Scattering and inverse scattering for a second-order differential equation

Tuncay Aktosun
Department of Mathematics, North Dakota State University, Fargo, North Dakota 58105

(Received 23 October 1992; accepted for publication 29 January 1993)

The scattering and three inverse scattering problems for $(d/dx)[a(x)(d\psi/dx)]+k^2h(x)\psi=Q(x)\psi$ on the real axis are considered herein. In the first inverse scattering problem, $Q(x)$ is recovered when $a(x)$, $h(x)$, and the scattering data are given. In the second inverse problem $h(x)$ is recovered when $a(x)$, $Q(x)$, and the scattering data are known. In the third inverse problem, in case $Q(x)=0$, $a(x)$ is recovered when $h(x)$ and the scattering data are known. The inversion is illustrated with examples.

I. INTRODUCTION

Consider the second-order differential equation

$$\alpha(x)\varphi''+\beta(x)\varphi'+k^2\gamma(x)\varphi=\epsilon(x)\varphi,\quad x\in\mathbf{R}.$$

Here and throughout the article the prime denotes the derivative with respect to x. Using the integrating factor $(1/\beta)e^{\int_0^x(\alpha/\beta)}$, we can transform the above differential equation into

$$[a(x)\psi'(k,x)]'+k^2h(x)\psi(k,x)=Q(x)\psi(k,x), \tag{1.1}$$

where $x\in\mathbf{R}$ is the space coordinate and k^2 is the spectral parameter. The functions $a(x)$, $h(x)$, and $Q(x)$ are assumed to be real, $a(x)\to 1$, $h(x)\to 1$, and $Q(x)\to 0$ as $x\to\pm\infty$. Letting

$$y=\int_0^x d\xi\sqrt{\frac{h(\xi)}{a(\xi)}}, \tag{1.2}$$

and $\phi(k,y) = \sqrt[4]{a(x)h(x)}\psi(k,x)$, one can transform[1] Eq. (1.1) into the Schrödinger equation

$$\frac{d^2\phi}{dy^2}+k^2\phi=V(y)\phi, \tag{1.3}$$

where the potential $V(y)$ is given by

$$V(y)=\frac{1}{h}(ah)^{1/4}\{(ah)^{-1/4}Q-a'[(ah)^{-1/4}]'-a[(ah)^{-1/4}]''\}. \tag{1.4}$$

The scattering problem for Eq. (1.1) is to obtain the scattering matrix $S(k)$ defined in Eq. (2.3). We will study and solve three inverse scattering problems associated with Eq. (1.1). The first one is the recovery of $Q(x)$ when $a(x)$, $h(x)$, and the scattering data are known. The second inverse problem is the recovery of $h(x)$ when $a(x)$, $Q(x)$, and the scattering data are known. The third inverse problem is, in case $Q(x)=0$, the recovery of $a(x)$ when $h(x)$ and the scattering data are known. Note that $V(y)$ can be obtained by solving the inverse scattering problem for Eq. (1.3) by using one of the inverse scattering methods[2–5] for the 1-D Schrödinger equation such as the Marchenko method, the Gel'fand–Levitan method, and the Wiener–Hopf factorization method. Knowing $a(x)$ and $h(x)$, one can then obtain $Q(x)$ by

0022-2488/93/001619-16$06.00

using Eq. (1.2) and hence solve the first inverse scattering problem for Eq. (1.1). In this article we will use the spatial coordinate x directly because this will help us to solve also the second and third inverse problems. The sufficient conditions for all the results in this article to hold are a and h are bounded and positive, $1-a\in L^1$, $a',Q,h-a,\Upsilon\in L^1_1$, where Υ is the function defined in Eq. (2.31) and L^1_1 is the space of integrable functions having a finite first moment.

This article is organized as follows. In Sec. II we define the scattering solutions of Eq. (1.1), study their properties, and establish their asymptotics for small and large k. In Sec. III we study the properties of the scattering matrix. In Sec. IV we study the Riemann–Hilbert problem associated with Eq. (1.1). In Sec. V we give the solutions of the three inverse scattering problems for Eq. (1.1). Finally in Sec. VI, we illustrate the inversion procedure by explicitly solved examples.

II. SCATTERING SOLUTIONS

In this section we study the scattering solutions of Eq. (1.1) and establish their asymptotics for small k. The physical solutions ψ_l from the left and ψ_r from the right of Eq. (1.1) satisfy the boundary conditions

$$\psi_l(k,x)=\begin{cases}T_l(k)e^{ikx}+o(1), & x\to+\infty\\ e^{ikx}+L(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{2.1}$$

$$\psi_r(k,x)=\begin{cases}e^{-ikx}+R(k)e^{ikx}+o(1), & x\to+\infty\\ T_r(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{2.2}$$

where T_l and T_r are the transmission coefficients from the left and from the right, respectively, and L and R are the reflection coefficients from the left and from the right, respectively. The scattering matrix is defined as

$$S(k)=\begin{bmatrix}T_l(k) & R(k)\\ L(k) & T_r(k)\end{bmatrix}. \tag{2.3}$$

Let us write Eq. (1.1) in the form

$$[a(x)\psi'(k,x)]'+\frac{k^2}{a(x)}\,\psi(k,x)=F(k,x)\psi(k,x), \tag{2.4}$$

where

$$F(k,x)=Q(x)+k^2\left[\frac{1}{a(x)}-h(x)\right]. \tag{2.5}$$

Note that $F(0,x)=Q(x)$. Let

$$\Gamma(k;x-z)=\frac{1}{2ik}\exp\left(ik\int_{\min(x,z)}^{\max(x,z)}\frac{dt}{a(t)}\right). \tag{2.6}$$

The physical solutions ψ_l and ψ_r then satisfy the Lippmann–Schwinger equations

$$\psi_l(k,x)=e^{ik\int_0^x(1/a)}e^{-ik\int_{-\infty}^0[1-(1/a)]}+\int_{-\infty}^{+\infty}dz\,\Gamma(k;x-z)F(k,z)\psi_l(k,z), \tag{2.7}$$

$$\psi_r(k,x)=e^{-ik\int_0^x(1/a)}-e^{ik\int_0^\infty[1-(1/a)]}+\int_{-\infty}^{\infty} dz\,\Gamma(k;x-z)F(k,z)\psi_r(k,z). \tag{2.8}$$

The Jost solutions of Eq. (1.1), f_l from the left and f_r from the right, are defined by

$$f_l(k,x)=\frac{1}{T_l(k)}\,\psi_l(k,x) \quad \text{and} \quad f_r(k,x)=\frac{1}{T_r(k)}\,\psi_r(k,x). \tag{2.9}$$

Let us define

$$G_l(k;x-z)=\frac{1}{k}\sin\left(k\int_x^z \frac{dt}{a(t)}\right),\quad G_r(k;x-z)=\frac{1}{k}\sin\left(k\int_z^x \frac{dt}{a(t)}\right).$$

The Jost solutions of Eq. (1.1) then satisfy the integral equations

$$f_l(k,x)=e^{ik\int_0^x(1/a)}e^{ik\int_0^\infty[1-(1/a)]}+\int_x^{\infty} dz\,G_l(k;x-z)F(k,z)f_l(k,z), \tag{2.10}$$

$$f_r(k,x)=e^{-ik\int_0^x(1/a)}e^{ik\int_{-\infty}^0[1-(1/a)]}+\int_{-\infty}^{x} dz\,G_r(k;x-z)F(k,z)f_r(k,z). \tag{2.11}$$

The Faddeev functions for Eq. (1.1) are defined as

$$m_l(k,x)=\frac{1}{T_l(k)}\,e^{-ikx}\psi_l(k,x) \quad \text{and} \quad m_r(k,x)=\frac{1}{T_r(k)}\,e^{ikx}\psi_r(k,x). \tag{2.12}$$

Then from Eqs. (1.1) and (2.4) it is seen that m_l and m_r satisfy the equations

$$a(x)m_l''(k,x)+[a'(x)+2ika(x)]m_l'(k,x)+E(k,x)m_l(k,x)=0, \tag{2.13}$$

$$a(x)m_r''(k,x)+[a'(x)-2ika(x)]m_r'(k,x)+E(-k,x)m_r(k,x)=0, \tag{2.14}$$

where we have defined

$$E(k,x)=k^2[h(x)-a(x)]+ika'(x)-Q(x), \tag{2.15}$$

with the boundary conditions

$$m_l(k,x)=1+o(1) \quad \text{and} \quad m_l'(k,x)=o(1), \quad x\to+\infty, \tag{2.16}$$

$$m_r(k,x)=1+o(1) \quad \text{and} \quad m_r'(k,x)=o(1), \quad x\to-\infty. \tag{2.17}$$

From Eqs. (2.10) and (2.11) it is seen that $m_l(k,x)$ and $m_r(k,x)$ satisfy the integral equations

$$m_l(k,x)=e^{ik\int_x^\infty[1-(1/a)]}+\frac{1}{2ik}\int_x^{\infty} dz[e^{ik\int_x^z[1+(1/a)]}-e^{ik\int_x^z[1-(1/a)]}]F(k,z)m_l(k,z),$$

$$m_r(k,x)=e^{ik\int_{-\infty}^x[1-(1/a)]}+\frac{1}{2ik}\int_{-\infty}^{x} dz[e^{ik\int_z^x[1+(1/a)]}-e^{ik\int_z^x[1-(1/a)]}]F(k,z)m_r(k,z),$$

where $F(k,x)$ is the quantity defined in Eq. (2.5). From Eqs. (2.13) and (2.16), we also obtain

$$a(x)e^{2ikx}m_l'(k,x) = -\int_x^\infty dz\, E(k,z)e^{2ikz}m_l(k,z), \tag{2.18}$$

where $E(k,x)$ is the quantity defined in Eq. (2.15). Integrating Eq. (2.18) and using Eq. (2.16) again, we obtain

$$m_l(k,x) = 1 + \int_x^\infty dz \int_x^z dt \frac{e^{2ik(z-t)}}{a(t)} E(k,z)m_l(k,z). \tag{2.19}$$

Similarly, from Eqs. (2.14) and (2.17), we obtain

$$m_r(k,x) = 1 + \int_{-\infty}^x dz \int_z^x dt \frac{e^{2ik(t-z)}}{a(t)} E(-k,z)m_r(k,z). \tag{2.20}$$

Let $\mathbf{C}^+$ denote the upper half complex plane, $\mathbf{C}^-$ denote the lower half complex plane, and $\overline{C^\pm}$ denote $\mathbf{C}^\pm \cup \mathbf{R}$. The next theorem shows that the Faddeev functions defined in Eq. (2.12) can be extended analytically in k to $\mathbf{C}^+$.

Theorem 2.1: When a', Q, and $h-a$ belong to $L_1^1(\mathbf{R})$, for each $x\in\mathbf{R}$ the Faddeev functions $m_l(k,x)$ and $m_r(k,x)$ are analytic in k in C^+ and continuous in $\overline{\mathbf{C}^+}$.

Proof: From Eq. (2.19) we have $m_l(k,x) = \Sigma_{j=0}^\infty n_j(k,x)$ where $n_0(k,x) = 1$ and

$$n_j(k,x) = \int_x^\infty dz \int_x^z dt \frac{e^{2ik(z-t)}}{a(t)} E(k,z)n_{j-1}(k,z), \quad j \geqslant 1.$$

Define

$$N = \inf_{x\in\mathbf{R}} a(x), \quad M = \sup_{x\in\mathbf{R}} h(x). \tag{2.21}$$

Since for $t \geqslant x$ and $k \in \overline{\mathbf{C}^+}$ we have

$$\left| \int_x^z dt \frac{e^{2ik(z-t)}}{a(t)} \right| \leqslant \frac{1}{N}(z-x), \tag{2.22}$$

we obtain

$$|n_j(k,x)| \leqslant \frac{1}{j!N^j} \left[\int_x^\infty dz(z-x)|E(k,z)| \right]^j,$$

which implies

$$|m_l(k,x)| \leqslant e^{(1/N)\int_x^\infty dz(z-x)|E(k,z)|}, \quad k\in\overline{\mathbf{C}^+}, \tag{2.23}$$

where

$$|E(k,x)| \leqslant |k|^2|h(x)-a(x)| + |k|\,|a'(x)| + |Q(x)|.$$

Now, when $|k| \leqslant c$ each $n_j(k,x)$ is analytic in k in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, and thus by the Weierstrass theorem, $m_l(k,x)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$ for each $x\in\mathbf{R}$ whenever $h-a$, a', and Q belong to $L_1^1(\mathbf{R})$.

A repetition of the above argument with Eq. (2.20) gives us

$$|m_r(k,x)| \leqslant e^{(1/N)\int_{-\infty}^{x} dz(x-z)|E(k,z)|}, \quad k\in\overline{\mathbf{C}^+}, \tag{2.24}$$

and that $m_r(k,x)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. ■

Proposition 2.2: If $h-a$, a', and Q belong to $L_1^1(\mathbf{R})$, then for $k \in \overline{\mathbf{C}^+}$, the Faddeev functions $m_l(k,x)$ and $m_r(k,x)$ satisfy the inequality $|m(k,x)| \leqslant C_1(k)[1+|x|]$, where

$$C_1(k) = e^{(1/N)\int_{-\infty}^{\infty} dt(1+|t|)|E(k,t)|}\left[1+e^{(1/N)\int_{-\infty}^{\infty} dz|E(k,z)|}\int_{-\infty}^{\infty} dv\,\frac{|v|}{N}\,|E(k,v)|\right].$$

Proof: From Eqs. (2.19) and (2.22), for $k \in \overline{\mathbf{C}^+}$ we obtain

$$|m_l(k,x)| \leqslant 1+\int_x^{\infty} dz\,\frac{z-x}{N}\,|E(k,z)|\,|m_l(k,z)|$$

$$\leqslant 1+\int_0^{\infty} dz\,\frac{z}{N}\,|E(k,z)|\,|m_l(k,z)| - \frac{x}{N}\int_x^{\infty} dz|E(k,z)|\,|m_l(k,z)|. \tag{2.25}$$

Using Eq. (2.23) and letting

$$C_2(k) = 1+e^{(1/N)\int_{-\infty}^{\infty} dz|t||E(k,t)|}\left[\int_{-\infty}^{\infty} dt\,\frac{|t|}{N}\,|E(k,t)|\right],$$

from Eq. (2.25) we obtain

$$\frac{|m_l(k,x)|}{C_2(k)[1+|x|]} \leqslant 1+\int_x^{\infty} dz\,\frac{1+|z|}{N}\,|E(k,z)|\,\frac{|m_l(k,z)|}{C_2(k)[1+|z|]}.$$

Hence, using iteration we get

$$\frac{|m_l(k,x)|}{C_2(k)[1+|x|]} \leqslant e^{\int_x^{\infty} dz[(1+|z|)/N]|E(k,z)|},$$

which gives the result stated in the proposition with

$$C_1(k) = C_2(k)e^{\int_{-\infty}^{\infty} dz[(1+|z|)/N]|E(k,z)|}.$$

Proceeding similarly, we also obtain $|m_r(k,x)| \leqslant C_1(k)[1+|x|]$. ■

From Eqs. (2.18) and (2.20) we obtain

$$m_l'(k,x) = -\frac{1}{a(x)}\int_x^{\infty} dz\,e^{2ik(z-x)}E(k,z)m_l(k,z),$$

$$m_r'(k,x) = \frac{1}{a(x)}\int_{-\infty}^{x} dz\,e^{2ik(x-z)}E(k,z)m_r(k,z).$$

Hence, using Proposition 2.2 we obtain

$$|m_l'(k,x)| \leqslant C_1(k)\int_{-\infty}^{\infty} dz\,\frac{1+|z|}{N}\,|E(k,z)|, \quad k\in\overline{\mathbf{C}^+},$$

$$|m_r'(k,x)| < C_1(k) \int_{-\infty}^{\infty} dz \frac{1+|z|}{N} |E(k,z)|, \quad k \in \overline{\mathbf{C}^+},$$

where $C_1(k)$ is as specified in Proposition 2.2. Thus, if $h-a$, h', and Q belong to $L_1^1(\mathbf{R})$, the functions $m_l'(k,x)$ and $m_r'(k,x)$ are analytic in $k \in \mathbf{C}^+$ and continuous in $k \in \overline{\mathbf{C}^+}$ for each $x \in \mathbf{R}$.

Assume a solution of Eq. (1.1) of the form $\psi(k,x) = Y(k,x)Z(k,x)$, where $Y(k,x)$ stands for either of the two functions defined by

$$Y_l(k,x) = \frac{1}{\sqrt[4]{a(x)h(x)}} e^{ik\int_0^x dz \sqrt{h(z)/a(z)}}, \tag{2.26}$$

$$Y_r(k,x) = \frac{1}{\sqrt[4]{a(x)h(x)}} e^{-ik\int_0^x dz \sqrt{h(z)/a(z)}}. \tag{2.27}$$

Proceeding as in Ref. 6 one can show that $Z(k,x)$ satisfies

$$(aY^2Z')' + \Upsilon Y^2 Z = 0, \tag{2.28}$$

where we have defined

$$\Upsilon = a' \frac{Y'}{Y} + \frac{aY''}{Y} + k^2 h - Q. \tag{2.29}$$

Using Eqs. (2.26) and (2.27) one can directly verify that Υ is a function of x only and is independent of k. It can also be shown that $V(y) = -\Upsilon(x)/h(x)$, where $V(y)$ is the quantity given in Eq. (1.4) and y is the quantity given in Eq. (1.2).

Integrating Eq. (2.28) twice with the boundary condition $Z'(k,x_0)=0$ and $Z(k,x_0)=1$, we obtain

$$Z'(k,x) = -\frac{1}{a(x)} \int_{x_0}^{x} dz \frac{Y(k,z)^2}{Y(k,x)^2} \Upsilon(z) Z(k,z), \tag{2.30}$$

$$Z(k,x) = 1 - \int_{x_0}^{x} \frac{d\xi}{a(\xi)Y(k,\xi)^2} \int_{x_0}^{\xi} dz\, Y(k,z)^2 \Upsilon(z) Z(k,z). \tag{2.31}$$

We can write Eq. (2.31) as

$$Z(k,x) = 1 - \int_{x_0}^{x} dz\, \mathscr{L}(k;x,z) Z(k,z), \tag{2.32}$$

where

$$\mathscr{L}(k;x,z) = \int_z^x d\xi \frac{Y(k,z)^2 \Upsilon(z)}{a(\xi) Y(k,\xi)^2}.$$

In Eq. (2.32) choosing $x_0 = \pm\infty$, we obtain two linearly independent solutions of Eq. (2.28) denoted by Z_l and Z_r, respectively, satisfying

$$Z_l(k,x) = 1 + \int_x^{\infty} dz\, \mathscr{L}_l(k;x,z) Z_l(k,z), \tag{2.33}$$

$$Z_r(k,x)=1-\int_{-\infty}^{x} dz\, \mathscr{L}_r(k;x,z)Z_r(k,z), \tag{2.34}$$

where

$$\mathscr{L}_l(k;x,z)=\frac{1}{2ik}\,[1-e^{2ik\int_x^z d\xi\sqrt{h(\xi)/a(\xi)}}]\Theta(z),$$

$$\mathscr{L}_r(k;x,z)=-\frac{1}{2ik}\,[1-e^{-2ik\int_x^z d\xi\sqrt{h(\xi)/a(\xi)}}]\Theta(z),$$

with

$$\Theta(x)=\frac{\Upsilon(x)}{\sqrt{a(x)h(x)}}. \tag{2.35}$$

For $k\in\overline{\mathbf{C}^+}\setminus\{0\}$, we have

$$|\mathscr{L}_l(k;x,z)|\leqslant\frac{1}{|k|}\,|\Theta(z)| \quad \text{and} \quad |\mathscr{L}_r(k;x,z)|\leqslant\frac{1}{|k|}\,|\Theta(z)|$$

in the domains of integration given in Eqs. (2.33) and (2.34), respectively. Thus, iterating Eqs. (2.33) and (2.34) we obtain

$$|Z_l(k,x)|\leqslant e^{(1/|k|)\int_x^{\infty} dz|\Theta(z)|}\leqslant e^{(1/|k|)\int_{-\infty}^{\infty} dz|\Theta(z)|}, \quad k\in\overline{\mathbf{C}^+}\setminus\{0\}, \tag{2.36}$$

$$|Z_r(k,x)|\leqslant e^{(1/|k|)\int_{-\infty}^{x} dz|\Theta(z)|}\leqslant e^{(1/|k|)\int_{-\infty}^{\infty} dz|\Theta(z)|}, \quad k\in\overline{\mathbf{C}^+}\setminus\{0\}. \tag{2.37}$$

Hence, by the Weierstrass theorem, when $\Upsilon\in L^1(\mathbf{R})$, for each x both $Z_l(k,x)$ and $Z_r(k,x)$ have continuous extensions in k to $\overline{\mathbf{C}^+}\setminus\{0\}$ which are analytic on $\mathbf{C}^+$. Iterating Eqs. (2.33) and (2.34), we obtain $Z_l(k,x)=1+O(1/k)$ and $Z_r(k,x)=1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Using Eqs. (2.30), (2.36), and (2.37), for $k\in\overline{\mathbf{C}^+}\setminus\{0\}$, we obtain

$$|Z_l'(k,x)|\leqslant\frac{M}{N}\,e^{(1/|k|)\int_{-\infty}^{\infty} dz|\Theta(z)|}\int_{-\infty}^{\infty} d\xi|\Theta(\xi)|,$$

$$|Z_r'(k,x)|\leqslant\frac{M}{N}\,e^{(1/|k|)\int_{-\infty}^{\infty} dz|\Theta(z)|}\int_{-\infty}^{\infty} d\xi|\Theta(\xi)|,$$

where M and N are the constants defined in Eq. (2.21) and $\Theta(x)$ is the function defined in Eq. (2.35). Hence, by the Weierstrass theorem, both $Z_l'(k,x)$ and $Z_r'(k,x)$ have continuous extensions to $k\in\overline{\mathbf{C}^+}\setminus\{0\}$ which are analytic on $\mathbf{C}^+$, and $Z_l'(k,x)=O(1)$ and $Z_r'(k,x)=O(1)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$.

Since $Z_l'(k,+\infty)=0$, $Z_l(k,+\infty)=1$, $Z_r'(k,-\infty)=0$, and $Z_r(k,-\infty)=1$, using

$$Y_l(k,x)Z_l(k,x)=e^{ikx-ikA_+}+o(1), \quad x\to+\infty,$$

$$Y_r(k,x)Z_r(k,x)=e^{-ikx-ikA_-}+o(1), \quad x\to-\infty,$$

we see that the Jost solutions defined in Eq. (2.9) are given by

$$f_l(k,x)=e^{ikA}+Y_l(k,x)Z_l(k,x), \tag{2.38}$$

$$f_r(k,x)=e^{ikA}-Y_r(k,x)Z_r(k,x), \tag{2.39}$$

where

$$A_\pm=\pm\int_0^{\pm\infty} dz\left[1-\sqrt{\frac{h(z)}{a(z)}}\right]. \tag{2.40}$$

Hence, from Eq. (2.12) we have

$$m_l(k,x)=\frac{1}{\sqrt[4]{a(x)h(x)}}\, e^{ik\int_x^\infty[1-\sqrt{h/a}]}Z_l(k,x), \tag{2.41}$$

$$m_r(k,x)=\frac{1}{\sqrt[4]{a(x)h(x)}}\, e^{ik\int_{-\infty}^x[1-\sqrt{h/a}]}Z_r(k,x). \tag{2.42}$$

Since $m_l(k,x)$ and $m_r(k,x)$ are continuous in k also at $k=0$, it follows that $Z_l(k,x)$ and $Z_r(k,x)$ are also continuous in k at $k=0$ in $\overline{\mathbf{C}^+}$.

III. SCATTERING MATRIX

In this section we show that the scattering matrix $S(k)$ is unitary and continuous for $k\in\mathbf{R}$ and study its asymptotics for small and large k.

From Eqs. (2.1), (2.2), (2.6), (2.7), and (2.8) we obtain

$$\frac{1}{T_l(k)}=e^{ik\int_{-\infty}^\infty[1-(1/a)]}-\frac{1}{2ik}\int_{-\infty}^\infty dz\, e^{ik\int_{-\infty}^z[1-(1/a)]}F(k,z)m_l(k,z), \tag{3.1}$$

$$\frac{1}{T_r(k)}=e^{ik\int_{-\infty}^\infty[1-(1/a)]}-\frac{1}{2ik}\int_{-\infty}^\infty dz\, e^{ik\int_z^\infty[1-(1/a)]}F(k,z)m_r(k,z),$$

$$\frac{L(k)}{T_l(k)}=\frac{1}{2ik}\int_{-\infty}^\infty dz\, e^{2ikz}e^{-ik\int_{-\infty}^z[1-(1/a)]}F(k,z)m_l(k,z), \tag{3.2}$$

$$\frac{R(k)}{T_r(k)}=\frac{1}{2ik}\int_{-\infty}^\infty dz\, e^{-2ikz}e^{ik\int_z^\infty[1-(1/a)]}F(k,z)m_r(k,z), \tag{3.3}$$

where $F(k,x)$ is the quantity defined in Eq. (2.5).

Let $[f;g]=fg'-f'g$ denote the Wronskian of f and g; it can be shown that for any two solutions f and g of Eq. (1.1), $a(x)[f;g]$ is independent of x. Using Eqs. (2.1) and (2.2) from $a(x)[\psi_l(k,x);\psi_r(k,x)]$ we have

$$a(x)[\psi_l(k,x);\psi_r(k,x)]=-2ikT_l(k)=-2ikT_r(k),$$

and hence $T_l(k)=T_r(k)$; this common value will be denoted by $T(k)$. From $a(x)[\psi_l(-k,x);\psi_l(k,x)]$ we obtain

$$T(k)T(-k)+L(k)L(-k)=1,\quad k\in\mathbf{R},$$

from $a(x)[\psi_r(-k,x);\psi_r(k,x)]$ we obtain

$$T(k)T(-k)+R(k)R(-k)=1, \quad k\in\mathbf{R},$$

and from $a(x)[\psi_l(k,x);\psi_r(-k,x)]$ we obtain

$$T(k)R(-k)+L(k)T(-k)=0, \quad k\in\mathbf{R},$$

and hence the scattering matrix $S(k)$ is unitary and

$$S(-k)^t=\overline{S(k)^t}=S(k)^{-1}, \quad k\in\mathbf{R},$$

where $S(k)^t$ is the transpose and $S(k)^{-1}$ is the inverse of the matrix $S(k)$. Consequently, the transmission and reflection coefficients cannot exceed one in absolute value for $k\in\mathbf{R}$. From Eqs. (2.13) and (2.14) we have

$$a(x)[m_r(k,x);m_l(k,x)]=-2ika(x)m_r(k,x)m_l(k,x)+\frac{2ik}{T(k)}. \tag{3.4}$$

Since $m_l(k,x)$, $m_r(k,x)$, $m_l'(k,x)$, and $m_r'(k,x)$ are continuous in k for $k\in\overline{\mathbf{C}^+}$ and analytic in k for $k\in\mathbf{C}^+$, it follows from Eq. (3.4) that $k/T(k)$ is continuous in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$ and that for $k\in\mathbf{R}\setminus\{0\}$ we have $T(k)\neq 0$ and $|R(k)|=|L(k)|<1$.

Let us now study the small k asymptotics of $S(k)$. There are two cases to consider; namely, the case $\int_{-\infty}^{\infty} dz\, Q(z)m_l(0,z)=0$, which is the exceptional case, and the case $\int_{-\infty}^{\infty} dz\, Q(z)m_l(0,z)\neq 0$, which is the generic case. In the exceptional case $T(0)\neq 0$. From Eq. (2.5), we have $F(0,x)=Q(x)$, and thus in the generic case, as $k\to 0$ from $\overline{\mathbf{C}^+}$, using Eqs. (3.1), (3.2), and (3.3) we obtain

$$T(k)=\frac{-2ik}{\int_{-\infty}^{\infty} dz\, Q(z)m_l(0,z)}+o(k), \quad k\in\overline{\mathbf{C}^+}.$$

$$\frac{2ikL(k)}{T(k)}=\int_{-\infty}^{\infty} dz\, Q(z)m_l(0,z)+o(1),$$

$$\frac{2ikR(k)}{T(k)}=\int_{-\infty}^{\infty} dz\, Q(z)m_r(0,z)+o(1),$$

and hence as $k\to 0$, in the generic case, we have

$$L(k)=-1+O(k), \quad k\in\mathbf{R},$$

$$R(k)=-1+O(k), \quad k\in\mathbf{R}.$$

Now let us study the large k asymptotics of the scattering matrix. Using Eqs. (2.33) and (2.34), from Eqs. (2.38) and (2.39) as $x\to\pm\infty$, we obtain

$$\frac{1}{T(k)}=e^{ikA}\left[1+\frac{1}{2ik}\int_{-\infty}^{\infty} dz\, \Theta(z)Z_l(k,z)\right], \tag{3.5}$$

$$\frac{1}{T(k)}=e^{ikA}\left[1+\frac{1}{2ik}\int_{-\infty}^{\infty} dz\, \Theta(z)Z_r(k,z)\right], \tag{3.6}$$

$$\frac{L(k)}{T(k)}=-\frac{1}{2ik}e^{ik(A_{+}-A_{-})}\int_{-\infty}^{\infty}dz\,\Theta(z)Z_{l}(k,z)e^{2ikz-2ik\int_{0}^{z}[1-\sqrt{h/a}]},\tag{3.7}$$

$$\frac{R(k)}{T(k)}=-\frac{1}{2ik}e^{ik(A_{-}-A_{+})}\int_{-\infty}^{\infty}dz\,\Theta(z)Z_{r}(k,z)e^{-2ikz+2ik\int_{0}^{z}[1-\sqrt{h/a}]},\tag{3.8}$$

where

$$A=A_{-}+A_{+}=\int_{-\infty}^{\infty}[1-\sqrt{h/a}],\tag{3.9}$$

with $A_{\pm}$ being the constants defined in Eq. (2.40) and $\Theta(x)$ is the function in Eq. (2.35). Note that it is possible to obtain the small k behavior of the scattering matrix also from Eqs. (3.5)–(3.8).

From Eqs. (3.5)–(3.8) as $|k|\to\infty$ we obtain

$$T(k)=e^{-ikA}\left[1-\frac{1}{2ik}\int_{-\infty}^{\infty}dz\,\Theta(z)+O(1/k^{2})\right],\quad k\in\overline{\mathbf{C}^{+}},\tag{3.10}$$

$$L(k)=-\frac{1}{2ik}e^{-2ikA_{-}}\int_{-\infty}^{\infty}dz\,\Theta(z)e^{2ikz-2ik\int_{0}^{z}[1-\sqrt{h/a}]}+O(1/k^{2}),\quad k\in\mathbf{R},$$

$$R(k)=-\frac{1}{2ik}e^{-2ikA_{+}}\int_{-\infty}^{\infty}dz\,\Theta(z)e^{-2ikz+2ik\int_{0}^{z}[1-\sqrt{h/a}]}+O(1/k^{2}),\quad k\in\mathbf{R}.$$

IV. RIEMANN–HILBERT PROBLEM

In this section we formulate and solve a Riemann–Hilbert problem that will help us to solve the three inverse scattering problems for Eq. (1.1). The scattering data consist of the bound state information, one of the reflection coefficients, and the delay time caused by the nonhomogeneity when the signal travels from a fixed point to either of $\pm\infty$. Since there is no loss of generality in choosing the origin as that fixed point, such a delay time can be specified by giving either A_{+} or A_{-} defined in Eq. (2.40).

When $h(x)\equiv 1$ in Eq. (1.1), we obtain the differential equation

$$\frac{d}{dx}\left[a(x)\frac{d\psi^{[0]}(k,x)}{dx}\right]+k^{2}\psi^{[0]}(k,x)=Q(x)\psi^{[0]}(k,x).\tag{4.1}$$

Let $m_{l}^{[0]}(k,x)$ and $m_{r}^{[0]}(k,x)$ denote the Faddeev functions of Eq. (4.1), i.e., let

$$m_{l}^{[0]}(k,x)=\frac{1}{T_{0}(k)}e^{-ikx}\psi_{l}^{[0]}(k,x)\quad\text{and}\quad m_{r}^{[0]}(k,x)=\frac{1}{T_{0}(k)}e^{ikx}\psi_{r}^{[0]}(k,x),$$

where the scattering matrix for Eq. (4.1) is denoted by

$$S_{0}(k)=\begin{bmatrix}T_{0}(k) & R_{0}(k)\\ L_{0}(k) & T_{0}(k)\end{bmatrix},$$

and $\psi_{l}^{[0]}(k,x)$ and $\psi_{r}^{[0]}(k,x)$ are the physical solutions of Eq. (4.1). Then $m_{l}^{[0]}(k,x)$ and $m_{r}^{[0]}(k,x)$ satisfy Eqs. (2.13) and (2.14), respectively, in case $h(x)\equiv 1$. When we solve the inverse scattering problems, we will use the fact that the Faddeev functions of Eqs. (1.1) and

(4.1) satisfy $m_l(0,x)=m_l^{[0]}(0,x)$ and $m_r(0,x)=m_r^{[0]}(0,x)$ because these functions satisfy $[a(x)\eta']'=Q(x)\eta$ with the boundary conditions $\eta(\pm\infty)=1$ and $\eta'(\pm\infty)=0$, where the plus sign is used for the Faddeev functions from the left identified with the subscript l and the minus sign is used for those from the right identified with the subscript r, respectively.

Since k appears as k^2 in Eq. (1.1), $\psi_l(-k,x)$ and $\psi_r(-k,x)$ are also solutions of Eq. (1.1) whenever $\psi_l(k,x)$ and $\psi_r(k,x)$ are the physical solutions. Using Eqs. (2.1) and (2.2) it follows that the vectors

$$\psi(\pm k,x)=\begin{bmatrix}\psi_l(\pm k,x)\\ \psi_r(\pm k,x)\end{bmatrix}$$

satisfy

$$\psi(-k,x)=S(-k)^t q\psi(k,x), \quad k\in\mathbf{R}, \tag{4.2}$$

where $q=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Letting

$$Z(k,x)=\begin{bmatrix}Z_l(k,x)\\ Z_r(k,x)\end{bmatrix},$$

where $Z_l(k,x)$ and $Z_r(k,x)$ are the functions defined in Eqs. (2.33) and (2.34), respectively, we can write Eq. (4.2) as

$$Z(-k,x)=\Lambda(k,x)qZ(k,x), \quad k\in\mathbf{R}, \tag{4.3}$$

with

$$\Lambda(k,x)=\begin{bmatrix}T(k)e^{iAk} & -R(k)e^{2iky+2ikA_+}\\ -L(k)e^{-2iky+2ikA_-} & T(k)e^{iAk}\end{bmatrix}, \tag{4.4}$$

where A is the constant defined in Eq. (3.9) and y is the quantity in Eq. (1.2). When the scattering matrix $S(k)$ is given, A can be obtained from $T(k)$ as seen from Eq. (3.10).

Let $\hat{1}=\begin{bmatrix}1\\1\end{bmatrix}$. As shown in Sec. II, the vector $Z(k,x)$ is continuous in $k\in\overline{\mathbf{C}^+}$, has an analytic extension in k to $\mathbf{C}^+$ for each x, and $Z(k,x)-\hat{1}=O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Hence $Z(-k,x)$ is continuous in $k\in\overline{\mathbf{C}^-}$, has an analytic extension in k to $\mathbf{C}^-$ for each x, and $Z(-k,x)-\hat{1}=O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^-}$. Hence, when the matrix $\Lambda(k,x)$ is known, solving Eq. (4.3) for $Z(k,x)$ constitutes a Riemann–Hilbert problem. There are various methods to solve this Riemann–Hilbert problem such as the Marchenko method, Gel'fand–Levitan method, and the Wiener–Hopf factorization method. When there are bound states, the normalization constant for each bound state must also be specified as a part of the scattering data. The scattering data consists of one of the reflection coefficients, one of $A_\pm$, the bound state energies, and the normalization constants for each bound state.[7] The solution of Eq. (4.3) can be found in Refs. 6 and 7.

V. INVERSE PROBLEMS

In this section we give the solutions of the three inverse scattering problems. Note that if $h(x)=a(x)$ on either half line, $A_\pm$ can be obtained from $S(k)$. This is because, as seen from Eq. (3.9), if one of $A_\pm$ vanishes, the other must be equal to A, and A is known when $S(k)$ is given.

The first inverse scattering problem is to recover $Q(x)$ when the scattering data, $a(x)$, and $h(x)$ are known. This inverse problem is solved as follows. When the scattering data are known, one can obtain the matrix $\Lambda(k,y)$ given in Eq. (4.4). Then the Riemann–Hilbert problem (4.3) is solved,[6] and $V(y)$ given in Eq. (1.4) is recovered. Knowing $a(x)$ and $h(x)$, one obtains y in terms of x in Eq. (1.2) and from Eq. (1.4) one then recovers $Q(x)$. Thus, the first inverse problem is solved.

The second inverse scattering problem is to recover $h(x)$ when the scattering data, $a(x)$, and $Q(x)$ are known; this inverse problem is solved as follows. After solving the Riemann–Hilbert problem (4.3), from Eqs. (2.41) and (2.44), we have

$$Z_l(0,x)=\sqrt[4]{a(x)h(x)}\,m_l(0,x) \tag{5.1}$$

$$Z_r(0,x)=\sqrt[4]{a(x)h(x)}\,m_r(0,x). \tag{5.2}$$

Using $m_l(0,x)=m_l^{[0]}(0,x)$ and $m_r(0,x)=m_r^{[0]}(0,x)$, from Eqs. (5.1) and (5.2) we then obtain

$$Z_l(0,x)=\sqrt[4]{a(x)h(x)}\,m_l^{[0]}(0,x) \tag{5.3}$$

$$Z_r(0,x)=\sqrt[4]{a(x)h(x)}\,m_r^{[0]}(0,x), \tag{5.4}$$

where $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ are the Faddeev functions of Eq. (4.1). Note that as seen from Eq. (4.4), x enters $\Lambda(k,x)$ only in the form $y=\int_0^x\sqrt{h/a}$, and as a result the solution $Z(k,x)$ of Eq. (4.3) contains x only through y. Thus, both $Z_l(0,x)$ and $Z_r(0,x)$ are functions of y only; let us use the notation $\tilde{Z}_l(y)=Z_l(0,x)$ and $\tilde{Z}_r(y)=Z_r(0,x)$. Furthermore, knowing $a(x)$ and $Q(x)$, one can obtain $m_l^{[0]}(0,x)$ and $m_r^{[0]}(0,x)$ and these two are functions of x only. Therefore, using $dy/dx=\sqrt{h(x)/a(x)}$, we see that Eqs. (5.3) and (5.4) are first-order separable ordinary differential equations, and thus we can write them in the separated form as

$$\frac{dy}{\tilde{Z}_l(y)^2}=\frac{dx}{a(x)m_l^{[0]}(0,x)^2}, \tag{5.5}$$

$$\frac{dy}{\tilde{Z}_r(y)^2}=\frac{dx}{a(x)m_r^{[0]}(0,x)^2}. \tag{5.6}$$

Using the initial condition $y=0$ when $x=0$, we can obtain y in terms of x by integrating either Eqs. (5.5) or (5.6). Once y is obtained in terms of x, replacing it in $\tilde{Z}_l(y)=Z_l(0,x)$ or in $\tilde{Z}_r(y)=Z_r(0,x)$ by its equivalent in terms of x, using

$$h(x)=a(x)\left(\frac{dy}{dx}\right)^2=\frac{Z_l(0,x)^4}{a(x)m_l^{[0]}(0,x)^4}=\frac{Z_r(0,x)^4}{a(x)m_r^{[0]}(0,x)^4},$$

we obtain $h(x)$ in terms of x only. Hence, the second inverse problem is solved.

The third inverse scattering problem is, in case $Q(x)=0$, to recover $a(x)$ when $h(x)$ and the scattering data are given. This inverse scattering problem is solved as follows. Defining

$$P(y)=\sqrt[4]{a(x)h(x)}, \tag{5.7}$$

where y is the coordinate defined in Eq. (1.2), we can write Eq. (1.4) in case $Q(x)=0$ in the form

$$V(y)=\frac{1}{P(y)}\frac{d^2P(y)}{dy^2}. \tag{5.8}$$

Note that Eq. (5.8) is obtained by using

$$\frac{dy}{dx}=\frac{h(x)}{P(y)^2}. \tag{5.9}$$

Since we assume that $a(\pm\infty)=h(\pm\infty)=1$, we also need $P(\pm\infty)=1$. Thus, $P(y)$ can be obtained from the solution of the Schrödinger equation (1.3) with $k=0$ using the boundary conditions $P(\pm\infty)=1$. After obtaining $P(y)$, we can write Eq. (5.9) as a separable first-order differential equation

$$P(y)^2dy=h(x)dx. \tag{5.10}$$

Integrating Eq. (5.10) with the boundary condition $y=0$ when $x=0$, we obtain

$$\int_0^y dt\,P(t)^2=\int_0^x dz\,h(z). \tag{5.11}$$

Once y is obtained in terms of x from Eq. (5.11), from Eq. (5.7) we recover $a(x)$ as

$$a(x)=\frac{P(y)^4}{h(x)}. \tag{5.12}$$

VI. EXAMPLES

In this section we present three examples to illustrate the method described in the previous section.

Consider the scattering matrix

$$S(k)=\begin{bmatrix} \tau(k)e^{-ikA} & \rho(k)e^{-2ikA_+} \\ \ell(k)e^{-2ikA_-} & \tau(k)e^{-ikA} \end{bmatrix}, \tag{6.1}$$

where we have

$$\tau(k)=\frac{k+i}{k+2i},\quad \rho(k)=\frac{\sqrt{3}i}{k+2i},\quad \ell(k)=\frac{\sqrt{3}i}{k+2i}\frac{k+i}{k-i}. \tag{6.2}$$

The Riemann–Hilbert problem (4.3) can be solved explicitly as in Refs. 6 and 7. We obtain

$$Z_l(k,x)=1,\quad x\geqslant 0,$$

$$Z_r(k,x)=\frac{1}{\tau(k)}+\frac{\rho(k)}{\tau(k)}e^{2iky},\quad x\geqslant 0,$$

$$Z_l(k,x)=\frac{1}{\tau(k)}Z_r(-k,x)+\frac{\ell(k)}{\tau(k)}Z_r(k,x)e^{-2iky},\quad x\leqslant 0,$$

$$Z_r(k,x)=1+\frac{i}{k+i}\frac{2}{\sqrt{3}e^{-2y}-1},\quad x\leqslant 0.$$

Hence, $Z_r(0,x)=2+\sqrt{3}$ for $x\geqslant 0$ and $Z_r(0,x)=(\sqrt{3}+e^{2y})/(\sqrt{3}-e^{2y})$ for $x\leqslant 0$. The potential in Eq. (1.4) is then given by

$$V(y) = -2\sqrt{3}\delta(y) + \theta(-y)\,\frac{8\sqrt{3}e^{2y}}{(\sqrt{3}-e^{2y})^2},$$

where $\delta(y)$ is the delta function and $\theta(y)$ is the Heaviside function.

As an example for the inverse problem of the recovery of $Q(x)$ when $a(x)$ and $h(x)$ are known, let us use the scattering matrix specified in Eqs. (6.1) and (6.2) with

$$a(x) = \frac{1}{1+e^{-|x|}}\left(\frac{x^2+2}{x^2+1}\right)^2, \quad h(x) = (1+e^{-|x|})\left(\frac{x^2+2}{x^2+1}\right)^2.$$

Thus, we have $\sqrt[4]{ah} = (x^2+2)/(x^2+1)$, and $\sqrt{h/a} = 1 + e^{-|x|}$. Hence from Eq. (1.2) we obtain

$$y = x + \operatorname{sgn}(x)[1-e^{-|x|}], \tag{6.3}$$

where sgn denotes the sign function. Thus, using Eq. (6.3) we can write $V(y)$ in terms of x as

$$V(y) = -\sqrt{3}\delta(x) + \theta(-x)\,\frac{8\sqrt{3}e^{2(x-1+e^x)}}{[\sqrt{3}-e^{2(x-1+e^x)}]^2}.$$

Solving Eq. (1.4) for $Q(x)$, we obtain

$$Q(x) = hV + (ah)^{1/4}[a[(ah)^{-1/4}]']',$$

or equivalently

$$Q(x) = -8\sqrt{3}\delta(x) + \theta(-x)Q_-(x) + \frac{x^2+2}{x^2+1}\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2(1+e^{|x|})}\right],$$

where

$$Q_-(x) = (1+e^x)\left(\frac{x^2+2}{x^2+1}\right)^2 \frac{8\sqrt{3}e^{2(x-1+e^x)}}{[\sqrt{3}-e^{2(x-1+e^x)}]^2}.$$

As an example for the inverse problem of the recovery of $h(x)$ when $a(x)$ and $Q(x)$ are known, let us use the scattering matrix given in Eqs. (6.1) and (6.2) with

$$Q(x) = \frac{2\sqrt{3}+4}{\sqrt{3}}\delta(x) - \theta(x)\,\frac{2\sqrt{3}e^x+2}{3e^{2x}-\sqrt{3}e^x},$$

$$a(x) = \left(\frac{2\sqrt{3}+1+e^{2x}}{2\sqrt{3}+1-e^{2x}}\right)^2\theta(-x) + \frac{(\sqrt{3}e^x+1)^2}{3e^{2x}}\,\theta(x).$$

Note $a(x)$ is continuous at $x=0$ and $a(0) = \frac{1}{3}(\sqrt{3}+1)^2$. For these values of $a(x)$ and $Q(x)$, we have

$$m_r^{[0]}(0,x) = 1, \quad x \leqslant 0,$$

$$m_r^{[0]}(0,x) = (2+\sqrt{3})\,\frac{\sqrt{3}e^x-1}{\sqrt{3}e^x+1}, \quad x \geqslant 0.$$

In this example, for $x \geqslant 0$, Eq. (5.6) becomes

$$dy = dx\,\frac{3e^{2x}}{(\sqrt{3}e^{x}-1)^2}. \tag{6.4}$$

Integrating Eq. (6.4) with the initial condition $y=0$ when $x=0$, we obtain

$$y = \ln\left(\frac{\sqrt{3}e^{x}-1}{\sqrt{3}-1}\right) + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}e^{x}-1}, \quad x \geqslant 0,$$

$$h(x) = \frac{3e^{2x}(\sqrt{3}e^{x}+1)^2}{(\sqrt{3}e^{x}-1)^4}, \quad x \geqslant 0.$$

Note that $A_{+} = \ln((\sqrt{3}-1)/\sqrt{3}) - 1/(\sqrt{3}-1)$. On the other hand, for $x \leqslant 0$, Eq. (2.32) becomes

$$dy\left(\frac{\sqrt{3}-e^{2y}}{\sqrt{3}+e^{2y}}\right)^2 = dx\left(\frac{2\sqrt{3}+1-e^{2x}}{2\sqrt{3}+1+e^{2x}}\right)^2. \tag{6.5}$$

After using the initial condition $y=0$ when $x=0$, we can solve Eq. (6.5) and obtain

$$y + \frac{2\sqrt{3}}{\sqrt{3}+e^{2y}} - \frac{2\sqrt{3}}{\sqrt{3}+1} = x + \frac{2(2\sqrt{3}+1)}{2\sqrt{3}+1+e^{2x}} - \frac{2\sqrt{3}+1}{\sqrt{3}+1}, \quad x \leqslant 0. \tag{6.6}$$

We then have

$$h(x) = \left(\frac{\sqrt{3}+e^{2y}}{\sqrt{3}-e^{2y}}\right)^4 \left(\frac{2\sqrt{3}+1-e^{2x}}{2\sqrt{3}+1+e^{2x}}\right)^2, \quad x \leqslant 0. \tag{6.7}$$

In Eq. (6.6) y can be computed numerically in terms of x to any desired accuracy. We can then also obtain $h(x)$ in Eq. (6.7) to any desired accuracy.

As an example to the third inverse problem of the recovery of $a(x)$ when $Q(x)=0$ and $h(x)$ and the scattering data are given, let us use the scattering matrix corresponding to the potential

$$V(y) = \frac{6y^2-2}{(y^2+1)^2(y^2+2)},$$

and assume that $A=0$. Let us also assume that $A_{-}=0$ and thus we have $A_{+}=0$. Let us use

$$h(x) = \left(\frac{x^2+2}{x^2+1}\right)^2.$$

Then solving Eq. (5.8) we obtain

$$P(y) = \frac{y^2+2}{y^2+1}.$$

From Eq. (5.11) we then obtain $y=x$. Thus, from Eq. (5.12) we obtain

$$a(x) = \left(\frac{x^2+2}{x^2+1}\right)^2.$$

ACKNOWLEDGMENT

The research leading to this article was supported in part by the National Science Foundation under Grant Nos. DMS 9096268 and DMS 9217627.

[1] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[2] L. D. Faddeev, Am. Math. Soc. Transl. **2**, 139 (1964) [Tr. Mat. Inst. Stekl. **73**, 314 (1964) (Russian)].
[3] R. G. Newton, J. Math. Phys. **21**, 493 (1980).
[4] V. A. Marchenko, *Sturm–Liouville Operators and Applications* (Birkhäuser, Basel, 1986).
[5] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer-Verlag, New York, 1989).
[6] T. Aktosun, M. Klaus, and C. van der Mee, J. Math. Phys. **33**, 1717 (1992).
[7] T. Aktosun, M. Klaus, and C. van der Mee, J. Math. Phys. **33**, 1395 (1992).