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On the number of bound states for the one-dimensional Schrödinger equation

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The number of bound states of the one-dimensional Schrödinger equation is analyzed in terms of the number of bound states corresponding to ''fragments'' of the potential. When the potential is integrable and has a finite first moment, the sharp inequalities $1-p+\Sigma_{j=1}^{p} N_j \leqslant N \leqslant \Sigma_{j=1}^{p} N_j$ are proved, where p is the number of fragments, N is the total number of bound states, and N_j is the number of bound states for the jth fragment. When $p=2$ the question of whether $N=N_1+N_2$ or $N=N_1+N_2-1$ is investigated in detail. An illustrative example is also provided. © *1998 American Institute of Physics.* [S0022-2488(98)03109-0]

I. INTRODUCTION

Consider the one-dimensional Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=V(x)\psi(k,x), \tag{1.1}$$

where the potential V is real valued and belongs to $L_1^1(\mathbf{R})$, the class of measurable functions for which $\int_{-\infty}^{\infty} dx(1+|x|)|V(x)|$ is finite. The prime denotes the derivative with respect to the spatial coordinate x. Let us partition the real axis as $\mathbf{R}=\cup_{j=1}^{p}(x_{j-1},x_j)$, with $x_{j-1}<x_j$ for $j=1,...,p$. Here we use the convention $x_0=-\infty$ and $x_p=+\infty$. We obtain a fragmentation of the potential by setting $V(x)=\Sigma_{j=1}^{p} V_j(x)$, where

$$V_j(x)=\begin{cases} V(x), & x\in(x_{j-1},x_j)\\ 0, & \text{elsewhere.}\end{cases} \tag{1.2}$$

In this paper we analyze the relationship between the number of bound states of V and the number of bound states of its fragments. In Sec. II we prove a pair of sharp inequalities relating these numbers (Theorem 2.1); we also study the case $p=2$ in more detail, and in Theorems 2.2 and 2.3 we present criteria that tell us when $N=N_1+N_2$ or $N=N_1+N_2-1$. In Sec. III we give another proof of Theorem 2.1 by using a factorization formula for the scattering matrix and exploiting its small-k asymptotics. We also briefly discuss what happens if we increase the separation distance between two fragments (Theorem 3.1). In Sec. IV we give an example which illustrates various aspects of our results.

The inequality (2.5) in Theorem 2.1 has been proved before by different methods and under stronger assumptions on the potential. In Ref. 1, (2.5) was proved when $p=2$ and the fragments have compact support. In Refs. 2 and 3, some special cases of (2.5) were proved for parity invariant, compactly supported fragments, but, as already mentioned in those references, the parity invariance is not an essential restriction. The method used in Ref. 1 was based on the nodal properties of the zero-energy solutions of the Schrödinger equation but was fairly contrived, while

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the method used in Refs. 2 and 3 relied on a factorization formula[4] for the scattering matrix and the small-k behavior of the scattering coefficients. In the proofs of Theorems 2.1 and 2.2 we use certain properties of the Jost solutions, especially the interlacing property of zeros, in a very straightforward way. As a result, we are able to establish the connection with the factorization method used in Sec. III. Furthermore, no additional technical restrictions are imposed on the potential besides $V\in L_1^1(\mathbf{R})$.

At various places in this paper we need to distinguish between ''generic'' and ''exceptional'' potentials. Recall that a potential is called generic if the corresponding transmission coefficient T vanishes at $k=0$, and that a potential is called exceptional if $T(0)\neq 0$. Equivalently, a potential is generic (exceptional) if for $k=0$ the two Jost solutions are linearly independent (dependent)[5–7].

II. INEQUALITY FOR THE NUMBER OF BOUND STATES

In preparation of this section we first establish some notation and collect some results about the Jost solutions and their nodal properties. Let $f_{l;j}(k,x)$ and $f_{r;j}(k,x)$ denote the Jost solutions from the left and right, respectively, for the fragment V_j. Recall that $f_{l;j}(k,x)=e^{ikx}[1+o(1)]$ as $x\to+\infty$ and $f_{r;j}(k,x)=e^{-ikx}[1+o(1)]$ as $x\to-\infty$. Furthermore, let n_j denote the number of zeros of $f_{r;j}(0,x)$ lying in $(-\infty,x_j)$, m_j the number of zeros of $f_{l;j}(0,x)$ lying in $(x_{j-1},+\infty)$, and N_j the number of bound states of the fragment V_j. Since N_j is equal[8,9] to the number of zeros of either $f_{l;j}(0,x)$ or $f_{r;j}(0,x)$, we conclude that

$$N_j=\begin{cases} n_j, & \text{if } f_{r;j}(0,x_j)f'_{r;j}(0,x_j)\geq 0 \quad \text{and} \quad f_{r;j}(0,x_j)\neq 0 \\ n_j+1, & \text{if } f_{r;j}(0,x_j)f'_{r;j}(0,x_j)\leq 0 \quad \text{and} \quad f'_{r;j}(0,x_j)\neq 0, \end{cases} \tag{2.1}$$

$$N_j=\begin{cases} m_j, & \text{if } f_{l;j}(0,x_{j-1})f'_{l;j}(0,x_{j-1})\leq 0 \quad \text{and} \quad f_{l;j}(0,x_{j-1})\neq 0 \\ m_j+1, & \text{if } f_{l;j}(0,x_{j-1})f'_{l;j}(0,x_{j-1})\geq 0 \quad \text{and} \quad f'_{l;j}(0,x_{j-1})\neq 0. \end{cases} \tag{2.2}$$

Note that on $(x_j,+\infty)$ the function $f_{r;j}(0,x)$ is equal to $f'_{r;j}(0,x_j)(x-x_j)+f_{r;j}(0,x_j)$ and that this linear function has the root $x=x_j-f_{r;j}(0,x_j)/f'_{r;j}(0,x_j)$ which lies in $[x_j,+\infty)$ precisely if $f_{r;j}(0,x_j)=0$ or $f_{r;j}(0,x_j)f'_{r;j}(0,x_j)<0$; in this case we have $N_j=n_j+1$. On the other hand, if $f'_{r;j}(0,x_j)=0$ or $f_{r;j}(0,x_j)f'_{r;j}(0,x_j)>0$, then $f_{r;j}(0,x)$ has no zeros in $[x_j,+\infty)$, i.e., all its zeros are in $(-\infty,x_j)$; thus $N_j=n_j$. This proves (2.1). We obtain (2.2) by applying a similar argument to $f_{l;j}(0,x)$. We will also need the Jost solutions for the potential V, which we denote by $f_l(k,x)$ and $f_r(k,x)$, respectively. In the generic case when $k=0$ the following asymptotic relations hold[10] as $x\to+\infty$:

$$f_l(0,x)=1+o(1), \quad f'_l(0,x)=o(1/x), \tag{2.3}$$

$$f_r(0,x)=c_r x+o(x), \quad f'_r(0,x)=c_r+o(1), \tag{2.4}$$

with some constant $c_r\neq 0$.

Theorem 2.1: Suppose that $V\in L_1^1(\mathbf{R})$. Let N denote the number of bound states of V. Then

$$1-p+\sum_{j=1}^{p} N_j\leq N\leq \sum_{j=1}^{p} N_j, \quad p=1,2,..., \tag{2.5}$$

where both inequalities are sharp.

Proof: It suffices to prove (2.5) for $p=2$ because the general case follows by induction. Let $u(x)$ denote the solution of (1.1) for $k=0$ satisfying the initial conditions $u(x_1)=1$ and $u'(x_1)=0$. Then $u(x)=f_{r;2}(0,x)$ on $x\geq x_1$ and $u(x)=f_{l;1}(0,x)$ on $x\leq x_1$. Hence $u(x)$ has N_1 zeros on $(-\infty,x_1)$ and N_2 zeros on $(x_1,+\infty)$, i.e., N_1+N_2 zeros in all. Hence, by the interlacing property of zeros, $f_l(0,x)$ has either N_1+N_2 or N_1+N_2-1 zeros. This proves (2.5). To see that the inequalities are sharp, note that a square-well potential of depth $-H^2$ and width w has exactly N bound states, where N is the positive integer satisfying $(N-1)\pi<wH\leq N\pi$. Choosing V to be a square-well potential of depth $-\pi^2$ with support $(0,1)$, we obtain $N=1$. Let us partition the interval $(0,1)$ into p nonempty subintervals and hence obtain a fragmentation of V; each fragment still contains exactly one bound state and hence the lower bound in (2.5) becomes equal to N. On

the other hand, consider the square-well potential of depth $-\pi^2$ with support $(0,p)$, and partition $(0,p)$ into the p subintervals $(j-1,j)$ for $j=1,...,p$. Then $N_j=1$, $N=p$, and hence the upper bound in (2.5) becomes equal to N. ■

We remark that the short proof of (2.5) given here was suggested by the referee. Inequality (2.5) also follows from the next theorem that gives us, in case of two fragments, the precise information on whether $N=N_1+N_2$ or $N=N_1+N_2-1$. Let

$$Z(x_1)=\frac{f_{l;2}(0,x_1)}{f'_{l;2}(0,x_1)}-\frac{f_{r;1}(0,x_1)}{f'_{r;1}(0,x_1)}. \tag{2.6}$$

Theorem 2.2: Assume that V is partitioned into two fragments. Then:
(a) If $f'_{r;1}(0,x_1)\neq 0$, $f'_{l;2}(0,x_1)\neq 0$, and $Z(x_1)\geqslant 0$, then $N=N_1+N_2-1$; if $Z(x_1)<0$, then $N=N_1+N_2$.
(b) If $f'_{r;1}(0,x_1)=0$ or $f'_{l;2}(0,x_1)=0$, then $N=N_1+N_2$.

Proof: (a) In order to determine N we will count the zeros of $f_r(0,x)$ that lie in $[x_1,+\infty)$. We do this by using the interlacing property of the zeros of $f_r(0,x)$ and $f_l(0,x)$, noting that $f_l(0,x)=f_{l;2}(0,x)$ on $[x_1,+\infty)$ and $f_r(0,x)=f_{r;1}(0,x)$ on $(-\infty,x_1]$. We already know that n_1 zeros of $f_r(0,x)$ lie in $(-\infty,x_1)$, where n_1 is related to N_1 by (2.1). Upon multiplying $f_{l;2}(0,x)$ and $f_{r;1}(0,x)$ by suitable constants α and β, we can achieve that $\varphi_{l;2}(0,x)=\alpha f_{l;2}(0,x)$ and $\varphi_{r;1}(0,x)=\beta f_{r;1}(0,x)$ satisfy $\varphi'_{l;2}(0,x_1)=\varphi'_{r;1}(0,x_1)=1>0$. Then $Z(x_1)$ in (2.6) becomes

$$Z(x_1)=\varphi_{l;2}(0,x_1)-\varphi_{r;1}(0,x_1).$$

First suppose that $Z(x_1)>0$, which is equivalent to assuming $W[\varphi_l,\varphi_r](x_1)>0$, where $W[g,h](x)=g(x)h'(x)-g'(x)h(x)$ denotes the Wronskian. We first consider the case when $\varphi_{l;2}(0,x)$ has at least one zero on $(x_1,+\infty)$. Suppose that $\varphi_{l;2}(0,x)$ has its zeros at z_j for $j=1,...,m_2$, where $x_1<z_1<z_2<\cdots<z_{m_2}$. If $\varphi_{l;2}(0,x_1)>\varphi_{r;1}(0,x_1)>0$, then $\varphi_{l;2}(0,x)$ has m_2 zeros in $(x_1,+\infty)$ because, by a Wronskian argument, there are no zeros in (x_1,z_1) and there is exactly one zero in each of the intervals (z_1,z_2), $(z_2,z_3),\dots,(z_{m_2},+\infty)$. To see that there is a zero in $(z_{m_2},+\infty)$, note that by (2.3) and (2.4), $W[\varphi_l,\varphi_r](x)=\alpha\beta c_r>0$. Hence α and βc_r have the same sign. Moreover, if $\alpha>0$, then $\varphi'_{l;2}(0,z_{m_2})>0$ and hence $\varphi_r(0,z_{m_2})<0$. Similarly, if $\alpha<0$, then $\varphi'_{l;2}(0,z_{m_2})<0$ and hence $\varphi_r(0,z_{m_2})>0$. Because $\varphi_r(0,x)=\beta c_r x+\mathrm{o}(x)$ as $x\to+\infty$, it follows that $\varphi_r(0,x)$ must have a zero in $(x_1,+\infty)$ and, by the interlacing property, this is the only zero on this interval. Hence, using (2.1) and (2.2), we have $n_1=N_1$, $m_2=N_2-1$, and $N=n_1+m_2=N_1+N_2-1$. The same result holds when $\varphi_{l;2}(0,x)$ has no zeros on $(x_1,+\infty)$. Then $\varphi_r(0,x)$ has no zeros on $(x_1,+\infty)$ either and we have $m_2=0$. If $\varphi_{l;2}(0,x_1)>\varphi_{r;1}(0,x_1)=0$, then the previous argument goes through with only a minor change in counting the zeros because now $\varphi_r(0,x)$ also has a zero at $x=x_1$. We have $n_1=N_1-1$, $m_2=N_2-1$, and $N=n_1+m_2+1=N_1+N_2-1$. If $\varphi_{l;2}(0,x_1)\geqslant 0>\varphi_{r;1}(0,x_1)$, then $\varphi_r(0,x)$ has m_2+1 zeros on $(x_1,+\infty)$ because now there is also a zero in (x_1,z_1). Thus $n_1=N_1-1$, $m_2=N_2-1$, and $N=n_1+m_2+1=N_1+N_2-1$. If $0>\varphi_{l;2}(0,x_1)>\varphi_{r;1}(0,x_1)$, then $n_1=N_1-1$, $m_2=N_2$, and $N=N_1+N_2-1$ because $\varphi_r(0,x)$ has no zeros in (x_1,z_1). All the possibilities with $Z(x_1)>0$ have now been exhausted. If $Z(x_1)<0$, we can apply similar arguments and find that $N=N_1+N_2$. Finally, if $Z(x_1)=0$ because $\varphi_{l;2}(0,x_1)=\varphi_{r;1}(0,x_1)>0$, then $n_1=N_1$, $m_2=N_2-1$, and $N=n_1+m_2=N_1+N_2-1$. If $Z(x_1)=0$ because $\varphi_{l;2}(0,x_1)=\varphi_{r;1}(0,x_1)=0$, then $n_1=N_1-1$, $m_2=N_2-1$, and $N=n_1+m_2+1=N_1+N_2-1$. If $Z(x_1)=0$ because $\varphi_{l;2}(0,x_1)=\varphi_{r;1}(0,x_1)<0$, then $n_1=N_1-1$, $m_2=N_2$, and $N=n_1+m_2=N_1+N_2-1$. This concludes the proof of part (a). The proof of (b) is similar, using (2.1), (2.2), and the Wronskian. The details are omitted. ■

Theorem 2.3: Assume that V is partitioned into two fragments, and let $W(x_1)$ denote the Wronskian $W[f_{l;2}(0,\cdot);f_{r;1}(0,\cdot)](x_1)$. Then:
(i) Suppose N_1 and N_2 are either both even or both odd. If $W(x_1)>0$ $(W(x_1)<0)$, then $N=N_1+N_2$ $(N=N_1+N_2-1)$.
(ii) Suppose N_1 is even and N_2 is odd or vice versa. If $W(x_1)>0$ $(W(x_1)<0)$, then $N=N_1+N_2-1$ $(N=N_1+N_2)$.

Proof: The proof is a consequence of the following observation. If $W(x_1)>0$, then the constant c_r in (2.4) is positive and so N is even, while if $W(x_1)<0$, then c_r is negative and N is odd. ■

If $W(x_1)=0$, then Theorem 2.3 gives no information as to which possibility is realized. However, Theorem 2.2 (a) says that if $W(x_1)=0$ because $Z(x_1)=0$, then $N=N_1+N_2-1$. The only other possibility is that $W(x_1)=0$ because $f'_{r;1}(0,x_1)=f_{l;2}(0,x_1)=0$, in which case $N=N_1+N_2$ by Theorem 2.2 (b).

III. FURTHER OBSERVATIONS

In this section we analyze the result of Theorem 2.1 in conjunction with the scattering matrices corresponding to the fragments of this potential. For simplicity let us consider the fragmentation of V as $V=V_1+V_2$, where V_1 has support in $(-\infty,x_1]$ and V_2 has support in $[x_1,+\infty)$. The analysis for three or more fragments can be carried out by using induction. Let $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}$ be the scattering matrices corresponding to the potentials V_1, V_2, and V, respectively. The scattering coefficients appear in the scattering matrix as follows:

$$\mathbf{S}(k)=\begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \tag{3.1}$$

where T is the transmission coefficient, and L and R are the reflection coefficients from the left and from the right, respectively. Similarly, T_j, R_j, and L_j denote the corresponding entries of $\mathbf{S}_j$ for $j=1,2$. Let us define the so-called transition matrix associated with $\mathbf{S}$ as follows:

$$\Lambda(k)=\begin{bmatrix} \dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)} \\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*} \end{bmatrix}=\begin{bmatrix} \dfrac{1}{T(k)} & \dfrac{L(k)^*}{T(k)^*} \\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*} \end{bmatrix}, \tag{3.2}$$

where the asterisk denotes complex conjugation. Similarly, let Λ_1 and Λ_2 be the transition matrices corresponding to $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. It is known[4] that

$$\Lambda(k)=\Lambda_1(k)\Lambda_2(k). \tag{3.3}$$

From the (1,1) entry of the matrix product in (3.3) we get

$$\frac{1}{T(k)}=\frac{1-R_1(k)L_2(k)}{T_1(k)T_2(k)}. \tag{3.4}$$

Let $\mathbf{R}^+=(0,+\infty)$. For $k\in\mathbf{R}^+$, let us define the phases $\phi(k)$, $\phi_1(k)$, and $\phi_2(k)$ of the transmission coefficients as follows:

$$T(k)=|T(k)|e^{i\phi(k)}, \quad T_1(k)=|T_1(k)|e^{i\phi_1(k)}, \quad T_2(k)=|T_2(k)|e^{i\phi_2(k)}, \tag{3.5}$$

where it is understood that ϕ, ϕ_1, and ϕ_2 are continuous in $k\in\mathbf{R}^+$ and normalized such that

$$\phi(+\infty)=\phi_1(+\infty)=\phi_2(+\infty)=0. \tag{3.6}$$

Similarly, let

$$1-R_1(k)L_2(k)=|1-R_1(k)L_2(k)|e^{i\omega(k)}, \tag{3.7}$$

where ω is assumed continuous in $k\in\mathbf{R}^+$ and to satisfy $\omega(+\infty)=0$. From (3.4) we obtain

$$\phi(k)=\phi_1(k)+\phi_2(k)-\omega(k), \quad k\in\mathbf{R}^+. \tag{3.8}$$

From Levinson's theorem[11] we have

$$\phi(0+)=\left[N-\frac{d}{2}\right]\pi, \quad \phi_1(0+)=\left[N_1-\frac{d_1}{2}\right]\pi, \quad \phi_2(0+)=\left[N_2-\frac{d_2}{2}\right]\pi, \tag{3.9}$$

where N, N_1, and N_2 denote the number of bound states corresponding to the potentials V, V_1, and V_2, respectively; $d=1$ if V is a generic potential and $d=0$ if V is exceptional; in a similar manner, d_1 and d_2 take values 1 or 0 depending on whether V_1 and V_2 are generic or exceptional. Using (3.9) in (3.8) we obtain

$$N=N_1+N_2+\frac{1}{2}\left[d-d_1-d_2\right]-\frac{1}{\pi}\,\omega(0+). \tag{3.10}$$

Now let us analyze ω further. Note that R_1 and L_2 are continuous and nonzero and strictly less than one in absolute value for $k\in\mathbf{R}^+$ and that, as $k\to+\infty$, both R_1 and L_2 vanish.

In the following we need to distinguish between the generic case and the exceptional case. When V_1 and V_2 are both generic we have

$$R_1(k)=-1-2ika_{r;1}+\mathrm{o}(k), \quad L_2(k)=-1-2ika_{l;2}+\mathrm{o}(k), \quad k\to 0, \tag{3.11}$$

where

$$a_{r;1}=\left(1-\int_{-\infty}^{x_1}dx\ xV_1(x)f_{r;1}(0,x)\right)\Bigg/\int_{-\infty}^{x_1}dx\ V_1(x)f_{r;1}(0,x), \tag{3.12}$$

$$a_{l;2}=\left(1+\int_{x_1}^{\infty}dx\ xV_2(x)f_{l;2}(0,x)\right)\Bigg/\int_{x_1}^{\infty}dx\ V_2(x)f_{l;2}(0,x). \tag{3.13}$$

In the exceptional case we define

$$\gamma_1=\frac{f_{l;1}(0,x)}{f_{r;1}(0,x)}=\frac{1}{f_{r;1}(0,x_1)}, \quad \gamma_2=\frac{f_{l;2}(0,x)}{f_{r;2}(0,x)}=f_{l;2}(0,x_1), \tag{3.14}$$

and note that, if V_1, resp. V_2, is exceptional, then

$$R_1(k)=-b_1+\mathrm{o}(1), \quad \text{resp.} \quad L_2(k)=b_2+\mathrm{o}(1), \quad k\to 0, \tag{3.15}$$

where

$$b_j=\frac{\gamma_j^2-1}{\gamma_j^2+1}, \quad j=1,2. \tag{3.16}$$

The relations (3.11)–(3.13) follow from p. 146 of Ref. 6; (3.15) was proved in Ref. 12. We remark that the validity of (3.11) depends on the property that V_1 and V_2 are each supported on a semi-infinite interval; this guarantees the convergence of the integrals in the numerators in (3.12) and (3.13). In general, for potentials in $L_1^1(\mathbf{R})$ one can only conclude[12] that the reflection coefficients behave like $-1+\mathrm{o}(1)$ as $k\to 0$ in the generic case.

When both V_1 and V_2 are generic we have

$$1-R_1(k)L_2(k)=-2ik[a_{r;1}+a_{l;2}]+\mathrm{o}(k), \quad k\to 0. \tag{3.17}$$

When both V_1 and V_2 are exceptional we get

$$1-R_1(k)L_2(k)=1+b_1b_2+\mathrm{o}(1), \quad k\to 0. \tag{3.18}$$

When V_1 is generic and V_2 is exceptional we have

$$1-R_1(k)L_2(k)=1+b_2+\mathrm{o}(1), \quad k\to 0, \tag{3.19}$$

and finally, when V_1 is exceptional and V_2 is generic, we have

$$1-R_1(k)L_2(k)=1-b_1+\mathrm{o}(1), \quad k\to 0. \tag{3.20}$$

From (3.15) and (3.18)–(3.20) we see that if at least one of V_1 and V_2 is exceptional, then the quantity $[1-R_1(0)L_2(0)]$ is strictly positive, and hence $\omega(0+)=0$.

If both V_1 and V_2 are generic, the analysis is slightly more complicated: If $a_{r;1}<-a_{l;2}$, then $\omega(0+)=\pi/2$; if $a_{r;1}>-a_{l;2}$, then $\omega(0+)=-\pi/2$. If $a_{r;1}=-a_{l;2}$, then, as $k\to 0$, we get $1-R_1(k)L_2(k)=o(k)$, where we have used (3.17). As a result, (3.4) implies that $k/T(k)=o(1)$ as $k\to 0$, and this, in turn, implies that V is exceptional. Therefore the left-hand side of (3.4) has a limit as $k\to 0$, which means that in fact $1-R_1(k)L_2(k)=O(k^2)$, from which we obtain $\omega(0+)=0$.

It is known[13] that when V_1 and V_2 are both exceptional, then V is exceptional. If exactly one of V_1 and V_2 is exceptional, then V is generic. If both V_1 and V_2 are generic, then V can be exceptional or generic. By using these facts along with the value of $\omega(0+)$ and (3.10), we arrive at the following conclusions:

(i) If both V_1 and V_2 are exceptional, then $N=N_1+N_2$.
(ii) If exactly one of V_1 and V_2 is exceptional and the other is generic, then $N=N_1+N_2$.
(iii) If both V_1 and V_2 are generic and V is also generic, then $\omega(0+)=\pm\pi/2$. In this case, we have $N=N_1+N_2-1$ if $\omega(0+)=\pi/2$, and this happens if $a_{r;1}<-a_{l;2}$ in (3.17); or we have $N=N_1+N_2$ if $\omega(0+)=-\pi/2$, and this happens if $a_{r;1}>-a_{l;2}$.
(iv) If both V_1 and V_2 are generic and V is exceptional, then we must have $\omega(0+)=0$ and $N=N_1+N_2-1$. This happens if $a_{r;1}=-a_{l;2}$ in (3.17).

Summarizing, if $a_{r;1}\leqslant -a_{l;2}$ in (3.17) and both $a_{r;1}$ and $a_{l;2}$ are finite, then we have $N=N_1+N_2-1$; if at least one of $a_{r;1}$ and $a_{l;2}$ is infinite or if $a_{r;1}>-a_{l;2}$, then we have $N=N_1+N_2$.

There is a direct connection between cases (i)–(iv) above and cases (a) and (b) of Theorem 2.2 because the coefficients $a_{r;1}$ and $a_{l;2}$ are related to the quantity $Z(x_1)$ defined in (2.6). To see this recall that $f_{r;1}(0,x)$ and $f_{l;2}(0,x)$ obey the integral equations

$$f_{r;1}(0,x)=1+\int_{-\infty}^{x}dy(x-y)V_1(y)f_{r;1}(0,y), \tag{3.21}$$

$$f_{l;2}(0,x)=1+\int_{x}^{\infty}dy(y-x)V_2(y)f_{l;2}(0,y). \tag{3.22}$$

Hence from (3.21) and (3.22) we obtain

$$f_{r;1}(0,x)=c_{r;1}x+d_{r;1},\quad x>x_1,$$

$$f_{l;2}(0,x)=-c_{l;2}x+d_{l;2},\quad x<x_1,$$

with

$$c_{r;1}=\int_{-\infty}^{x_1}dy\ V_1(y)f_{r;1}(0,y),\quad d_{r;1}=1-\int_{-\infty}^{x_1}dy\ yV_1(y)f_{r;1}(0,y), \tag{3.23}$$

$$c_{l;2}=\int_{x_1}^{\infty}dy\ V_2(y)f_{l;2}(0,y),\quad d_{l;2}=1+\int_{x_1}^{\infty}dy\ yV_2(y)f_{l;2}(0,y). \tag{3.24}$$

Thus from (3.12), (3.13), (3.23), and (3.24) we conclude that

$$a_{r;1}=\frac{d_{r;1}}{c_{r;1}},\quad a_{l;2}=\frac{d_{l;2}}{c_{l;2}}. \tag{3.25}$$

Moreover,

$$\frac{f_{r;1}(0,x_1)}{f'_{r;1}(0,x_1)}=x_1+a_{r;1},\quad \frac{f_{l;2}(0,x_1)}{f'_{l;2}(0,x_1)}=x_1-a_{l;2},$$

and hence

$$Z(x_1) = -a_{r,1} - a_{l;2}.$$

Thus (i) and (ii) above correspond to the possibilities of Theorem 2.2 (b); (i) is the case when $f'_{r;1}(0,x_1) = f'_{l;2}(0,x_1) = 0$, and (ii) is the case when exactly one of $f'_{r;1}(0,x_1)$ and $f'_{l;2}(0,x_1)$ is zero. Case (iii) corresponds to (a) of Theorem 2.2 with $Z(x_1) \neq 0$ and case (iv) corresponds to (a) with $Z(x_1) = 0$.

We conclude this section with a brief look at families of potentials of the form

$$V_\xi(x) = V_1(x) + V_2(x-\xi), \tag{3.26}$$

where ξ is a non-negative parameter and V_1 and V_2 are the two fragments of V. In other words, the parameter ξ controls the separation distance between the two fragments. The next result shows that the number of bound states can only increase if ξ is increased. By virtue of (2.5) it can only increase by one. Since the proof is short we present two versions, one using the method of Sec. II and the other using the method of this section. In the case of compactly supported fragments the result is already known from Refs. 1 and 3.

Theorem 3.1: Let N_ξ denote the number of bound states of V_ξ given in (3.26). Then either $N_\xi = N_1 + N_2$ for all $\xi \geqslant 0$ or there is a unique $\xi_0 \geqslant 0$ such that $N_\xi = N_1 + N_2 - 1$ for $0 \leqslant \xi \leqslant \xi_0$ and $N_\xi = N_1 + N_2$ for $\xi > \xi_0$.

Proof: (a) First, if one of the fragments is exceptional, then we have $N_\xi = N_1 + N_2$ for all $\xi \geqslant 0$. If both fragments are generic, then we let $f_{l;2;\xi}(k,x)$ denote the Jost solution from the left for the potential $V_2(x-\xi)$. Then $f_{l;2;\xi}(0,x) = -c_{l;2}(x-\xi) + d_{l;2}$ for $x < x_1 + \xi$, and thus, by using (2.6) and (3.26), we obtain $Z_\xi(x_1) = -\xi - a_{l;2} - a_{r;1}$. Thus if $Z_0(x_1) < 0$, then, for all $\xi \geqslant 0$, $Z_\xi(x_1) < 0$ and hence $N_\xi = N_1 + N_2$. If $Z_0(x_1) \geqslant 0$, then $Z_{\xi_0}(x_1) = 0$ when $\xi_0 = Z_0(x_1) = -a_{l;2} - a_{r;1}$ and the assertion follows.

(b) Replacing $L_2(k)$ by $e^{2ik\xi}L_2(k)$ in (3.18) we obtain

$$1 - R_1(k)L_{2;\xi}(k) = -2ik[a_{r;1} + a_{l;2} + \xi] + o(k), \quad k \to 0.$$

Now the conclusion follows using (iii) and (iv) above. ■

IV. AN EXAMPLE

The following example illustrates Theorems 2.1 and 3.1. Let

$$V(x) = \begin{cases} A^2, & x \in (0,1) \\ -B^2, & x \in (1,2) \\ 0, & \text{elsewhere,} \end{cases} \tag{4.1}$$

where A and B are some positive constants. We can fragment V as $V = V_1 + V_2$, where V_1 is a square potential barrier of height A^2 with support (0,1) and V_2 is a square well of depth $-B^2$ with support (1,2). Then, a straightforward computation using (3.23)–(3.25) yields $c_{r;1} = A \sinh A$, $d_{r;1} = \cosh A - A \sinh A$, $c_{l;2} = -B \sin B$, $d_{l;2} = \cos B - B \sin B$, and thus

$$a_{r;1} = \frac{1}{A} \coth A - 1, \quad a_{l;2} = -\frac{1}{B} \cot B + 1.$$

Let us demonstrate that by choosing A and B suitably, we can have $N_1 = 0$, $N_2 = 1$, $N = 0$. In other words, the positive fragment V_1 may cancel the bound state caused by the negative fragment V_2, resulting in no bound states for V. Unless B is a multiple of π, both V_1 and V_2 are generic. If we let, for example, $B = \pi/4$, then from (2.6) we get $Z(x_1) = -a_{r;1} - a_{l;2} \geqslant 0$ whenever $A \geqslant A_0$, where A_0 satisfies $A_0 \tanh A_0 = \pi/4$, i.e., $A_0 = 1.0201. \ldots$ For $A = A_0$ the potential V is exceptional with no bound state and for $A > A_0$ it is generic with no bound state. Now let us consider the family V_ξ defined in (3.26). If $A < A_0$ and $B = \pi/4$, then $Z_0(x_1) < 0$ and we have $N_\xi = N_1 + N_2 = 1$ for all ξ. If $A = A_0$, then we have $N_0 = 0$ but $N_\xi = 1$ for $\xi > 0$, i.e., $\xi_0 = 0$. If $A > A_0$, then ξ_0 is given by

$$\xi_0 = \frac{1}{B \tan B} - \frac{1}{A \tanh A},$$

and we have $N_\xi = 0$ for $\xi \leqslant \xi_0$ and $N_\xi = 1$ for $\xi > \xi_0$.

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[1] M. Klaus, ''Some remarks on double-wells in one and three dimensions,'' Ann. Inst. Henri Poincaré, Sect. A **34**, 405–417 (1981).
[2] M. Sassoli de Bianchi, ''Levinson's theorem, zero-energy resonances, and time delay in one-dimensional scattering systems,'' J. Math. Phys. **35**, 2719–2733 (1994).
[3] M. Sassoli de Bianchi and M. Di Ventra, ''On the number of bound states by one-dimensional finite periodic potentials,'' J. Math. Phys. **36**, 1753–1764 (1995).
[4] T. Aktosun, ''A factorization of the scattering matrix for the Schrödinger equation and for the wave equation in one dimension,'' J. Math. Phys. **33**, 3865–3869 (1992).
[5] L. D. Faddeev, ''Properties of the *S*-matrix of the one-dimensional Schrödinger equation,'' Am. Math. Soc. Trans. **2**, 139–166 (1964) [Tr. Mat. Inst. Steklova **73**, 314–336 (1964) (Russian)].
[6] P. Deift and E. Trubowitz, ''Inverse scattering on the line,'' Commun. Pure Appl. Math. **32**, 121–251 (1979).
[7] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer, New York, 1989).
[8] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators* (Academic, New York, 1978).
[9] T. Aktosun, M. Klaus, and C. van der Mee, ''Wave scattering in one dimension with absorption,'' J. Math. Phys. **39**, 1957–1992 (1998).
[10] T. Aktosun, M. Klaus, and C. van der Mee, ''On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation,'' J. Math. Phys. **34**, 2651–2690 (1993).
[11] R. G. Newton, ''Inverse scattering. I. One dimension,'' J. Math. Phys. **21**, 493–505 (1980).
[12] M. Klaus, ''Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line,'' Inverse Probl. **4**, 505–512 (1988).
[13] T. Aktosun, M. Klaus, and C. van der Mee, ''Factorization of scattering matrices due to partitioning of potentials in one-dimensional Schrödinger-type equations,'' J. Math. Phys. **37**, 5897–5915 (1996).