Factorization and small-energy asymptotics for the radial Schrödinger equation

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The radial Schrödinger equation is considered when the potential is real valued, is integrable, and has a finite first moment. The Jost function, the scattering matrix, the number of bound states for the potential are expressed in terms of the corresponding quantities associated with the fragments of the potential. An improved expansion on the small-energy asymptotics of the Jost solution is presented. © *2000 American Institute of Physics.* [S0022-2488(00)05307-X]

I. INTRODUCTION

Consider the radial Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=V(x)\psi(k,x),\quad x\in(0,+\infty),\tag{1.1}$$

where the potential V is real valued and belongs to $L_1^1(\mathbf{R}^+)$, i.e., $\int_0^\infty dx(1+x)|V(x)|$ is finite. The prime denotes the derivative with respect to the spatial coordinate x. We refer the reader to Ref. 1 for the analysis of the scattering theory for (1.1). We choose our notations and conventions to conform with those given in Ref. 2. For the quantities associated with V, we use $S(k)$ for the scattering matrix, $F(k)$ for the Jost function, $f(k,x)$ for the Jost solution, $\varphi(k,x)$ for the regular solution, $\psi(k,x)$ for the physical solution, $\delta(k)$ for the phase shift, and N for the number of bound states. We define

$$d:=\begin{cases}0, & F(0)\neq 0,\\ 1, & F(0)=0,\end{cases}\tag{1.2}$$

and say that V is generic if $F(0)\neq 0$ and is exceptional if $F(0)=0$.

We recall the definition of these quantities below. The regular solution of (1.1), $\varphi(k,x)$, satisfies the boundary conditions

$$\varphi(k,0)=0,\quad \varphi'(k,0)=1,\tag{1.3}$$

and the Jost solution $f(k,x)$ satisfies

$$e^{-ikx}f(k,x)=1+o(1),\quad e^{-ikx}f'(k,x)=ik+o(1),\quad x\to+\infty.\tag{1.4}$$

The Jost function $F(k)$ is defined as

$$F(k):=f(k,0),\tag{1.5}$$

and the phase shift $\delta(k)$ is defined in terms of the phase of the Jost function as

$$F(k):=|F(k)|e^{-i\delta(k)},$$

where $\delta(k)$ is the continuous branch of the phase function such that $\delta(+\infty)=0$. The scattering matrix is defined as

$$S(k):=\frac{F(-k)}{F(k)}, \tag{1.6}$$

and the physical solution of (1.1) satisfies

$$\psi(k,x)=e^{i\delta(k)}\sin(kx+\delta(k))+o(1), \quad x\to+\infty. \tag{1.7}$$

Let us use an asterisk to denote complex conjugation. The following are known.[1,2]

$$F(-k)=F(k)^*, \quad k\in\mathbf{R}, \tag{1.8}$$

$$\psi(k,x)=\frac{k}{F(k)}\varphi(k,x), \tag{1.9}$$

$$\varphi(k,x)=\frac{1}{2ik}[F(-k)f(k,x)-F(k)f(-k,x)], \tag{1.10}$$

$$S(k)=e^{2i\delta(k)}. \tag{1.11}$$

The bound states of V correspond to the zeros of $F(k)$ on the positive imaginary axis in $\mathbf{C}^+$, and according to the Levinson theorem

$$\delta(0^+)=\left(N+\frac{d}{2}\right)\pi,$$

where d is the quantity defined in (1.2).

In this paper we study the small-k asymptotics of the Jost solution by fragmenting the potential into two pieces and using the small-k properties related to the fragments. The reader is referred to Refs. 3 and 4 for the history and further references on the small-k limits of the Jost solution of (1.1). The derivation of the ''factorization formulas'' in Sec. II has been motivated by similar formulas (see, e.g., Ref. 5) for the one-dimensional Schrödinger equation. It should be stated, however, that the formulas in the radial case are somewhat different from the corresponding formulas on the full line; this is not surprising because the factorization formulas on the full line possess certain symmetries, e.g., under a reflection through the origin or an interchange of the two fragments, whereas such symmetries are missing in the radial case. Nevertheless, such factorization formulas are useful because in general the properties related to the fragments are easier to obtain than the properties related to the whole potential; the factorization formulas allow us to obtain the properties related to the whole potential in terms of those related to its fragments.

This paper is organized as follows. In Sec. II we fragment the potential V into two pieces and express its Jost function and scattering matrix in terms of the corresponding quantities associated with the two fragments. In Sec. III we analyze the small-k asymptotics of the Jost solution and show that for each fixed $x\in\mathbf{R}^+$ the quantity $f'(k,x)/f(k,x)$ or its reciprocal has a derivative with respect to k at $k=0$ and we explicitly find that derivative. In Sec. IV we study the relation between the number of bound states of V and the corresponding numbers for its fragments; we show that the sum of the number of bound states for the two fragments is either equal to or one larger than the number of bound states of V. In Sec. IV we also investigate exactly when V is generic or exceptional depending on its fragments being generic or exceptional.

II. FACTORIZATION

Let us fragment the potential as $V=V_1+V_2$ such that V_1 is supported in $(0,a)$ and V_2 is supported in $(a,+\infty)$ for some positive constant a. Our purpose in this section is to relate Jost function and the scattering matrix of V to the Jost functions and scattering matrices of V_1 and V_2.

Let us use the subscripts 1 and 2 to identify the quantities related to V_1 and V_2, respectively. Thus, for example, S_1 and S_2 are the scattering matrices, F_1 and F_2 are the Jost functions, f_1 and f_2 are the Jost solutions, φ_1 and φ_2 are the regular solutions, and δ_1 and δ_2 are the phase shifts for V_1 and V_2, respectively.

Theorem 2.1: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$. Let V_1 and V_2 be the fragments of V with supports in $(0,a)$ and $(a,+\infty)$, respectively, for some $a>0$. Then

$$\frac{F(k)}{F_1(k)F_2(k)}=\frac{1}{2}[1+S_1(k)]+\frac{1}{2}[1-S_1(k)]\chi_2(k), \tag{2.1}$$

where

$$\chi_2(k):=\frac{f_2'(k,0)}{ikf_2(k,0)}. \tag{2.2}$$

Consequently

$$S(k)=S_2(k)\,\frac{ik[1+S_1(k)]f_2(-k,0)+[1-S_1(k)]f_2'(-k,0)}{ik[1+S_1(k)]f_2(k,0)+[1-S_1(k)]f_2'(k,0)}. \tag{2.3}$$

Proof: The regular solutions for V and V_1 satisfy the same equation on $(0,a)$ and the same boundary conditions at $x=0$; thus

$$\varphi(k,x)=\varphi_1(k,x),\quad \varphi'(k,x)=\varphi_1'(k,x),\quad x\in[0,a]. \tag{2.4}$$

Similarly, the Jost solutions for V and V_2 satisfy the same equation on $(a,+\infty)$ and the same boundary conditions at $x=+\infty$, and hence

$$f(k,x)=f_2(k,x),\quad f'(k,x)=f_2'(k,x),\quad x\in[a,+\infty). \tag{2.5}$$

Let $[f;g]:=fg'-f'g$ denote the Wronskian. The Wronskian of any two solutions of (1.1) is independent of x. For example, from (1.4) we see that

$$[f(k,x);f(-k,x)]=-2ik,$$

and hence from (1.5) and (1.10) we get

$$F(k)=[f(k,x);\varphi(k,x)]. \tag{2.6}$$

Evaluating the Wronskian in (2.6) at $x=a$ and using (2.4) and (2.5), we get

$$F(k)=f_2(k,a)\varphi_1'(k,a)-f_2'(k,a)\varphi_1(k,a). \tag{2.7}$$

On the other hand, since $V_1\equiv 0$ for $x>a$, using (1.7) and (1.9) we obtain

$$\varphi_1(k,x)=\frac{F_1(k)}{k}e^{i\delta_1}\sin(kx+\delta_1),\quad x\in[a,+\infty), \tag{2.8}$$

and similarly, since $V_2\equiv 0$ for $x<a$, we have

$$f_2(k,x)=f_2(k,0)\cos kx+f_2'(k,0)\frac{\sin kx}{k},\quad x\in[0,a],$$

or equivalently, by using (1.5) and (2.2), we get

$$f_2(k,x)=F_2(k)[\cos kx+i\chi_2(k)\sin kx],\quad x\in[0,a]. \tag{2.9}$$

Using (2.8) and (2.9) in (2.7), we obtain

$$F=F_1F_2e^{i\delta_1}[\cos ka+i\chi_2\sin ka]\cos(ka+\delta_1)-F_1F_2e^{i\delta_1}[-\sin ka+i\chi_2\cos ka]\sin(ka+\delta_1),$$

which simplifies to

$$F=F_1F_2e^{i\delta_1}[\cos\delta_1-i\chi_2\sin\delta_1]. \tag{2.10}$$

Converting the trigonometric functions in (2.10) into complex exponentials and using the analog of (1.11) for S_1, we obtain (2.1). Then, with the help of (1.6), (1.8), (1.11), (2.1), and (2.2), we get (2.3). ■

III. SMALL-ENERGY ESTIMATES

In this section we consider the small-energy asymptotics of the Jost solution. Our main result is given in Theorem 3.5, where we show that for $V\in L_1^1(\mathbf{R}^+)$, at each fixed x the quantity $f'(k,x)/f(k,x)$ or its reciprocal can be differentiated with respect to k at $k=0$.

The following result is well known.[1]

Theorem 3.1: If V is real valued and belongs to $L_1^1(\mathbf{R}^+)$, then as $k\to 0$ in $\mathbf{R}$ we have $S(k)=1+o(1)$ generically and $S(k)=-1+o(1)$ in the exceptional case. If V_1 is real valued, it has support in $(0,a)$ for some finite $a>0$, and $V_1\in L^1(0,a)$, then the corresponding Jost function F_1 is entire in the complex plane $\mathbf{C}$ and hence

$$F_1(k)=F_1(0)+k\dot{F}_1(0)+O(k^2),\quad k\to 0\ \text{ in }\ \mathbf{C},$$

where the overdot denotes the derivative with respect to k. In the generic case we have $F_1(0)\neq 0$ and

$$S_1(k)=1-2k\frac{\dot{F}_1(0)}{F_1(0)}+O(k^2),\quad k\to 0\ \text{ in }\ \mathbf{C}. \tag{3.1}$$

In the exceptional case, $F_1(0)=0$ and $\dot{F}_1(0)\neq 0$, and we have $S_1(k)=-1+O(k)$ as $k\to 0$ in $\mathbf{C}$.

Let $g(k,x)$ be the solution of (1.1) satisfying

$$g(k,0)=1,\quad g'(k,0)=0. \tag{3.2}$$

We have

$$g(k,x)=\cos kx+\frac{1}{k}\int_0^x dy\,\sin k(x-y)V(y)g(k,y). \tag{3.3}$$

The regular solution $\varphi(k,x)$ satisfies

$$\varphi(k,x)=\frac{\sin kx}{k}+\frac{1}{k}\int_0^x dy\,\sin k(x-y)V(y)\varphi(k,y). \tag{3.4}$$

Let $\phi(k,x)$ be the solution of (1.1) satisfying

$$\phi(k,0)=f(0,0),\quad \phi'(k,0)=f'(0,0). \tag{3.5}$$

Thus

$$\phi(0,x)=f(0,x),\quad x\in[0,+\infty), \tag{3.6}$$

$$\phi(0,x)=f(0,0)+xf'(0,0)+\int_0^x dy\ (x-y)V(y)\phi(0,y). \tag{3.7}$$

Since $f(0,x)=1+o(1)$ as $x\to+\infty$, with the help of (3.6), by letting $x\to+\infty$ in (3.7) we get

$$f'(0,0)=-\int_0^\infty dy\ V(y)\phi(0,y), \tag{3.8}$$

$$f(0,0)=1+\int_0^\infty dy\ yV(y)\phi(0,y). \tag{3.9}$$

Moreover, from (1.3), (3.2), and (3.5) it follows that

$$\phi(k,x)=f(0,0)g(k,x)+f'(0,0)\varphi(k,x). \tag{3.10}$$

Proposition 3.2: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$. Then,

$$|\phi(k,x)-\phi(0,x)|\leqslant C\left(\frac{|kx|}{1+|kx|}\right)^2,\quad x\in\mathbf{R}^+,\quad k\in[-\epsilon,\epsilon], \tag{3.11}$$

for any fixed positive ϵ, where C denotes a constant independent of x and k.

Proof: The proof can be found in Lemma 2.2 of Ref. 6. ■

Using (1.3) and (3.2) we get

$$f(k,0)=[f(k,x);\varphi(k,x)],\quad f'(k,0)=-[f(k,x);g(k,x)]. \tag{3.12}$$

Evaluating the Wronskians in (3.12) as $x\to+\infty$, with the help of (3.3) and (3.4) we get

$$f(k,0)=1+\int_0^\infty dy\ e^{iky}V(y)\varphi(k,y), \tag{3.13}$$

$$f'(k,0)=ik-\int_0^\infty dy\ e^{iky}V(y)g(k,y). \tag{3.14}$$

Proposition 3.3: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$. Then $P(k)=-ik+o(k)$ as $k\to0$ in $\overline{\mathbf{C}^+}$, where $P(k)$ is the quantity defined as

$$P(k):=-f'(k,0)f(0,0)+f'(0,0)f(k,0). \tag{3.15}$$

Proof: From (3.5) and (3.15) it follows that $P(k)=[f(k,x);\phi(k,x)]$. Using (3.12) and (3.13) in (3.15) we get

$$P(k)=f(0,0)\left[-ik+\int_0^\infty dy\ e^{iky}V(y)g(k,y)\right]+f'(0,0)\left[1+\int_0^\infty dy\ e^{iky}V(y)\varphi(k,y)\right]. \tag{3.16}$$

Using (3.10) in (3.16) we have

$$P(k)=-ikf(0,0)+f'(0,0)+\int_0^\infty dy\ e^{iky}V(y)\phi(k,y). \tag{3.17}$$

Evaluating (3.13) and (3.14) at $k=0$ and using the result on the right-hand side of (3.17), with the help of (3.8) and (3.9) we get $P(k)=-ik+J_1+J_2$, where

$$J_1 := \int_0^\infty dy\ [e^{iky} - 1 - iky] V(y) \phi(0,y), \tag{3.18}$$

$$J_2 := \int_0^\infty dy\ e^{iky} V(y) [\phi(k,y) - \phi(0,y)]. \tag{3.19}$$

Let us use C to denote a constant not necessarily assuming the same value at different appearances. Using the inequality

$$|e^{iz} - iz - 1| \leqslant \frac{Cz^2}{1+z}, \quad z \geqslant 0,$$

from (3.18) we get

$$|J_1| \leqslant C|k| \int_0^\infty dy\, \frac{|ky|}{1+|ky|} y |V(y)|,$$

and hence $J_1 = o(k)$ as $k \to 0$. Similarly, using (3.11) in (3.19) we get

$$|J_2| \leqslant C|k| \int_0^\infty dy\, \frac{|ky|}{1+|ky|} y |V(y)|,$$

and hence $J_2 = o(k)$. Thus, the theorem is proved when $k \to 0$ in $\mathbf{R}$. With the help of the Phragmén–Lindelöf theorems it follows that the limit is valid also when $k \to 0$ in $\overline{C^+}$. ■

Theorem 3.4: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$. Then, if $f(0,0) \neq 0$ we have

$$\frac{f'(k,0)}{f(k,0)} = \frac{f'(0,0)}{f(0,0)} + \frac{ik}{f(0,0)^2} + o(k), \quad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{3.20}$$

and if $f'(0,0) \neq 0$ we have

$$\frac{f(k,0)}{f'(k,0)} = \frac{f(0,0)}{f'(0,0)} - \frac{ik}{f'(0,0)^2} + o(k), \quad k \to 0 \text{ in } \overline{\mathbf{C}^+}. \tag{3.21}$$

Proof: When $f(0,0) \neq 0$, from (3.15) we get

$$\frac{f'(k,0)}{f(k,0)} - \frac{f'(0,0)}{f(0,0)} = \frac{P(k)}{f(k,0) f(0,0)}.$$

Thus, using Proposition 3.3 and the continuity of $f(k,0)$ at $k=0$, we get (3.20). On the other hand, if $f'(0,0) \neq 0$, we obtain (3.21) by using

$$\frac{f(k,0)}{f'(k,0)} - \frac{f(0,0)}{f'(0,0)} = -\frac{P(k)}{f'(k,0) f'(0,0)},$$

and by applying Proposition 3.3 and the continuity of $f'(k,0)$ at $k=0$. ■

Next, we show that the result in Theorem 3.4 holds not only at $x=0$ but for any $x \in \mathbf{R}^+$.

Theorem 3.5: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$. Then, for each fixed $x \in \mathbf{R}^+$, if $f(0,x) \neq 0$ we have

$$\frac{f'(k,x)}{f(k,x)} = \frac{f'(0,x)}{f(0,x)} + \frac{ik}{f(0,x)^2} + o(k), \quad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{3.22}$$

and if $f'(0,x) \neq 0$ we have

$$\frac{f(k,x)}{f'(k,x)}=\frac{f(0,x)}{f'(0,x)}-\frac{ik}{f'(0,x)^2}+o(k), \quad k\to 0 \text{ in } \overline{\mathbf{C}^+}. \tag{3.23}$$

Proof: The proof is similar to the proof of Theorem 3.4. For any fixed $a\geqslant 0$, define the solutions φ_a, g_a, and ϕ_a of (1.1) satisfying

$$\varphi_a(k,a)=0, \quad \varphi_a'(k,a)=1,$$

$$g_a(k,a)=1, \quad g_a'(k,a)=0,$$

$$\phi_a(k,a)=f(0,a), \quad \phi_a'(k,a)=f'(0,a).$$

Similar to (3.6)–(3.10) we get

$$\phi_a(0,x)=f(0,x), \quad x\in[0,+\infty),$$

$$\phi_a(0,x)=f(0,a)+(x-a)f'(0,a)+\int_a^x dy\ (x-y)V(y)\phi_a(0,y),$$

$$f'(0,a)=-\int_a^\infty dy\ V(y)\phi_a(0,y),$$

$$f(0,a)=1+af'(0,a)+\int_a^\infty dy\ yV(y)\phi_a(0,y),$$

$$\phi_a(k,x)=f(0,a)g_a(k,x)+f'(0,a)\varphi_a(k,x).$$

Proposition 3.2 still holds[6] if we use ϕ_a instead of ϕ in (3.11). Proceeding as in the proof of Proposition 3.3, we obtain

$$f(0,a)f'(k,a)-f'(0,a)f(k,a)=ik+o(k), \quad k\to 0 \text{ in } \overline{\mathbf{C}^+}. \tag{3.24}$$

Imitating the proof of Theorem 3.4, from (3.24) we get (3.22) and (3.23) holding at any $x=a$. ■

When $V\in L_1^1(\mathbf{R}^+)$, even though in general $f(k,x)$ and $f'(k,x)$ are not differentiable with respect to k at $k=0$, the above theorem shows that their ratio is indeed differentiable with respect to k at $k=0$. Note that (3.22) does not hold at the x values where $f(0,x)=0$. We will see in Proposition 4.2(iv) that the number of such x values is equal to the number of bound states of V.

IV. BOUND STATES

In this section we relate the number of bound states of V to the number of bound states of its fragments V_1 and V_2. We also analyze the circumstances of V being generic and exceptional depending on whether the fragments are generic or exceptional. A similar analysis on the whole line was given in Ref. 7.

The first two propositions contain known results.[8] A brief proof of Proposition 4.1 is included merely to remind the reader the oscillation properties of the Jost function when k is on the positive imaginary axis.

Proposition 4.1: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$, and let its bound states correspond to $k=i\kappa_j$ with $0<\kappa_1<\cdots<\kappa_N$. Then,

(i) $F(i\beta)$ has simple zeros at $\beta=\kappa_j$ for $j=1,...,N$.
(ii) $F(i\beta)>0$ when $\beta>\kappa_N$.
(iii) $(-1)^{N-j}F(i\beta)>0$ when $\beta\in(\kappa_j,\kappa_{j+1})$ for $j=1,...,N-1$.
(iv) $(-1)^N F(i\beta)>0$ when $\beta\in(0,\kappa_1)$.

(v) Generically $(-1)^N F(0)>0$ and in the exceptional case $F(0)=0$.

Proof: The proof is standard: (i) is known, and the rest follow from the fact that $F(k)=1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$, the only (simple) zeros of $F(k)$ in $\overline{\mathbf{C}^+}\setminus\{0\}$ occur at $k=i\kappa_j$, and that $F(i\beta)$ is real and continuous on $\beta\in\mathbf{R}^+$. ■

The number of bound states is also related to the zeros of $f(i\beta,x)$ on $x\in\mathbf{R}^+$, as summarized in the following proposition.

Proposition 4.2: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$, and let its bound states correspond to $k=i\kappa_j$ with $0<\kappa_1<\cdots<\kappa_N$. Then we have the following:

(i) For each $\beta\geqslant 0$, we have $f(i\beta,x)=e^{-\beta x}[1+o(1)]$ as $x\to+\infty$.
(ii) For each fixed $\beta\geqslant\kappa_N$, $f(i\beta,x)$ has no zeros on $x\in(0,+\infty)$.
(iii) For each fixed $\beta\in[\kappa_j,\kappa_{j+1})$ with $j=1,...,N-1$, the quantity $f(i\beta,x)$ has exactly $N-j$ zeros on $x\in(0,+\infty)$.
(iv) For each fixed $\beta\in[0,\kappa_1), f(i\beta,x)$ has N zeros on $x\in(0,+\infty)$.

Proposition 4.3: Assume V is real valued and belongs to $L_1^1(\mathbf{R}^+)$, and let $V=V_1+V_2$, where V_1 is supported in $(0,a)$ and V_2 in $(a,+\infty)$ for some $a>0$. If V_1 is generic, then

$$f(0,x)=\begin{cases} \dfrac{F(0)}{F_1(0)}f_1(0,x)+\left[f'(0,0)-\dfrac{F(0)}{F_1(0)}f_1'(0,0)\right]\varphi(0,x), & x\leqslant a,\\ f_2(0,x), & x\geqslant a. \end{cases} \tag{4.1}$$

If V_1 is exceptional, then

$$f(0,x)=\begin{cases} F(0)g_1(0,x)+\dfrac{f'(0,0)}{f_1'(0,0)}f_1(0,x), & x\leqslant a,\\ f_2(0,x), & x\geqslant a, \end{cases} \tag{4.2}$$

where $g_1(k,x)$ is the solution of (1.1) corresponding to the potential V_1 with the boundary conditions [cf. (3.2)]

$$g_1(k,0)=1, \quad g_1'(k,0)=0.$$

Proof: If V_1 is generic, from (2.6) we see that $f_1(0,x)$ and $\varphi_1(0,x)$ are linearly independent on $x\in\mathbf{R}^+$. Writing $f(0,x)$ as a linear combination of $f_1(0,x)$ and $\varphi_1(0,x)$, we determine the coefficients in terms of $f(0,0)$ and $f'(0,0)$ and get (4.1). In the exceptional case, writing $f(0,x)$ as a linear combination of $f_1(0,x)$ and $g_1(0,x)$ and determining the coefficients in terms of $f(0,0)$ and $f'(0,0)$, we get (4.2). ■

Proposition 4.3: Assume V_1 is real valued, has support in $(0,a)$ for some $a>0$, belongs to $L^1(0,a)$, and has N_1 bound states. If V_1 is generic, then $f_1(0,x)$ has N_1 zeros all located in $(0,a)$ with no zeros at $x=0$ and no zeros in $[a,+\infty)$. If V_1 is exceptional, then $f_1(0,x)$ has N_1 zeros all located in $(0,a)$, an additional zero at $x=0$, and no zeros in $[a,+\infty)$.

Proof: The proof follows from Proposition 4.2(iv) and the fact that $f_1(0,x)=1$ for all $x\in[a,+\infty)$. ■

Proposition 4.4: Assume that V_2 is real valued, has support in $(a,+\infty)$ for some $a>0$, belongs to $L_1^1(a,+\infty)$, and has N_2 bound states. If V_2 is exceptional, then $f_2(0,x)$ has N_2 zeros in $(a,+\infty)$, no zeros in $(0,a]$, and one zero at $x=0$. If V_2 is generic and $f_2'(0,0)/f_2(0,0)\geqslant 0$, then $f_2(0,x)$ has N_2 zeros in $(a,+\infty)$ and no zeros in $[0,a]$. If V_2 is generic and $f_2'(0,0)/f_2(0,0)<0$, then $f_2(0,x)$ has N_2-1 zeros in $(a,+\infty)$, and one zero in $(0,a]$, and no zeros at $x=0$.

Proof: The proof is obtained by using Proposition 4.2(iv) and the fact that

$$f_2(0,x)=F_2(0)+f_2'(0,0)x, \quad x\leqslant 0,$$

which implies that $f_2(0,x)$ has exactly one zero in $(0,a]$ if $f_2'(0,0)/f_2(0,0)<0$. ■

Theorem 4.5: Assume V is real valued, belongs to $L_1^1(\mathbf{R}^+)$, and has N bound states; let $V=V_1+V_2$, where V_1 has support in $(0,a)$ and V_2 in $(a,+\infty)$ for some $a>0$, and suppose V_1 has N_1 bound states and V_2 has N_2 bound states. Then we have the following:

(i) If V_2 is generic and $f_2'(0,0)/f_2(0,0)<0$, then $N=N_1+N_2-1$; in any other cases, we have $N=N_1+N_2$.
(ii) If both V_1 and V_2 are exceptional, then V is also exceptional.
(iii) If V_1 is exceptional, V_2 is generic, and $f_2'(0,0)\neq 0$, then V is generic.
(iv) If V_1 is exceptional, V_2 is generic, and $f_2'(0,0)=0$, then V is exceptional.
(v) If V_1 and V_2 are both generic and $f_2'(0,0)=0$, then V is also generic.
(vi) If V_1 and V_2 are both generic and $f_2'(0,0)\neq 0$, then V is exceptional if $F_1(0)F_2(0)=i\dot{F}_2(0)f_2'(0,0)$ and otherwise generic.

Proof: According to the Sturm–Liouville theory,[8] $f(0,x)$ and $f_1(0,x)$ must have the same number of zeros in $(0,a)$; hence, from Proposition 4.3 it follows that $f(0,x)$ has N_1 zeros in $(0,a)$; on the other hand, the number of zeros of $f(0,x)$ in $(a,+\infty)$ is determined in terms of N_2 by Proposition 3.4. Thus, (i) is proved. Recall from Theorem 3.1 that $S(0)=1$ generically and $S(0)=-1$ in the exceptional case. When both V_1 and V_2 are exceptional, we have $S_1(0)=-1$ and $f_2'(0,0)\neq 0$; thus, letting $k\to 0$ in (2.3), we see that $S(0)=S_2(0)$, and hence $S(0)=-1$, which proves (ii). The proof of (iii) is obtained similarly as in the proof of (ii); from $S(0)=S_2(0)$ it follows that V is generic as V_2 is. To get (iv) note that (2.2) and (3.20) imply that $\chi_2(k)=1/F_2(0)^2+o(1)$ as $k\to 0$, and hence from (2.3) we get $S(0)=-S_2(0)$, which implies that V is exceptional because V_2 is generic. The proof of (V) is obtained from (2.1) as $k\to 0$, i.e., from $F(0)=F_1(0)F_2(0)$, which is obtained by using $S_1(k)=1+o(1)$ and $\chi_2(k)=1/F_2(0)^2+o(1)$ as $k\to 0$. Finally, to prove (vi), using (3.1) and (3.21) in (2.1), we get $F(0)=F_1(0)F_2(0)-i\dot{F}_1(0)f_2'(0,0)$, from which the conclusion follows. ■

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[1] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Springer, New York, 1982).
[2] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer, New York, 1989).
[3] R. G. Newton, ''Low-energy scattering for medium-range potentials,'' J. Math. Phys. **27**, 2720–2730 (1986).
[4] M. Klaus, ''Exact behavior of Jost functions at low energy,'' J. Math. Phys. **29**, 148–154 (1988).
[5] T. Aktosun, M. Klaus, and C. van der Mee, ''Factorization of scattering matrices due to partitioning of potentials in one-dimensional Schrödinger-type equations,'' J. Math. Phys. **37**, 5897–5915 (1996).
[6] M. Klaus, ''Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line,'' Inverse Probl. **4**, 505–512 (1988).
[7] T. Aktosun, M. Klaus, and C. van der Mee, ''On the number of bound states for the one-dimensional Schrödinger equation,'' J. Math. Phys. **39**, 4249–4256 (1998).
[8] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).