}}(x_0,x_0^{-})+K_{\mathrm{r}}(x_0-t,x_0^{-}-t)-K_{\mathrm{r}}(x_0-t,x_0^{+}-t)\\&+P_{\mathrm{r}}(x_0-t,x_0^{-}-t)-P_{\mathrm{r}}(x_0-t,x_0^{+}-t).\end{aligned} \tag{4.33}$$

Now using (4.4), (4.5), (4.28), and (4.29) in (4.32) and (4.33) and the fact that U_{l} and U_{r} are causal, we establish (4.30) and (4.31). ■

As an application of Theorem 4.5, let us show how one can recover the value of $V(x_0)$ by using waves focusing to x_0 and its vicinity. Consider the left-hand side of (4.30) at some fixed time t in the interval $(-\infty,-x_0)$; in fact, one can even consider it when $t\to-\infty$. Let

$$\Gamma_{\mathrm{l}}(x_0,t):=\Upsilon_{\mathrm{l}}(x_0^{-}+t,t;x_0). \tag{4.34}$$

Thus, $\Gamma_{\mathrm{l}}(x_0,t)$ indicates the height of the tail of U_{l} at the wavefront at some fixed time $t<-x_0$. From (4.30), if V is continuous at x_0 and x_0+t, we see that

$$V(x_0)-V(x_0+t)=2\,\frac{\partial\Gamma_{\mathrm{l}}(x_0,t)}{\partial x_0}. \tag{4.35}$$

Note that $V(x_0+t)$ can be made as small as we want by choosing t so that either x_0+t lies to the left of the support of V (if V is supported in a right-half line) or by letting $t\to-\infty$ (if the support of V extends to $x=-\infty$). Clearly, $\partial\Gamma_{\mathrm{l}}(x_0,t)/\partial x_0$ can be obtained by using waves focusing to x_0 and its vicinity. An explicit example in Sec. VIII illustrates the recovery of $V(x_0)$ by using the technique described here.

Note that in the recovery technique outlined above, we have not made any other assumptions on V besides $V \in L_1^1(\mathbf{R})$, its realness, and its continuity at x_0 and x_0+t for some fixed $t<-x_0$. In fact, even when V is not continuous but only sectionally continuous, this technique still holds provided we replace (4.35) by

$$V(x_0^-)-V(x_0^+ +t)=2\,\frac{\partial \Gamma_l(x_0,t)}{\partial x_0},\quad t<-x_0.$$

Under some stronger assumptions on V in Sec. VII, we will see that we can recover $V(x_0)$ by only using the wave that focuses to x_0 without needing any waves focusing near x_0.

V. CONNECTION WITH THE MARCHENKO METHOD

In this section we explore the connection between wave focusing for (1.1) and the Marchenko method to solve the inverse scattering problem for (1.2). By presenting certain representations for U_l and U_r, we show that their focusing is a direct consequence of the Marchenko method.

Let us define

$$M_r(t):=\hat{L}(-t)+\sum_{j=1}^{N} c_{rj}^2 e^{\kappa_j t},\quad M_l(t):=\hat{R}(t)+\sum_{j=1}^{N} c_{lj}^2 e^{-\kappa_j t}, \tag{5.1}$$

where $\hat{L}$ and $\hat{R}$ are as in (4.6), and c_{rj} and c_{lj} are as in (2.5). Using (4.3), (4.4), (4.12)–(4.15), and (5.1) we get the two Marchenko equations

$$K_r(x,t)+M_r(x+t)+\int_{-\infty}^{x} ds M_r(t+s)K_r(x,s)=0,\quad t<x, \tag{5.2}$$

$$K_l(x,t)+M_l(x+t)+\int_{x}^{\infty} ds M_l(t+s)K_l(x,s)=0,\quad t>x, \tag{5.3}$$

and using (4.4), (4.14), and (4.15) we obtain the two complementary equations

$$\Phi_l(x,t)=\hat{L}(-x-t)+\int_{-\infty}^{x} ds \hat{L}(-t-s)K_r(x,s),\quad t>x, \tag{5.4}$$

$$\Phi_r(x,t)=\hat{R}(x+t)+\int_{x}^{\infty} ds \hat{R}(t+s)K_l(x,s),\quad t<x. \tag{5.5}$$

Let

$$F_l(x,t):=K_l(x,t)+M_l(x+t)+\int_{x}^{\infty} ds M_l(t+s)K_l(x,s), \tag{5.6}$$

$$F_r(x,t):=K_r(x,t)+M_r(x+t)+\int_{-\infty}^{x} ds M_r(t+s)K_r(x,s), \tag{5.7}$$

$$Z_l(x,t):=\Phi_l(x,t)-\hat{L}(-x-t)-\int_{-\infty}^{x} ds \hat{L}(-t-s)K_r(x,s), \tag{5.8}$$

$$Z_r(x,t):=\Phi_r(x,t)-\hat{R}(x+t)-\int_{x}^{\infty} ds \hat{R}(t+s)K_l(x,s). \tag{5.9}$$

Using (4.4) we can write the Marchenko equations (5.2) and (5.3) as

$$F_{\mathrm{r}}(x,t)=0, \quad t<x; \quad K_{\mathrm{r}}(x,t)=0, \quad t>x, \tag{5.10}$$

$$F_{\mathrm{l}}(x,t)=0, \quad t>x; \quad K_{\mathrm{l}}(x,t)=0, \quad t<x, \tag{5.11}$$

and the complementary equations (5.4) and (5.5) as

$$Z_{\mathrm{l}}(x,t)=0, \quad t>x; \quad Z_{\mathrm{r}}(x,t)=0, \quad t<x. \tag{5.12}$$

The following proposition shows that the focusing wave U_{l} can be constructed from the scattering data $\{L,\{\kappa_j\},\{c_{\mathrm{r}j}\}\}$ via the solution K_{r} of the Marchenko equation (5.2); similarly, the focusing wave U_{r} can be constructed from the data $\{R,\{\kappa_j\},\{c_{\mathrm{l}j}\}\}$ via the solution K_{l} of the Marchenko equation (5.3).

Proposition 5.1: The waves U_{l} and U_{r} defined in (4.18) and (4.19) can be expressed in terms of the quantities defined in (4.1), (4.2), (5.6), and (5.7) as

$$U_{\mathrm{l}}(x,t;x_0)=\delta(x-x_0-t)+K_{\mathrm{r}}(x_0,x-t)+F_{\mathrm{r}}(x,x_0+t)+\int_{-\infty}^{x_0} ds F_{\mathrm{r}}(x,t+s)K_{\mathrm{r}}(x_0,s), \tag{5.13}$$

$$U_{\mathrm{r}}(x,t;x_0)=\delta(x-x_0+t)+K_1(x_0\,,x+t)F_{\mathrm{l}}(x,x_0-t)+\int_{x_0}^{\infty} ds F_{\mathrm{l}}(x,s-t)K_{\mathrm{l}}(x_0,s). \tag{5.14}$$

Proof: Using (4.3), (4.14), (4.22), (5.1), and (5.7), we obtain (5.13). Similarly, using (4.3), (4.15), (4.23), (5.1), and (5.6), we obtain (5.14). ∎

Let us define

$$\begin{aligned}A_{\mathrm{r}}(x,t;x_0):=&K_{\mathrm{r}}(x_0,x-t)-K_{\mathrm{r}}(x_0,x+t)-F_{\mathrm{r}}(x_0,x-t)+F_{\mathrm{r}}(x_0,x+t)\\&-\int_{-\infty}^{x} ds K_{\mathrm{r}}(x,s)[F_{\mathrm{r}}(x_0,s-t)-F_{\mathrm{r}}(x_0,s+t)]\\&+\int_{-\infty}^{\max\{x,x_0\}} ds[K_{\mathrm{r}}(x_0,s)K_{\mathrm{r}}(x,s+t)-K_{\mathrm{r}}(x_0,s+t)K_{\mathrm{r}}(x,s)],\end{aligned} \tag{5.15}$$

$$\begin{aligned}A_{\mathrm{l}}(x,t;x_0):=&K_{\mathrm{l}}(x_0,x+t)-K_{\mathrm{l}}(x_0,x-t)+F_{\mathrm{l}}(x_0,x-t)-F_{\mathrm{l}}(x_0,x+t)\\&-\int_{x}^{\infty} ds K_{\mathrm{l}}(x,s)[F_{\mathrm{l}}(x_0,s+t)-F_{\mathrm{l}}(x_0,s-t)]\\&+\int_{\min\{x,x_0\}}^{\infty} ds[K_{\mathrm{l}}(x_0,s)K_{\mathrm{l}}(x,s-t)-K_{\mathrm{l}}(x_0,s-t)K_{\mathrm{l}}(x,s)].\end{aligned} \tag{5.16}$$

Note that at $t=0$ both $A_{\mathrm{r}}(x,t;x_0)$ and $A_{\mathrm{l}}(x,t;x_0)$ vanish.

Proposition 5.2: The waves U_{l} and U_{r} defined in (4.18) and (4.19) can be expressed in terms of the quantities defined in (4.1), (4.2), (5.6)–(5.9), (5.15), and (5.16) as

$$U_{\mathrm{l}}(x,t;x_0)=\delta(x-x_0-t)+Z_{\mathrm{l}}(x,x_0+t)+F_{\mathrm{r}}(x_0,x-t)+\int_{x_0+t}^{x} ds F_{\mathrm{r}}(x_0,s-t)K_{\mathrm{r}}(x,s)+A_{\mathrm{r}}(x,t;x_0), \tag{5.17}$$

$$\begin{aligned}U_{\mathrm{r}}(x,t;x_0)=&\delta(x-x_0+t)+Z_{\mathrm{r}}(x,x_0-t)+F_{\mathrm{l}}(x_0,x+t)\\&+\int_{x}^{x_0-t} ds F_{\mathrm{l}}(x_0,s+t)K_{\mathrm{l}}(x,s)+A_{\mathrm{l}}(x,t;x_0).\end{aligned} \tag{5.18}$$

Proof: We obtain (5.17) by using (4.3), (4.11), (4.14), (5.1), (5.7), and (5.8) in (4.22). Similarly, (5.18) is obtained by using (4.3), (4.11), (4.15), (5.6), and (5.9) in (4.23). ∎

Theorem 5.3: *Assume that V is real valued and belongs to $L_1^1(\mathbf{R})$. Then we have the following.*

(i) The causality of U_{l} is a consequence of the Marchenko equation (5.2).
(ii) The causality of U_{r} is a consequence of the Marchenko equation (5.3).
(iii) The focusing of U_{l} to x_0 at $t=0$ is a consequence of (5.2) and (5.4).
(iv) The focusing of U_{r} to x_0 at $t=0$ is a consequence of (5.3) and (5.5).

Proof: Recall that (5.2) and (5.3) are equivalent to (5.10) and (5.11), respectively; similarly, (5.4) and (5.5) are equivalent to the first and second equations in (5.12), respectively. Note that if (5.10) holds, then the right-hand side of (5.13) vanishes when $x>x_0+t$ and hence (i) is proved. Similarly, if (5.11) holds, then the right-hand side of (5.14) vanishes for $x<x_0-t$ and hence (ii) is proved. If (5.10) and the first equation in (5.12) hold, then each of the first four terms on the right-hand side of (5.17) vanishes when $x<x_0+t$; moreover, the last term $A_{\mathrm{r}}(x,t;x_0)$ vanishes at $t=0$ and hence (iii) is proved. In the same manner, if (5.11) and the second equation in (5.12) hold, then each of the first four terms on the right-hand side of (5.18) vanishes when $x>x_0-t$; moreover, the last term $A_{\mathrm{l}}(x,t;x_0)$ vanishes at $t=0$ and hence (iv) is also proved. ∎

Let us comment on the roles that the complementary equations (5.4) and (5.5) play in the Marchenko method. For the recovery of V by solving (5.2), one uses the "right" scattering data $\{L,\{\kappa_j\},\{c_{\mathrm{r}j}\}\}$ as the input and obtains the "right" quantity K_{r}, from which the "right" Jost solution f_{r} is obtained with the help of (4.1). The complementary equation (5.4) is a means to construct the "left" quantity Φ_{l}, from which the "left" physical solution ψ_{l} can be obtained via the inverse Fourier transform on (4.9). In a similar manner, the complementary equation (5.5) is a means to construct the "right" quantities such as ψ_{r} by using only the "left" scattering data $\{R,\{\kappa_j\},\{c_{\mathrm{l}j}\}\}$ as the input to the Marchenko procedure. Hence, Theorem 5.3(iii) is equivalent to the statement that the focusing of U_{l} is a consequence of the Marchenko method using the "right" scattering data as the input. Similarly, Theorem 5.3(iv) is equivalent to saying that focusing of U_{r} is a consequence of the Marchenko method using the "left" scattering data as the input.

The following result and its proof are used in the proof of Theorems 6.4 and 6.5. Even though this result is already known, we include a brief proof for convenience because we later refer to the facts stated in it.

Proposition 5.4: Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. If $V\equiv 0$ for $x>0$, then $M_{\mathrm{l}}(t)=0$ for $t>0$, and if $V\equiv 0$ for $x<0$, then $M_{\mathrm{r}}(t)=0$ for $t<0$, where M_{l} and M_{r} are the quantities defined in (5.1).

Proof: If $V\equiv 0$ for $x>0$, then R is meromorphic in $\mathbf{C}^+$ with simple poles at the bound states $k=i\kappa_j$ having the residues $\mathrm{Res}(R,i\kappa_j)=ic_{\mathrm{l}j}^2$, and $R(k)e^{2ikx}=o(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for each $x\geqslant 0$. Similarly, if $V\equiv 0$ for $x<0$, then L is meromorphic in $\mathbf{C}^+$ with simple poles at the bound states $k=i\kappa_j$ having the residues $\mathrm{Res}(L,i\kappa_j)=ic_{\mathrm{r}j}^2$, and $L(k)e^{-2ikx}=o(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for each $x\leqslant 0$. Thus, from (5.1) with the help of a contour integration, we directly get the conclusion of our proposition. ∎

Let us make a contrast between an incident focusing wave and an incident plane wave; the latter is often used to probe an inhomogeneous medium. For simplicity, consider the special case $V\equiv 0$ for $x<0$. In this case, from (4.2) it follows that $K_{\mathrm{r}}(x,t)=0$ for $x<0$. Thus, using (5.7) and Proposition 5.4 we get $F_{\mathrm{r}}(x,x_0+t)=0$ for $x<0$ and $t<-x_0$. Hence, from (5.13) it follows that our focusing wave incident from the left is given by

$$U_{\mathrm{l}}(x,t;x_0)=\delta(x-x_0-t)+K_{\mathrm{r}}(x_0,x-t), \quad x<0, \quad t<-x_0. \tag{5.19}$$

From (5.19) we see that, if $x_0>0$, U_{l} contains some information about V even before the incident wave first encounters the potential at $x=0$ and $t=-x_0$. Using $x_0+t<0$ and $V(z)=0$ for $z<0$, from (4.30) we get $\Upsilon_{\mathrm{l}}(x_0^-+t,t;x_0)=(1/2)\int_0^{x_0}dz\,V(z)$. This is in contrast to the case where a pure plane wave is sent onto V from $x=-\infty$, in which case the incident wave consists solely of the Dirac-delta wavefront alone and a tail is nonexistent until the wave encounters the potential at $x=0$.

VI. TEMPORAL ANTISYMMETRIES AND SUPPORT PROPERTIES

In this section we are interested in showing that the focusing waves U_{l} and U_{r} satisfy certain temporal antisymmetries and support properties that are useful in understanding their focusing. We also show that for potentials vanishing on a half line, a gap may develop between the wavefront and the tail of these focusing waves. We present our results only for U_{l} because the corresponding results for U_{r} are obtained in a similar manner.

In terms of the Jost solutions of (1.2), let us define the Faddeev functions m_{l} and m_{r} as follows:

$$m_{\mathrm{l}}(k,x):=e^{-ikx}f_{\mathrm{l}}(k,x), \quad m_{\mathrm{r}}(k,x):=e^{ikx}f_{\mathrm{r}}(k,x).$$

Then, we can write Υ_{l} and Υ_{r} defined in (4.20) and (4.21), respectively, as

$$\Upsilon_{\mathrm{l}}(x,t;x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[T(k)m_{\mathrm{l}}(k,x)m_{\mathrm{r}}(k,x_0)-1]e^{ik(x-x_0-t)}+\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}, \tag{6.1}$$

$$\Upsilon_{\mathrm{r}}(x,t;x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[T(k)m_{\mathrm{r}}(k,x)m_{\mathrm{l}}(k,x_0)-1]e^{-ik(x-x_0+t)}+\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}. \tag{6.2}$$

From Theorem 4.3 we know that U_{l} and U_{r} are causal. Hence, from (6.1) and (6.2) we conclude the following.

Corollary 6.1: Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then for $x>x_0+t$ *we have*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[T(k)m_{\mathrm{l}}(k,x)m_{\mathrm{r}}(k,x_0)-1]e^{ik(x-x_0-t)}=-\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}, \tag{6.3}$$

and for $x<x_0-t$ we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[T(k)m_{\mathrm{r}}(k,x)m_{\mathrm{l}}(k,x_0)-1]e^{-ik(x-x_0+t)}=-\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}. \tag{6.4}$$

Theorem 6.2: *Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then, for $t<x_0-x$, the tail Υ_{l} defined in (4.20) satisfies*

$$\Upsilon_{\mathrm{l}}(x,t;x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[m_{\mathrm{l}}(k,x)m_{\mathrm{l}}(-k,x_0)-1][e^{ik(x-x_0-t)}-e^{ik(x-x_0+t)}]. \tag{6.5}$$

Similarly, for $t<x-x_0$, the tail Υ_{r} defined in (4.21) satisfies

$$\Upsilon_{\mathrm{r}}(x,t;x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[m_{\mathrm{r}}(k,x)m_{\mathrm{r}}(-k,x_0)-1][e^{-ik(x-x_0+t)}-e^{-ik(x-x_0-t)}]. \tag{6.6}$$

Proof: From (3.3) we get

$$T(k)m_{\mathrm{r}}(k,x_0)=m_{\mathrm{l}}(-k,x_0)+R(k)e^{2ikx_0}m_{\mathrm{l}}(k,x_0), \quad k\in\mathbf{R},$$

and, hence, for real k, we have

$$T(k)m_{\mathrm{l}}(k,x)m_{\mathrm{r}}(k,x_0)=m_{\mathrm{l}}(k,x)m_{\mathrm{l}}(-k,x_0)+R(k)m_{\mathrm{l}}(k,x)e^{2ikx_0}m_{\mathrm{l}}(k,x_0). \tag{6.7}$$

From (3.3) we see that

$$R(k)m_{\mathrm{l}}(k,x)=e^{-2ikx}[T(k)m_{\mathrm{r}}(k,x)-m_{\mathrm{l}}(-k,x)], \quad k\in\mathbf{R}. \tag{6.8}$$

Thus, using (6.8) in the second term on the right-hand side of (6.7), we obtain

$$\begin{aligned}T(k)m_{\mathrm{l}}(k,x)m_{\mathrm{r}}(k,x_0)=&[T(k)m_{\mathrm{r}}(k,x)-m_{\mathrm{l}}(-k,x)]m_{\mathrm{l}}(k,x_0)e^{2ik(x_0-x)}\\&+m_{\mathrm{l}}(k,x)m_{\mathrm{l}}(-k,x_0), \quad k\in\mathbf{R}.\end{aligned} \tag{6.9}$$

Using (6.9) in (6.1) we get

$$\Upsilon_{\mathrm{l}}(x,t;x_0)=I_1-I_2+I_3-I_4+I_5+I_6,$$

where we have defined

$$I_1:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, m_{\mathrm{l}}(k,x)m_{\mathrm{l}}(-k,x_0)e^{ik(x-x_0-t)},$$

$$I_2:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, m_{\mathrm{l}}(-k,x)m_{\mathrm{l}}(k,x_0)e^{-ik(x-x_0+t)},$$

$$I_3:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{-ik(x-x_0+t)}, \quad I_4:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{ik(x-x_0-t)},$$

$$I_5:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk[T(k)m_{\mathrm{r}}(k,x)m_{\mathrm{l}}(k,x_0)-1]e^{-ik(x-x_0+t)},$$

$$I_6:=\sum_{j=1}^{N}\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}.$$

Because of (6.4) we have $I_5+I_6=0$ when $x_0-t-x>0$. Changing the dummy integration variable k to $-k$ in I_2 and I_3 we obtain (6.5). The proof of (6.6) is similar to that of (6.5) and is obtained with the help of (3.4) and (6.3). ■

From (6.5) and (6.6) we get the following antisymmetry properties for Υ_{l} and Υ_{r}.

Corollary 6.3: Assume V is real valued and belongs to $L_1^1(\mathbf{R})$. Then

$$\Upsilon_{\mathrm{l}}(x,-t;x_0)=-\Upsilon_{\mathrm{l}}(x,t;x_0), \quad t\in(x-x_0,x_0-x),$$

$$\Upsilon_{\mathrm{r}}(x,-t;x_0)=-\Upsilon_{\mathrm{r}}(x,t;x_0), \quad t\in(x_0-x,x-x_0),$$

$$\Upsilon_{\mathrm{l}}(x,-t;x_0)-\Upsilon_{\mathrm{r}}(x,-t;x_0)=\Upsilon_{\mathrm{r}}(x,t;x_0)-\Upsilon_{\mathrm{l}}(x,t;x_0), \quad x,t,x_0\in\mathbf{R}.$$

We remark that the temporal antisymmetry is a key part of the physics underlying wave focusing. It is immediate from Corollary 6.3 that both Υ_{l} and Υ_{r} vanish at $t=0$.

Next, we present some results related to the support properties of Υ_{l} when the potential vanishes on a half line. Similar results hold for Υ_{r} although they are not listed here. In the next theorem, we show that, if the incident wave focuses to a point lying behind the support of the potential, a gap develops between the wavefront and the tail of the wave.

Theorem 6.4: *Assume V is real valued, belongs to $L_1^1(\mathbf{R})$, and vanishes for $x>0$; let $x_0\geqslant 0$. Then we have the following.*

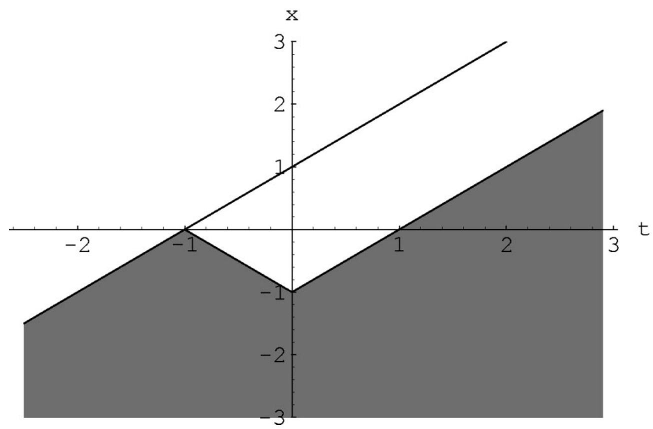

FIG. 1. The support of Υ_1 with $x_0=1$ in Theorem 6.4 is shaded in the tx-plane.

(i) *When* $t\geqslant 0$, *we have* $\Upsilon_1(x,t;x_0)=0$ *for* $x\geqslant -x_0+t$.
(ii) *When* $t\in[-x_0,0]$, *we have* $\Upsilon_1(x,t;x_0)=0$ *for* $x\geqslant -x_0-t$.
(iii) *Consequently, the wavefront* $\delta(x-x_0-t)$ *is in distance* $2x_0$ *ahead of the tail* Υ_1 *for* $t\geqslant 0$.
(iv) *Similarly, the wavefront* $\delta(x-x_0-t)$ *is in distance* $2(x_0+t)$ *ahead of the tail* Υ_1 *for* $t\in[-x_0,0]$.

Proof: The proof will be given by showing that Υ_1 vanishes on the closure of the rectangular region lying below the wavefront and above the shaded region exemplified in Fig. 1. If $V\equiv 0$ for $x>0$, then $m_1(k,x)=1$ for $x\geqslant 0$ and $T(k)m_{\mathrm{r}}(k,x_0)=1+R(k)e^{2ikx_0}$. Thus, from (6.1), with the help of (2.7), we get

$$\Upsilon_1(x,t;x_0)=\hat{R}(x+x_0-t)+\sum_{j=1}^{N} c_{1j}^2 e^{-\kappa_j(x+x_0-t)}, \quad x\geqslant 0. \tag{6.10}$$

Comparing (6.10) with (5.1), we see that $\Upsilon_1(x,t;x_0)=M_1(x+x_0-t)$; hence, with the help of Proposition 5.4, we conclude that $\Upsilon_1(x,t;x_0)=0$ for $x>-x_0+t$ and $x\geqslant 0$. By the continuity of Υ_1 on the line $x=-x_0+t$, as indicated in Theorem 4.5, we get $\Upsilon_1(x,t;x_0)=0$ for $x\geqslant -x_0+t$.

On the other hand, when $x\leqslant 0$, using $T(k)m_{\mathrm{r}}(k,x_0)=1+R(k)e^{2ikx_0}$, from (6.1) we get $\Upsilon_1(x,t;x_0)=I_7+I_8$, where we have defined

$$I_7:=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk[m_1(k,x)-1]e^{ik(x-x_0-t)}, \tag{6.11}$$

$$I_8:=\frac{1}{2\pi}\int_{-\infty}^{\infty} dkR(k)m_1(k,x)e^{ik(x+x_0-t)}+\sum_{j=1}^{N} \varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}. \tag{6.12}$$

For $x>-x_0+t$, from (6.12) we obtain

$$I_8=i\sum_{j=1}^{N} \mathrm{Res}(R,i\kappa_j)m_1(i\kappa_j,x)e^{-\kappa_j(x+x_0-t)}+\sum_{j=1}^{N} c_{1j}^2 m_1(i\kappa_j,x)e^{-\kappa_j(x+x_0-t)}.$$

As mentioned in the proof of Proposition 5.4, we have $\mathrm{Res}(R,i\kappa_j)=ic_{1j}^2$, and hence $I_8=0$ for $x\leqslant 0$ and $x>-x_0+t$. Thus, from (4.1) and (6.11), we see that $\Upsilon_1(x,t;x_0)=K_1(x,x_0+t)$ for $x\leqslant 0$ and $x>-x_0+t$. From the Marchenko equation (5.3), we get

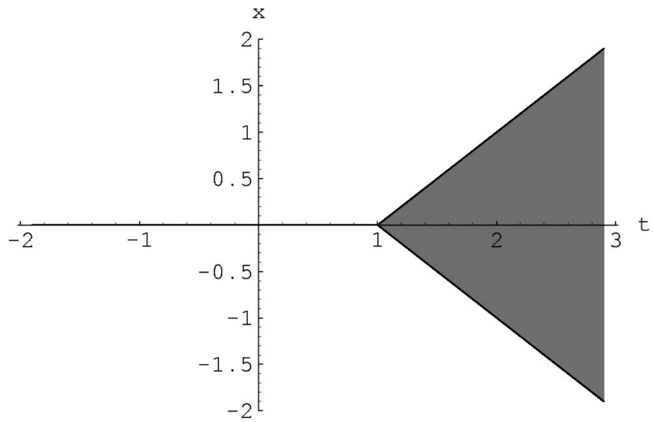

FIG. 2. The support of Υ_1 with $x_0=-1$ in Theorem 6.5 is shaded in the tx-plane.

$$K_1(x,x_0+t)+M_1(x+x_0+t)+\int_x^\infty ds M_1(x_0+t+s)K_1(x,s)=0, \quad x_0+t>x. \tag{6.13}$$

Note that in our region of interest, we have $x+x_0+t>0$, and hence by Proposition 5.4 each of the second and third terms on the left-hand side of (6.13) vanishes. Thus, from (6.13) we conclude $K_1(x,x_0+t)=0$, which in turn gives us $\Upsilon_1(x,t;x_0)=0$ for $x\leqslant 0$ and $x>\max\{-x_0-t,-x_0+t\}$. Again by the continuity of Υ_1 everywhere except at the wavefront of U_1, we get $\Upsilon_1(x,t;x_0)=0$ also on the line segment $x=-x_0+t$ for $t\in[0,x_0]$ and on the segment $x=-x_0-t$ for $t\in[-x_0,0]$. Therefore, we have proved (i) and (ii). Note that (iii) and (iv) directly follow from (i) and (ii). ■

In the next two theorems, when $V\equiv 0$ for $x<0$, we investigate the support properties of the focusing wave U_1.

Theorem 6.5: *Assume V is real valued, belongs to $L_1^1(\mathbf{R})$, and vanishes for $x<0$; let $x_0\leqslant 0$. Then we have the following.*

- *(i) At each time $t\leqslant -x_0$, we have $\Upsilon_1(x,t;x_0)=0$.*
- *(ii) At each time $t\geqslant -x_0$, we have $\Upsilon_1(x,t;x_0)=0$ for $x\leqslant -x_0-t$.*
- *(iii) Consequently, at each fixed time $t>-x_0$, the support of $\Upsilon_1(\cdot,t;x_0)$ is the finite interval $(-x_0-t,x_0+t)$.*

Proof: The proof will be given by showing that the support of Υ_1 is confined to the interior of the region exemplified in Fig. 2. Because U_1 is a causal wave, in the region $x\leqslant \pm x_0\pm t$ we have $x\leqslant 0$; hence $T(k)m_1(k,x)=1+L(k)e^{-2ikx}$, and $m_{\mathrm{r}}(k,x_0)=1$. Thus, from (6.3), with the help of (2.7), we get

$$\Upsilon_1(x,t;x_0)=\hat{L}(-x-x_0-t)+\sum_{j=1}^N c_{\mathrm{r}j}^2 e^{\kappa_j(x+x_0+t)}, \quad x\leqslant 0. \tag{6.14}$$

From (5.1) and (6.14) we see that $\Upsilon_1(x,t;x_0)=M_{\mathrm{r}}(x+x_0+t)$, and by Proposition 5.4 we conclude $\Upsilon_1(x,t;x_0)=0$ for $x<-x_0-t$. Because of the continuity of Υ_1 in the tx-plane off the wavefront, we also get $\Upsilon_1(x,t;x_0)=0$ on the line segment $x=-x_0-t$ for $t\geqslant -x_0$. ■

Theorem 6.6: *Assume V is real valued, belongs to $L_1^1(\mathbf{R})$, and vanishes for $x<0$; let $x_0\geqslant 0$. Then we have the following:*

- *(i) At each time $t\leqslant 0$, we have $\Upsilon_1(x,t;x_0)=0$ for $x\leqslant -x_0+t$.*
- *(ii) At each time $t\geqslant 0$, we have $\Upsilon_1(x,t;x_0)=0$ for $x\leqslant -x_0-t$.*

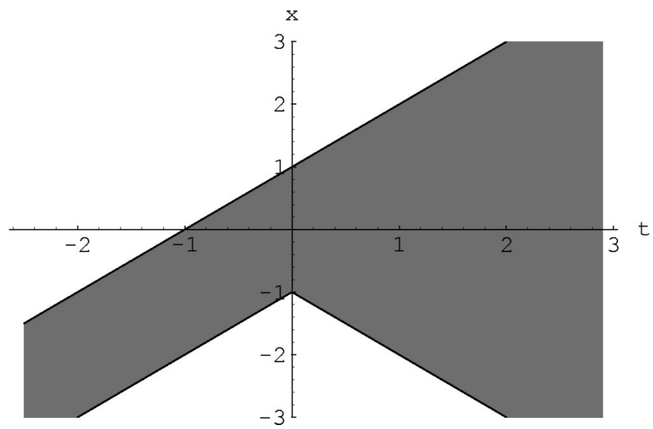

FIG. 3. The support of Υ_1 with $x_0 = 1$ in Theorem 6.6 is shaded in the tx-plane.

(iii) Consequently, at each fixed $t \leqslant 0$, *the support of* $\Upsilon_1(\cdot, t; x_0)$ *is the finite interval* $(-x_0 + t, x_0 + t)$. *Similarly, at each fixed* $t \geqslant 0$, *the support of* $\Upsilon_1(\cdot, t; x_0)$ *is the finite interval* $(-x_0 - t, x_0 + t)$.

Proof: The proof is obtained by using arguments similar to those used in the proof of Theorem 6.4 and by showing that the support of Υ_1 is confined to the interior of the shaded region exemplified in Fig. 3. ■

VII. DISCONTINUITIES IN DERIVATIVES OF FOCUSING WAVES

In this section, under more restrictions on V, we analyze the discontinuities in the spatial and temporal derivatives of our focusing waves, and we present some corollaries of our analysis. First, we show that we can recover $V(x_0)$ remotely by using only the wave that focuses to x_0 with a measurement taken at any fixed specified time $t < -x_0$; this complements our result in Sec. IV on the recovery of $V(x_0)$ from measurements on waves focusing to x_0 and its vicinity. Next, we examine the t-derivative of our focusing waves at $t = 0$. Finally, for potentials vanishing on a half line, we derive an identity involving the temporal and spatial derivatives of the solutions to the Marchenko equations. We present the results mainly for the focusing wave incident from the left because the results for the incidence from the right can be obtained analogously.

In our analysis in the section, we put some or all of the following restrictions on V:

(H1) V is real valued and belongs to $L_1^1(\mathbf{R})$.

(H2) V is sectionally continuous with jump discontinuities at $x = a_j$ for $j = 1, \ldots, n$.

(H3) V is piecewise continuously differentiable in each of the intervals $(-\infty, a_1)$, $(a_n, +\infty)$, and (a_j, a_{j+1}) with $j = 1, \ldots, n-1$.

(H4) V' is integrable in each interval of its continuity.

(H5) V' is piecewise differentiable, and V'' is integrable in each interval it exists.

Paraphrasing, hypothesis (H2) states that V is continuous on $\mathbf{R}$ except perhaps at a finite number of points, and V has finite left and right limits at those points; (H3) is a similar statement for V'. Hypotheses (H4) and (H5) state that V' and V'' exist everywhere except perhaps at a finite number of points; moreover, if we remove those points from $\mathbf{R}$, V' and V'' are integrable on the resulting set.

Define

$$\alpha_l(x) := \int_x^\infty dy\, V(y), \quad \alpha_r(x) := \int_{-\infty}^x dy\, V(y), \quad \beta := \int_{-\infty}^\infty dy\, V(y), \tag{7.1}$$

$$q_l(k,x) := V(x) + \sum_{a_j > x} [V(a_j^+) - V(a_j^-)] e^{2ik(a_j - x)}, \tag{7.2}$$

$$q_{\mathrm{r}}(k,x):=-V(x)+\sum_{a_j<x}[V(a_j^+)-V(a_j^-)]e^{2ik(x-a_j)}. \tag{7.3}$$

In the next proposition we list the large-k asymptotics of the transmission coefficient and the Faddeev functions and their x-derivatives up to the orders needed in (7.15) and (7.16) [cf. p. 163 of Deift and Trubowitz(1979) where some expansions are given up to $o(1/k^2)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$].

Proposition 7.1: (i) Assume V satisfies hypotheses (H1)–(H4). Then, as $k\to\infty$ *in* $\overline{\mathbf{C}^+}$ *we have*

$$T(k)=1+\frac{\beta}{2ik}-\frac{\beta^2}{8k^2}+O(1/k^3), \tag{7.4}$$

$$m_{\mathrm{l}}(k,x)=1-\frac{\alpha_{\mathrm{l}}(x)}{2ik}-\frac{1}{8k^2}[\alpha_{\mathrm{l}}(x)^2-2q_{\mathrm{l}}(k,x)]+O(1/k^3), \tag{7.5}$$

$$m_{\mathrm{r}}(k,x)=1-\frac{\alpha_{\mathrm{r}}(x)}{2ik}-\frac{1}{8k^2}[\alpha_{\mathrm{r}}(x)^2+2q_{\mathrm{r}}(k,x)]+O(1/k^3). \tag{7.6}$$

(ii) In addition, if V satisfies also hypothesis (H5), then as $k\to\infty$ *in* $\overline{\mathbf{C}^+}$ *we have*

$$m_{\mathrm{l}}'(k,x)=\frac{q_{\mathrm{l}}(k,x)}{2ik}+O(1/k^2), \quad m_{\mathrm{r}}'(k,x)=\frac{q_{\mathrm{r}}(k,x)}{2ik}+O(1/k^2). \tag{7.7}$$

Proof: The proof is straightforward. For example, (7.5) is obtained by iterating the integral representation for m_{l}, see, e.g., Deift and Trubowitz (1979),

$$m_{\mathrm{l}}(k,x)=1+\frac{1}{2ik}\int_x^\infty dy[e^{2ik(y-x)}-1]V(y)m_{\mathrm{l}}(k,y), \tag{7.8}$$

and using integration by parts on $\int_x^\infty dy e^{2ik(y-x)}V(y)$. Differentiating (7.8) with respect to x and using iteration and integration by parts, we obtain the asymptotics for m_{l}'. Similarly, with the help of the integral representations

$$m_{\mathrm{r}}(k,x)=1+\frac{1}{2ik}\int_{-\infty}^x dy[e^{2ik(x-y)}-1]V(y)m_{\mathrm{r}}(k,y),$$

$$\frac{1}{T(k)}=1-\frac{1}{2ik}\int_{-\infty}^\infty dy V(y)m_{\mathrm{l}}(k,y),$$

we obtain the large-k asymptotics for m_{r}, m_{r}', and T. ∎

Let $\theta(x)$ denote the Heaviside function, and with the help of (7.1)–(7.3) let us define

$$D(x,x_0):=\beta[\alpha_{\mathrm{l}}(x)+\alpha_{\mathrm{r}}(x_0)]-\alpha_{\mathrm{l}}(x)\alpha_{\mathrm{r}}(x_0)-\frac{1}{2}[\beta^2+\alpha_{\mathrm{l}}(x)^2+\alpha_{\mathrm{r}}(x_0)^2]. \tag{7.9}$$

Using (7.1) we simplify the right-hand side in (7.9) to obtain

$$D(x,x_0)=-\frac{1}{2}\left[\int_x^{x_0} dy V(y)\right]^2. \tag{7.10}$$

Theorem 7.2: *Assume that V satisfies hypotheses (H1)–(H5). Then, the discontinuous part of* $\partial\Upsilon_{\mathrm{l}}(x,t;x_0)/\partial x$ *is given by*

$$\frac{1}{4}[V(x_0)-V(x)+D(x,x_0)]\theta(x_0+t-x)-\frac{1}{4}\theta(x_0+t-x)\sum_{a_j>x}[V(a_j^+)-V(a_j^-)]$$
$$\times\theta(x-2a_j+x_0+t)-\frac{1}{4}\theta(x_0+t-x)\sum_{a_j<x_0}[V(a_j^+)-V(a_j^-)]\theta(-x+2a_j-x_0+t), \tag{7.11}$$

where $D(x,x_0)$ is the quantity in (7.10).

Proof: Consider the representation of Υ_1 given in (6.1), from which we get

$$\frac{\partial\Upsilon_1(x,t;x_0)}{\partial x}=\theta(x_0+t-x)\left[I_{10}+I_{11}+\sum_{j=1}^{N}\varphi_j'(x)\varphi_j(x_0)e^{\kappa_j t}\right], \tag{7.12}$$

where we have defined

$$I_{10}:=\frac{i}{2\pi}\int_{-\infty}^{\infty}dk\ k[T(k)m_1(k,x)m_{\mathrm{r}}(k,x_0)-1]e^{ik(x-x_0-t)}, \tag{7.13}$$

$$I_{11}:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dkT(k)m_1'(k,x)m_{\mathrm{r}}(k,x_0)e^{ik(x-x_0-t)}. \tag{7.14}$$

When $V\in L_1^1(\mathbf{R})$, it is known that φ_j' is continuous; hence, the summation term within the brackets in (7.12) is continuous in (x,t). From the integrands in (7.13) and (7.14) let us separate the terms that are continuous in (k,x) and integrable in k; by the Lebesgue dominated convergence theorem, the integrals of such terms are continuous in (x,t). Using (7.1)–(7.7) and (7.9), as $k\to\infty$ in $\overline{\mathbf{C}^+}$ we obtain

$$ik[T(k)m_1(k,x)m_{\mathrm{r}}(k,x_0)-1]=-\frac{1}{2}\int_x^{x_0}dyV(y)+\frac{i}{4(k+i0^+)}[D(x,x_0)+q_1(k,x)-q_{\mathrm{r}}(k,x_0)]$$
$$+O(1/k^2), \tag{7.15}$$

$$T(k)m_1'(k,x)m_{\mathrm{r}}(k,x_0)=\frac{q_1(k,x)}{2i(k+i0^+)}+O(1/k^2), \tag{7.16}$$

where the $O(1/k^2)$-terms are continuous in (k,x) and integrable in k. When $x<x_0+t$, the $O(1)$-term in (7.15) does not contribute to the integral in (7.13). Using $\int_{-\infty}^{\infty}dke^{ikz}/(k+i0^+)=-2\pi i\theta(-z)$, we evaluate the contribution of the $O(1/k)$-terms in (7.13) and (7.14) to I_{10} and I_{11}, respectively, which, with the help of (7.10), results in (7.11). ∎

Let us define

$$G(x,t;x_0):=-\frac{1}{4}[V(x_0)+V(x)+D(x,x_0)]\theta(x_0+t-x)$$
$$-\frac{1}{4}\theta(x_0+t-x)\sum_{a_j>x}[V(a_j^+)-V(a_j^-)]\theta(x-2a_j+x_0+t)$$
$$+\frac{1}{4}\theta(x_0+t-x)\sum_{a_j<x_0}[V(a_j^+)-V(a_j^-)]\theta(-x+2a_j-x_0+t), \tag{7.17}$$

where $D(x,x_0)$ is the quantity in (7.10).

Theorem 7.3: *Assume that V satisfies hypotheses (H1)–(H5). Then, the discontinuous part of $\partial\Upsilon_1(x,t;x_0)/\partial t$ is equal to $G(x,t;x_0)$ defined in (7.17).*

Proof: As in (7.12) we have

$$\frac{\partial \Upsilon_{\mathrm{l}}(x,t;x_0)}{\partial t}=\theta(x_0+t-x)\left[-I_{10}+\sum_{j=1}^{N}\kappa_j\varphi_j(x)\varphi_j(x_0)e^{\kappa_j t}\right]. \tag{7.18}$$

Proceeding as in the proof of Theorem 7.2, we see that the only contribution to the discontinuities comes from the $O(1/k)$-term in (7.13), which gives us (7.17). ∎

As a corollary of (7.18), we see that $U_{\mathrm{l}}(x,t;x_0)$ is the solution to (1.1) satisfying the initial conditions $U_{\mathrm{l}}(x,0;x_0)=\delta(x-x_0)$ and

$$\frac{\partial U_{\mathrm{l}}(x,0;x_0)}{\partial t}=\delta'(x-x_0)+\sum_{j=1}^{N}\kappa_j\varphi_j(x)\varphi_j(x_0)$$
$$-\frac{i}{2\pi}\int_{-\infty}^{\infty}dk\; k[T(k)m_{\mathrm{l}}(k,x)m_{\mathrm{r}}(k,x_0)-1]e^{ik(x-x_0)}. \tag{7.19}$$

With the help of (7.11)–(7.13), it is possible to identify the discontinuities on the right-hand side of (7.19) and hence also in $\partial U_{\mathrm{l}}(x,0;x_0)/\partial t$. The initial value $\partial U_{\mathrm{r}}(x,0;x_0)/\partial t$ can be obtained similarly and its discontinuities can be evaluated explicitly in an analogous manner.

Next, we turn our attention to the inverse scattering problem and describe the recovery of $V(x_0)$ by using only the wave $U_{\mathrm{l}}(x,t;x_0)$ focusing to x_0. Assume that V satisfies hypotheses (H1)–(H5) and that x_0 is a point of continuity of V. As stated below (4.34), we know that the height of the tail Υ_{l} at the wavefront at any fixed time $t<-x_0$ is given by (4.30). Furthermore, from (7.11) we see that the x-derivative from the left for the tail Υ_{l} at the wavefront is given by

$$\frac{\partial \Upsilon_{\mathrm{l}}(x_0^-+t,t;x_0)}{\partial x}=\frac{1}{4}[V(x_0)-V(x_0+t)]-\frac{1}{8}\left(\int_{x_0+t}^{x_0}dz V(z)\right)^2. \tag{7.20}$$

Eliminating the integral term in (4.30) and (7.20), we get

$$V(x_0)=V(x_0+t)+2\Upsilon_{\mathrm{l}}(x_0^-+t,t;x_0)^2+4\,\frac{\partial \Upsilon_{\mathrm{l}}(x_0^-+t,t;x_0)}{\partial x}. \tag{7.21}$$

Note that all of the three terms on the right-hand side of (7.21) can be measured at some moment $t<-x_0$, where x_0+t is a point of continuity for V; in fact, we can even make our measurement when $t\to-\infty$, in which case $V(x_0+t)\to 0$. Thus, $V(x_0)$ can be remotely determined by using only the wave that focuses to x_0.

As a corollary of the arguments leading to (5.19) and (7.21) we obtain the following property for the solution to the Marchenko equation (5.2).

Theorem 7.4: *Assume V satisfies hypotheses (H1)–(H5) and $V\equiv 0$ for $x<0$. Then, for any point $x\in\mathbf{R}$ at which V is continuous, the solution $K_{\mathrm{r}}(x,t)$ of the Marchenko equation (5.2) satisfies*

$$\frac{\partial K_{\mathrm{r}}(x,x^-)}{\partial x}-\frac{\partial K_{\mathrm{r}}(x,x^-)}{\partial t}=K_{\mathrm{r}}(x,x^-)^2,\quad x\in\mathbf{R}. \tag{7.22}$$

Proof: Note that (7.22) holds trivially for all $x\leqslant 0$ because $K_{\mathrm{r}}(x,t)=0$ for $x\leqslant 0$ as indicated above (5.19). From the Marchenko method, it is known that

$$V(x_0)=2\,\frac{\partial K_{\mathrm{r}}(x_0,x_0^-)}{\partial x}+2\,\frac{\partial K_{\mathrm{r}}(x_0,x_0^-)}{\partial t},\quad x\in\mathbf{R}. \tag{7.23}$$

From (5.19) we see that $\Upsilon_l(x,t;x_0)=K_r(x_0,x-t)$ for all $x<0$ and $t<-x_0$; hence, we can write (7.21) with $x_0>0$ and $t<-x_0$ as

$$V(x_0)=2K_r(x_0,x_0^-)^2+4\,\frac{\partial K_r(x_0,x_0^-)}{\partial t},\quad x_0>0. \tag{7.24}$$

Finally, comparing (7.23) and (7.24), we see that (7.22) also holds for $x>0$. ■

We next present the analog of Theorem 7.4 for K_l without a proof. It can also be obtained from (7.22) by changing the signs of x and t.

Theorem 7.5: *Assume V satisfies hypotheses (H1)–(H5) and* $V\equiv 0$ *for* $x>0$. *Then, for any point* $x\in\mathbf{R}$ *at which V is continuous, the solution* $K_l(x,t)$ *of the Marchenko equation (5.3) satisfies*

$$\frac{\partial K_l(x,x^+)}{\partial x}-\frac{\partial K_l(x,x^+)}{\partial t}=K_l(x,x^+)^2,\quad x\in\mathbf{R}. \tag{7.25}$$

VIII. EXAMPLES

In this section we illustrate wave focusing, various properties of focusing waves, and the recovery of a potential via wave focusing. An animated example of wave focusing can be found at the web site http://www.msstate.edu/~aktosun/publications.html.

Example 8.1: Consider the wave focusing for

$$V(x)=\theta(-x)\,\frac{16(\sqrt{2}+1)^2e^{-2\sqrt{2}x}}{[(\sqrt{2}+1)^2e^{-2\sqrt{2}x}-1]^2}.$$

Note that V, being non-negative, does not support any bound states; moreover, it is supported in $(-\infty,0)$ and hence the region $(0,+\infty)$ is the "free region." The corresponding scattering coefficients are rational functions of k, and one can obtain explicitly the scattering coefficients and Jost solutions of (1.2). We have $f_l(k,x)=e^{ikx}$ for $x\geqslant 0$,

$$f_r(k,x)=e^{-ikx}\left[1+\frac{i}{k+\sqrt{2}i}\,\frac{2\sqrt{2}}{(\sqrt{2}+1)^2e^{-2\sqrt{2}x}-1}\right],\quad x\leqslant 0,$$

$$T(k)=\frac{k(k+\sqrt{2}i)}{(k+i)^2},\quad L(k)=-\frac{1}{(k+i)^2}\,\frac{k+\sqrt{2}i}{k-\sqrt{2}i},\quad R(k)=\frac{1}{(k+i)^2}.$$

Using a contour integration in (4.18), we can evaluate U_l explicitly and verify that U_l focuses to x_0 at time $t=0$. When $x_0>0$, as indicated in Theorem 6.4, a gap develops between the wavefront and the tail, which is illustrated in Fig. 4. When $x_0<0$, the corresponding wave is illustrated in Fig. 5. It can be verified directly that the discontinuity in the tail at the wavefront, the discontinuities in the x-derivative of the tail, and the discontinuities in the t-derivative of the tail agree with the results in (4.30), (7.11), and (7.17), respectively.

Next, we illustrate the recovery of $V(x_0)$ by using the technique described in Sec. IV utilizing (4.35).

Example 8.2: Consider the wave focusing for the potential

$$V(x)=\theta(x)\,\frac{80(\sqrt{5}+1)(\sqrt{5}+2)e^{2\sqrt{5}x}}{[(\sqrt{5}+1)(\sqrt{5}+2)e^{2\sqrt{5}x}-2]^2}. \tag{8.1}$$

The corresponding scattering coefficients and Jost solutions can be evaluated explicitly as

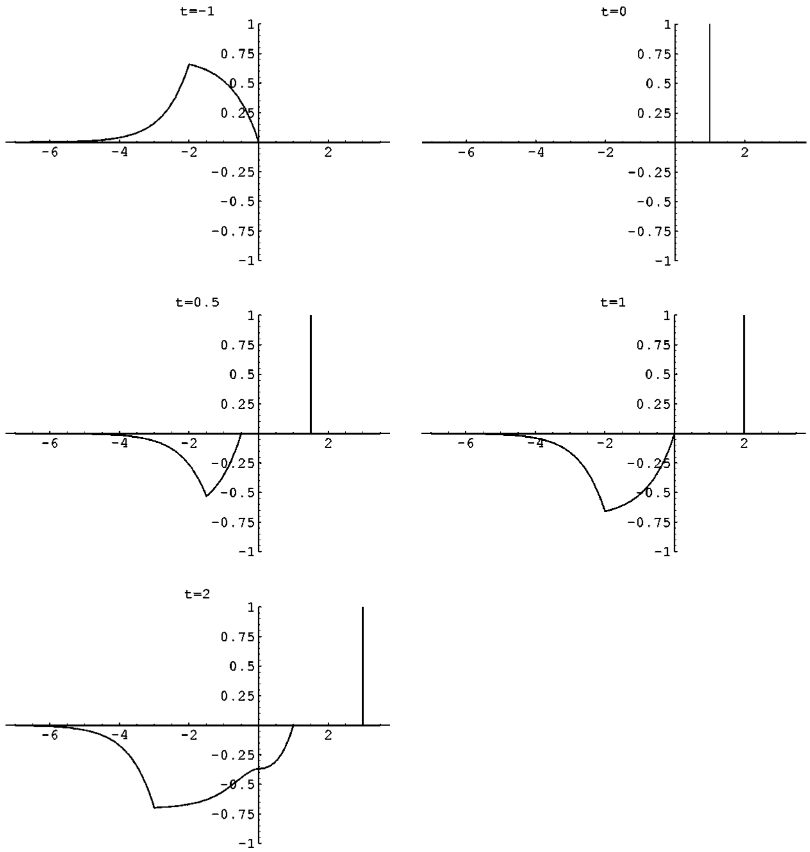

FIG. 4. The focusing wave of Example 8.1 with $x_0=1$ is shown at $t=-1,0,0.5,1,2$.

$$T(k)=\frac{k(k+\sqrt{5}i)}{(k+i)(k+2i)},\quad L(k)=\frac{2}{(k+i)(k+2i)},\quad R(k)=\frac{-2(k+\sqrt{5}i)}{(k+i)(k+2i)(k-\sqrt{5}i)},$$

$$f_{\mathrm{l}}(k,x)=e^{ikx}\left[1+\frac{iE(x)}{k+\sqrt{5}i}\right],\quad x\geqslant 0;\quad f_{\mathrm{r}}(k,x)=e^{-ikx},\quad x\leqslant 0,$$

$$f_{\mathrm{l}}(k,x)=\frac{(k+i)(k+2i)}{k(k+\sqrt{5}i)}e^{ikx}+\frac{2}{k(k+\sqrt{5}i)}e^{-ikx},\quad x\leqslant 0,$$

$$f_{\mathrm{r}}(k,x)=e^{-ikx}\left[1-\frac{iE(x)}{k-\sqrt{5}i}\right]-\frac{2(k+\sqrt{5}i)e^{ikx}}{(k+i)(k+2i)(k-\sqrt{5}i)}\left[1+\frac{iE(x)}{k+\sqrt{5}i}\right],\quad x\geqslant 0,$$

where

$$E(x):=\frac{4\sqrt{5}}{(1+\sqrt{5})(2+\sqrt{5})e^{2\sqrt{5}x}-2}.$$

Using (4.18) it is possible to construct $U_{\mathrm{l}}(x,t;x_0)$ explicitly. Since $V\equiv 0$ when $x<0$, we see that $V(x_0+t)=0$ for any fixed $t<-x_0$. Note that $\Gamma_{\mathrm{l}}(x_0,t)$ given in (4.34) can be computed as $\sqrt{5}-3+E(x_0)$. Then, using (4.35), we obtain $V(x_0)=2E'(x_0)$, agreeing with (8.1).

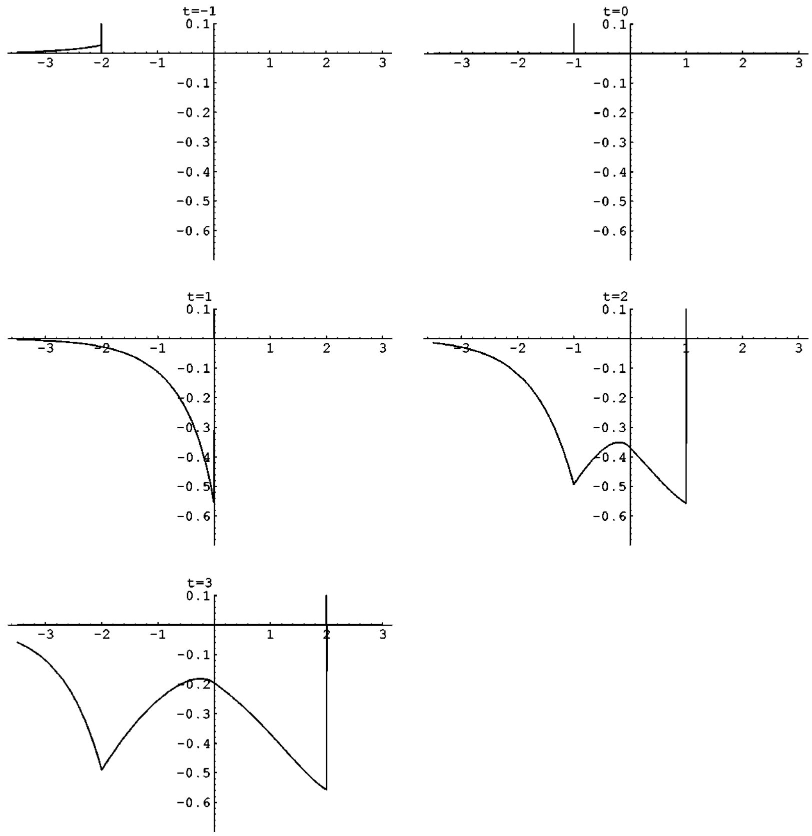

FIG. 5. The focusing wave of Example 8.1 with $x_0=-1$ is shown at $t=-1,0,1,2,3$.

Finally, we illustrate the recovery of $V(x_0)$ by using the technique described in Sec. VII utilizing (7.21).

Example 8.3: Consider the wave focusing for the potential

$$V(x)=-\frac{16c^2e^{2x}}{(2e^{2x}+c^2)^2}, \tag{8.2}$$

where $c>0$ is the bound-state norming constant. The corresponding scattering coefficients are $T(k)=(k+i)/(k-i)$, $L(k)=R(k)=0$, and the Jost solutions for $x\in\mathbf{R}$ are given by

$$f_{\rm l}(k,x)=e^{ikx}\left[1-\frac{2i}{k+i}\,\frac{c^2}{2e^{2x}+c^2}\right],\quad f_{\rm r}(k,x)=e^{-ikx}\left[1-\frac{4i}{k+i}\,\frac{e^{2x}}{2e^{2x}+c^2}\right].$$

Using (4.18) we get

$$U_{\rm l}(x,t;x_0)=\delta(x-x_0-t)+\theta(-x+x_0+t)\,\frac{4c^2e^{x+x_0}(e^t-e^{-t})}{(2e^{2x}+c^2)(2e^{2x_0}+c^2)}.$$

Hence,

$$\Upsilon_{\rm l}(x_0^-+t,t;x_0)=\frac{4c^2e^{t+x_0}(e^t-e^{-t})}{(2e^{2x_0}+c^2)(2e^{2(t+x_0)}+c^2)},$$

$$\lim_{t\to-\infty} \Upsilon_l(x_0^-+t,t;x_0) = -\frac{4e^{x_0}}{2e^{2x_0}+c^2}, \tag{8.3}$$

$$\frac{\partial \Upsilon_l(x_0^-+t,t;x_0)}{\partial x} = \frac{4c^2 e^{2x_0+t}(e^t-e^{-t})(c^2-e^{2(t+x_0)})}{(2e^{2x_0}+c^2)(2e^{2(t+x_0)}+c^2)^2},$$

$$\lim_{t\to-\infty} \frac{\partial \Upsilon_l(x_0^-+t,t;x_0)}{\partial x} = \frac{4e^{2x_0}}{2e^{2x_0}+c^2}. \tag{8.4}$$

Since $\lim_{t\to-\infty} V(x_0+t)=0$, using (8.3) and (8.4) in (7.21), we can construct $V(x_0)$ explicitly, agreeing with its value obtained from (8.2).

IX. FOCUSING FOR THE VARIABLE-SPEED WAVE EQUATION

In this section we analyze focusing for the variable-speed wave equation given in (9.14) by using the corresponding results for (1.1).

Consider the generalized Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2} + k^2 H(x)^2 \psi(k,x) = Q(x)\psi(k,x), \quad x\in\mathbf{R}, \tag{9.1}$$

where Q is real valued and belongs to $L_1^1(\mathbf{R})$, and H is bounded, strictly positive, $H-1 \in L^1(\mathbf{R})$, and $2HH''-3(H')^2 \in L_1^1(\mathbf{R})$. Via the Liouville transformation

$$y=y(x) := \int_0^x dz H(z), \quad \phi(k,y(x)) := \sqrt{H(x)}\,\psi(k,x), \tag{9.2}$$

we can transform (9.1) into the Schrödinger equation

$$\frac{d^2\phi(k,y)}{dy^2} + k^2\phi(k,y) = V(y)\phi(k,y), \quad y\in\mathbf{R}, \tag{9.3}$$

with

$$V(y)=V(y(x)) := \frac{Q(x)}{H(x)^2} + \frac{H''(x)}{2H(x)^3} - \frac{3H'(x)^2}{4H(x)^4}. \tag{9.4}$$

The aforementioned conditions on Q and H guarantee that V is real valued and belongs to $L_1^1(\mathbf{R})$, for which the direct and inverse scattering problems are well understood.

Let $f_l(k,x)$ and $f_r(k,x)$ denote the Jost solutions of (9.1) satisfying the boundary conditions (2.1) and (2.2), respectively. The scattering coefficients for (9.1) are obtained as in (2.3) and (2.4). For the analysis of the scattering and inverse scattering problems for (9.1), see, e.g., Aktosun *et al.* (1992a,b). It is known that the potential $V(y)$ has bound states if and only if $Q(x)$ has bound states. Since $H(x)$ is strictly positive, the mapping $x\mapsto y$ is one-to-one. Thus, for any x_0 there is a unique $y_0 := y(x_0)$, and conversely.

Let us denote the Jost solutions of (9.3) by $g_l(k,y)$ and $g_r(k,y)$, from the left and right, respectively. Let us use $\tau(k)$, $\rho(k)$, and $\ell(k)$ to denote the transmission coefficient and the reflection coefficients from the right and left, respectively, for (9.3). We have, see, e.g., Aktosun *et al.* (1992a),

$$g_l(k,y) = e^{-ikA_+}\sqrt{H(x)}\,f_l(k,x), \quad g_r(k,y) = e^{-ikA_-}\sqrt{H(x)}\,f_r(k,x), \tag{9.5}$$

$$\tau(k) = T(k)e^{ikA}, \quad \ell(k) = L(k)e^{2ikA_-}, \quad \rho(k) = R(k)e^{2ikA_+}, \tag{9.6}$$

where

$$A_\pm := \pm \int_0^{\pm\infty} dt[1-H(t)], \quad A := A_- + A_+ .$$

As seen from the first formula in (9.6) the bound states for (9.1) and (9.3) occur simultaneously at the same k-value on the positive imaginary axis, i.e., at the common poles of $T(k)$ and $\tau(k)$ in $\mathbf{C}^+$. We let N denote the number of bound states for (9.1) and let the bound states occur at $k = i\kappa_j$ with $0<\kappa_1<\cdots<\kappa_N$.

In terms of the Jost solutions of (9.3), as in (4.1) and (4.2), let us define

$$\tilde{K}_{\mathrm{l}}(y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[g_{\mathrm{l}}(k,y)-e^{iky}]e^{-ikt},$$

$$\tilde{K}_{\mathrm{r}}(y,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[g_{\mathrm{r}}(k,y)-e^{-iky}]e^{ikt}.$$

As in (4.4) we have

$$\tilde{K}_{\mathrm{l}}(y,t)=0, \quad t<y; \quad \tilde{K}_{\mathrm{r}}(y,t)=0, \quad t>y.$$

For each fixed $y\in\mathbf{R}$, $\tilde{K}_{\mathrm{l}}(y,\cdot)$ and $\tilde{K}_{\mathrm{r}}(y,\cdot)$ belong to $L^2(\mathbf{R})\cap L^1(\mathbf{R})$. They are discontinuous at $t=y$, and as in (4.5) the jumps there are related to V as

$$\tilde{K}_{\mathrm{l}}(y,y^+) = \frac{1}{2}\int_y^{\infty} dz V(z), \quad \tilde{K}_{\mathrm{r}}(y,y^-) = \frac{1}{2}\int_{-\infty}^{y} dz V(z).$$

Next we present the analog of Theorem 3.1 for (9.1).

Theorem 9.1: *Assume Q and H satisfy the conditions stated below (9.1). Then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk T(k) f_{\mathrm{l}}(k,x)\, f_{\mathrm{r}}(k,x_0) + \sum_{j=1}^{N} \frac{\varphi_j(x)\varphi_j(x_0)}{H(x)H(x_0)} = \frac{\delta(x-x_0)}{H(x)H(x_0)}, \tag{9.7}$$

where $\varphi_j(x)$ are the normalized bound-state wave functions for (9.1) corresponding to the bound states at $k=i\kappa_j$ with $j=1,\ldots,N$.

Proof: Using (9.5) and (9.6), from (3.6) we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \tau(k) g_{\mathrm{l}}(k,y) g_{\mathrm{r}}(k,y_0) + \sum_{j=1}^{N} \xi_j(y)\xi_j(y_0) = \delta(y-y_0), \tag{9.8}$$

where $\xi_j(y)$ are the normalized bound-state wave functions for (9.3). Using (9.5) and (9.6), we can write the first term on the left-hand side of (9.8) as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \tau(k) g_{\mathrm{l}}(k,y) g_{\mathrm{r}}(k,y_0) = \frac{\sqrt{H(x)H(x_0)}}{2\pi}\int_{-\infty}^{\infty} dk T(k) f_{\mathrm{l}}(k,x) f_{\mathrm{r}}(k,x_0). \tag{9.9}$$

As in (2.5) and (2.7), for $j=1,\ldots,N$ we have

$$\xi_j(y) = \frac{g_{\mathrm{l}}(i\kappa_j,y)}{\sqrt{\int_{-\infty}^{\infty} dz g_{\mathrm{l}}(i\kappa_j,z)^2}}, \quad \varphi_j(x) = \frac{H(x) f_{\mathrm{l}}(i\kappa_j,x)}{\sqrt{\int_{-\infty}^{\infty} dz H(z)^2 f_{\mathrm{l}}(i\kappa_j,z)^2}}. \tag{9.10}$$

With the help of (9.5) and (9.10), we see that $\xi_j(y)$ and $\varphi_j(x)$ are related to each other as

$$\xi_j(y)=\frac{\varphi_j(x)}{\sqrt{H(x)}}, \tag{9.11}$$

and thus the summation term on the left-hand side of (9.8) is expressed as

$$\sum_{j=1}^{N} \xi_j(y)\xi_j(y_0)=\sum_{j=1}^{N} \frac{\varphi_j(x)\varphi_j(x_0)}{\sqrt{H(x)H(x_0)}}. \tag{9.12}$$

Moreover, from $dy/dx=H(x)$ and the fact that $y=y_0$ if and only if $x=x_0$, we get

$$\delta(y-y_0)=\frac{\delta(x-x_0)}{\sqrt{H(x)H(x_0)}}. \tag{9.13}$$

Hence, using (9.9), (9.12), and (9.13) in (9.8), we obtain (9.7). ∎

Using the Fourier transformation (1.3), we can transform (9.1) into the variable-speed wave equation

$$\frac{\partial^2 w(x,t)}{\partial x^2}-H(x)^2\frac{\partial^2 w(x,t)}{\partial t^2}=Q(x)w(x,t), \quad x,t\in\mathbf{R}, \tag{9.14}$$

where $1/H(x)$ corresponds to the variable wave speed. We are interested in wave focusing for (9.14); in other words, we would like to construct causal solutions to (9.14) incident either from the left or right such that they focus at time $t=0$ to any specified point x_0. In particular, we want to construct solutions to (9.14) satisfying $w(x,0)=\delta(x-x_0)/H(x_0)$.

Let us define

$$W_{\mathrm{l}}(x,t;x_0):=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,T(k)f_{\mathrm{l}}(k,x)f_{\mathrm{r}}(k,x_0)e^{-ikt}+\sum_{j=1}^{N}\frac{\varphi_j(x)\varphi_j(x_0)}{H(x)H(x_0)}e^{\kappa_j t}, \tag{9.15}$$

$$W_{\mathrm{r}}(x,t;x_0):=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,T(k)f_{\mathrm{r}}(k,x)f_{\mathrm{l}}(k,x_0)e^{-ikt}+\sum_{j=1}^{N}\frac{\varphi_j(x)\varphi_j(x_0)}{H(x)H(x_0)}e^{\kappa_j t}, \tag{9.16}$$

where φ_j are the normalized bound-state wave functions given in (9.10).

Theorem 9.2: Assume that Q and H satisfy the conditions stated below (9.1). Then W_{l} is a causal solution to (9.14) that is incident from the left and that focuses to $x=x_0$ when $t=0$. Similarly, W_{r} is a causal solution to (9.14) that is incident from the right and that focuses to $x=x_0$ when $t=0$.

Proof: Since $g_{\mathrm{l}}(k,y)$ is a solution to (9.3), with the help of (9.2), (9.10), and (9.11) we see that W_{l} defined in (9.15) is a solution to (9.14). Using (9.5), (9.6), and (9.12) in (9.15) we get

$$\sqrt{H(x)H(x_0)}W_{\mathrm{l}}(x,t;x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\tau(k)g_{\mathrm{l}}(k,y)g_{\mathrm{r}}(k,y_0)e^{-ikt}+\sum_{j=1}^{N}\xi_j(y)\xi_j(y_0)e^{\kappa_j t}. \tag{9.17}$$

By comparing the right-hand sides of (4.18) and (9.17) and applying Theorem 4.3, we see that the theorem is proved for W_{l}. The proof for W_{r} defined in (9.16) is similarly obtained. ∎

We see from Theorem 9.2 that W_{l} consists of the wavefront $\delta(y-y_0-t)/\sqrt{H(x)H(x_0)}$ followed by a tail on the left and that it is incident from the left. Similarly, W_{r} consists of the wavefront $\delta(y-y_0+t)/\sqrt{H(x)H(x_0)}$ followed by a tail on the right and that it is incident from the right.

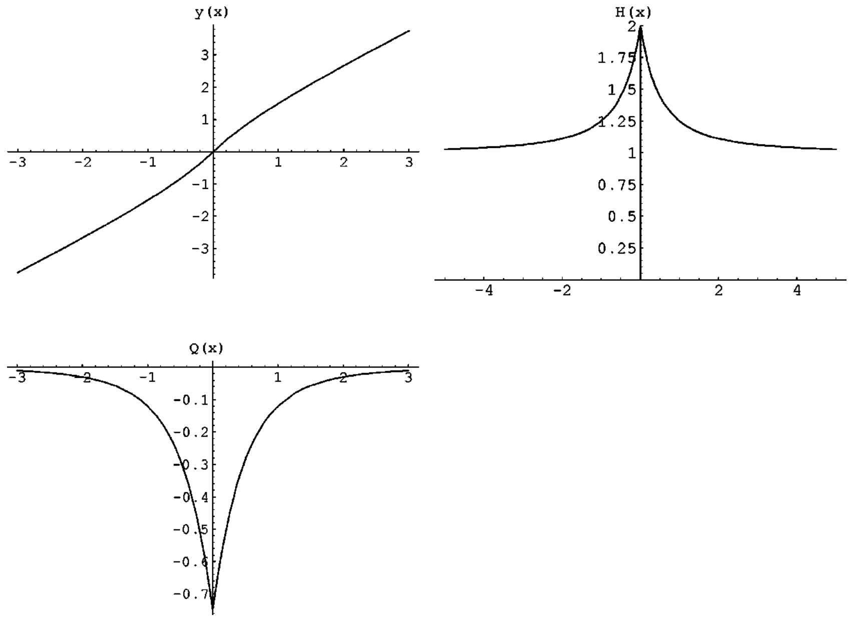

FIG. 6. The plots of $y(x)$, $H(x)$, and $Q(x)$ in Example 9.3.

The Marchenko equations associated with (9.1) were analyzed in Aktosun *et al.* (1992a). Using the results given in Secs. III and IV, it is possible to give various representations for W_{l} and W_{r}, similar to those in (4.22), (4.23), (5.13), (5.14), (5.17), and (5.18), and thus obtain the analogs of Theorems 4.5 and 5.3. We let the interested reader work out the details.

Next we present an example of focusing for (9.14).

Example 9.3: Consider

$$y(x)=\theta(-x)\left[x-1+\frac{1}{1-x}\right]+\theta(x)\left[x+1-\frac{1}{x+1}\right],$$

and hence

$$H(x)=\theta(-x)\,\frac{1+(1-x)^2}{(1-x)^2}+\theta(x)\,\frac{1+(1+x)^2}{(1+x)^2}.$$

Note that $H(x)$ is continuous, but $H'(x)$ has a discontinuity at $x=0$. Even though the focusing theory we outlined above is developed under the assumption that H'' exists, it can be extended in a straightforward manner when H'' contains some Dirac delta distributions. We have $H(0)=2$, $H'(0^+)=-2$, $H'(0^-)=2$, and hence H'' contains a delta distribution at $x=0$. Letting

$$Q(x)=\theta(-x)\,\frac{-3}{[1+(1-x)^2]^2}+\theta(x)\,\frac{-3}{[1+(1+x)^2]^2}, \tag{9.18}$$

from (9.4) we get $V(y)=-\delta(y)/2$. In Fig. 6 we show $y(x)$, $H(x)$, and $Q(x)$.

Using (9.5) and (9.6) we obtain

$$T(k)=\frac{4ke^{2ik}}{4k-i},\quad R(k)=\frac{4ie^{2ik}}{4k-i},\quad L(k)=\frac{4ie^{2ik}}{4k-i},$$

$$f_l(k,x)=\begin{cases}\dfrac{e^{ik(y-1)}}{\sqrt{H(x)}}, & x\geqslant 0,\\ \left(1+\dfrac{1}{4ik}\right)\dfrac{e^{-ik(y+1)}}{\sqrt{H(x)}}-\dfrac{1}{4ik}\,\dfrac{e^{ik(y-1)}}{\sqrt{H(x)}}, & x\leqslant 0,\end{cases}$$

$$f_r(k,x)=\begin{cases}\left(1+\dfrac{1}{4ik}\right)\dfrac{e^{-ik(y+1)}}{\sqrt{H(x)}}-\dfrac{1}{4ik}\,\dfrac{e^{ik(y-1)}}{\sqrt{H(x)}}, & x\geqslant 0,\\ \dfrac{e^{-ik(y+1)}}{\sqrt{H(x)}}, & x\leqslant 0.\end{cases}$$

Note that there is a bound state at $k=i/4$. Using (2.7) and (9.11) we get the normalized bound-state wave function as

$$\varphi(x)=\frac{\sqrt{H(x)}}{2}e^{-|y|/4}.$$

If the focusing point x_0 occurs in $[0,+\infty)$, with $y_0:=y(x_0)$ from (9.15) we get

$$W_l(x,t;x_0)=\frac{\delta(y-y_0-t)}{\sqrt{H(x)H(x_0)}}+\frac{\theta(-x)w_-(x,t;x_0)+\theta(x)w_+(x,t;x_0)}{4\sqrt{H(x)H(x_0)}},$$

where

$$w_-(x,t;x_0):=-\theta(-y+y_0+t)+\theta(y+y_0+t)+\theta(-y-y_0+t)-\theta(y-y_0+t)\\ +\theta(y-y_0+t)e^{(y-y_0+t)/4},$$

$$w_+(x,t;x_0):=\theta(-y-y_0+t)e^{-(y+y_0-t)/4}.$$

It can directly be verified that $W_l(x,0;x_0)=\delta(x-x_0)/[H(x)H(x_0)]$. This wave is illustrated in Fig. 7.

Now let us consider a slight modification of the above example.

Example 9.4: Suppose that $y(x)$ and hence $H(x)$ are as in Example 9.3. Let us assume that $Q(x)$ is given by

$$Q(x)=\delta(x)+\theta(-x)\frac{-3}{[1+(1-x)^2]^2}+\theta(x)\frac{-3}{[1+(1+x)^2]^2},$$

and hence differs from (9.18) by a delta distribution at $x=0$. From (9.4) it follows that $V(y)=0$ for all $y\in\mathbf{R}$, and hence $\tau(k)=1$ and $\rho(k)=\ell(k)=0$. Thus, we have a reflectionless case and there are no bound states. In this case, from (9.5) and (9.6) we get $T(k)=e^{2ik}$, $R(k)=L(k)=0$, and

$$f_l(k,x)=\frac{e^{ik(y-1)}}{\sqrt{H(x)}},\quad f_r(k,x)=\frac{e^{-ik(y+1)}}{\sqrt{H(x)}}.$$

Since Q contains a delta distribution, f_l' and f_r' are discontinuous in x at $x=0$. Using (9.15) we obtain

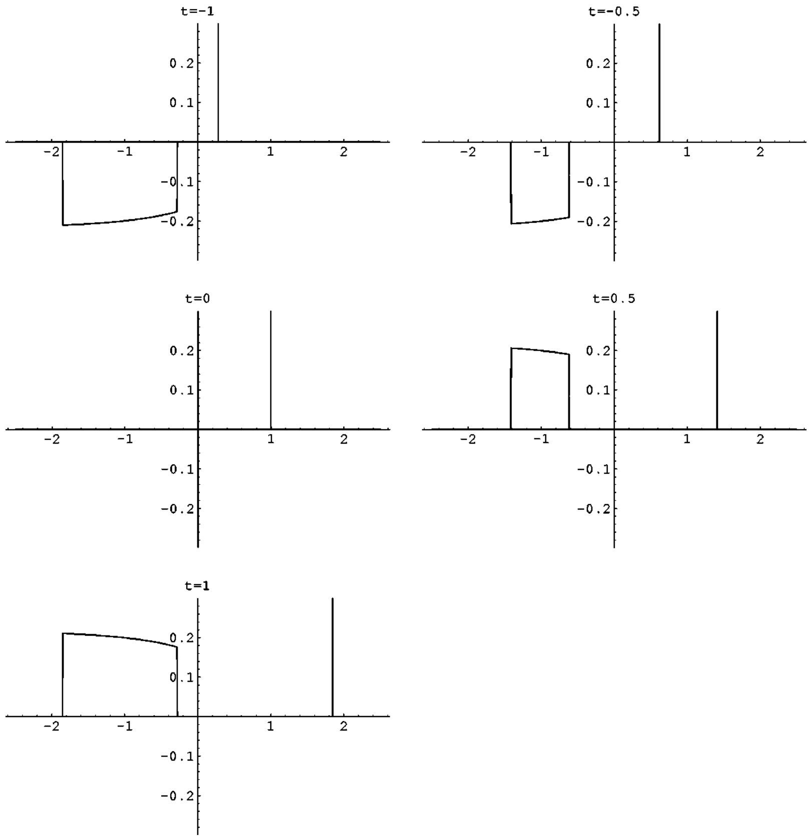

FIG. 7. The focusing wave of Example 9.3 with $x_0=1$ is shown at $t=-1,-0.5,0,0.5,1$.

$$W_1(x,t;x_0)=\frac{\delta(y-y_0-t)}{\sqrt{H(x)H(x_0)}}.$$

Thus, the wave W_1 is always focused, and there is no tail following the wave front due to the fact that there is no reflection.

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