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Asymptotics of the scattering coefficients for a generalized Schrödinger equation

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The generalized Schrödinger equation $d^2\psi/dx^2+F(k)\psi=[ikP(x)+Q(x)]\psi$ is considered, where P and Q are integrable potentials with finite first moments and F satisfies certain conditions. The behavior of the scattering coefficients near zeros of F is analyzed. It is shown that in the so-called exceptional case, the values of the scattering coefficients at a zero of F may be affected by $P(x)$. The location of the k-values in the complex plane where the exceptional case can occur is studied. Some examples are provided to illustrate the theory. © *1999 American Institute of Physics.* [S0022-2488(99)03007-8]

I. INTRODUCTION

In this paper we consider the generalized Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2}+F(k)\psi(k,x)=[ikP(x)+Q(x)]\psi(k,x), \quad x\in\mathbf{R}, \tag{1.1}$$

where the properties of F will be detailed below. The functions P and Q satisfy

$$P\in L_1^1(\mathbf{R}), \quad Q\in L_1^1(\mathbf{R}), \tag{1.2}$$

where $L_1^1(\mathbf{R})$ is the class of measurable functions f such that $\int_{-\infty}^{\infty}dx\,|f(x)|(1+|x|)<+\infty$. For the majority of the paper, P and Q need not be real valued; if they are, this will be stated explicitly. In applications, k may correspond to a wave number while $F(k)$ may represent energy. The coefficient $P(x)$ may represent the absorptive properties of a medium, and $Q(x)$ may be a restoring force density or a potential for an external force. Some special cases of (1.1) are

(A) $F(k)=k^2$ with $P(x)\equiv 0$,
(B) $F(k)=k^2$ with $P(x)\not\equiv 0$,
(C) $F(k)=k^2+1/(4\beta^2)$ with $\beta>0$.

Case (A) corresponds to the well-known quantum-mechanical case of the Schrödinger equation on the line with potential $Q(x)$. Case (B) was studied by Jean and Jaulent,[1–4] and more recently by Sattinger and Szmigielski,[5] and by us[6] when P is real valued. Case (C) has been investigated by Kaup[7] in connection with the inverse scattering transform for an evolution equation (a long-wave water equation resembling the Boussinesq equation) by Tsutsumi[8] and, more recently, under the assumption that $\int_{-\infty}^{\infty}dx\,P(x)=0$, by Sattinger and Szmigielski.[9]

Our interest in (1.1) is motivated by various inverse problems associated with (1.1). In studying such problems, one needs to know the asymptotics of various quantities as the parameter k approaches certain special values, in particular as $F(k)\to\infty$ or as $k\to k_0$, where k_0 is a zero of F. In this paper we will only be concerned with the second situation. We will call $k_0\in\mathbf{C}$ a *critical* value of (1.1) if $F(k_0)=0$. Here, $\mathbf{C}$ denotes the complex plane. The quantities whose asymptotics we will study are the transmission and reflection coefficients associated with (1.1). Before we define these quantities we list the assumptions on F:

(H1) Supposing k_0 is a critical value of (1.1), there exists a set $\mathcal{S}\subset\mathbf{C}$ such that $F(k)$ is continuous on $\mathcal{S}$, $F(k)\neq 0$ on $\mathcal{S}\setminus\{k_0\}$, and the map $k\mapsto\mu(k)=\sqrt{F(k)}$ is one-to-one for $k\in S$. Here the branch of the square root is such that $0\leqslant\arg\mu<\pi$, where $\mu=\mu(k)$.

(H2) There is a path $\mathcal{P}(k_0)$ in $\mathcal{S}$ containing k_0 on which μ takes on real non-negative values.

Note that, by (H1), $\mathcal{D}=\mu(\mathcal{S})$ is a subset of the closed upper-half complex plane $\overline{\mathbf{C}^+}$. (H2) indicates that there is an $\epsilon>0$ so that $[0,\epsilon]\in\mathcal{D}$. In cases (A) and (B), $k_0=0$ is the only critical value. We may then choose $\mathcal{S}=\{k:0\leqslant\arg k<\pi\}\cup\{0\}$, so that $\mu(k)=k$ and $\mathcal{D}=\mathcal{S}$. For the path $\mathcal{P}(k_0)$ we may take the interval $[0,+\infty)$. In case (C) the critical values are $k_0=\pm i/(2\beta)$. The disk $\{k:|k-i/(2\beta)|\leqslant 1/(2\beta)\}$ can then be used as $\mathcal{S}$ near the critical point $+i/(2\beta)$ and we have $\mathcal{D}=\{\mu:|\mu|\leqslant 1/(2\beta),\ 0\leqslant\arg\mu<\pi\}\cup\{0\}$. As the path $\mathcal{P}(k_0)$ we can take the imaginary interval $i[0,1/(2\beta)]$. The modifications for the other critical point are obvious.

For $k\in\mathcal{S}$, (1.1) possesses the solutions $f_l(k,x)$ and $f_r(k,x)$, the so-called Jost solutions from the left and from the right, respectively, that are uniquely defined by their spatial asymptotics, namely,

$$f_l(k,x)=e^{i\mu x}[1+o(1)],\quad f_l'(k,x)=i\mu e^{i\mu x}[1+o(1)],\quad x\to+\infty, \tag{1.3}$$

$$f_r(k,x)=e^{-i\mu x}[1+o(1)],\quad f_r'(k,x)=-i\mu e^{-i\mu x}[1+o(1)],\quad x\to-\infty, \tag{1.4}$$

where the prime indicates the derivative with respect to the spatial variable x. For $k\in\mathcal{P}(k_0)\setminus\{k_0\}$ the Jost solutions obey

$$f_l(k,x)=\frac{1}{T(k)}e^{i\mu x}+\frac{L(k)}{T(k)}e^{-i\mu x}+o(1),\quad x\to-\infty, \tag{1.5}$$

$$f_r(k,x)=\frac{1}{T(k)}e^{-i\mu x}+\frac{R(k)}{T(k)}e^{i\mu x}+o(1),\quad x\to+\infty, \tag{1.6}$$

which define the transmission coefficient T and the reflection coefficients R from the right and L from the left, respectively. These quantities will collectively be referred to as scattering coefficients. It is also possible to define the scattering coefficients in terms of certain Wronskians of the Jost solutions. For example, letting $[f;g]=fg'-f'g$ denote the Wronskian, from (1.1) and (1.3)–(1.6) we get

$$\frac{2i\mu}{T(k)}=[f_r(k,\cdot);f_l(k,\cdot)]. \tag{1.7}$$

In analogy with the usual Schrödinger equation, given a critical value k_0 we will distinguish between two cases: We say that the generic (exceptional) case occurs at $k=k_0$ if and only if $f_l(k_0,x)$ and $f_r(k_0,x)$ are linearly independent (dependent). In the exceptional case, we let γ denote the nonzero constant defined as,

$$\gamma=\frac{f_l(k_0,x)}{f_r(k_0,x)}. \tag{1.8}$$

From (1.7) we see that k_0 corresponds to the exceptional case if and only if

$$F(k_0)=0,\quad \lim_{k\to k_0}\frac{\mu(k)}{T(k)}=0.$$

In short, we will say that k_0 is an exceptional value if it corresponds to the exceptional case for (1.1).

The behavior of the scattering coefficients of (1.1) at the critical values k_0 does not seem to have been studied in detail before, except in cases (A) and (B). In these two cases it is known[6,10–13] that there are two ways in which $T(k)$ can behave as $k\to 0$: either $T(k)=ick+o(k)$ for some nonzero c, or $T(k)=T(0)+o(1)$ with $T(0)\neq 0$. The former corresponds to the generic case and the latter corresponds to the exceptional case. In case (C) a detailed investigation of the behavior of the scattering coefficients near $k_0=\pm 1/(2\beta)$ does not seem to have been done before. This is one of our goals in this paper, and particular attention will be paid to the exceptional case. In connection with a statement made in Theorem 2.6 of Ref. 9 regarding case (C) with $\beta=\frac{1}{2}$, we would like to comment that while it is true that for reflectionless potentials only the exceptional case can occur at $k=\pm i$, there are also potentials $P(x)$ and $Q(x)$, in particular real ones, which are not reflectionless and for which the exceptional case occurs. This will be discussed in more detail in Sec. III.

This paper is organized as follows. In Sec. II we prove our main result concerning the behavior of the scattering coefficients at a critical value (Theorem 2.2) and apply it to cases (A)–(C) (Corollary 2.3). We also present some information about the location of the exceptional k-values in the complex plane. In Sec. III we consider case (C) in more detail, show that one must not identify the exceptional case with the reflectionless case, and provide four examples illustrating the location of the exceptional k-values and other aspects of the theory.

II. ASYMPTOTICS OF THE SCATTERING COEFFICIENTS

In this section we study the asymptotic behavior of the scattering coefficients as $k\to k_0$, where k_0 is a critical value of (1.1). In doing so we will only be concerned with the leading terms of the asymptotic expansions. Our main result is presented in Theorem 2.2. For its proof, we first need some results about the usual Schrödinger equation.

Consider the pair of Schrödinger equations

$$\frac{d^2\phi_j(\mu,x)}{dx^2}+\mu^2\phi_j(\mu,x)=V_j(x)\phi_j(\mu,x),\quad j=1,2, \tag{2.1}$$

where $V_j\in L^1(\mathbf{R})$. Here μ is allowed to range over all of $\overline{\mathbf{C}^+}$; it is not restricted to $\mathcal{D}$ defined earlier. Let t_j denote the transmission coefficient and r_j and l_j denote the reflection coefficients from the right and left, respectively, for the potential V_j. Let $g_{j;l}(\mu,x)$ and $g_{j;r}(\mu,x)$ denote the corresponding Jost solutions of (2.1) from the left and right, respectively. It is known[10,11,13] that

$$g_{j;l}(-\mu,x)=t_j(\mu)g_{j;r}(\mu,x)-r_j(\mu)g_{j;l}(\mu,x),\quad \mu\in\mathbf{R},$$
$$g_{j;r}(-\mu,x)=t_j(\mu)g_{j;l}(\mu,x)-l_j(\mu)g_{j;r}(\mu,x),\quad \mu\in\mathbf{R}. \tag{2.2}$$

Since μ appears as μ^2 in (2.1), $g_{j;l}(-\mu,x)$ and $g_{j;r}(-\mu,x)$ are also solutions of (2.1), and $g_{j;l}(-\mu,x)=e^{-i\mu x}[1+o(1)]$ as $x\to+\infty$ and $g_{j;r}(-\mu,x)=e^{i\mu x}[1+o(1)]$ as $x\to-\infty$.

Proposition 2.1: Suppose that $V_j\in L^1(\mathbf{R})$ for $j=1,2$. Then the scattering coefficients of (2.1) satisfy

$$\frac{1}{t_2(\mu)}=\frac{1}{t_1(\mu)}+\frac{i}{2\mu}\int_{-\infty}^{\infty}dx\,[V_2(x)-V_1(x)]g_{2;l}(\mu,x)g_{1;r}(\mu,x),\quad \mu\in\overline{\mathbf{C}^+}\setminus\{0\}, \tag{2.3}$$

$$\frac{l_2(\mu)}{t_2(\mu)}=\frac{l_1(\mu)}{t_1(\mu)}-\frac{i}{2\mu}\int_{-\infty}^{\infty}dx\,[V_2(x)-V_1(x)]g_{2;l}(\mu,x)g_{1;r}(-\mu,x),\quad \mu\in\mathbf{R}\setminus\{0\}, \tag{2.4}$$

$$\frac{1}{t_2(\mu)}=\frac{1}{t_1(\mu)}+\frac{i}{2\mu}\int_{-\infty}^{\infty}dx\,[V_2(x)-V_1(x)]g_{2;r}(\mu,x)g_{1;l}(\mu,x),\quad \mu\in\overline{\mathbf{C}^+}\setminus\{0\}, \tag{2.5}$$

$$\frac{r_2(\mu)}{t_2(\mu)}=\frac{r_1(\mu)}{t_1(\mu)}-\frac{i}{2\mu}\int_{-\infty}^{\infty}dx\,[V_2(x)-V_1(x)]g_{2;r}(\mu,x)g_{1;l}(-\mu,x),\quad \mu\in\mathbf{R}\setminus\{0\}. \tag{2.6}$$

Proof: First, let us note that (2.3) and (2.5) are given on p. 329 of Ref. 13, and some formulas related to (2.4) and (2.6) can be also found there. We will give a different proof which yields both (2.3) and (2.4) simultaneously. The proof of (2.5) and (2.6) is similar and hence will be omitted. By the variation of parameters formula, $g_{2;l}(\mu,x)$ obeys the integral equation

$$g_{2;l}(\mu,x)=g_{1;l}(\mu,x)+\int_x^\infty dy\,\mathcal{G}(\mu;x,y)[V_2(y)-V_1(y)]g_{2;l}(\mu,y), \tag{2.7}$$

where

$$\mathcal{G}(\mu;x,y)=\frac{g_{1;l}(\mu,x)g_{1;r}(\mu,y)-g_{1;r}(\mu,x)g_{1;l}(\mu,y)}{[g_{1;l}(\mu,\cdot);g_{1;r}(\mu,\cdot)]}. \tag{2.8}$$

Note that the Wronskian in (2.8) is related to the transmission coefficient as

$$t_j(\mu)=-\frac{2i\mu}{[g_{j;l}(\mu,\cdot);g_{j;r}(\mu,\cdot)]}. \tag{2.9}$$

Now (2.3) and (2.4) follow by letting $x\to-\infty$ in (2.7) and using (1.4), (1.5), (2.2), and (2.9). ■

In the next theorem, the behavior of the scattering coefficients of (1.1) is analyzed at critical k-values.

Theorem 2.2: Suppose $P,Q\in L_1^1(\mathbf{R})$ and $F(k)$ satisfies (H1) and (H2). If $k_0\in\mathbf{C}$ is a critical value of (1.1), then we have the following.

(i) In the generic case we have

$$T(k)=-\frac{2i\mu}{[f_l(k_0,\cdot);f_r(k_0,\cdot)]}+o(\mu),\quad k\to k_0\quad\text{in}\quad\mathcal{S}, \tag{2.10}$$

$$L(k)=-1+o(1),\quad R(k)=-1+o(1),\quad k\to k_0\quad\text{in}\quad\mathcal{P}(k_0).$$

(ii) In the exceptional case, using the constants α and ω defined by

$$\alpha=\lim_{k\to k_0}\frac{k-k_0}{\mu(k)},\quad \omega=\gamma^2+1-\alpha\int_{-\infty}^\infty dx\,P(x)f_l(k_0,x)^2,$$

we distinguish two subcases: (a) If α exists and is finite and $\omega\neq0$, then

$$T(k)=\frac{2\gamma}{\omega}+o(1),\quad k\to k_0\quad\text{in}\quad\mathcal{S}, \tag{2.11}$$

$$L(k)=\frac{2\gamma^2-\omega}{\omega}+o(1),\quad k\to k_0\quad\text{in}\quad\mathcal{P}(k_0), \tag{2.12}$$

$$R(k)=\frac{2-\omega}{\omega}+o(1),\quad k\to k_0\quad\text{in}\quad\mathcal{P}(k_0). \tag{2.13}$$

(b) If $\lim_{k\to k_0}|(k-k_0)/\mu(k)|=+\infty$ and $\int_{-\infty}^\infty dx\,P(x)f_l(k_0,x)^2\neq0$, then

$$T(k_0)=0,\quad L(k_0)=-1,\quad R(k_0)=-1.$$

In the exceptional case, if α exists and $\omega=0$, then the scattering coefficients are not continuous at k_0; if α does not exist, then, in general, the scattering coefficients are not continuous at k_0.

Proof: In (2.3)–(2.6) we replace $V_1(x)$ by $ik_0P(x)+Q(x)$ and $V_2(x)$ by $ikP(x)+Q(x)$ and note that because of (1.2) we have $V_j\in L_1^1(\mathbf{R})$ for $j=1,2$ instead of just $V_j\in L^1(\mathbf{R})$. The stronger assumption allows us to take the limit $\mu\to 0$ in (2.3)–(2.6). Thanks to Proposition 2.1 we can make full use of the results[12] known in the case $P(x)\equiv 0$. For $g_{1;l}(\mu,x)$ and $g_{2;r}(\mu,x)$ in (2.3)–(2.6), we substitute $f_l^{[0]}(\mu,x)$ and $f_r^{[0]}(\mu,x)$, respectively, where the latter two are the Jost solutions of

$$\frac{d^2\varphi(\mu,x)}{dx^2}+\mu^2\varphi(\mu,x)=[ik_0P(x)+Q(x)]\varphi(\mu,x). \tag{2.14}$$

Let $T^{[0]}(\mu)$, $R^{[0]}(\mu)$, and $L^{[0]}(\mu)$ denote the scattering coefficients associated with (2.14). Then from (2.3) we get

$$\frac{1}{T(k)}=\frac{1}{T^{[0]}(\mu)}\left[1-\frac{k-k_0}{2\mu}T^{[0]}(\mu)\int_{-\infty}^{\infty}dx\,P(x)f_r^{[0]}(\mu,x)f_l(k,x)\right],\quad \mu\in\mathcal{D}\setminus\{0\}. \tag{2.15}$$

When $P,Q\in L_1^1(\mathbf{R})$, we have

$$|f_r^{[0]}(\mu,x)|\leqslant C(1+\max\{0,x\})e^{(\mathrm{Im}\,\mu)x},\quad \mu\in\overline{C^+}, \tag{2.16}$$

$$|f_l(k,x)|\leqslant C(1+\max\{0,-x\})e^{-(\mathrm{Im}\,\mu)x},\quad k\in\mathcal{S}, \tag{2.17}$$

where C is a constant independent of x and k. Hence, by the Lebesgue dominated convergence theorem, the integral on the right-hand side in (2.15) converges as $k\to k_0$. Now (2.10) follows from (2.9), (2.15), and the fact that in the generic case we have

$$[f_l(k_0,\cdot);f_r(k_0,\cdot)]=[f_l^{[0]}(k_0,\cdot);f_r^{[0]}(k_0,\cdot)]\neq 0.$$

In the exceptional case we obtain (2.11) by using (2.15)–(2.17) along with the fact that

$$f_r^{[0]}(0,x)=f_r(k_0,x)=\frac{1}{\gamma}f_l(k_0,x), \tag{2.18}$$

and (cf. Ref. 12)

$$T^{[0]}(0)=\frac{2\gamma}{\gamma^2+1},$$

where γ is the constant in (1.8). The statement $T(k_0)=0$ in part (b) follows directly from (2.15). Turning to $L(k)$, from (2.4) we get

$$\frac{L(k)}{T(k)}=\frac{L^{[0]}(\mu)}{T^{[0]}(\mu)}+\frac{k-k_0}{2\mu}\int_{-\infty}^{\infty}dx\,P(x)f_r^{[0]}(-\mu,x)f_l(k,x). \tag{2.19}$$

Using (2.16) and (2.17) one can show that the integral in (2.19) has a finite limit as $k\to k_0$. In the generic case, we have $L^{[0]}(0)=-1$ and

$$\lim_{k\to k_0}\frac{T(k)}{T^{[0]}(\mu)}=\frac{[f_l^{[0]}(0,\cdot);f_r^{[0]}(0,\cdot)]}{[f_l(k_0,\cdot);f_r(k_0,\cdot)]}=1,$$

$$\lim_{k\to k_0}\frac{T(k)}{\mu}=\frac{-2i}{[f_l(k_0,\cdot);f_r(k_0,\cdot)]}.$$

Thus (2.19) implies that $L(k_0)=-1$. To prove (2.12) we use (2.18), (2.19), and the fact that in the exceptional case we have

$$L^{[0]}(0)=\frac{\gamma^2-1}{\gamma^2+1}.$$

The arguments leading to (2.13) and in case (b) are similar. ■

The implications of Theorem 2.2 for the special cases (A)–(C) are as follows.

Corollary 2.3: Suppose $P,Q\in L^1_1(\mathbf{R})$ and $F(k)$ satisfies (H1) and (H2). If $k_0\in\mathbf{C}$ is an exceptional value, then we have the following.

(i) In case (A), we have $k_0=0$ and

$$T(k_0)=\frac{2\gamma}{\gamma^2+1},\quad L(k_0)=\frac{\gamma^2-1}{\gamma^2+1},\quad R(k_0)=\frac{1-\gamma^2}{\gamma^2+1}. \tag{2.20}$$

(ii) In case (B), we have $k_0=0$, $\mu(k)=k$, $\alpha=1$, and (2.11)–(2.13) hold.

(iii) In case (C), $k_0=\pm i/(2\beta)$ and $F(k)$ vanishes linearly at k_0; hence $\alpha=0$. In this case (2.20) holds.

Next we address the question of where in the complex plane the possible exceptional k-values can occur. Of course, in order for a k-value to correspond to the exceptional case, it must first be a critical value, and this depends on $F(k)$. In the next proposition, without referring to any specific form of $F(k)$, we present some sufficient conditions which ensure that the exceptional k-values cannot occur off the imaginary axis.

Proposition 2.4: Assume $P(x)\not\equiv 0$, $Q(x)$ and $P(x)$ are real valued, and $P,Q\in L^1_1(\mathbf{R})$. If k_0 is an exceptional value but not purely imaginary, then $\int_{-\infty}^{\infty}dx\,P(x)|f_1(k_0,x)|^2=0$. If $Q(x)\geqslant 0$, or $P(x)\leqslant 0$, or $P(x)\geqslant 0$, then the exceptional k-values for (1.1) can occur only on the imaginary axis.

Proof: Recall that in the exceptional case the Jost solutions of (1.1), $f_l(k_0,x)$ and $f_r(k_0,x)$, are linearly dependent and hence $f_l(k_0,x)$ remains bounded as $x\to\pm\infty$. Moreover, since $F(k_0)=0$, one can show [cf. (2.11) of Ref. 14] that $f_l'(k_0,x)=o(1/x)$ as $x\to\pm\infty$. Thus, from (1.1), after integrating by parts and using

$$\lim_{x\to\pm\infty} f_l'(k_0,x)f_l(k_0,x)^*=0,$$

where $*$ denotes complex conjugation, we obtain

$$\int_{-\infty}^{\infty}dx\,|f_l'(k_0,x)|^2+\int_{-\infty}^{\infty}dx\,Q(x)|f_l(k_0,x)|^2=-ik_0\int_{-\infty}^{\infty}dx\,P(x)|f_l(k_0,x)|^2.$$

Since the right-hand side has to be real, both assertions follow. ■

III. SPECIAL CASE (C) AND EXAMPLES

We first consider case (C) in some more detail and discuss the implications of our results for the work of Sattinger and Szmigielski.[9] To establish the connection between the notation used here and that used in Ref. 9, we note that in Ref. 9 the special case $F(k)=k^2+1$ was considered with the notation $E^2=k^2+1$ (i.e., E in Ref. 9 corresponds to μ here), and a complex uniformization parameter z was used to express E and k as

$$E=\frac{1}{2}\left(z+\frac{1}{z}\right),\quad k=\frac{1}{2}\left(z-\frac{1}{z}\right).$$

Then two sets of solutions of (1.1), $\psi_\pm(x,z)$ and $\phi_\pm(x,z)$, having specific asymptotic behaviors were defined. We state here only their connection with the Jost solutions of (1.1). We have

$$\psi_+(x,z)=f_l(k,x),\quad \psi_-(x,z)=T(k)f_r(k,x)-R(k)f_l(k,x),$$

$$\phi_+(x,z)=f_r(k,x),\quad \phi_-(x,z)=T(k)f_l(k,x)-L(k)f_r(k,x).$$

These definitions imply that

$$\phi_+(x,z)=a(z)\psi_-(x,z)+b(z)\psi_+(x,z),$$

$$\phi_-(x,z)=c(z)\psi_-(x,z)+d(z)\psi_+(x,z),$$

with

$$a(z)=\frac{1}{T(k)},\quad b(z)=\frac{R(k)}{T(k)},\quad c(z)=-\frac{L(k)}{T(k)},\quad d(z)=\frac{T(k)^2-L(k)R(k)}{T(k)}.$$

The quantities

$$r_+(z)=\frac{b(z)}{a(z)},\quad r_-(z)=\frac{c(z)}{d(z)},$$

were called generalized reflection coefficients in Ref. 9. In terms of our scattering coefficients we have

$$r_+(z)=R(k),\quad r_-(z)=\frac{L(k)}{L(k)R(k)-T(k)^2}.$$

Now let us apply Theorem 2.2 and Corollary 2.3 to the problem studied in Ref. 9. The critical points are $k=\pm i$, corresponding to $z=\pm i$. In the notation of Ref. 9, generically one has $r_\pm(i)=r_\pm(-i)=-1$; on the other hand, in the exceptional case, one has

$$r_+(\pm i)=\frac{1-\gamma_\pm^2}{\gamma_\pm^2+1},\quad r_-(\pm i)=\frac{1-\gamma_\pm^2}{\gamma_\pm^2+1}.$$

Here γ_+ and γ_- are the constants in (1.8) at the critical points i and $-i$, respectively. Thus, we see that potentials need not necessarily be reflectionless in order to violate $r_+(\pm i)=-1$ or $r_-(\pm i)=-1$. In fact, in the next example we show that even rather simple potentials may cause nontrivial reflection in the exceptional case. The following examples involve potentials of the form

$$P(x)=\begin{cases} b_+, & 0<x<1,\\ b_-, & -1<x<0,\\ 0, & \text{elsewhere},\end{cases}\quad Q(x)=\begin{cases} a_+, & 0<x<1,\\ a_-, & -1<x<0,\\ 0, & \text{elsewhere},\end{cases}\tag{3.1}$$

where $a_\pm$ and $b_\pm$ are parameters.

Example 3.1: In (3.1) let us use $b_+=2$, $b_-=b$ with $b\geqslant 0$, $a_+=1$, $a_-=0$, and choose $F(k)=k^2+1$. We can solve (1.1) and evaluate the scattering coefficients explicitly. The critical points are $k=\pm i$. Letting $k=i(1-\epsilon)$, as $k\to i$ so that $\epsilon\to 0$ through positive values, we obtain

$$\frac{2i\mu}{T(k)}=\sqrt{b}\cos 1\sin\sqrt{b}+\sin 1\cos b+O(\sqrt{\epsilon}).\tag{3.2}$$

There are an infinite number of positive b-values that cause the leading term in (3.2) to vanish, and each such b-value causes $k=i$ to yield the exceptional case. The smallest is $b=6.77194$, the next two values are $b=36.3663$ and $b=85.7127$ (the overline means that the last digit may have been affected by round-off). For $b=6.77194$ we get $T(i)=-0.901744$ and $L(i)=-0.432166$; for $b=36.3663$ we get $T(i)=0.851046$ and $L(i)=-0.525091$; and for $b=85.7127$ we get $T(i)=-0.842793$ and $L(i)=-0.538224$.

Example 3.2: In (3.1) let $b_+=b_-=b$, $a_+=a_-=0$, and choose $F(k)=k^2$. Then we are in the exceptional case for every $b\geqslant 0$. The only critical value is $k_0=0$ and we have $f_l(k_0,x)=1$ and $\gamma=1$ [cf. (1.8)], and

$$\frac{1}{T(k)}=e^{2ik}\left[\cos(2\sigma)-\frac{2ik+b}{2\sigma}\sin(2\sigma)\right], \tag{3.3}$$

where $\sigma=\sqrt{k^2-ikb}$. Hence

$$\frac{1}{T(0)}=1-b,$$

which is in agreement with (2.11). If $b=1$, then $\omega=0$ and $\alpha=1$ in Theorem 2.2, and (3.3) gives

$$T(k)=\frac{3i}{2k}+O(1), \quad k\rightarrow 0.$$

This shows that $T(k)$ can be discontinuous at a critical value.

We conclude with two examples illustrating the location of possible exceptional k-values; in these examples, unless otherwise indicated, $F(k)$ is not assumed to have any special form.

Example 3.3: In (3.1) let $b_+=1$ and $b_-=a_+=a_-=-1$. Setting $k=k_0$ and $F(k_0)=0$ we solve (1.1) to find the Jost solution $f_l(k_0,x)$ and then impose the condition that $f_l(k_0,x)$ be bounded as $x\rightarrow-\infty$; that is, we demand that $f_l'(k_0,-1)=0$. This is a necessary condition for k_0 to be an exceptional value for any given function $F(k)$. A straightforward calculation shows that the (possibly) exceptional values are given by the solutions of the equation

$$\sqrt{-1+ik_0}\tanh\sqrt{-1+ik_0}+\sqrt{-1-ik_0}\tanh\sqrt{-1-ik_0}=0.$$

This equation has infinitely many roots on the imaginary axis located symmetrically about the origin and, as can be seen numerically, one symmetric pair of roots on the real axis. The two real roots are $k_0=\pm 1.355$, and the imaginary roots closest to zero are $k_0=\pm 14.139i$. The corresponding Jost solution $f_l(k_0,x)$ is given by

$$f_l(k_0,x)=\begin{cases}\cosh(\sqrt{-1+ik_0}(1-x)), & 0\leqslant x\leqslant 1,\\ \dfrac{\cosh\sqrt{-1+ik_0}}{\cosh\sqrt{-1-ik_0}}\cosh(\sqrt{-1-ik_0}(x+1)), & -1\leqslant x\leqslant 0,\end{cases}$$

and on each of the intervals $(-\infty,-1)$ and $(1,+\infty)$, $f_l(k_0,x)$ is constant and obtained by continuity. This example shows the possibility of real as well as purely imaginary exceptional values. The two imaginary roots above would be critical values for case (C) if $\beta=0.035$. The two real roots are not critical values for any of cases (A)–(C). In accordance with Proposition 2.4, one can verify that $\int_{-1}^{1}dx\,P(x)|f_l(k_0,x)|^2=0$.

Example 3.4: In (3.1) let $b_+=b_-=1$, $a_+=0$, and $a_-=-1$. Then the (possibly) exceptional values satisfy

$$\sqrt{ik_0}\tanh\sqrt{ik_0}+\sqrt{-1-ik_0}\tanh\sqrt{-1-ik_0}=0.$$

There are again infinitely many purely imaginary roots; there are also complex roots, one pair of which is $k_0=\pm 1.1008+0.5i$. The corresponding Jost solution $f_l(k_0,x)$ is given by

$$f_l(k_0,x)=\begin{cases}\cosh(\sqrt{ik_0}(1-x)), & 0\leqslant x\leqslant 1,\\ \dfrac{\cosh\sqrt{ik_0}}{\cosh\sqrt{-1-ik_0}}\cosh(\sqrt{-1-ik_0}(x+1)), & -1\leqslant x\leqslant 0,\end{cases}$$

and for $|x|>1$, $f_l(k_0,x)$ is constant and obtained by continuity. This example shows the possibility of exceptional values that are neither real nor purely imaginary. In case (C) only the purely imaginary roots could be critical values for suitable β.

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[1] M. Jaulent and C. Jean, ''The inverse *s*-wave scattering problem for a class of potentials depending on energy,'' Commun. Math. Phys. **28**, 177–220 (1972).
[2] M. Jaulent and C. Jean, ''The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. I,'' Ann. Inst. Henri Poincaré, Sect. A **25**, 105–118 (1976).
[3] M. Jaulent and C. Jean, ''The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. II,'' Ann. Inst. Henri Poincaré, Sect. A **25**, 119–137 (1976).
[4] M. Jaulent, ''Inverse scattering problems in absorbing media,'' J. Math. Phys. **17**, 1351–1360 (1976).
[5] D. H. Sattinger and J. Szmigielski, ''Energy dependent scattering theory,'' Differ. Integral Eqs. **8**, 945–959 (1995).
[6] T. Aktosun, M. Klaus, and C. van der Mee, ''Wave scattering with absorption,'' J. Math. Phys. **39**, 1957–1992 (1998).
[7] J. Kaup, ''A higher-order water-wave equation and the method for solving it,'' Prog. Theor. Phys. **54**, 396–408 (1975).
[8] M. Tsutsumi, ''On the inverse scattering problem for the one-dimensional Schrödinger equation with an energy dependent potential,'' J. Math. Anal. Appl. **83**, 316–350 (1981).
[9] D. H. Sattinger and J. Szmigielski, ''A Riemann-Hilbert problem for an energy dependent Schrödinger operator,'' Inverse Probl. **12**, 1003–1025 (1996).
[10] L. D. Faddeev, ''Properties of the *S*-matrix of the one-dimensional Schrödinger equation,'' Am. Math. Soc. Transl. **2**, 139–166 (1964) [Trudy Mat. Inst. Steklova **73**, 314–336 (1964) (in Russian)].
[11] P. Deift and E. Trubowitz, ''Inverse scattering on the line,'' Commun. Pure Appl. Math. **32**, 121–251 (1979).
[12] M. Klaus, ''Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line,'' Inverse Probl. **4**, 505–512 (1988).
[13] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer-Verlag, New York, 1989).
[14] T. Aktosun, M. Klaus, and C. van der Mee, ''On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation,'' J. Math. Phys. **34**, 2651–2690 (1993).