A factorization of the scattering matrix for the Schrödinger equation and for the wave equation in one dimension

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For the one-dimensional Schrödinger equation with a potential decaying at both ends of the real axis, the scattering matrix of the potential is given explicitly in terms of the scattering matrices corresponding to the fragments of this potential. A similar result also holds for the wave equation in a nonhomogeneous, nondispersive medium, where the wave speed has the same asymptotics at both ends of the real line.

I. INTRODUCTION

Consider the one-dimensional Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=V(x)\psi(k,x), \tag{1.1}$$

where $x\in\mathbf{R}$ is the space coordinate, $k^2\in\mathbf{R}$ is energy, and $V(x)$ is a potential that vanishes as $x\to\pm\infty$ in such a way that a scattering matrix for this potential is meaningful. Note that throughout the paper we use the prime to denote the derivative with respect to x. The physical solutions ψ_l from the left and ψ_r from the right of (1.1) satisfy the boundary conditions

$$\psi_l(k,x)=\begin{cases}T(k)e^{ikx}+o(1), & x\to+\infty\\ e^{ikx}+L(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases}$$

and

$$\psi_r(k,x)=\begin{cases}e^{-ikx}+R(k)e^{ikx}+o(1), & x\to+\infty\\ T(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases}$$

where $T(k)$ is the transmission coefficient, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively. The scattering matrix $S(k)$ is defined as

$$S(k)=\begin{bmatrix}T(k) & R(k)\\ L(k) & T(k)\end{bmatrix}. \tag{1.2}$$

Let us partition the real axis as $\mathbf{R}=\cup_{n=1}^{N+2}(x_{n-1},x_n)$, where $x_0=-\infty$ and $x_{N+2}=+\infty$. Thus we can write the potential $V(x)$ appearing in (1.1) as

$$V(x)=\sum_{n=0}^{N+1}V_{n,n+1}(x),$$

where we have defined

$$V_{n,n+1}(x)=V(x)\chi_{(x_n,x_{n+1})}, \tag{1.3}$$

with χ_I being the characteristic function of the interval I; i.e., $\chi_I(x)=1$ if $x\in I$ and $\chi_I(x)=0$ if $x\notin I$. Let $S_{n,n+1}(k)$ be the scattering matrix corresponding to the potential $V_{n,n+1}(x)$. We then have

$$S_{n,n+1}(k)=\begin{bmatrix}T_{n,n+1}(k) & R_{n,n+1}(k)\\ L_{n,n+1}(k) & T_{n,n+1}(k)\end{bmatrix}, \tag{1.4}$$

where $T_{n,n+1}(k)$ as the corresponding transmission coefficient, and $L_{n,n+1}(k)$ and $R_{n,n+1}(k)$ are the corresponding reflection coefficients from the left and from the right, respectively.

The main result in this paper is the relation between $S(k)$ and the matrices $S_{n,n+1}(k)$ for $n=1,...,N$; that is, we show that the scattering matrix from a potential can be expressed in a rather simple way in terms of the scattering matrices from the fragments of that potential. To be precise, we have

$$\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*}\end{bmatrix}=\prod_{n=1}^{N}\begin{bmatrix}\dfrac{1}{T_{n,n+1}(k)} & -\dfrac{R_{n,n+1}(k)}{T_{n,n+1}(k)}\\ \dfrac{L_{n,n+1}(k)}{T_{n,n+1}(k)} & \dfrac{1}{T_{n,n+1}(k)^*}\end{bmatrix}, \tag{1.5}$$

where the asterisk denotes the complex conjugation. The matrix product in (1.5) is an ordered (noncommutative) product. Thus, we have a decomposition or a factorization of a scattering matrix that corresponds to a potential vanishing as $x\to\pm\infty$. The proof of (1.5) is rather elementary and is given in Sec. II. In Sec. III we show that (1.5) also holds for the wave equation in nonhomogeneous media, where the wave speed and the restoring force depend on position. In Sec. IV we discuss some corollaries of the decomposition formula given in (1.5).

II. DECOMPOSITION OF THE SCATTERING MATRIX

In this section we give an elementary proof of (1.5). Let us define

$$\Lambda(k)=\begin{bmatrix}1 & 0\\ L(k) & T(k)\end{bmatrix}\begin{bmatrix}T(k) & R(k)\\ 0 & 1\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*}\end{bmatrix}. \quad (2.1)$$

From the properties[1–3] of the scattering matrix, it follows that, when $V(x)$ is real, we have $\det\Lambda(k)=1$ and $\Lambda(k)^*=q\Lambda(k)q$, where

$$q=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Similarly, let us define

$$\Lambda_{n,n+1}=\begin{bmatrix}\dfrac{1}{T_{n,n+1}(k)} & -\dfrac{R_{n,n+1}(k)}{T_{n,n+1}(k)}\\ \dfrac{L_{n,n+1}(k)}{T_{n,n+1}(k)} & \dfrac{1}{T_{n,n+1}(k)^*}\end{bmatrix}. \quad (2.2)$$

Note that we can write (1.5) as

$$\Lambda(k)=\prod_{j=1}^{N}\Lambda_{j,j+1}(k). \quad (2.3)$$

Theorem 2.1: Let $V(x)$ be a potential that vanishes as $x\rightarrow\pm\infty$ and let $S(k)$ given in (1.2) be the scattering matrix for the potential $V(x)$. Let $S_{n,n+1}$ given in (1.4) be the scattering matrix for the potential $V_{n,n+1}(x)$ defined in (1.3). Let $\Lambda(k)$ be the matrix given in (2.1) associated with $S(k)$ and $\Lambda_{n,n+1}(k)$ be the matrix given in (2.2) associated with $S_{n,n+1}(k)$. Then (2.3) holds, or equivalently we have (1.5).

Proof: Let $\phi_{j,j+1}(k,x)$ and $\xi_{j,j+1}(k,x)$ be two linearly independent solutions of (1.1) in the interval $[x_j,x_{j+1}]$. The physical solutions of (1.1) in this interval are then given by

$$\psi_l(k,x)=a_{j,j+1}(k)\phi_{j,j+1}(k,x)+b_{j,j+1}(k)\xi_{j,j+1}(k,x),$$

$$\psi_r(k,x)=c_{j,j+1}(k)\phi_{j,j+1}(k,x)+d_{j,j+1}(k)\xi_{j,j+1}(k,x),$$

where $a_{j,j+1}(k)$, $b_{j,j+1}(k)$, $c_{j,j+1}(k)$, and $d_{j,j+1}(k)$ are the coefficients to be determined. Let us define

$$\Gamma_{j,j+1}(k,x)=\begin{bmatrix}\phi_{j,j+1}(k,x) & \xi_{j,j+1}(k,x)\\ \phi'_{j,j+1}(k,x) & \xi'_{j,j+1}(k,x)\end{bmatrix}, \quad (2.4)$$

$$M(k,x)=\begin{bmatrix}e^{ikx} & e^{-ikx}\\ ike^{ikx} & -ike^{-ikx}\end{bmatrix}, \quad (2.5)$$

$$A_{j,j+1}(k)=\begin{bmatrix}a_{j,j+1}(k) & c_{j,j+1}(k)\\ b_{j,j+1}(k) & d_{j,j+1}(k)\end{bmatrix}.$$

Then, from the continuity of ψ_l, ψ_r, ψ'_l, and ψ'_r, at x_j and x_{j+1}, we obtain the matrix equations

$$\Gamma_{j-1,j}(k,x_j)A_{j-1,j}(k)=\Gamma_{j,j+1}(k,x_j)A_{j,j+1}(k), \quad (2.6)$$

$$\Gamma_{j,j+1}(k,x_{j+1})A_{j,j+1}(k)=\Gamma_{j+1,j+2}(k,x_{j+1})A_{j+1,j+2}(k). \quad (2.7)$$

Thus, from (2.6) and (2.7) we obtain

$$A_{0,1}(k)=\left[\prod_{j=1}^{N+1}\Gamma_{j-1,j}(k,x_j)^{-1}\Gamma_{j,j+1}(k,x_j)\right]\times A_{N+1,N+2}(k). \quad (2.8)$$

If $V(x)$ has compact support, we can choose x_1 to the left and x_{N+1} to the right of the compact support. If $V(x)$ does not have compact support, we can let $x_1\rightarrow-\infty$ and $x_{N+1}\rightarrow+\infty$, and use $\phi_{0,1}(k,x)=\phi_{N+1,N+2}(k,x)=e^{ikx}$ and $\xi_{0,1}(k,x)=\xi_{N+1,N+2}(k,x)=e^{-ikx}$, in which case we have

$$A_{0,1}(k)=\begin{bmatrix}1 & 0\\ L(k) & T(k)\end{bmatrix}, \quad (2.9)$$

$$A_{N+1,N+2}(k)=\begin{bmatrix}T(k) & R(k)\\ 0 & 1\end{bmatrix}. \quad (2.10)$$

Using (2.1), (2.8), (2.9), and (2.10), we obtain

$$\Lambda(k)=\prod_{j=1}^{N+1}\Gamma_{j-1,j}(k,x_j)^{-1}\Gamma_{j,j+1}(k,x_j). \quad (2.11)$$

For the potential $V_{n,n+1}(x)$ given in (1.3), we have

$$\Gamma_{0,n}(k,x)=M(k,x),$$

$$\Gamma_{n+1,N+2}(k,x)=M(k,x),$$

where $M(k,x)$ is the matrix defined in (2.5). Inserting the matrices $M(k,x_n)M(k,x_n)^{-1}$ in (2.11) in the appropriate places and using

$$\Lambda_{j,j+1}(k)$$

$$=M(k,x_j)^{-1}\Gamma_{j,j+1}(k,x_j)\Gamma_{j,j+1}(k,x_{j+1})^{-1}M(k,x_{j+1}),$$

we see that (2.11) gives us (2.3). ■

III. WAVE EQUATION IN NONHOMOGENEOUS MEDIA

The propagation of waves in a one-dimensional nonhomogeneous, nondispersive medium is governed by

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2}=Q(x)u, \tag{3.1}$$

where $c(x)$ is the wave speed and $Q(x)$ is the restoring force. We assume that $Q(x)$ vanishes as $x\to\pm\infty$, $c(x)$ is positive and bounded above, and $c(x)\to c_\pm$, where $c_\pm$ are positive constants. The Fourier transformation from the time t domain into the frequency k domain changes (3.1) into the Schrödinger equation with an energy-dependent potential

$$\frac{d^2\psi(k,x)}{dx^2}+k^2H(x)^2\psi(k,x)=Q(x)\psi(k,x), \tag{3.2}$$

where $x\in\mathbf{R}$ is the space coordinate, k^2 is the energy, and $H(x)=1/c(x)$ is the so-called slowness. There are two solutions of (3.2), which we will call the physical solutions[4] ψ_l from the left and ψ_r from the right, which satisfy

$$\psi_l(k,x)=\begin{cases}\dfrac{T(k)}{\sqrt{H_+}}e^{ikH_+x}+o(1), & x\to+\infty\\ \dfrac{1}{\sqrt{H_-}}[e^{ikH_-x}+L(k)e^{-ikH_-x}]+o(1), & x\to-\infty,\end{cases}$$

$$\psi_r(k,x)=\begin{cases}\dfrac{1}{\sqrt{H_+}}[e^{-ikH_+x}+R(k)e^{ikx}]+o(1), & x\to+\infty\\ \dfrac{T(k)}{\sqrt{H_-}}e^{-ikH_-x}+o(1), & x\to-\infty,\end{cases}$$

where $T(k)$ is the transmission coefficient, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively, and $H_\pm=\lim_{x\to\pm\infty}H(x)$. The scattering matrix $S(k)$ is defined as in (1.2).

Let us use the same partition of the real axis defined in the Introduction. We would like to express the scattering matrix $S(k)$ corresponding to the pair $\{H(x),Q(x)\}$ given in (3.2) in terms of the scattering matrices $S_{n,n+1}(k)$ defined in (1.4) corresponding to $\{H_{n,n+1}(x),Q_{n,n+1}(x)\}$, where we have defined

$$Q_{n,n+1}(x)=Q(x)\chi_{(x_n,x_{n+1})}, \tag{3.3}$$

$$H_{n,n+1}(x)=H_-\chi_{(-\infty,x_n)}+H(x)\chi_{(x_n,x_{n+1})}+H_+\chi_{(x_{n+1},+\infty)}, \tag{3.4}$$

for $n=1,...,N$. As in the Introduction, we assume $x_0=-\infty$ and $x_{N+2}=+\infty$. Associated with $S(k)$ we define

$$\Lambda(k)=\sqrt{\frac{H_+}{H_-}}\begin{bmatrix}\dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)}\\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*}\end{bmatrix},$$

and similarly, associated with $S_{n,n+1}(k)$ we define

$$\Lambda_{n,n+1}=\sqrt{\frac{H_+}{H_-}}\begin{bmatrix}\dfrac{1}{T_{n,n+1}(k)} & -\dfrac{R_{n,n+1}(k)}{T_{n,n+1}(k)}\\ \dfrac{L_{n,n+1}(k)}{T_{n,n+1}(k)} & \dfrac{1}{T_{n,n+1}(k)^*}\end{bmatrix}.$$

When $H_+=H_-$, the following theorem gives us the analog of Theorem 2.1.

Theorem 3.1: Let $H(x)$ be the slowness in (3.2) such that $H(x)>0$ and is bounded above, and assume that $\lim_{x\to\pm\infty}H(x)$ are both equal to the same constant and $Q(x)$ vanishes as $x\to\pm\infty$. Then (1.5) holds, where $S(k)$ given in (1.2) is the scattering matrix of the pair $\{H(x),Q(x)\}$ and $S_{n,n+1}$ given in (1.4) is the scattering matrix of the pair $\{H_{n,n+1}(x),Q_{n,n+1}(x)\}$ defined in (3.3) and (3.4).

Proof: The proof is similar to that of Theorem 2.1.■

IV. CONCLUSION

The decomposition given in (1.5) can be used to obtain many useful results, and in this section we mention a few examples. For brevity, we will confine our remarks to the case of the Schrödinger equation (1.1), but similar conclusions can easily be obtained in the case of the wave equation (3.2).

Example 4.1: Let $V(x)$ be a symmetric potential with the corresponding scattering matrix $S(k)$ given in (1.2). Let $\sigma(k)$ be the scattering matrix for the potential $V(x)\chi_{(0,+\infty)}$ defined by

$$\sigma(k)=\begin{bmatrix}\tau(k) & \rho(k)\\ l(k) & \tau(k)\end{bmatrix}, \tag{4.1}$$

where $\tau(k)$ is the transmission coefficient, and $l(k)$ and $\rho(k)$ are the reflection coefficients from the left and from the right, respectively. From (1.5), we obtain

$$\begin{bmatrix} \frac{1}{T(k)} & -\frac{R(k)}{T(k)} \\ \frac{L(k)}{T(k)} & \frac{1}{T(k)^*} \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau(k)} & -\frac{l(k)}{\tau(k)} \\ \frac{\rho(k)}{\tau(k)} & \frac{1}{\tau(k)^*} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\tau(k)} & -\frac{\rho(k)}{\tau(k)} \\ \frac{l(k)}{\tau(k)} & \frac{1}{\tau(k)^*} \end{bmatrix},$$

and hence we have

$$T(k)=\frac{\tau(k)^2}{1-l(k)^2},$$

$$L(k)=R(k)=\frac{\tau(k)}{\tau(k)^*}\frac{l(k)-l(k)^*}{1-l(k)^2}.$$

Example 4.2: It is known that when the coordinate axis is shifted by p; i.e., under the transformation $V(x)\to V(x+p)$, the scattering matrix is transformed[2] as $L(k)\to L(k)e^{-2ikp}$, $R(k)\to R(k)e^{2ikp}$, and $T(k)\to T(k)$. Let $v(x)$ be a potential that vanishes outside the interval $[a,a+p]$, where a is a real constant and p is a positive constant. Let $\sigma(k)$ defined in (4.1) be the scattering matrix corresponding to the potential $v(x)$. Then the scattering matrix $S(k)$ corresponding to the compactly supported, periodic potential $V(x)=\Sigma_{n=0}^{N}v(x-np)$ can be found from $\sigma(k)$ by using (1.5) as

$$\begin{bmatrix} \frac{1}{T(k)} & -\frac{R(k)}{T(k)} \\ \frac{L(k)}{T(k)} & \frac{1}{T(k)^*} \end{bmatrix} = \prod_{n=0}^{N} \begin{bmatrix} \frac{1}{\tau(k)} & -\frac{\rho(k)e^{-2iknp}}{\tau(k)} \\ \frac{l(k)e^{2iknp}}{\tau(k)} & \frac{1}{\tau(k)^*} \end{bmatrix}.$$

Example 4.3: Let $V(x)$ and $\tilde{V}(x)$ be two potentials that are equal when $x>a$ and different when $x<a$, for some real number a. Using $x_0=x_1=-\infty$, $x_2=a$, and $x_3=+\infty$, from (1.5) we obtain

$$\Lambda(k)\tilde{\Lambda}(k)^{-1}=\Lambda_{1,2}(k)\tilde{\Lambda}(k)_{1,2}^{-1}, \tag{4.2}$$

where $\Lambda(k)$ is the matrix defined in (2.1) associated with the scattering matrix $S(k)$ for the potential $V(x)$, and $\tilde{\Lambda}(k)$ is the corresponding matrix associated with the scattering matrix $\tilde{S}(k)$ for the potential $\tilde{V}(x)$; the matrix $\Lambda_{1,2}(k)$ is the matrix defined as in (2.2) that is related to the scattering matrix $S_{1,2}(k)$ for the potential $V(x)\chi_{(-\infty,a)}$ and the matrix $\tilde{\Lambda}_{1,2}(k)$ is the matrix defined as in (2.2) that is related to the scattering matrix $\tilde{S}_{1,2}(k)$ for the potential $\tilde{V}(x)\chi_{(-\infty,a)}$. Thus, (4.2) gives us, in a way, the relationship between the scattering matrices for the potentials $V(x)-\tilde{V}(x)$ and $[V(x)-\tilde{V}(x)]\chi_{(-\infty,a)}$, respectively.

Example 4.4: The inverse scattering problem for (1.1) consists of the recovery of $V(x)$ when $S(k)$ is known. We can use (1.5) in order to solve the inverse scattering problem by a layer-stripping method and obtain some estimates for the error in the potential in terms of the error in the scattering matrix. Let $\Lambda(k)$ be the matrix in (2.1) corresponding to the potential $V(x)$ with the scattering matrix $S(k)$ and let $\tilde{\Lambda}(k)$ be the matrix corresponding to the potential $\tilde{V}(x)$ with the scattering matrix $\tilde{S}(k)$. We then have

$$\Lambda(k)-\tilde{\Lambda}(k)=[\Lambda(k)\tilde{\Lambda}(k)^{-1}-\mathbf{I}]\tilde{\Lambda}(k),$$

where

$$\Lambda(k)\tilde{\Lambda}(k)^{-1}-\mathbf{I}=\begin{bmatrix} \frac{\tilde{T}-T}{T}+(\tilde{R}-R)\frac{\tilde{R}^*}{T\tilde{T}^*} & (\tilde{R}-R)\frac{1}{T\tilde{T}} \\ (\tilde{R}^*-R^*)\frac{1}{T^*\tilde{T}^*} & \frac{\tilde{T}^*-T^*}{T^*}+(\tilde{R}^*-R^*)\frac{\tilde{R}^*}{T^*\tilde{T}} \end{bmatrix},$$

or equivalently,

$$\Lambda(k)\tilde{\Lambda}(k)^{-1}-\mathbf{I}=\begin{bmatrix} \frac{T^*-\tilde{T}^*}{\tilde{T}^*}+(R^*-\tilde{R}^*)\frac{R}{T\tilde{T}^*} & (\tilde{R}-R)\frac{1}{T\tilde{T}} \\ (\tilde{R}^*-R^*)\frac{1}{T^*\tilde{T}^*} & \frac{T-\tilde{T}}{\tilde{T}}+(R-\tilde{R})\frac{R^*}{T^*\tilde{T}} \end{bmatrix}.$$

In order to recover $V(x)$ when the scattering matrix $S(k)$ is known, one can use (1.5) by approximating $S_{n,n+1}(k)$ by the scattering matrix $\widetilde{S}_{n,n+1}(k)$, corresponding to a known potential $\widetilde{V}_{n,n+1}(x)$ in such a way that $\Lambda_{n,n+1}(k)\widetilde{\Lambda}_{n,n+1}(k)^{-1}$ is as close as possible to the identity matrix **I**.

Finally, let us mention that in the special case of a compactly supported potential consisting of flat barriers, a decomposition similar to (2.8) is given in Ref. 5, where the authors used their decomposition formula in order to approximate the scattering matrix of a compactly supported, nonconstant, continuous potential by a compactly supported potential consisting of flat barriers. In the special case of a compactly supported potential consisting of flat barriers, we have $V_{j,j+1}(x)=E_{j,j+1}$ are constants, and in (2.4) one can use $\phi_{j,j+1}(k,x) = e^{ik_{j,j+1}x}$ and $\xi_{j,j+1}(k,x) = e^{-ik_{j,j+1}x}$, where $k_{j,j+1} = \sqrt{k^2-E_{j,j+1}}$. In this special case, one can even write (1.5) explicitly, by using in (2.11)

$$\Gamma_{n,n+1}(k,x_n)\Gamma_{n,n+1}(k,x_{n+1})^{-1}=\begin{bmatrix} \cos k_{n,n+1}(x_{n+1}-x_n) & -\dfrac{1}{k_{n,n+1}}\sin k_{n,n+1}(x_{n+1}-x_n) \\ k_{n,n+1}\sin k_{n,n+1}(x_{n+1}-x_n) & \cos k_{n,n+1}(x_{n+1}-x_n) \end{bmatrix}.$$

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