JOURNAL OF MATHEMATICAL PHYSICS VOLUME 42, NUMBER 10 OCTOBER 2001

Small-energy asymptotics of the scattering matrix for the matrix Schrödinger equation on the line

Tuncay Aktosun[a)]
Department of Mathematics and Statistics, Mississippi State University, Mississippi State, Mississippi 39762

Martin Klaus[b)]
Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061

Cornelis van der Mee[c)]
Department of Mathematics, University of Cagliari, Cagliari, Italy

(Received 1 June 1999; accepted for publication 10 July 2001)

The one-dimensional matrix Schrödinger equation is considered when the matrix potential is self-adjoint with entries that are integrable and have finite first moments. The small-energy asymptotics of the scattering coefficients are derived, and the continuity of the scattering coefficients at zero energy is established. When the entries of the potential have also finite second moments, some more detailed asymptotic expansions are presented. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1398059]

I. INTRODUCTION

Consider the matrix Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=Q(x)\psi(k,x), \quad x\in\mathbf{R}, \tag{1.1}$$

where $x\in\mathbf{R}$ is the spatial coordinate, the prime denotes the derivative with respect to x, k^2 is the energy, $Q(x)$ is an $n\times n$ self-adjoint matrix potential, i.e., $Q(x)^\dagger=Q(x)$ with the dagger standing for the matrix conjugate transpose, and $\psi(k,x)$ is either an $n\times 1$ or an $n\times n$ matrix function. We use $\|\cdot\|$ to denote the (Euclidean) norm of a vector or the operator norm of a matrix. Let $L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ with $m\geqslant 0$ denote the Banach space of all measurable $n\times n$ matrix functions f for which $(1+|x|)^m\|f(x)\|$ is integrable on $\mathbf{R}$. If $n=1$, we denote this space by $L^1_m(\mathbf{R})$. In this paper we always assume that Q is self-adjoint and belongs to $L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Certain results will be obtained under the assumption that $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, but we will clearly indicate when this stronger assumption is needed. We use $\mathbf{C}^+$ to denote the upper-half complex plane and write $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+\cup\mathbf{R}$.

Among the $n\times n$ solutions of (1.1) are the so-called Jost solution from the left, $f_1(k,x)$, and the Jost solution from the right, $f_{\mathrm{r}}(k,x)$, satisfying the asymptotic boundary conditions

$$e^{-ikx}f_1(k,x)=I_n+o(1) \ \text{ and } \ e^{-ikx}f_1'(k,x)=ikI_n+o(1), \ \ x\to+\infty, \tag{1.2}$$

$$e^{ikx}f_{\mathrm{r}}(k,x)=I_n+o(1) \ \text{ and } \ e^{ikx}f_{\mathrm{r}}'(k,x)=-ikI_n+o(1), \ \ x\to-\infty, \tag{1.3}$$

where I_n denotes the identity matrix of order n. The existence of the Jost solutions can be established as in the scalar ($n=1$) case[1,2] by using the appropriate integral equations[3,4] [cf. (2.2), (2.3), and Theorem 2.1 in our paper].

For each $k\in\mathbf{R}\setminus\{0\}$ we have

$$f_1(k,x)=a_1(k)e^{ikx}+b_1(k)e^{-ikx}+o(1), \quad x\to-\infty, \tag{1.4}$$

[a)]Electronic mail: aktosun@math.msstate.edu
[b)]Electronic mail: klaus@math.vt.edu
[c)]Electronic mail: cornelis@krein.sc.unica.it

$$f_r(k,x) = a_r(k)e^{-ikx} + b_r(k)e^{ikx} + o(1), \quad x \to +\infty, \tag{1.5}$$

where $a_l(k)$, $b_l(k)$, $a_r(k)$, and $b_r(k)$ are some $n\times n$ matrix functions of k. These matrix functions enter the scattering matrix $\mathbf{S}(k)$ defined in (2.22), and our primary aim is the analysis of the small-k behavior of $\mathbf{S}(k)$.

The motivation for this paper comes from our interest in the inverse scattering problem for (1.1), namely the recovery of Q from an appropriate set of data involving the scattering matrix. As is known from the scalar case, it is important to have detailed information about the behavior of $\mathbf{S}(k)$ for small k. For example,[1,2] this information is used to characterize the scattering data, so as to ensure that the potential Q constructed from the data at hand belongs to a certain class of functions such as $L_1^1(\mathbf{R})$ or $L_2^1(\mathbf{R})$. The inverse scattering problem for (1.1) when $n>1$ has been considered by several authors,[4–10] but we are not aware of any in-depth study of the small-k behavior of $\mathbf{S}(k)$. Not even the continuity of the scattering matrix at $k=0$ seems to have been established when $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$; for example, in Ref. 6 (p. 294), the continuity at $k=0$ of the transmission coefficients is *assumed*. In the scalar case it is well known[1,2,11,12] that the continuity of $\mathbf{S}(k)$ at $k=0$ is easy to establish if $Q\in L_2^1(\mathbf{R})$, but not if only $Q\in L_1^1(\mathbf{R})$. In the matrix case, the situation is somewhat different. The decay of $Q(x)$ as $x\to\pm\infty$ plays an important role, but there are further complications due to the particular structure of the solution space of (1.1) at $k=0$. From the scalar case it is known[1,2,11] that the behavior of the solutions of (1.1) at $k=0$ makes it necessary to distinguish between two cases, the *generic case* and the *exceptional case*, and that the small-k behavior of $\mathbf{S}(k)$ is different in each case. If $n>1$, the situation is more complicated because the exceptional case gives rise to a variety of possibilities depending on the Jordan structure of a certain matrix associated with the solution space of (1.1) at $k=0$. In this paper we clarify the connection between the solutions of (1.1) at $k=0$ and the behavior of $\mathbf{S}(k)$ near $k=0$. As a result, we are able to prove the continuity of the scattering matrix at $k=0$ when $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ and to obtain more detailed asymptotic expansions when $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. The inverse problem is not considered here; we may report on it elsewhere.

This paper is organized as follows. In Sec. II we establish our notations and review some basic known results on the solutions of (1.1). Since this material is standard, we refer the reader to the literature for proofs and more details. In Sec. II we also give various characterizations of the generic and exceptional cases. In Sec. III we prove the continuity of the scattering matrix at $k=0$ in the generic case, and we obtain some more detailed asymptotic results when $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. The exceptional case is treated in Sec. IV; the main results are contained in Theorem 4.6 when $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ and in Theorem 4.7 when $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, where we prove the continuity and differentiability of $\mathbf{S}(k)$ at $k=0$, respectively. In Sec. V we discuss some special cases that illustrate the results of Sec. IV. Finally, the Appendix contains the proof of Proposition 4.2, which is a key result needed to establish Theorems 4.6 and 4.7.

II. SCATTERING COEFFICIENTS AND A CASE DISTINCTION

In this section we review some basic results about those solutions of (1.1) that are relevant to scattering theory, and we define the scattering coefficients and some related quantities. We also elaborate on the distinction between the generic case and the exceptional case which will play an important role in the subsequent sections.

We define the Faddeev functions $m_l(k,x)$ and $m_r(k,x)$ by

$$m_l(k,x) = e^{-ikx}f_l(k,x), \quad m_r(k,x) = e^{ikx}f_r(k,x). \tag{2.1}$$

From (1.2), (1.3), and (2.1) it follows that

$$m_l(k,x) = I_n + \frac{1}{2ik}\int_x^\infty dy\,[e^{2ik(y-x)}-1]Q(y)m_l(k,y), \tag{2.2}$$

$$m_{\mathrm{r}}(k,x)=I_n+\frac{1}{2ik}\int_{-\infty}^{x}dy\,[e^{2ik(x-y)}-1]Q(y)m_{\mathrm{r}}(k,y). \tag{2.3}$$

Some properties of the matrix functions $m_{\mathrm{l}}(k,x)$ and $m_{\mathrm{r}}(k,x)$ are summarized in the next theorem and its corollary. The proofs of these results can be obtained as in the scalar case and we refer the reader to the literature;[2–4,11] in particular, see Theorem 1.4.1 in Ref. 3 and Theorem 1 in Ref. 4. We denote differentiation with respect to k by an overdot and use C for suitable constants that do not depend on x or k.

Theorem 2.1: *If $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$, then, for each $x\in\mathbf{R}$, the functions $m_{\mathrm{l}}(k,x)$, $m_{\mathrm{r}}(k,x)$, $m_{\mathrm{l}}'(k,x)$, and $m_{\mathrm{r}}'(k,x)$ are analytic in $k\in\mathbf{C}^+$* and *continuous in $k\in\overline{\mathbf{C}^+}$; moreover*

$$m_{\mathrm{l}}(k,x)=I_n+o(1),\quad m_{\mathrm{l}}'(k,x)=o(1/x),\quad x\to+\infty, \tag{2.4}$$

$$m_{\mathrm{r}}(k,x)=I_n+o(1),\quad m_{\mathrm{r}}'(k,x)=o(1/x),\quad x\to-\infty,$$

$$\|m_{\mathrm{l}}(k,x)\|\leqslant C[1+\max\{0,-x\}],\quad \|m_{\mathrm{r}}(k,x)\|\leqslant C[1+\max\{0,x\}],\quad k\in\overline{\mathbf{C}^+}. \tag{2.5}$$

In addition, if $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, then $\dot m_{\mathrm{l}}(k,x)$ and $\dot m_{\mathrm{r}}(k,x)$ exist, are analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and satisfy the estimates

$$\|\dot m_{\mathrm{l}}(k,x)\|\leqslant C(1+x^2),\quad \|\dot m_{\mathrm{r}}(k,x)\|\leqslant C(1+x^2),\quad k\in\overline{\mathbf{C}^+}.$$

In the following an asterisk will be used to denote complex conjugation. From (2.1) and Theorem 2.1 we get the following.

Corollary 2.2: Assume $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. Then, for each fixed $x\in\mathbf{R}$, the four matrix functions $f_{\mathrm{l}}(-k^*,x)^\dagger$, $f_{\mathrm{r}}(-k^*,x)^\dagger$, $f_1'(-k^*,x)^\dagger$, and $f_{\mathrm{r}}'(-k^*,x)^\dagger$ are analytic in $k\in\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Moreover, if $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, then these functions are differentiable with respect to $k\in\overline{\mathbf{C}^+}$.

The scattering coefficients will be defined in terms of certain Wronskians involving the Jost solutions. We first state a standard result about such Wronskians, which is a consequence of the selfadjointness of Q. Let $[F;G]=FG'-F'G$ denote the Wronskian of two square matrix functions $F(x)$ and $G(x)$.

Proposition 2.3: For $k\in\mathbf{C}$, let $\phi(k,x)$ be any $n\times p$ solution and $\psi(k,x)$ any $n\times q$ solution of (1.1). Then the $p\times q$ Wronskian matrix $[\phi(\pm k^*,x)^\dagger;\psi(k,x)]$ is independent of x.

As a result of Proposition 2.3 the matrices $a_{\mathrm{l}}(k)$, $b_{\mathrm{l}}(k)$, $a_{\mathrm{r}}(k)$, and $b_{\mathrm{r}}(k)$ appearing in (1.4) and (1.5) can be expressed in terms of certain Wronskians of the Jost solutions as follows:

$$a_{\mathrm{l}}(k)=\frac{1}{2ik}[f_{\mathrm{r}}(-k^*,x)^\dagger;f_{\mathrm{l}}(k,x)],\quad k\in\overline{\mathbf{C}^+}\backslash\{0\}, \tag{2.6}$$

$$a_{\mathrm{r}}(k)=-\frac{1}{2ik}[f_1(-k^*,x)^\dagger;f_{\mathrm{r}}(k,x)],\quad k\in\overline{\mathbf{C}^+}\backslash\{0\}, \tag{2.7}$$

$$b_{\mathrm{l}}(k)=-\frac{1}{2ik}[f_{\mathrm{r}}(k,x)^\dagger;f_{\mathrm{l}}(k,x)],\quad k\in\mathbf{R}\backslash\{0\}, \tag{2.8}$$

$$b_{\mathrm{r}}(k)=\frac{1}{2ik}[f_1(k,x)^\dagger;f_{\mathrm{r}}(k,x)],\quad k\in\mathbf{R}\backslash\{0\}. \tag{2.9}$$

Alternatively, it is sometimes convenient to use the integral representations

$$a_{\mathrm{l}}(k)=I_n-\frac{1}{2ik}\int_{-\infty}^{\infty}dx\,Q(x)m_{\mathrm{l}}(k,x), \tag{2.10}$$

$$a_{\mathrm{r}}(k)=I_n-\frac{1}{2ik}\int_{-\infty}^{\infty}dx\,Q(x)m_{\mathrm{r}}(k,x), \tag{2.11}$$

$$b_{\mathrm{l}}(k)=\frac{1}{2ik}\int_{-\infty}^{\infty}dx\,e^{2ikx}\,Q(x)m_{\mathrm{l}}(k,x), \tag{2.12}$$

$$b_{\mathrm{r}}(k)=\frac{1}{2ik}\int_{-\infty}^{\infty}dx\,e^{-2ikx}\,Q(x)m_{\mathrm{r}}(k,x), \tag{2.13}$$

which follow from (1.4), (1.5), and (2.1)–(2.5). Also, with the help of (1.2)–(1.5) and (2.6)–(2.9), we obtain

$$a_{\mathrm{r}}(-k^*)^\dagger=a_{\mathrm{l}}(k),\quad k\in\overline{\mathbf{C}^+}\setminus\{0\}, \tag{2.14}$$

$$b_{\mathrm{r}}(k)=-b_{\mathrm{l}}(k)^\dagger,\quad k\in\mathbf{R}\setminus\{0\}, \tag{2.15}$$

$$a_{\mathrm{l}}(k)^\dagger a_{\mathrm{l}}(k)=b_{\mathrm{l}}(k)^\dagger b_{\mathrm{l}}(k)+I_n,\quad k\in\mathbf{R}\setminus\{0\}, \tag{2.16}$$

$$a_{\mathrm{r}}(k)^\dagger a_{\mathrm{r}}(k)=b_{\mathrm{r}}(k)^\dagger b_{\mathrm{r}}(k)+I_n,\quad k\in\mathbf{R}\setminus\{0\}, \tag{2.17}$$

$$a_{\mathrm{l}}(-k)^\dagger b_{\mathrm{l}}(k)=b_{\mathrm{l}}(-k)^\dagger a_{\mathrm{l}}(k),\quad k\in\mathbf{R}\setminus\{0\}, \tag{2.18}$$

$$a_{\mathrm{r}}(-k)^\dagger b_{\mathrm{r}}(k)=b_{\mathrm{r}}(-k)^\dagger a_{\mathrm{r}}(k),\quad k\in\mathbf{R}\setminus\{0\}. \tag{2.19}$$

We define the transmission coefficient from the left, $T_{\mathrm{l}}(k)$, and the transmission coefficient from the right, $T_{\mathrm{r}}(k)$, by

$$T_{\mathrm{l}}(k)=a_{\mathrm{l}}(k)^{-1},\quad T_{\mathrm{r}}(k)=a_{\mathrm{r}}(k)^{-1}, \tag{2.20}$$

provided the inverses on the right-hand sides exist, and we define the reflection coefficient from the left, $L(k)$, and the reflection coefficient from the right, $R(k)$, by

$$L(k)=b_{\mathrm{l}}(k)a_{\mathrm{l}}(k)^{-1},\quad R(k)=b_{\mathrm{r}}(k)a_{\mathrm{r}}(k)^{-1}. \tag{2.21}$$

From (2.16) and (2.17) we see that $a_{\mathrm{l}}(k)$ and $a_{\mathrm{r}}(k)$ are nonsingular for $k\in\mathbf{R}\setminus\{0\}$. In $\mathbf{C}^+$, $a_{\mathrm{l}}(k)$ and $a_{\mathrm{r}}(k)$ are nonsingular except possibly at a finite number of points on the positive imaginary axis where[4] both $\det a_{\mathrm{l}}(k)=0$ and $\det a_{\mathrm{r}}(k)=0$; at these points, $T_{\mathrm{l}}(k)$ and $T_{\mathrm{r}}(k)$ have simple poles[8] corresponding to the bound states of (1.1). For $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$ the finiteness of the number of bound states has already been established in Refs. 4 and 13. We note that even if $Q\notin L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$ but $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$, the finiteness follows from the (operator) inequality $Q(x)\geqslant-\|Q(x)\|I_n$ and the fact that in one dimension a scalar potential in $L_1^1(\mathbf{R})$ can support at most a finite number of bound states. Alternatively, the finiteness of the number of bound states will follow from the results of this paper (cf. Theorems 3.1 and 4.6), which show that $k=0$ cannot be an accumulation point for poles of either $T_{\mathrm{l}}(k)$ or $T_{\mathrm{r}}(k)$. Because of this latter property we will study the asymptotic behavior of the transmission coefficients as $k\to0$ through values in $\overline{\mathbf{C}^+}$. The reflection coefficients, on the other hand, in general do not have analytic extensions off the real axis, so their asymptotics will be studied for real k only. The $n\times n$ matrix functions $T_{\mathrm{l}}(k)$, $T_{\mathrm{r}}(k)$, $R(k)$, and $L(k)$ are referred to as scattering coefficients, and the $2n\times2n$ matrix

$$\mathbf{S}(k)=\begin{bmatrix}T_{\mathrm{l}}(k) & R(k)\\ L(k) & T_{\mathrm{r}}(k)\end{bmatrix}, \tag{2.22}$$

is called the scattering matrix.

From (2.15)–(2.17), we get

$$T_{\mathrm{l}}(k)^{\dagger}R(k)+L(k)^{\dagger}T_{\mathrm{r}}(k)=0, \quad k\in\mathbf{R}\backslash\{0\},$$

$$T_{\mathrm{l}}(k)^{\dagger}T_{\mathrm{l}}(k)+L(k)^{\dagger}L(k)=I_n, \quad k\in\mathbf{R}\backslash\{0\}, \tag{2.23}$$

$$T_{\mathrm{r}}(k)^{\dagger}T_{\mathrm{r}}(k)+R(k)^{\dagger}R(k)=I_n, \quad k\in\mathbf{R}\backslash\{0\}, \tag{2.24}$$

and hence, for $k\in\mathbf{R}\backslash\{0\}$, $\mathbf{S}(k)$ is unitary. Using (2.14) we obtain

$$T_{\mathrm{r}}(k)=T_{\mathrm{l}}(-k^*)^{\dagger}, \quad k\in\overline{\mathbf{C}^+}\backslash\{0\}, \tag{2.25}$$

whenever $a_{\mathrm{r}}(k)$ is nonsingular, and from (2.18) and (2.19) we get

$$L(-k)^{\dagger}=L(k), \quad R(-k)^{\dagger}=R(k), \quad k\in\mathbf{R}\backslash\{0\}.$$

In order to study $\mathbf{S}(k)$ in the small-energy limit, we need to make an important case distinction which involves the solutions to (1.1) with $k=0$, i.e., the solutions to

$$\phi''(x)=Q(x)\phi(x), \quad x\in\mathbf{R}. \tag{2.26}$$

We already know from (2.1) and Theorem 2.1 that $f_{\mathrm{l}}(0,x)$ is a solution of (2.26) satisfying $f_{\mathrm{l}}(0,x)=I_n+o(1)$ and $f_{\mathrm{l}}'(0,x)=o(1/x)$ as $x\to+\infty$. According to basic asymptotic results for systems of linear differential equations (Theorem 1.5.1 of Ref. 3), (2.26) also has an $n\times n$ matrix solution, $\phi_{\mathrm{l}}(x)$, satisfying

$$\phi_{\mathrm{l}}(x)=xI_n+o(x), \quad \phi_{\mathrm{l}}'(x)=I_n+o(1), \quad x\to+\infty.$$

Thus the columns of $f_{\mathrm{l}}(0,x)$ together with the columns of $\phi_{\mathrm{l}}(x)$ form a fundamental set of $2n$ vector solutions for (2.26). Any vector solution $\phi(x)$ of (2.26) can be written as

$$\phi(x)=f_{\mathrm{l}}(0,x)\,\eta_1+\phi_{\mathrm{l}}(x)\,\eta_2, \tag{2.27}$$

where $\eta_1,\eta_2\in\mathbf{C}^n$ are uniquely determined by $\phi(x)$. It follows from (2.27) that a vector solution of (2.26) is bounded as $x\to+\infty$ if and only if $\eta_2=0$, i.e., if and only if $\phi(x)=f_{\mathrm{l}}(0,x)\,\eta_1$ for some $\eta_1\in\mathbf{C}^n$. Moreover, in this case $\lim_{x\to+\infty}\phi(x)=\eta_1$ exists. This means that if a solution is bounded at $+\infty$, then it also has a limit as $x\to+\infty$. Also, (2.27) implies that any solution of (2.26) that is $o(x)$ as $x\to+\infty$ is necessarily bounded at $x=+\infty$ and any solution that is $o(1)$ as $x\to+\infty$ must be the zero solution. Similar results hold at $x=-\infty$; in particular, any solution of (2.26) that is $o(x)$ as $x\to-\infty$ is necessarily bounded at $x=-\infty$ and has a limit as $x\to-\infty$, and any solution that is $o(1)$ as $x\to-\infty$ must be the zero solution.

From (2.1)–(2.3) we see that $f_{\mathrm{l}}(0,x)$ and $f_{\mathrm{r}}(0,x)$ obey the integral equations

$$f_{\mathrm{l}}(0,x)=I_n+\int_x^{\infty}dy\,(y-x)Q(y)f_{\mathrm{l}}(0,y), \tag{2.28}$$

$$f_{\mathrm{r}}(0,x)=I_n-\int_{-\infty}^{x}dy\,(y-x)Q(y)f_{\mathrm{r}}(0,y). \tag{2.29}$$

In the subsequent analysis the two Wronskian matrices

$$\Delta_{\mathrm{l}}=[f_{\mathrm{r}}(0,x)^{\dagger};f_{\mathrm{l}}(0,x)], \quad \Delta_{\mathrm{r}}=-[f_{\mathrm{l}}(0,x)^{\dagger};f_{\mathrm{r}}(0,x)], \tag{2.30}$$

will play a key role. By Proposition 2.3, Δ_{l} and Δ_{r} are independent of x, and from (2.30) it follows that

$$\Delta_{\mathrm{l}}=\Delta_{\mathrm{r}}^{\dagger}. \tag{2.31}$$

The importance of these Wronskians lies in the fact that they are related to the transmission coefficients via (2.20) and

$$\Delta_l = \lim_{k\to 0} 2ika_l(k), \quad \Delta_r = \lim_{k\to 0} 2ika_r(k), \tag{2.32}$$

where the limits are taken from within $\overline{\mathbf{C}^+}$; (2.32) follows from (2.6), (2.7), and Corollary 2.2.

Evaluating the first Wronskian in (2.30) as $x\to -\infty$ and using (2.28) we obtain

$$\Delta_l = \lim_{x\to -\infty} f_l'(0,x) = -\int_{-\infty}^{\infty} dy\, Q(y) f_l(0,y). \tag{2.33}$$

Similarly, from (2.29) and (2.30), letting $x\to +\infty$, we get

$$\Delta_r = -\lim_{x\to +\infty} f_r'(0,x) = -\int_{-\infty}^{\infty} dy\, Q(y) f_r(0,y). \tag{2.34}$$

From (2.28) and (2.29) we also infer that

$$f_l(0,x) = x\Delta_l + o(x), \quad x\to -\infty,$$

$$f_r(0,x) = -x\Delta_r + o(x), \quad x\to +\infty. \tag{2.35}$$

Now we are ready to introduce the distinction between the exceptional case and the generic case. Let

$$\mathcal{N} = \{\xi \in \mathbf{C}^n : f_l(0,x)\xi \text{ is bounded on } \mathbf{R}\}. \tag{2.36}$$

Then we say that the generic case occurs if $\mathcal{N}=\{0\}$ and we say that the exceptional case occurs if $\mathcal{N}\neq\{0\}$. These two cases can be characterized in other ways. We choose the above definition as our starting point and will arrive at some other characterizations as we go along.

We observe that the generic case occurs if and only if (2.26) has no bounded nontrivial solution. The exceptional case occurs if and only if there exists at least one nontrivial bounded solution. As the next theorem shows, we can alternatively characterize the two cases by means of the subspace

$$\mathcal{M} = \{\chi \in \mathbf{C}^n : f_r(0,x)\chi \text{ is bounded on } \mathbf{R}\}. \tag{2.37}$$

Then the generic (exceptional) case occurs if and only if $\mathcal{M}=\{0\}$ $(\mathcal{M}\neq\{0\})$.

We mention that when $n=1$ the exceptional case occurs if and only if $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent, i.e., the Wronskian $[f_r(0,x);f_l(0,x)]$ is zero. In our paper we generalize this characterization to the matrix case. In the scalar case it is also known that the generic (exceptional) case occurs if $T_l(0)=0$ $(T_l(0)\neq 0)$. This will also turn out be true in the matrix case, but we do not use this property as our primary characterization because it is implicitly based on the assumption that $T_l(k)$ is continuous at $k=0$, something we first need to prove.

The next theorem further clarifies the relations among the two cases, the Wronskians in (2.31), and the subspaces $\mathcal{N}$ and $\mathcal{M}$.

Theorem 2.4: *Assume* $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. *Then we have*

(i) *The generic case occurs if and only if* Δ_l, *or equivalently* Δ_r, *is nonsingular.*
(ii) $\mathcal{N}=\operatorname{Ker}\ \Delta_l$ *and* $\mathcal{M}=\operatorname{Ker}\ \Delta_r$.
(iii) $\dim\ \mathcal{N}=\dim\ \mathcal{M}$.

Proof: If Δ_l is nonsingular, then (2.35) implies that every solution of (2.26) of the form $f_l(0,x)\eta$, with some nonzero vector $\eta\in\mathbf{C}^n$, becomes unbounded as $x\to -\infty$. Hence, if Δ_l is nonsingular, then (2.26) has no bounded nontrivial solutions; so the generic case occurs. Conversely, suppose the generic case (i.e., $\mathcal{N}=\{0\}$) occurs and Δ_{l} is singular. Then, by (2.35), for any nonzero $\xi\in\mathrm{Ker}\ \Delta_{\mathrm{l}}$, we have $f_{\mathrm{l}}(0,x)\xi=o(x)$ as $x\to-\infty$. Hence, by the remarks following (2.26) and (2.27), $f_{\mathrm{l}}(0,x)\xi$ is bounded, i.e., $\xi\in\mathcal{N}$ and thus $\mathcal{N}\neq\{0\}$. This is a contradiction. Therefore, in the generic case, Δ_{l} cannot be singular. This proves (i) for Δ_{l}. In view of (2.31), the assertion also holds if Δ_{l} is replaced by Δ_{r}. To prove (ii), suppose that $\xi\in\mathrm{Ker}\ \Delta_{\mathrm{l}}$. Then, by (2.35), $f_{\mathrm{l}}(0,x)\xi=o(x)$ as $x\to-\infty$. Hence $f_{\mathrm{l}}(0,x)\xi$ is bounded and so $\xi\in\mathcal{N}$. Conversely, if $\xi\in\mathcal{N}$, then $f_{\mathrm{l}}(0,x)\xi$ is bounded and, therefore, again by (2.35), $\Delta_{\mathrm{l}}\xi=0$. This proves the first equality in (ii). The second equality is proved similarly. Finally, (iii) follows immediately from (2.31). ■

III. SMALL-k BEHAVIOR IN THE GENERIC CASE

In this section we analyze the behavior of the scattering coefficients near $k=0$ in the generic case. In order to state the next theorem, which is the main result of this section, we introduce the matrices

$$E_{\mathrm{l}}=\int_{-\infty}^{\infty}dx\,xQ(x)m_{\mathrm{l}}(0,x),\quad E_{\mathrm{r}}=\int_{-\infty}^{\infty}dx\,xQ(x)m_{\mathrm{r}}(0,x),\tag{3.1}$$

$$G_{\mathrm{l}}=\int_{-\infty}^{\infty}dx\,Q(x)\dot{m}_{\mathrm{l}}(0,x),\quad G_{\mathrm{r}}=\int_{-\infty}^{\infty}dx\,Q(x)\dot{m}_{\mathrm{r}}(0,x).$$

The quantities E_{l} and E_{r} will also play a role in Sec. IV.

Theorem 3.1: *Assume Q is a generic potential in $L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ for $m=1$ or 2. Then the scattering coefficients satisfy the following:*

(i) *If $m=1$, then*

$$T_{\mathrm{l}}(k)=2ik\Delta_{\mathrm{l}}^{-1}+o(k),\quad T_{\mathrm{r}}(k)=2ik\Delta_{\mathrm{r}}^{-1}+o(k),\quad k\to0\ \text{ in }\ \overline{\mathbf{C}^{+}},$$
$$R(k)=-I_n+o(1),\quad L(k)=-I_n+o(1),\quad k\to0\ \text{ in }\ \mathbf{R}.$$

(ii) *If $m=2$, then*

$$T_{\mathrm{l}}(k)=2ik\Delta_{\mathrm{l}}^{-1}+k^2\Delta_{\mathrm{l}}^{-1}[4I_n+2iG_{\mathrm{l}}]\Delta_{\mathrm{l}}^{-1}+o(k^2),\quad k\to0\ \text{ in }\ \overline{\mathbf{C}^{+}},$$
$$T_{\mathrm{r}}(k)=2ik\Delta_{\mathrm{r}}^{-1}+k^2\Delta_{\mathrm{r}}^{-1}[4I_n+2iG_{\mathrm{r}}]\Delta_{\mathrm{r}}^{-1}+o(k^2),\quad k\to0\ \text{ in }\ \overline{\mathbf{C}^{+}},$$
$$L(k)=-I_n+2ik[I_n+E_{\mathrm{l}}]\Delta_{\mathrm{l}}^{-1}+o(k),\quad k\to0\ \text{ in }\ \mathbf{R},$$
$$R(k)=-I_n+2ik[I_n-E_{\mathrm{r}}]\Delta_{\mathrm{r}}^{-1}+o(k),\quad k\to0\ \text{ in }\ \mathbf{R}.$$

Proof: Using the fact that in the generic case Δ_{l} and Δ_{r} are invertible, (i) is a consequence of (2.6)–(2.9), (2.20), (2.21), (2.32), and Corollary 2.2. When $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$, expanding the integrals in (2.10)–(2.13) as

$$a_{\mathrm{l}}(k)=\frac{1}{2ik}\Delta_{\mathrm{l}}+I_n+\frac{i}{2}G_{\mathrm{l}}+o(1),\quad k\to0\ \text{ in }\ \overline{\mathbf{C}^{+}},\tag{3.2}$$

$$b_{\mathrm{l}}(k)=-\frac{1}{2ik}\Delta_{\mathrm{l}}+E_{\mathrm{l}}-\frac{i}{2}G_{\mathrm{l}}+o(1),\quad k\to0\ \text{ in }\ \mathbf{R},\tag{3.3}$$

$$a_{\mathrm{r}}(k)=\frac{1}{2ik}\Delta_{\mathrm{r}}+I_n+\frac{i}{2}G_{\mathrm{r}}+o(1),\quad k\to0\ \text{ in }\ \overline{\mathbf{C}^{+}},\tag{3.4}$$

$$b_{\mathrm{r}}(k)=-\frac{1}{2ik}\Delta_{\mathrm{r}}-E_{\mathrm{r}}-\frac{i}{2}G_{\mathrm{r}}+o(1),\quad k\to0\ \text{ in }\ \mathbf{R},\tag{3.5}$$

and using (2.20) and (2.21) we obtain (ii). ■

For later use we remark that when $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, E_l and E_r can be expressed in terms of certain Wronskians, namely

$$I_n + E_\mathrm{l} = i[\dot{f}_\mathrm{r}(0,x)^\dagger; f_\mathrm{l}(0,x)], \quad I_n - E_\mathrm{r} = -i[\dot{f}_\mathrm{l}(0,x)^\dagger; f_\mathrm{r}(0,x)]. \tag{3.6}$$

Note that the Wronskians in (3.6) are independent of x because $\dot{f}_\mathrm{l}(0,x)$ and $\dot{f}_\mathrm{r}(0,x)$ are also solutions of (2.26). The expressions in (3.6) follow easily from (2.28), (2.29), and the corresponding integral equations for $\dot{f}_\mathrm{l}(0,x)$ and $\dot{f}_\mathrm{r}(0,x)$ [cf. (A.20)]. Moreover, we have $G_\mathrm{r} = -G_\mathrm{l}^\dagger$ and $E_\mathrm{r} = E_\mathrm{l}^\dagger + iG_\mathrm{l}^\dagger$, as can be seen by using (2.14), (2.15), and (3.2)–(3.5).

Theorem 3.1 shows that if the generic case occurs, then $T_\mathrm{l}(0) = 0$. In the next section we will see that the converse is also true.

IV. SMALL-*k* BEHAVIOR IN THE EXCEPTIONAL CASE

Recall that in the exceptional case (2.26) has at least one bounded nontrivial solution. In this section we analyze how this affects the small-k properties of $\mathbf{S}(k)$, and we prove in the exceptional case the continuity of $\mathbf{S}(k)$ at $k=0$ when $Q \in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ and its differentiability when $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. It turns out that when $Q \in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ the exceptional case gives rise to certain technical complications that necessitate a careful study of certain asymptotic expansions. Since the proof of one result, namely Proposition 4.2, is especially long, that proof is given in the Appendix.

Recall the definitions of the subspaces $\mathcal{N}$ and $\mathcal{M}$ given in (2.36) and (2.37), respectively. There is a natural mapping from $\mathcal{N}$ to $\mathcal{M}$, which we denote by Γ, defined as follows. For every $\xi \in \mathcal{N}$, let

$$\chi = \lim_{x\to-\infty} f_\mathrm{l}(0,x)\xi, \tag{4.1}$$

and put

$$\chi = \Gamma\xi. \tag{4.2}$$

Note that, by (2.36), $f_\mathrm{l}(0,x)\xi$ is bounded and hence, by the discussion below (2.27), the limit in (4.1) exists. To see that Γ maps $\mathcal{N}$ into $\mathcal{M}$, we note that (4.1) implies

$$\lim_{x\to-\infty} [f_\mathrm{l}(0,x)\xi - f_\mathrm{r}(0,x)\chi] = 0.$$

Hence $f_\mathrm{l}(0,x)\xi - f_\mathrm{r}(0,x)\chi$ is a solution of (2.26) which approaches zero as $x\to-\infty$; therefore, it must be identically zero and we have

$$f_\mathrm{l}(0,x)\xi = f_\mathrm{r}(0,x)\chi, \quad x \in \mathbf{R}. \tag{4.3}$$

Hence $f_\mathrm{r}(0,x)\chi$ is bounded, which implies $\chi \in \mathcal{M}$.

Proposition 4.1: Assume $Q \in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. *Then* Γ *is a bijection between* $\mathcal{N}$ *and* $\mathcal{M}$.

Proof: We have already seen that Γ maps $\mathcal{N}$ into $\mathcal{M}$. The map Γ is injective, for if $\Gamma\xi = 0$, then, by (4.2) and (4.3), $f_\mathrm{l}(0,x)\xi = 0$ for all $x \in \mathbf{R}$ and hence $\xi = 0$. It is also onto, because for every $\chi \in \mathcal{M}$, $\lim_{x\to+\infty} f_\mathrm{r}(0,x)\chi = \xi$ exists, and hence (4.3) holds; thus $\chi = \Gamma\xi$. ■

The mapping Γ will make its appearance as a restriction to $\mathcal{N}$ of certain linear transformations defined on all of $\mathbf{C}^n$. One such representation immediately follows from (4.3). We can pick any x_0 for which $f_\mathrm{r}(0,x_0)$ is invertible and write

$$\Gamma = [f_\mathrm{r}(0,x_0)^{-1} f_\mathrm{l}(0,x_0)]|_{\mathcal{N}}, \tag{4.4}$$

where the symbol $|_{\mathcal{N}}$ denotes the restriction to the subspace $\mathcal{N}$. Recall that when $n=1$, Γ becomes a constant, so that (4.4) expresses the fact that, in the exceptional case, the two Jost solutions at k$k=0$ are linearly dependent. Clearly, (4.4) is valid whenever $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. Another representation of Γ that will play a role in this section is only valid when $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. It follows from (2.28) which, for any $\xi\in\mathcal{N}$, implies

$$\chi=\lim_{x\to-\infty} f_{\mathrm{l}}(0,x)\xi=\xi+\int_{-\infty}^{\infty}dy\, yQ(y)[f_{\mathrm{l}}(0,y)\xi], \tag{4.5}$$

where we have also used (2.33) and the fact that $\Delta_{\mathrm{l}}\xi=0$. Note that the integral on the right-hand side of (4.5) exists when $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ because $f_{\mathrm{l}}(0,y)\xi$ is bounded. However, without the vector ξ in the integrand, the integral in general does not exist as a matrix-valued integral, because some column vectors of the matrix $f_{\mathrm{l}}(0,y)$ may grow linearly as $y\to-\infty$. In fact, according to (2.35), this is always the case unless $\Delta_{\mathrm{l}}=0$. On the other hand, if $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, then the integral in (4.5) without the vector ξ in it exists as a matrix-valued integral and, in view of (3.1), we can write $\chi=(I_n+E_{\mathrm{l}})\xi$. In other words, we have

$$\Gamma=(I_n+E_{\mathrm{l}})|_{\mathcal{N}} \text{ provided } Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n}). \tag{4.6}$$

We will also need representations for Γ^{-1}. To this end we assume, without loss of generality, that $f_{\mathrm{l}}(0,0)$ is invertible. If not, we can perform a shift of the origin and use the fact that $f_{\mathrm{l}}(0,x)$ is invertible for x sufficiently large. We define

$$\mathcal{R}=f_{\mathrm{l}}(0,0)^{-1}f_{\mathrm{r}}(0,0), \tag{4.7}$$

and note that, by (4.3),

$$\mathcal{R}|_{\mathcal{M}}=\Gamma^{-1}. \tag{4.8}$$

Another representation for Γ^{-1} is obtained by using the integral relation for $f_{\mathrm{r}}(0,x)$ given in (2.29). If $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, then, for any $\chi\in\mathcal{M}$, by using (2.29), (2.34), and the fact that $\Delta_{\mathrm{r}}\chi=0$, we obtain

$$\xi=\lim_{x\to+\infty} f_{\mathrm{r}}(0,x)\chi=\chi-\left[\int_{-\infty}^{\infty}dy\, yQ(y)f_{\mathrm{r}}(0,y)\right]\chi,$$

and thus, by (3.1), $\xi=(I_n-E_{\mathrm{r}})\chi$. Therefore,

$$\Gamma^{-1}=(I_n-E_{\mathrm{r}})|_{\mathcal{M}} \text{ provided } Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n}).$$

After these preparations we are ready to begin the analysis of the small-k asymptotics of $\mathbf{S}(k)$ in the exceptional case. We first consider the Wronskian

$$W(k)=[f_{\mathrm{r}}(-k^*,x)^\dagger;f_{\mathrm{l}}(k,x)],\quad k\in\overline{\mathbf{C}^+},$$

which appears in (2.6) and, as seen from (2.20), is related to the transmission coefficient $T_{\mathrm{l}}(k)$ by

$$T_{\mathrm{l}}(k)=2ikW(k)^{-1}. \tag{4.9}$$

The method employed here to study $W(k)$ is patterned after that used in Ref. 12 in the scalar case. Unless otherwise stated, we will assume that k is real. This suffices for all the auxiliary results leading up to our main result given in Theorem 4.6. There we will extend the asymptotics from the real axis to $\mathbf{C}^+$ with the help of a Phragmén–Lindelöf theorem.

Using $[f_{\mathrm{l}}(0,x)^\dagger;f_{\mathrm{l}}(0,x)]=0$ we first write $W(k)$ in the form

$$W(k)=f_{\mathrm{r}}(-k,0)^\dagger[f_{\mathrm{l}}(0,0)^\dagger]^{-1}\Omega_1+\Omega_2 f_{\mathrm{l}}(0,0)^{-1}f_{\mathrm{l}}(k,0),$$

where we have defined

$$\Omega_1 = f_1(0,0)^\dagger f_1'(k,0) - f_1'(0,0)^\dagger f_1(k,0),$$

$$\Omega_2 = f_{\rm r}(-k,0)^\dagger f_1'(0,0) - f_r'(-k,0)^\dagger f_1(0,0).$$

The quantities Ω_1 and Ω_2 can be written as Wronskians by means of a new solution, $\varphi(k,x)$, of (1.1), which is defined by the initial conditions

$$\varphi(k,0) = f_1(0,0), \quad \varphi'(k,0) = f_1'(0,0), \tag{4.10}$$

so that

$$\varphi(0,x) = f_1(0,x). \tag{4.11}$$

Then we have

$$W(k) = f_{\rm r}(-k,0)^\dagger [f_1(0,0)^\dagger]^{-1}[\varphi(k,x)^\dagger; f_1(k,x)] + [f_{\rm r}(-k,x)^\dagger; \varphi(k,x)] f_1(0,0)^{-1} f_1(k,0). \tag{4.12}$$

We mention that the particular choice of the solution $\varphi(k,x)$ is motivated by the fact that there is a crucial estimate, namely (A8) of the Appendix, for the difference $[\varphi(k,x) - \varphi(0,x)]\xi$ with $\xi \in \mathcal{N}$, which plays a key role in the proof of the next proposition. Since the proof of this proposition is lengthy, it is given in the Appendix.

Proposition 4.2: Assume $Q \in L^1_m(\mathbf{R}; \mathbf{C}^{n\times n})$ *for* $m=1$ *or* 2. *Then, as* $k \to 0$ *in* $\mathbf{R}$ *we have*

$$[\varphi(k,x)^\dagger; f_1(k,x)] = \sum_{j=1}^{m} k^j Y_j + o(k^m), \tag{4.13}$$

where

$$Y_1 = iI_n, \quad Y_2 = \int_0^\infty dz\, [f_1(0,z)^\dagger f_1(0,z) - I_n],$$

and

$$[f_{\rm r}(-k,x)^\dagger; \varphi(k,x)] = \sum_{j=0}^{m-1} k^j X_j + o(k^{m-1}), \tag{4.14}$$

with

$$X_0 = \Delta_1, \quad X_1 = i[I_n + E_1].$$

For $\xi \in \mathcal{N}$ *we have*

$$[f_{\rm r}(-k,x)^\dagger; \varphi(k,x)]\xi = \sum_{j=1}^{m} k^j \check{X}_j \xi + o(k^m), \tag{4.15}$$

$$\check{X}_1 = i\Gamma, \quad \check{X}_2 = \int_{-\infty}^0 dz\, [f_{\rm r}(0,z)^\dagger f_{\rm r}(0,z) - I_n]\Gamma.$$

The notational differences between (4.14) and (4.15) are justified by the fact that in (4.15) the coefficient $\check{X}_1$ is used when $m=1$, while in (4.14) the corresponding coefficient X_1 is used only when $m=2$. Of course, if $m=2$, then $\check{X}_1 = X_1|_{\mathcal{N}}$, by (4.6).

Our first goal is to find the leading terms in the asymptotics of $W(k)^{-1}$ as $k \to 0$. For this purpose it is convenient to temporarily replace the factors multiplying the Wronskians in (4.12) by their limits as $k \to 0$. That is, we consider the simpler expression

$$Z(k)=\mathcal{R}^\dagger[\varphi(k,x)^\dagger;f_l(k,x)]+[f_r(-k,x)^\dagger;\varphi(k,x)],\tag{4.16}$$

where we have used (4.7) via its adjoint. In order to further motivate the use of $Z(k)$, we note that on account of (4.12) and (4.16) we can write

$$W(k)^{-1}=f_l(k,0)^{-1}f_l(0,0)[Z(k)+\Theta_1(k)+\Theta_2(k)]^{-1},\tag{4.17}$$

where

$$\Theta_1(k)=\mathcal{R}^\dagger[\varphi(k,x)^\dagger;f_l(k,x)][f_l(k,0)^{-1}f_l(0,0)-I_n],\tag{4.18}$$

$$\Theta_2(k)=\{f_r(-k,0)^\dagger[f_l(0,0)^\dagger]^{-1}-\mathcal{R}^\dagger\}[\varphi(k,x)^\dagger;f_l(k,x)]f_l(k,0)^{-1}f_l(0,0),\tag{4.19}$$

provided the second inverse on the right-hand side of (4.17) exists. The existence of this inverse will be established below, where we show that $Z(k)^{-1}$ exists for sufficiently small k and satisfies $Z(k)^{-1}=O(1/k)$ as $k\to 0$. This, together with the fact that, in view of (4.13) and Corollary 2.2, $\Theta_1(k)$ and $\Theta_2(k)$ are both $o(k)$ as $k\to 0$, implies

$$W(k)^{-1}=f_l(k,0)^{-1}f_l(0,0)Z(k)^{-1}\{I_n+[\Theta_1(k)+\Theta_2(k)]Z(k)^{-1}\}^{-1},\tag{4.20}$$

where the inverse of the matrix inside the braces exists provided k is sufficiently small. This explains why we focus on $Z(k)$ in the next result, which is an immediate consequence of (4.16) and Proposition 4.2.

Corollary 4.3: Suppose that $Q\in L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ *for* $m=1$ *or* 2. *Then, as* $k\to 0$ *in* $\mathbf{R}$

$$Z(k)=\sum_{j=0}^{m-1}k^jV_j+o(k^{m-1}),\tag{4.21}$$

$$V_0=\Delta_1,\quad V_1=i[I_n+E_1+\mathcal{R}^\dagger].$$

Moreover, for $\xi\in\mathcal{N}$, *we have*

$$Z(k)\xi=\sum_{j=1}^{m}k^j\check{V}_j\xi+o(k^m),\tag{4.22}$$

$$\check{V}_1=i[\Gamma+\mathcal{R}^\dagger],$$

$$\check{V}_2=\mathcal{R}^\dagger\int_0^\infty dz\,[f_l(0,z)^\dagger f_l(0,z)-I_n]+\int_{-\infty}^0 dz\,[f_r(0,z)^\dagger f_r(0,z)-I_n]\Gamma.$$

Now our task is to identify those matrix elements of $Z(k)^{-1}$ that dominate as $k\to 0$. To do this we choose a Jordan basis for Δ_1 as follows. We assume that there are κ Jordan chains indexed by α for $\alpha=1,...,\kappa$, each consisting of n_α vectors $u_{\alpha j}$, with $j=1,...,n_\alpha$, satisfying the relations

$$\begin{cases}(\Delta_1-\lambda_\alpha)u_{\alpha 1}=0,\\ (\Delta_1-\lambda_\alpha)u_{\alpha j}=u_{\alpha(j-1)}, & j=2,...,n_\alpha.\end{cases}\tag{4.23}$$

Here λ_α is an eigenvalue of $\Delta_1, u_{\alpha 1}$ is the corresponding eigenvector belonging to the αth chain, and the vectors $u_{\alpha j}$ with $j\neq 1$ are the generalized eigenvectors. We assume that the eigenvalue 0 of Δ_1 has geometric multiplicity μ and algebraic multiplicity ν; thus $\Sigma_{\alpha=1}^{\mu}n_\alpha=\nu$ and $\mu=\dim\ \mathcal{N}\geqslant 1$. We arrange the vectors of the Jordan basis in a list which is ordered according to the rule that $u_{\alpha j}$ comes before $u_{\beta s}$ if and only if $\alpha<\beta$ or $\alpha=\beta$ and $j<s$. In other words, this is the "dictionary order" of the two two-letter words αj and βs. It is necessary to specify an order on the Jordan basis because later we will have to perform certain permutations on these basis vectors.

We further assume that the first μ Jordan chains belong to the eigenvalue 0 of Δ_1 so that $\{u_{11},u_{21},\dots,u_{\mu 1}\}$ forms a basis for the kernel of Δ_1. We will also need the adjoint Jordan basis $\{w_{\alpha j}\}$ whose vectors, for $\alpha=1,...,\kappa$, satisfy $w_{\alpha j}^\dagger u_{\rho t}=\delta_{\alpha\rho}\delta_{jt}$, where $\delta_{\alpha\beta}$ denotes the Kronecker delta, and

$$\begin{cases}(\Delta_{\rm r}-\lambda_\alpha^*)w_{\alpha n_\alpha}=0,\\ (\Delta_{\rm r}-\lambda_\alpha^*)w_{\alpha j}=w_{\alpha(j+1)}, \quad j=1,...,n_\alpha-1.\end{cases}$$

Thus the set $\{w_{1n_1},\dots,w_{\mu n_\mu}\}$ forms a basis for the kernel of $\Delta_{\rm r}$. The transition matrix from the standard basis to the Jordan basis will be denoted by $\mathcal{S}$. Given any $n\times n$ matrix M in the standard basis, we use $\tilde{M}$, where $\tilde{M}=\mathcal{S}^{-1}M\mathcal{S}$, to denote the matrix representation of M in the Jordan basis $\{u_{\alpha j}\}$. Then from (4.23) it follows that $\tilde{\Delta}_1$ has the appearance

$$\tilde{\Delta}_1=\bigoplus_{\alpha=1}^{\kappa} J_{n_\alpha}(\lambda_\alpha), \tag{4.24}$$

where $J_{n_\alpha}(\lambda_\alpha)$ is the Jordan block with λ_α appearing on the diagonal and 1 on the first superdiagonal.

In the notation introduced above we can view the pair αj as a ‘‘block index’’ in the sense that α indicates the Jordan block (resp. the Jordan chain) to which the vector $u_{\alpha j}$ belongs, and j indicates the position within that block. Generalizing this notation, we will sometimes use block indices to designate the matrix elements of matrices represented in the Jordan basis $\{u_{\alpha j}\}$. Then the matrix elements of $\tilde{M}=\mathcal{S}^{-1}M\mathcal{S}$ in block index notation are given by

$$\tilde{M}_{\beta s;\alpha j}=w_{\beta s}^\dagger M u_{\alpha j}. \tag{4.25}$$

An important observation about $\tilde{Z}(k)$ is that it has μ columns, namely those with ‘‘addresses’’ $\alpha 1$ for $\alpha=1,...,\mu$ which are $O(k)$, and these are the only columns with this property. Any other column contains at least one element that tends to a nonzero limit as $k\to 0$. Now, as we shall see below, the entries of $\tilde{Z}(k)$ which determine the leading asymptotic behavior of $\tilde{Z}(k)^{-1}$ as $k\to 0$ form a submatrix of $\tilde{Z}(k)$ consisting of columns $\alpha 1$ and rows βn_β, where α and β both belong to $\{1,...,\mu\}$. It is, therefore, convenient to perform suitable permutations of the columns and rows of $\tilde{Z}(k)$ in order to collect these particular matrix elements in a $\mu\times\mu$ diagonal block of a new matrix, called $\mathcal{Z}(k)$. The formal definition of these permutations and their implementation are as follows. Let π_1 be the permutation

$$\pi_1:(1,...,\nu)\mapsto(q_1,\dots,q_\nu),$$

where

$$q_\tau=\begin{cases}n_1+\cdots+n_{\tau-1}+1, & \tau=1,...,\mu,\\ \tau-\mu+\alpha, & \tau=\mu+1,...,\nu,\end{cases} \tag{4.26}$$

and $\alpha\in\{1,...,\mu\}$ is the unique integer such that, for given τ and μ,

$$n_1+n_2+\cdots+n_{\alpha-1}-\alpha+j=\tau-\mu,$$

for some $j\in\{2,...,n_\alpha\}$. Note that, since $n_\alpha\geqslant 1$, the quantity $n_1+n_2+\cdots+n_{\alpha-1}-\alpha$ is a nondecreasing function of α. Similarly, let π_2 be the permutation

$$\pi_2:(1,...,\nu)\mapsto(\sigma_1,\dots,\sigma_\nu),$$

where

$$\sigma_\alpha=\begin{cases} n_1+\cdots+n_\alpha, & \alpha=1,...,\mu, \\ \alpha-\mu+\rho-1, & \alpha=\mu+1,...,\nu, \end{cases} \tag{4.27}$$

and $\rho\in\{1,...,\mu\}$ is the unique integer such that, for given α and μ

$$n_1+n_2+\cdots+n_{\rho-1}-\rho+s=\alpha-\mu,$$

for some $s\in\{2,...,n_\rho\}$. To implement these permutations we let $\hat{e}_j$ for $j=1,...,\nu$ denote the column vectors of the standard basis in $\mathbf{C}^\nu$ and let Π_1 be the $\nu\times\nu$ permutation matrix whose jth column vector is $\hat{e}_{q_j}$, and let Π_2 be the $\nu\times\nu$ permutation matrix whose kth row vector is $\hat{e}^\dagger_{\sigma_k}$. Now observe that, if M is any $\nu\times\nu$ matrix, then the matrix $\Pi_2 M\Pi_1$ can be thought of as being obtained from M by a permutation of the columns according to π_1 and a permutation of the rows according to π_2. In order to apply these operations to $\tilde{Z}(k)$ we define

$$P_1=\mathrm{diag}\{\Pi_1,I_{n-\nu}\},\quad P_2=\mathrm{diag}\{\Pi_2,I_{n-\nu}\},$$

$$\mathcal{Z}(k)=P_2\tilde{Z}(k)P_1=P_2\mathcal{S}^{-1}Z(k)\mathcal{S}P_1, \tag{4.28}$$

and we partition $\mathcal{Z}(k)$ as

$$\mathcal{Z}(k)=\begin{bmatrix}\mathcal{A}(k) & \mathcal{B}(k)\\ \mathcal{C}(k) & \mathcal{D}(k)\end{bmatrix}, \tag{4.29}$$

where $\mathcal{A}(k)$ has size $\mu\times\mu$ and, consequently, $\mathcal{D}(k)$ has size $(n-\mu)\times(n-\mu)$. Then $\mathcal{A}(k)$ coincides with the submatrix of $\tilde{Z}(k)$ consisting of the elements in columns $\alpha 1$ and rows sn_s, where $1\leqslant\alpha\leqslant\mu$ and $1\leqslant s\leqslant\mu$. As we have already indicated above, the matrix $\mathcal{A}(k)$ determines the leading asymptotic behavior of $\mathcal{Z}(k)^{-1}$ as $k\to 0$. The next two propositions provide the necessary information about the behavior of the four matrix blocks in (4.29).

Proposition 4.4: Assume $Q\in L^1_m(\mathbf{R};\mathbf{C}^{n\times n})$ *for* $m=1$ *or* 2. *Then the matrices* $\mathcal{A}(k)$, $\mathcal{B}(k)$, $\mathcal{C}(k)$, *and* $\mathcal{D}(k)$ *appearing in (4.29) behave near* $k=0$, *with* $k\in\mathbf{R}$, *as*

$$\mathcal{A}(k)=\sum_{j=1}^{m} k^j\mathcal{A}_j+o(k^m),\quad \mathcal{B}(k)=\sum_{j=1}^{m-1} k^j\mathcal{B}_j+o(k^{m-1}), \tag{4.30}$$

$$\mathcal{C}(k)=\sum_{j=1}^{m} k^j\mathcal{C}_j+o(k^m),\quad \mathcal{D}(k)=\sum_{j=0}^{m-1} k^j\mathcal{D}_j+o(k^{m-1}), \tag{4.31}$$

where in the expansion for $\mathcal{B}(k)$ *the sum is absent when* $m=1$. *Moreover,* $\mathcal{A}_1$ *and* $\mathcal{D}_0$ *are invertible.*

Proof: We give the proof only for $\mathcal{B}(k)$; the proofs for the other matrices are similar. Let e_j for $j=1,...,n$ denote the standard basis vectors in $\mathbf{C}^n$. Let $s\in\{1,...,\mu\}$ and first suppose that $p\in\{1,...,\nu-\mu\}$. Then we have

$$\mathcal{B}(k)_{sp}=e_s^\dagger\mathcal{Z}(k)e_{\mu+p}=e_s^\dagger P_2\tilde{Z}(k)P_1e_{\mu+p}=[\hat{e}^\dagger_{\sigma_s}\ \ 0]\tilde{Z}(k)\begin{bmatrix}\hat{e}_{q_{\mu+p}}\\ 0\end{bmatrix}$$

$$=e^\dagger_{\sigma_s}\tilde{Z}(k)e_{q_{\mu+p}}=\tilde{Z}(k)_{\sigma_s q_{\mu+p}}=\tilde{Z}(k)_{sn_s;\alpha j},$$

where α and j are determined by (4.26) with $\tau=\mu+p\leqslant\nu$; hence $2\leqslant j\leqslant n_\alpha$ and $1\leqslant\alpha\leqslant\mu$. Thus it follows from (4.25) and Corollary 4.3 that $\mathcal{B}(k)_{sp}=o(1)$ if $m=1$ and $\mathcal{B}(k)_{sp}=k\mathcal{B}_{1,sp}+o(k)$ if

$m=2$. Specifically, we have $\mathcal{B}_{1,sp}=w_{sn_s}^\dagger V_1 u_{\alpha j}$, where V_1 is given in (4.21). It remains to consider the matrix elements with $p\in\{n-\nu+1,...,\nu-\mu\}$. Since $P_1 e_{\mu+p}=e_{\mu+p}$, we obtain

$$\mathcal{B}(k)_{sp}=\tilde{Z}(k)_{\sigma_s(\mu+p)}=\tilde{Z}(k)_{sn_s;\alpha j}=w_{sn_s}^\dagger Z(k)u_{\alpha j},$$

where α and j are determined by the equation $n_1+\cdots+n_{\alpha-1}+j=\mu+p$; note that $\mu+p>\nu$ and thus $\alpha\geqslant\mu+1$. Since $s\leqslant\mu$, by using Corollary 4.3, we conclude that $\mathcal{B}(k)_{sp}=o(1)$ if $m=1$, and $\mathcal{B}(k)_{sp}=k\mathcal{B}_{1,sp}+o(k)$ with $\mathcal{B}_{1,sp}=w_{sn_s}^\dagger V_1 u_{\alpha j}$ if $m=2$.

To prove that $\mathcal{A}_1$ is invertible we first note that for s and $j\in\{1,...,\mu\}$, we have

$$\mathcal{A}(k)_{sj}=\tilde{Z}(k)_{\sigma_s q_j}=\tilde{Z}(k)_{sn_s;j1}=w_{sn_s}^\dagger Z(k)u_{j1},$$

and thus, by (4.21)

$$\mathcal{A}_{1,sj}=w_{sn_s}^\dagger V_1 u_{j1}=i w_{sn_s}^\dagger[\Gamma+\mathcal{R}^\dagger]u_{j1}. \tag{4.32}$$

We show that the kernel of the transformation $\mathcal{A}_1:\mathbf{C}^\mu\mapsto\mathbf{C}^\mu$ is trivial. Suppose there is a vector $(c_1,\ldots,c_\mu)$ such that $\Sigma_{j=1}^\mu\mathcal{A}_{1,sj}c_j=0$ for $s=1,...,\mu$. Let $\xi=\Sigma_{j=1}^\mu c_j u_{j1}$ and $\chi=\Gamma\xi$ [cf. (4.2)]. Since $\chi\in\mathcal{M}$, it is a linear combination of the vectors $w_{1n_1},\ldots,w_{\mu n_\mu}$ and hence $\chi^\dagger V_1\xi=0$. On the other hand, by using (4.7), we obtain

$$\chi^\dagger V_1\xi=i\chi^\dagger[\Gamma+\mathcal{R}^\dagger]\xi=i(\|\chi\|^2+\|\xi\|^2),$$

which is nonzero unless $c_1=\cdots=c_\mu=0$. Hence $\mathcal{A}_1$ is invertible. Finally, from (4.28), (4.29), and Corollary 4.3, we get

$$\mathcal{D}_0=\mathrm{diag}\{I_{\nu-\mu},J_{n_{\mu+1}},\ldots,J_{n_\kappa}\}, \tag{4.33}$$

where J_{n_α} are the matrices appearing in (4.24). Clearly, $\mathcal{D}_0$ is invertible. ■

Next we study the behavior of the inverse of the matrix defined in (4.29) near $k=0$.

Proposition 4.5: Assume $Q\in L_m^1(\mathbf{R};\mathbf{C}^{n\times n})$ *for* $m=1$ *or* 2. *Then as* $k\to 0$ *in* $\mathbf{R}$ *we have the following:*

(i) *If* $m=1$, *then*

$$\mathcal{Z}(k)^{-1}=\begin{bmatrix}(1/k)\mathcal{A}_1^{-1}+o(1/k) & o(1/k)\\ -\mathcal{D}_0^{-1}\mathcal{C}_1\mathcal{A}_1^{-1}+o(1) & \mathcal{D}_0^{-1}+o(1)\end{bmatrix}. \tag{4.34}$$

(ii) *If* $m=2$, *then*

$$\mathcal{Z}(k)^{-1}=\frac{1}{k}\mathcal{Z}_{-1}+\mathcal{Z}_0+o(1), \tag{4.35}$$

where

$$\mathcal{Z}_{-1}=\mathrm{diag}\{\mathcal{A}_1^{-1},0\}, \tag{4.36}$$

$$\mathcal{Z}_0=\begin{bmatrix}-\mathcal{A}_1^{-1}\mathcal{A}_2\mathcal{A}_1^{-1}+\mathcal{A}_1^{-1}\mathcal{B}_1\mathcal{D}_0^{-1}\mathcal{C}_1\mathcal{A}_1^{-1} & -\mathcal{A}_1^{-1}\mathcal{B}_1\mathcal{D}_0^{-1}\\ -\mathcal{D}_0^{-1}\mathcal{C}_1\mathcal{A}_1^{-1} & \mathcal{D}_0^{-1}\end{bmatrix}. \tag{4.37}$$

Proof: We exploit the fact that

$$\begin{bmatrix}I_\mu & -\mathcal{B}(k)\mathcal{D}(k)^{-1}\\ 0 & I_{n-\mu}\end{bmatrix}\mathcal{Z}(k)\begin{bmatrix}I_\mu & 0\\ -\mathcal{D}(k)^{-1}\mathcal{C}(k) & I_{n-\mu}\end{bmatrix}=\mathrm{diag}\{\mathcal{U}(k),\mathcal{D}(k)\}, \tag{4.38}$$

where

$$\mathcal{U}(k)=\mathcal{A}(k)-\mathcal{B}(k)\mathcal{D}(k)^{-1}\mathcal{C}(k).$$

By (4.30), (4.31), and Proposition 4.4, we have

$$\mathcal{B}(k)\mathcal{D}(k)^{-1}\mathcal{C}(k)=o(k), \quad \mathcal{A}(k)=k\mathcal{A}_1+o(k),$$

with $\det \mathcal{A}_1\neq 0$, and hence we conclude that, for small enough nonzero k, $\mathcal{U}(k)$ is invertible and

$$\mathcal{U}(k)^{-1}=\begin{cases}(1/k)\mathcal{A}_1^{-1}+o(1/k), & m=1,\\ (1/k)\mathcal{A}_1^{-1}-\mathcal{A}_1^{-1}\mathcal{A}_2\mathcal{A}_1^{-1}+\mathcal{A}_1^{-1}\mathcal{B}_1\mathcal{D}_0^{-1}\mathcal{C}_1\mathcal{A}_1^{-1}+o(1), & m=2.\end{cases} \tag{4.39}$$

As a result, from (4.38) we obtain

$$\mathcal{Z}(k)^{-1}=\begin{bmatrix}\mathcal{U}(k)^{-1} & -\mathcal{U}(k)^{-1}\mathcal{B}(k)\mathcal{D}(k)^{-1}\\ -\mathcal{D}(k)^{-1}\mathcal{C}(k)\mathcal{U}(k)^{-1} & \mathcal{D}(k)^{-1}\mathcal{C}(k)\mathcal{U}(k)^{-1}\mathcal{B}(k)\mathcal{D}(k)^{-1}+\mathcal{D}(k)^{-1}\end{bmatrix},$$

and hence (4.34)–(4.37) follow by using (4.30), (4.31), and (4.39). ■

The primary conclusion of Proposition 4.5 is that $\mathcal{Z}(k)^{-1}$ has a $1/k$-singularity at $k=0$ if $\dim \mathcal{N}\geqslant 1$. Therefore, $\tilde{Z}(k)^{-1}$ and $Z(k)^{-1}$ have a similar behavior. Indeed, from (4.28) and (4.35) we infer that

$$Z(k)^{-1}=\sum_{j=0}^{m-1} k^{j-1}Z_{j-1}+o(k^{m-2}), \quad k\to 0 \text{ in } \mathbf{R}, \tag{4.40}$$

where

$$Z_{-1}=\mathcal{S}P_1\mathcal{Z}_{-1}P_2\mathcal{S}^{-1}, \quad Z_0=\mathcal{S}P_1\mathcal{Z}_0P_2\mathcal{S}^{-1}. \tag{4.41}$$

This leads us to the main result of this section. We will lift the restriction that k be real and allow $k\in\overline{\mathbf{C}^+}$ in the asymptotics of the transmission coefficients.

Theorem 4.6: *Assume* $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ *and* $\dim \mathcal{N}\geqslant 1$. *Then the scattering coefficients are continuous at* $k=0$, *and we have*

$$T_l(k)=2iZ_{-1}+o(1), \quad T_r(k)=-2iZ_{-1}^\dagger+o(1), \quad k\to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{4.42}$$

$$\operatorname{Im} T_l(0)=\operatorname{Ker} \Delta_1, \quad \operatorname{Ker} T_l(0)=\operatorname{Im} \Delta_1, \tag{4.43}$$

$$\operatorname{Im} T_r(0)=\operatorname{Ker} \Delta_r, \quad \operatorname{Ker} T_r(0)=\operatorname{Im} \Delta_r, \tag{4.44}$$

$$L(k)=-I_n+\Gamma T_l(0)+o(1), \quad R(k)=-I_n+\Gamma^{-1}T_l(0)^\dagger+o(1), \quad k\to 0 \text{ in } \mathbf{R}, \tag{4.45}$$

$$\operatorname{Ker}\{I_n+L(0)\}=\operatorname{Ker} T_l(0), \quad \operatorname{Ker} T_r(0)=\operatorname{Ker}\{I_n+R(0)\}, \tag{4.46}$$

$$\operatorname{Im}\{I_n+L(0)\}=\operatorname{Im} T_r(0), \quad \operatorname{Im}\{I_n+R(0)\}=\operatorname{Im} T_l(0). \tag{4.47}$$

Proof: For $k\in\mathbf{R}$, the continuity of the transmission coefficients and (4.42) follow immediately from (2.25), (4.9), (4.20), (4.40), and (4.41). To extend the asymptotic formulas in (4.42) from $k\in\mathbf{R}$ to $k\in\overline{\mathbf{C}^+}$ we first note that

$$\det W(k)=[\det Z(k)][1+o(1)]=[\det \mathcal{Z}(k)][1+o(1)]=C_0k^\mu[1+o(1)], \quad k\to 0 \text{ in } \mathbf{R},$$

where $C_0=(-1)^{\nu-\mu}(\det \mathcal{A}_1)(\det \mathcal{D}_0)\neq 0$. This follows from (4.20), (4.28), (4.30), (4.36), (4.38), Proposition 4.4, and the fact that $(\det P_1)(\det P_2)=(-1)^{\nu-\mu}$. It follows that $k^{-\mu}\det W(k)\to C_0$ as

$k \to 0$ along the real axis. Since $\det W(k)$ extends as an analytic function to $\mathbf{C}^+$, there is a constant C such that $|k^{-\mu} \det W(k)| \leqslant C|k|^{-\mu}$ for k near 0 in $\overline{\mathbf{C}^+}$. Appealing to some theorems of Phragmén–Lindelöf (e.g., Theorems 1.4.1–1.4.4 in Ref. 14) we conclude that $k^{-\mu} \det W(k) \to C_0$ as $k \to 0$ in $\overline{\mathbf{C}^+}$. Thus there is a set $\Sigma_\epsilon = \{k \in \overline{\mathbf{C}^+} : 0 < |k| < \epsilon\}$, with ϵ sufficiently small, on which $|\det W(k)| \geqslant C_1 |k|^\mu$ for some constant C_1. Recalling the cofactor representation of the inverse of a matrix we conclude that

$$\|W(k)^{-1}\| \leqslant C_2 |k|^{-\mu}, \quad k \in \Sigma_\epsilon,$$

for some constant C_2. Since $T_l(k) \to T_l(0)$ as $k \to 0$ along the real axis, we can apply a Phragmén–Lindelöf theorem to $2ikW(k)^{-1}$ and conclude that, by (4.9), $T_l(k) \to T_l(0)$ as $k \to 0$ in $\overline{\mathbf{C}^+}$. This, together with (2.25), completes the proof of (4.42).

To prove (4.43) we note that (4.24) and (4.26) imply $\operatorname{Ker} \tilde{\Delta}_l = \operatorname{Span}\{e_{q_1}, \ldots, e_{q_\mu}\}$. Thus, in view of the form of $\mathcal{Z}_{-1}$ given in (4.36), we have

$$\operatorname{Im}\{P_1 \mathcal{Z}_{-1} P_2\} = P_1 \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathbf{C}^\mu \right\} = P_1 \operatorname{Span}\{e_1, \ldots, e_\mu\} = \operatorname{Ker} \tilde{\Delta}_l.$$

Since $\Delta_l = \mathcal{S} \tilde{\Delta}_l \mathcal{S}^{-1}$, the first equality in (4.43) follows from (4.41) and (4.42). To prove the second equality we note that

$$\operatorname{Im} \tilde{\Delta}_l = \operatorname{Span}\{e_k : k \notin \{\sigma_1, \ldots, \sigma_\mu\}\},$$

which follows from (4.24) and (4.25). Therefore,

$$\begin{aligned} \operatorname{Ker}\{\mathcal{Z}_{-1} P_2\} &= \left\{ w \in \mathbf{C}^n : P_2 w = \begin{bmatrix} 0 \\ v \end{bmatrix}, \quad v \in \mathbf{C}^{n-\mu} \right\} \\ &= \{w \in \mathbf{C}^n : e_k^\dagger P_2 w = 0, \quad k = 1, \ldots, \mu\} \\ &= \{w \in \mathbf{C}^n : e_{\sigma_k}^\dagger w = 0, \quad k = 1, \ldots, \mu\} \\ &= \operatorname{Im} \tilde{\Delta}_l. \end{aligned}$$

This implies $\operatorname{Ker} T_l(0) = \operatorname{Im} \Delta_l$ and thus the second equality in (4.43) is proved. The equalities in (4.44) follow from (4.43) by taking adjoints and using the fact that $(\operatorname{Ker} M)^\perp = \operatorname{Im} M^\dagger$ for any $n \times n$ matrix M.

To prove the remaining assertions we use

$$f_l(k,x) T_l(k) = f_r(-k,x) + f_r(k,x) L(k), \quad k \in \mathbf{R} \setminus \{0\}, \tag{4.48}$$

$$f_r(k,x) T_r(k) = f_l(-k,x) + f_l(k,x) R(k), \quad k \in \mathbf{R} \setminus \{0\}, \tag{4.49}$$

which can be derived with the help of (1.4) and (1.5). From (4.48) and the continuity of $T_l(k)$ it immediately follows that $L(k)$ is continuous at $k=0$ and we have

$$f_r(0,x)[I_n + L(0)] = f_l(0,x) T_l(0). \tag{4.50}$$

Now choose x such that $f_r(0,x)$ is invertible and multiply (4.50) from the left by $f_r(0,x)^{-1}$. Owing to (4.4) and the first equation in (4.43), we can replace $f_r(0,x)^{-1} f_l(0,x)$ by Γ. Hence the first relation in (4.45) follows. Similarly, the second relation in (4.45) is obtained from (4.49). The two equalities in (4.46) are immediate consequences of (4.45). Finally, (4.47) follows from (4.43), (4.45), and Proposition 4.1. ■

From (4.43), (4.44), Proposition 2.4(i), and Theorem 3.1 we infer that the exceptional case occurs if and only if $T_{\rm l}(0)\neq 0$. Moreover, (4.43), (4.46), and Theorem 3.1 show that $L(0)$ and $R(0)$ each have eigenvalue -1 if and only if $\Delta_1\neq 0$. In view of (2.23) and (2.24) we also have $\|L(0)\|=\|R(0)\|=1$ if and only if $\Delta_1\neq 0$. The case $\Delta_1=0$ can be called the purely exceptional case because then we have $\mathcal{N}=\mathcal{M}=\mathbf{C}^n$. This case is further analyzed in Example 5.4 of the next section.

Theorem 4.7: *Assume* $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$ *and* $\dim\mathcal{N}\geqslant 1$. *Then the scattering coefficients are differentiable at* $k=0$ *and*

$$T_{\rm l}(k)=T_{\rm l}(0)+k\dot T_{\rm l}(0)+o(k),\quad k\to 0 \text{ in } \overline{\mathbf{C}^+},\tag{4.51}$$

with

$$\dot T_{\rm l}(0)=2i[Z_0-f_{\rm l}^{-1}(0,0)\dot f_{\rm l}(0,0)Z_{-1}+iH_1+iH_2],\tag{4.52}$$

where Z_{-1} *and* Z_0 *are given in (4.41) and*

$$H_1=Z_{-1}\mathcal{R}^\dagger f_{\rm l}^{-1}(0,0)\dot f_{\rm l}(0,0)Z_{-1},\quad H_2=Z_{-1}\dot f_{\rm r}(0,0)^\dagger[f_{\rm l}(0,0)^\dagger]^{-1}Z_{-1}.$$

Moreover,

$$T_{\rm r}(k)=T_{\rm l}(0)^\dagger-k^*\dot T_{\rm l}(0)^\dagger+o(k),\quad k\to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$L(k)=-I_n+(I_n+E_{\rm l})T_{\rm l}(0)+k\dot L(0)+o(k),\quad k\to 0 \text{ in } \mathbf{R},$$

$$R(k)=-I_n+(I_n-E_{\rm r})T_{\rm r}(0)+k\dot R(0)+o(k),\quad k\to 0 \text{ in } \mathbf{R},$$

where $E_{\rm l}$ *and* $E_{\rm r}$ *are as in (3.1) and*

$$\dot L(0)=[I_n+E_{\rm l}]\dot T_{\rm l}(0)+i[\dot f_{\rm r}(0,x)^\dagger;\dot f_{\rm l}(0,x)]T_{\rm l}(0),$$

$$\dot R(0)=[I_n-E_{\rm r}]\dot T_{\rm r}(0)-i[\dot f_{\rm l}(0,x)^\dagger;\dot f_{\rm r}(0,x)]T_{\rm r}(0).$$

Proof: To prove (4.51) and (4.52) for $k\to 0$ in $\mathbf{R}$, we first note the expansions

$$f_{\rm l}(k,0)^{-1}f_{\rm l}(0,0)=I_n-kf_{\rm l}(0,0)^{-1}\dot f_{\rm l}(0,0)+o(k),$$

$$f_{\rm r}(-k,0)^\dagger[f_{\rm l}(0,0)^\dagger]^{-1}=\mathcal{R}^\dagger-k\dot f_{\rm r}(0,0)^\dagger[f_{\rm l}(0,0)^\dagger]^{-1}+o(k),$$

$$\Theta_1(k)=-ik^2\mathcal{R}^\dagger f_{\rm l}(0,0)^{-1}\dot f_{\rm l}(0,0)+o(k^2),$$

$$\Theta_2(k)=-ik^2\dot f_{\rm r}(0,0)^\dagger[f_{\rm l}(0,0)^\dagger]^{-1}+o(k^2),$$

which follow from (4.18), (4.19), together with (4.7) and Proposition 4.2. Inserting these expansions in (4.20) and using (4.9) we obtain (4.51) and (4.52). As with (4.42) we can use a Phragmén–Lindelöf argument to extend the result to $\overline{\mathbf{C}^+}$. To find the expansions for $L(k)$ and $R(k)$ we first note that the existence of $\dot T_{\rm l}(0)$, together with (4.48) and (4.49), implies the existence of $\dot L(0)$ and $\dot R(0)$. Differentiating (4.48) with respect to k and taking the Wronskian with $\dot f_{\rm r}(0,x)^\dagger$, we obtain

$$[\dot f_{\rm r}(0,x)^\dagger;f_{\rm r}(0,x)]\dot L(0)=[\dot f_{\rm r}(0,x)^\dagger;\dot f_{\rm l}(0,x)]T_{\rm l}(0)+[\dot f_{\rm r}(0,x)^\dagger;f_{\rm l}(0,x)]\dot T_{\rm l}(0),\tag{4.53}$$

where we have used $[\dot{f}_{\mathrm{r}}(0,x)^{\dagger};\dot{f}_{\mathrm{r}}(0,x)]=0$. Using the integral relation (2.29) and that for $\dot{f}_{\mathrm{r}}(0,x)$ [cf. (A.20)] we obtain $[\dot{f}_{\mathrm{r}}(0,x)^{\dagger};f_{\mathrm{r}}(0,x)]=-iI_{\mathrm{n}}$. Inserting this together with (3.6) in (4.53) and using (4.47) we get the expansion for $L(k)$. The proof of the expansion for $R(k)$ is similar. ■

V. EXAMPLES

In this section we consider some special cases that illustrate various details of the analysis in Sec. IV. With the exception of Example 5.4 we only consider $T_{\mathrm{l}}(k)$.

Example 5.1: Let $n=1$ with $Q\in L_2^1(\mathbf{R})$ and assume the exceptional case occurs. Then $Z(k)=\tilde{Z}(k)=\mathcal{Z}(k)=\mathcal{A}(k)$, and these are all scalar functions. We choose $\xi=1=w$ and put $\gamma=\Gamma=f_{\mathrm{l}}(0,0)/f_{\mathrm{r}}(0,0)$, where now γ is a real nonzero constant. Since $T_{\mathrm{l}}(k)=T_{\mathrm{r}}(k)$, we denote the transmission coefficient by $T(k)$. By (4.32) we have $\mathcal{A}_1=i(\gamma^2+1)/\gamma$,

$$\mathcal{A}_2=\gamma^{-1}\int_0^{\infty}dz\,[f_{\mathrm{l}}(0,z)^{\dagger}f_{\mathrm{l}}(0,z)-I_n]+\gamma\int_{-\infty}^{0}dz\,[f_{\mathrm{r}}(0,z)^{\dagger}f_{\mathrm{r}}(0,z)-I_n],$$

so that

$$T(k)=\frac{2\gamma}{\gamma^2+1}+\frac{2ik\gamma\Xi}{(\gamma^2+1)^2}+o(k),\quad k\to 0\ \text{ in }\ \overline{\mathbf{C}^{+}}, \tag{5.1}$$

where we have defined

$$\Xi=\gamma[\dot{f}_{\mathrm{r}}(0,x);\dot{f}_{\mathrm{l}}(0,x)]+\int_0^{\infty}dz\,[f_{\mathrm{l}}(0,z)^2-1]+\gamma^2\int_{-\infty}^{0}dz\,[f_{\mathrm{r}}(0,z)^2-1].$$

In deriving (5.1) we have used the identity

$$\frac{\dot{f}_{\mathrm{l}}(0,x)}{f_{\mathrm{r}}(0,x)}+\frac{\dot{f}_{\mathrm{r}}(0,x)}{f_{\mathrm{l}}(0,x)}=-i[\dot{f}_{\mathrm{r}}(0,x);\dot{f}_{\mathrm{l}}(0,x)], \tag{5.2}$$

which can be verified as follows. Since $f_{\mathrm{r}}(0,x)$ and $\dot{f}_{\mathrm{r}}(0,x)$ are linearly independent solutions of (2.26), we can write

$$\dot{f}_{\mathrm{l}}(0,x)=c_1f_{\mathrm{r}}(0,x)+c_2\dot{f}_{\mathrm{r}}(0,x),$$

and evaluate c_1 and c_2 as

$$c_1=-i[\dot{f}_{\mathrm{r}}(0,0);\dot{f}_{\mathrm{l}}(0,0)],\quad c_2=-\frac{1}{\gamma},$$

so that (5.2) follows. It seems that the expansion (5.1) is new under the assumption $Q\in L_2^1(\mathbf{R})$.

Example 5.2: Assume $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ and suppose that $\tilde{\Delta}_1$ consists of one single Jordan block of size $n\geqslant 2$ associated with the eigenvalue 0. Thus $\kappa=1$, $\mu=1$, and $n=n_1=\nu$.

In this case we can simplify the notation by setting $u_{1j}=u_j$, for $j=1,...,n$. Then u_1 is the eigenvector for the eigenvalue 0 of Δ_1, that is $\mathcal{N}=\mathrm{Span}\{u_1\}$. The adjoint basis is $\{w_1,\ldots,w_n\}$ and we have $\mathcal{M}=\mathrm{Span}\{w_n\}$. The mapping Γ maps u_1 to a multiple of w_n, i.e., $\Gamma_{u1}=c_3w_n$ for some $c_3\neq 0$. Moreover, $\mathcal{A}(k)$ is a scalar function and from (4.32) we obtain

$$\mathcal{A}_1=\frac{i}{c_3^*}(|c_3|^2\|w_n\|^2+\|u_1\|^2),$$

where we have used (4.7) via $\mathcal{R}w_n=(1/c_3)u_1$. The permutation matrices appearing in (4.28) are given by

$$P_1 = I_n, \quad P_2 = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Using (4.41) and (4.42) we obtain

$$\tilde{T}_1(0) = \begin{bmatrix} 0 & 0 & \dots & 0 & 2/c_4 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where

$$c_4 = \frac{1}{c_3^*}(|c_3|^2 \|w_n\|^2 + \|u_1\|^2), \quad \tilde{T}_1(0) = \mathcal{S}^{-1} T_1(0) \mathcal{S}.$$

Example 5.3: This example illustrates the situation where $\tilde{\Delta}_1$ in (4.24) consists of two Jordan blocks. We assume $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$ and let $n=3$, $\mu=2$, $n_1=1$, $n_2=2$, $\nu=3$, and $\kappa=2$, so that Δ_1 has the Jordan form

$$\tilde{\Delta}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Jordan basis is $\{u_{11}, u_{21}, u_{22}\}$, where $\{u_{11}, u_{21}\}$ is a basis for $\mathcal{N}$, and the adjoint basis is $\{w_{11}, w_{21}, w_{22}\}$, where $\{w_{11}, w_{22}\}$ is a basis for $\mathcal{M}$. In this case the rows of $\tilde{Z}(k)$ need to be permuted according to $\pi_2: (1,2,3) \mapsto (1,3,2)$, whereas no permutation of the columns is required. Thus we have

$$P_1 = I_3, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\mathcal{A}(k)$ is a 2×2 matrix and

$$\mathcal{A}_1 = \begin{bmatrix} w_{11}^\dagger V_1 u_{11} & w_{11}^\dagger V_1 u_{21} \\ w_{22}^\dagger V_1 u_{11} & w_{22}^\dagger V_1 u_{21} \end{bmatrix},$$

where V_1 is given in (4.21). Hence we obtain

$$\tilde{T}_1(0) = \frac{1}{\det \mathcal{A}_1} \begin{bmatrix} w_{22}^\dagger V_1 u_{21} & 0 & -w_{11}^\dagger V_1 u_{21} \\ -w_{22}^\dagger V_1 u_{11} & 0 & w_{11}^\dagger V_1 u_{11} \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 5.4: This is the purely exceptional case mentioned above Theorem 4.7. We assume $Q\in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ with $n>1$. We have $\Delta_1=0$, which implies $\mu=\nu=\kappa=n$. Then $\mathcal{N}=\mathcal{M}=\mathbf{C}^n$, and thus no restrictions are necessary in (4.4) and (4.8); that is, we have $\mathcal{R}=\Gamma^{-1}$. Moreover, $P_1=P_2=I_n$. It follows that

$$\mathcal{A}_1=i\mathcal{S}^{-1}[\Gamma+(\Gamma^{-1})^\dagger]\mathcal{S},$$

and thus, since $\mathcal{Z}_{-1}=\mathcal{A}_1^{-1}$, we obtain

$$T_1(0)=2i\mathcal{S}(\mathcal{S}\mathcal{A}_1)^{-1}=2\Gamma^\dagger(\Gamma\Gamma^\dagger+I_n)^{-1}.$$

For the reflection coefficients, after some straightforward manipulations, we find

$$L(0)=(\Gamma\Gamma^\dagger-I_n)(\Gamma\Gamma^\dagger+I_n)^{-1},\quad R(0)=(I_n-\Gamma^\dagger\Gamma)(\Gamma^\dagger\Gamma+I_n)^{-1}.$$

Example 5.5: Suppose $Q(x)$ is even and belongs to $L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. This implies that $f_{\mathrm{r}}(0,x)=f_{\mathrm{l}}(0,-x)$ and from (2.31), (2.33), and (2.34) we conclude that Δ_1 is self-adjoint. Hence Δ_1 is diagonalizable and there are no Jordan chains of length greater than 1. We have $\mu=\nu$, $n_\alpha=1$ for $1\leqslant\alpha\leqslant\kappa$, and $\kappa=n$. We also have $P_1=P_2=I_n$. It is possible that Δ_1 has some nonzero eigenvalues, so $\mu<n$ in general. If $\xi\in\mathcal{N}$, then

$$f_{\mathrm{l}}(0,x)\xi=f_{\mathrm{r}}(0,-x)\xi,$$

which implies that $f_{\mathrm{r}}(0,x)\xi$ is bounded. This means $\xi\in\mathcal{M}$ and hence $\mathcal{N}=\mathcal{M}$. Furthermore, using (4.2) and (4.3) we conclude that

$$f_{\mathrm{l}}(0,x)\chi=f_{\mathrm{r}}(0,-x)\chi=f_{\mathrm{l}}(0,-x)\xi, \tag{5.3}$$

where $\xi\in\mathcal{N}$ and $\chi=\Gamma\xi$. Letting $x\to-\infty$, we see that $\Gamma\chi=\xi$, that is

$$\Gamma^2=I_\mu. \tag{5.4}$$

It follows that Γ is diagonalizable because $(I_\mu\pm\Gamma)^p=2^{p-1}(I_\mu\pm\Gamma)$ for $p\geqslant1$ and has eigenvalues ±1. Let $\epsilon_\pm$ denote the corresponding multiplicities $(\epsilon_++\epsilon_-=\mu)$. Since $n_\alpha=1$, we put $u_{\alpha1}=u_\alpha$ for the vectors of the Jordan basis for Δ_1 and assume that they are normalized and arranged such that

$$\Gamma u_\alpha=u_\alpha,\quad \alpha=1,...,\epsilon_+,$$

$$\Gamma u_\alpha=-u_\alpha,\quad \alpha=\epsilon_++1,...,\mu.$$

We also set $w_{sn_s}=w_s$, so that $w_s^\dagger u_\alpha=\delta_{s\alpha}$ for $s=1,...,n$ and $\alpha=1,...,n$. Note that as a consequence of (5.3), $\epsilon_+(\epsilon_-)$ is the number of linearly independent bounded even (odd) solutions of (2.26). Then from (4.8) and (5.4) we conclude that

$$w_s^\dagger\mathcal{R}^\dagger u_\alpha=(\Gamma^{-1}w_s)^\dagger u_\alpha=(\Gamma w_s)^\dagger u_\alpha=w_s^\dagger\Gamma^\dagger u_\alpha, \tag{5.5}$$

where $\Gamma^\dagger$ is the adjoint of Γ as a mapping from $\mathcal{N}$ to itself. Using (5.5) in (4.32) we obtain $\mathcal{A}_{1,sj}=iw_s^\dagger[\Gamma+\Gamma^\dagger]u_j$, and therefore

$$(\mathcal{A}_1^{-1})_{sj}=-iw_s^\dagger[\Gamma+\Gamma^\dagger]^{-1}u_j.$$

As a result, from (4.36), (4.41), and (4.42) we deduce that

$$T_1(0)=[2(\Gamma+\Gamma^\dagger)^{-1}]\oplus0,$$

where the direct sum refers to the direct decomposition $\mathbf{C}^n=\mathcal{N}\oplus\mathcal{N}'$ with

$$\mathcal{N}'=\mathrm{Span}\{u_{\mu+1},\dots,u_n\}.$$

ACKNOWLEDGMENTS

This material is based on work supported by the National Science Foundation under Grant No. DMS-9803219, by the Department of Energy under Grant No. DE-FG02-01ER45951, and by INDAM and MURST.

APPENDIX: PROOF OF PROPOSITION 4.2

Proof: Since the assertions of Proposition 4.2 concern the small-k asymptotics, we assume that k lies in a fixed interval $[-\delta,\delta]$ with $\delta>0$. In the following C is used to denote various constants that may depend on the choice of δ but not on k or x.

The solution $\varphi(k,x)$ of (1.1) defined by the initial conditions (4.10) satisfies the integral equation

$$\varphi(k,x)=f_1(0,0)\cos kx+f_1'(0,0)\left(\frac{\sin kx}{k}\right)+\frac{1}{k}\int_0^x dy\,\sin[k(x-y)]Q(y)\varphi(k,y),\tag{A1}$$

which can be solved by iteration. A standard Gronwall inequality shows that

$$\|\varphi(k,x)\|\leqslant C(1+|x|),\quad x\in\mathbf{R}.\tag{A2}$$

Therefore, by using (A1) and (A2), it follows that for each $k\in\mathbf{R}\setminus\{0\}$ we have

$$\varphi(k,x)=\alpha_\pm(k)e^{ikx}+\beta_\pm(k)e^{-ikx}+\epsilon_\pm(k,x),\tag{A3}$$

where $\epsilon_\pm(k,x)$ and $\epsilon_\pm'(k,x)$ are both $o(1)$ as $x\to\pm\infty$, and where

$$\alpha_\pm(k)=\frac{1}{2}f_1(0,0)+\frac{1}{2ik}f_1'(0,0)+\frac{1}{2ik}\int_0^{\pm\infty}dy\,e^{-iky}Q(y)\varphi(k,y),\tag{A4}$$

$$\beta_\pm(k)=\frac{1}{2}f_1(0,0)-\frac{1}{2ik}f_1'(0,0)-\frac{1}{2ik}\int_0^{\pm\infty}dy\,e^{iky}Q(y)\varphi(k,y).$$

From (A3) and (A4), together with (1.2) and (1.3), it follows that:

$$[\varphi(k,x)^\dagger;f_1(k,x)]=2ik\alpha_+(k)^\dagger=ikf_1(0,0)^\dagger-f_1'(0,0)^\dagger-\int_0^\infty dz\,e^{ikz}\varphi(k,z)^\dagger Q(z),\tag{A5}$$

$$[f_r(-k,x)^\dagger;\varphi(k,x)]=2ik\alpha_-(k)=ikf_1(0,0)+f_1'(0,0)-\int_{-\infty}^0 dz\,e^{-ikz}Q(z)\varphi(k,z).\tag{A6}$$

In order to control the remainder terms in the subsequent asymptotic expansions, we will need the estimates

$$\|\varphi(k,x)-\varphi(0,x)\|\leqslant C(1+\max\{0,-x\})\left(\frac{kx}{1+|k||x|}\right)^2,\tag{A7}$$

$$\|[\varphi(k,x)-\varphi(0,x)]\xi\|\leqslant C\left(\frac{kx}{1+|k||x|}\right)^2\|\xi\|,\quad \xi\in\mathcal{N}.\tag{A8}$$

The term $\max\{0,-x\}$ in (A7) accounts for the fact that $\varphi(0,x)$ is in general unbounded and $O(x)$ as $x\to-\infty$. In (A8), this term is absent because $\varphi(0,x)\xi$ is bounded when $\xi\in\mathcal{N}$. We omit the proofs of (A7) and (A8) here because (A7) follows from (A1) by some standard estimates and (A8) can be proved by mimicking the proof in the scalar case (see Lemma 2.2 in Ref. 12).

Now consider the integral on the right-hand side of (A5) and write it as

$$\int_0^\infty dz\, e^{ikz}\varphi(k,z)^\dagger Q(z)=A_1(k)+A_2(k), \tag{A9}$$

where

$$A_1(k)=\int_0^\infty dz\, e^{ikz}\varphi(0,z)^\dagger Q(z), \tag{A10}$$

$$A_2(k)=\int_0^\infty dz\, e^{ikz}[\varphi(k,z)^\dagger-\varphi(0,z)^\dagger]Q(z). \tag{A11}$$

When $m=1$, from (A.10) we get

$$\begin{cases} A_1(k)=\displaystyle\int_0^\infty dz\, \varphi(0,z)^\dagger Q(z)+ik\int_0^\infty dz\, z\varphi(0,z)^\dagger Q(z)+\mathcal{F}(k), \\ \qquad\quad =-f_1'(0,0)^\dagger+ik[f_1(0,0)^\dagger-I_n]+\mathcal{F}(k), \end{cases} \tag{A12}$$

where

$$\mathcal{F}(k)=\int_0^\infty dz\, (e^{ikz}-1-ikz)\varphi(0,z)^\dagger Q(z). \tag{A13}$$

Note that $\mathcal{F}(k)$ is $o(k)$ by (4.11), the boundedness of $f_1(0,z)$ on $[0,+\infty)$, and the estimate

$$|e^{ikz}-1-ikz|\leqslant \frac{Cz^2}{1+z},\quad z\geqslant 0.$$

In deriving (A12) we have also used the relations

$$\begin{cases} \displaystyle\int_0^\infty dz\, \varphi(0,z)^\dagger Q(z)=-f_1'(0,0)^\dagger, \\ \displaystyle\int_0^\infty dz\, z\varphi(0,z)^\dagger Q(z)=f_1(0,0)^\dagger-I_n, \end{cases} \tag{A14}$$

which follow from (2.28). Using (A7) in (A11) we see that

$$A_2(k)=o(k). \tag{A15}$$

Combining (A5), (A9), (A12), and (A15) we obtain

$$[\varphi(k,x)^\dagger;f_1(k,x)]=ikI_n+o(k),$$

which agrees with (4.13) for $m=1$.

Now consider (A5) for $m=2$, that is, $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. In this case we can expand the remainder $\mathcal{F}(k)$ in (A12) as

$$\mathcal{F}(k)=\int_0^\infty dz\, \frac{(ikz)^2}{2}\varphi(0,z)^\dagger Q(z)+o(k^2), \tag{A16}$$

where we have used (A7), (A13), and the estimate

$$\left| e^{ikz} - 1 - ikz + \frac{k^2 z^2}{2} \right| \leqslant \frac{C(|k|z)^3}{1+|k|z}, \quad z \geqslant 0.$$

The integral in (A16) can be expressed in a form that does not involve Q explicitly. To see this, substitute $\varphi''(0,z)^\dagger$ for $\varphi(0,z)^\dagger Q(z)$ and replace the upper limit of integration by N. Then integrate by parts twice and let $N \to +\infty$. This gives

$$-\frac{k^2}{2}\left[\int_0^\infty dz\, z^2 \varphi''(0,z)\right]^\dagger = -\frac{k^2}{2} \lim_{N\to+\infty} \mathcal{G}_N^\dagger = -k^2 \int_0^\infty dz\, [f_1(0,z)^\dagger - I_n],$$

where we have defined

$$\mathcal{G}_N = N^2 \varphi'(0,N) - 2N[\varphi(0,N) - I_n] + 2\int_0^N dz\, [\varphi(0,z) - I_n].$$

Thus

$$\mathcal{F}(k) = -k^2 \int_0^\infty dz\, [f_1(0,z)^\dagger - I_n] + o(k^2). \tag{A17}$$

In the derivation of (A17) we have also used

$$\varphi'(0,N) = o(1/N^2), \quad \varphi(0,N) - I_n = o(1/N), \quad \varphi(0,N) - I_n \in L^1(\mathbf{R}^+;\mathbf{C}^{n\times n}). \tag{A18}$$

These properties follow directly from (2.28) and (4.11). The expression (A17) for $\mathcal{F}(k)$ has the advantage that it allows us to combine $\mathcal{F}(k)$ with another term that arises from the expansion of $A_2(k)$. To see this we return to (A11). In order to expand the difference $\varphi(k,x) - \varphi(0,x)$, we use the variation of parameters formula in the form

$$\varphi(k,x) = \varphi(0,x) + ik^2 f_1(0,x) \int_0^x dz\, \dot{f}_1(0,z)^\dagger \varphi(k,z) + ik^2 \dot{f}_1(0,x) \int_0^x dz\, f_1(0,z)^\dagger \varphi(k,z). \tag{A19}$$

We briefly mention some details of the derivation of (A19) because there is a useful identity that falls out in the process. We write (2.26) as a first-order system with $2n$ components and note that a fundamental matrix $\Psi(x)$ for this system and its inverse $\Psi(x)^{-1}$ are given by

$$\Psi(x) = \begin{bmatrix} f_1(0,x) & \dot{f}_1(0,x) \\ f_1'(0,x) & \dot{f}_1'(0,x) \end{bmatrix}, \quad \Psi(x)^{-1} = i \begin{bmatrix} \dot{f}_1'(0,x)^\dagger & -\dot{f}_1(0,x)^\dagger \\ f_1'(0,x)^\dagger & -f_1(0,x)^\dagger \end{bmatrix}.$$

By taking $x \to +\infty$ and using (2.4), one can prove that $\det \Psi(x) = i$. For this and also later we need to use certain asymptotic information about the functions $\dot{f}_1(0,x)$ and $\dot{f}_1'(0,x)$. It suffices to mention that $\dot{f}_1(0,x)$ is the unique solution of the integral equation

$$\dot{f}_1(0,x) = ixI_n + \int_x^\infty dy\, (y-x) Q(y) \dot{f}_1(0,y), \tag{A20}$$

which, incidentally, shows that $\dot{f}_1(0,x)$ is also a matrix solution of (2.26). Moreover, a Gronwall inequality gives

$$\|\dot{f}_1(0,x)\| \leqslant C(1+|x|), \quad x \in \mathbf{R}. \tag{A21}$$

The identity $\Psi(x)^{-1}\Psi(x) = I_{2n}$ is easily verified by using the Wronskian relations

$$[f_1(0,x)^\dagger; f_1(0,x)] = [\dot{f}_1(0,x)^\dagger; \dot{f}_1(0,x)] = 0,$$

$$[f_1(0,x)^\dagger; \dot{f}_1(0,x)] = [\dot{f}_1(0,x)^\dagger; f_1(0,x)] = iI_n,$$

which follow from (2.28), (A20), and the first formula in (A18) which indicates that $f_1'(0,x) = o(1/x^2)$ as $x \to +\infty$. Then (A19) is an easy consequence of the variation of parameters formula for first-order systems. The useful identity alluded to above appears when we write out the identity $\Psi(x)\Psi(x)^{-1} = I_{2n}$ (in this order!) in terms of the entries of the matrices involved. Among the resulting identities we find

$$f_1'(0,x)\dot{f}_1(0,x)^\dagger + \dot{f}_1'(0,x) f_1(0,x)^\dagger = iI_n,$$

which will be useful later.

By iterating (A19) once and using (4.11) we obtain

$$\varphi(k,x) = \varphi(0,x) + ik^2 f_1(0,x) \int_0^x dz\, \dot{f}_1(0,z)^\dagger f_1(0,z) + ik^2 \dot{f}_1(0,x) \int_0^x dz\, f_1(0,z)^\dagger f_1(0,z) + \rho(k,x), \tag{A22}$$

where $\rho(k,x)$ obeys

$$\|\rho(k,x)\| \leqslant Ck^2 (1+|x|)^2 \left(\frac{kx}{1+|k|x|} \right)^2. \tag{A23}$$

This estimate follows by using (A7) and (A21). Taking the adjoint of $A_2(k)$ given in (A11) and expanding the exponential function there we get

$$A_2(k)^\dagger = \int_0^\infty dz\, Q(z)[\varphi(k,z) - \varphi(0,z)] + o(k^2), \tag{A24}$$

where we have used (A7) to determine the order of the error term. Now we insert (A22) into (A24) and proceed as in the derivation of (A17), using $\dot{f}_1''(0,x) = Q(x)\dot{f}_1(0,x)$ and two integrations by parts. We also use (A21), (A23), and the property $\dot{f}_1'(0,N) - iI_n = o(1/N)$ as $N \to +\infty$, which follows from (A20). The result is

$$\int_0^\infty dz\, Q(z)[\varphi(k,z) - \varphi(0,z)] = k^2 \int_0^\infty dz\, [f_1(0,z) - I_n] - k^2 \int_0^\infty dz\, [f_1(0,z)^\dagger f_1(0,z) - I_n] + o(k^2). \tag{A25}$$

Combining (A9), (A12), (A17), (A24), and (A25) we obtain

$$[\varphi(k,x)^\dagger; f_1(k,x)] = ikI_n + k^2 \int_0^\infty dz\, [f_1(0,z)^\dagger f_1(0,z) - I_n] + o(k^2),$$

which is the desired result in (4.13) for $m=2$.

To prove (4.14) we return to the Wronskian in (A6). If $m=1$, we have

$$\begin{cases} \displaystyle\int_{-\infty}^0 dz\, e^{-ikz} Q(z)\varphi(k,z) = \int_{-\infty}^0 dz\, Q(z)\varphi(0,z) + o(1), \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\Delta_1 + f_1'(0,0) + o(1), \end{cases} \tag{A26}$$

where we have used (2.28) and (A14). The order of the error term is again a consequence of (A7). Substituting (A26) in (A6) we get (4.14) for $m=1$. If $m=2$, we have

$$\int_{-\infty}^{0} dz\, e^{-ikz} Q(z)\varphi(k,z) = -\Delta_1 + f_1'(0,0) - ik\int_{-\infty}^{0} dz\, zQ(z)\varphi(0,z) + o(k), \tag{A27}$$

and, using (3.1), (A14), and (A27) we obtain

$$\int_{-\infty}^{0} dz\, zQ(z)\varphi(0,z) = I_n + E_1 - f_1(0,0). \tag{A28}$$

Substituting this in (A6) we get

$$[f_r(-k,x)^\dagger;\varphi(k,x)] = \Delta_1 + ik(I_n + E_1) + o(k),$$

proving (4.14) when $m=2$.

It remains to prove (4.15). So pick $\xi \in \mathcal{N}$ and assume $m=1$. Then $\varphi(0,x)\xi$ stays bounded as $x \to -\infty$, which has the same effect on the integral in (A6), when it acts on ξ, as if m were 2. In particular, (A28) now becomes

$$\int_{-\infty}^{0} dz\, zQ(z)[\varphi(0,z)\xi] = \Gamma\xi - f_1(0,0)\xi,$$

where we have used (2.28) and (4.5). Since $\Delta_1\xi = 0$, from (A27) we obtain

$$\int_{-\infty}^{0} dz\, e^{-ikz} Q(z)[\varphi(k,z)\xi] = f_1'(0,0)\xi - ik\Gamma\xi + ikf_1(0,0)\xi + o(k).$$

Substituting this expression in (A6) we get

$$[f_r(-k,x)^\dagger;\varphi(k,x)]\xi = ik\Gamma\xi + o(k),$$

which agrees with (4.15) for $m=1$. If $m=2$ and $\xi \in \mathcal{N}$, then we can carry the expansion in (A27) further as in the case of (A9) and (A11). To obtain the corresponding coefficients in the expansion we could proceed by using variation of parameters in terms of the solutions $f_r(0,x)$ and $\dot{f}_1(0,x)$. However, there is a simpler approach that exploits the connection between the left and right Jost solutions for (1.1) under the substitution $x \mapsto -x$, that is, under the transformation $Q(x) \mapsto Q^\#(x)$, where $Q^\#(x) = Q(-x)$. We use the superscript # to indicate that a given quantity pertains to (1.1) with potential $Q^\#$. It is straightforward to show that

$$f_r(k,x) = f_1^\#(k,-x), \quad f_1(k,x) = f_r^\#(k,-x). \tag{A29}$$

We now introduce a solution $\omega(k,x)$ of (1.1) satisfying the initial conditions

$$\omega(k,0) = f_r(0,0), \quad \omega'(k,0) = f_r'(0,0).$$

Then it follows from (4.3) and (4.10) that for $\xi \in \mathcal{N}$ we have

$$\varphi(k,x)\xi = \omega(k,x)\chi, \tag{A30}$$

where $\chi = \Gamma\xi$. Since, by (4.10) and (A29)

$$\varphi^\#(k,0) = f_1^\#(k,0) = f_r(k,0), \quad \varphi^{\#\prime}(k,0) = f_1^{\#\prime}(k,0) = -f_r'(k,0),$$

we get $\varphi^\#(k,x) = \omega(k,-x)$, which, together with (A30), yields $\varphi^\#(k,-x)\chi = \varphi(k,x)\xi$. In the following argument we use the more elaborate notation. $[G(k,x);H(k,x)]_{(x_0)}$ to denote the Wronskian of two matrix functions $G(k,x)$ and $H(k,x)$ evaluated at $x = x_0$. Then we have

$$\begin{cases} [f_{\mathrm{r}}(-k,x)^\dagger;\varphi(k,x)]_{(x)}\xi = [f_1^\#(-k,-x)^\dagger;\varphi^\#(k,-x)]_{(x)}\chi \\ \qquad = -[f_1^\#(-k,x)^\dagger;\varphi^\#(k,x)]_{(-x)}\chi \\ \qquad = [\varphi^\#(k,x)^\dagger;f_1^\#(-k,x)]_{(-x)}^\dagger\chi \\ \qquad = [\varphi^\#(-k,x)^\dagger;f_1^\#(-k,x)]_{(x)}^\dagger\chi, \end{cases} \tag{A31}$$

where in the the last step we have used the fact that the Wronskian is constant and that $\varphi(k,x)$ is an even function of k. The latter follows from the fact that the initial conditions in (4.10) are independent of k. Now the Wronskian on the right-hand side of (A31) is of the same form as that in (4.13). We can, therefore, apply the expansion given there. Then the integrand of Y_2 involves $f_1^\#(0,z)$ which can be rewritten in terms of $f_{\mathrm{r}}(0,z)$ by means of (A29). Using also (4.2), we obtain (4.15). The proof of Proposition 4.2 is now complete. ■

[1] V. A. Marchenko, *Sturm-Liouville Operators and Applications* (Birkhäuser, Basel, 1986).
[2] P. Deift and E. Trubowitz, Commun. Pure Appl. Math. **32**, 121 (1979).
[3] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963).
[4] L. Martínez Alonso and E. Olmedilla, J. Math. Phys. **23**, 2116 (1982).
[5] M. Wadati and T. Kamijo, Prog. Theor. Phys. **52**, 397 (1974).
[6] M. Wadati, ''Generalized matrix form of the inverse scattering method,'' in *Solitons, Topics in current physics, Vol. 17*, edited by R. K. Bullough and P. J. Caudry (Springer, Berlin, 1980), pp. 287–299.
[7] P. Schuur, ''Inverse scattering for the matrix Schrödinger equation with non-Hermitian potential,'' in *Nonlinear waves, Cambridge Monographs Mech. Appl. Math.*, edited by L. Debnath (Cambridge University Press, Cambridge-New York, 1983), pp. 285–297.
[8] E. Olmedilla, Inverse Probl. **1**, 219 (1985).
[9] D. Alpay and I. Gohberg, *Integral Equations and Operator Theory* **30**, 317 (1998).
[10] F. Calogero and A. Degasperis, Nuovo Cimento Soc. Ital. Fis., B **39**, 1 (1977).
[11] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer, New York, 1989).
[12] M. Klaus, Inverse Probl. **4**, 505 (1988).
[13] P. Šeba, J. Phys. A **19**, 2573 (1986).
[14] R. P. Boas, *Entire Functions* (Academic, New York, 1954).