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# Exact solutions of the Schrödinger equation and the non-uniqueness of inverse scattering on the line

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**Abstract.** Certain exact solutions to the Schrödinger equation on the line are given for each positive integer $N$ such that the corresponding transmission coefficient behaves like $O(k^N)$ as $k\to 0$. Such solutions form a one-parameter family for each $N$, and all the potentials in each family cause the same scattering at all energies. In each family there are infinitely many potentials that do not support any bound states.

## 1. Introduction

The one-dimensional Schrödinger equation

$$\frac{d^2\Psi}{dx^2}(k,x)+k^2\Psi(k,x)=V(x)\Psi(k,x),$$

assuming the potential $V(x)\sim 0$ in some appropriate sense, has two scattering solutions from the 'left' and 'right' respectively, which satisfy the boundary conditions

$$\Psi_l(k,x)=T(k)\exp(ikx)+o(1) \qquad \text{as } x\to\infty$$

$$\Psi_r(k,x)=T(k)\exp(-ikx)+o(1) \qquad \text{as } x\to-\infty$$

where $T(k)$ is the transmission coefficient and the subscripts $l$ and r are used for 'left' and 'right' respectively.

The functions $m_l(k,x)\equiv T^{-1}\exp(-ikx)\Psi_l(k,x)$ and $m_r(k,x)\equiv T^{-1}\exp(ikx)\Psi_r(k,x)$ satisfy

$$\frac{d^2m_l}{dx^2}(k,x)+2ik\frac{dm_l}{dx}(k,x)=V(x)m_l(k,x) \tag{1.1}$$

and

$$\frac{d^2m_r}{dx^2}(k,x)-2ik\frac{dm_r}{dx}(k,x)=V(x)m_r(k,x) \tag{1.2}$$

with the boundary conditions

$$m_l(k,x)=1+o(1) \qquad \frac{dm_l}{dx}(k,x)=o(1) \qquad \text{as } x\to\infty$$

and

$$m_r(k,x)=1+o(1) \qquad \frac{dm_r}{dx}(k,x)=o(1) \qquad \text{as } x\to-\infty.$$

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In an earlier paper [1], the solutions to (1.1) and (1.2) are obtained which have the form

$$m_l(k,x)=\sum_{n=0}^{\infty}\left(\frac{\mathrm{i}}{k}\right)^n f_n(x) \tag{1.3}$$

$$m_r(k,x)=\sum_{n=0}^{\infty}\left(\frac{-\mathrm{i}}{k}\right)^n g_n(x) \tag{1.4}$$

where

$$f_0(x)=1 \qquad f_n(x)=\frac{1}{2}\frac{\mathrm{d}f_{n-1}(x)}{\mathrm{d}x}-\frac{1}{2}\int_{\infty}^{x}\mathrm{d}y\, V(y)f_{n-1}(y),\quad n\geqslant 1 \tag{1.5}$$

$$g_0(x)=1 \qquad g_n(x)=\frac{1}{2}\frac{\mathrm{d}g_{n-1}(x)}{\mathrm{d}x}-\frac{1}{2}\int_{-\infty}^{x}\mathrm{d}y\, V(y)g_{n-1}(y),\quad n\geqslant 1.$$

As mentioned in [1], we cannot expect solutions of the form (1.3) and (1.4) to exist or to converge for arbitrary potentials. Note that if $f_N(x)=0$ for some positive integer $N$, then the series (1.3) terminates because $f_n(x)=0$ for $n\geqslant N$. The same remarks also apply to $m_r(k, x)$ of (1.4). If the series (1.3) and (1.4) terminate, we obtain exact solutions to the Schrödinger equation. In [1], examples of such solutions are given for which the series expansions in (1.3) and (1.4) terminate at the second, third, fourth and fifth terms. In this paper the form of the exact solutions for an arbitrary positive integer $N$ is given. For each $N$, such exact solutions form a one-parameter family, and all the potentials in a given family cause the same scattering at all energies. Explicit examples of such solutions are known for $N=1$, 2[1, 2] and $N=3$, 4[1, 3]. In [2] and [3], however, a different method is used to obtain such examples, namely the inverse scattering method of Newton [4, 5]. Recently Degasperis and Sabatier [6, 7] have investigated the scattering theory for potentials of type

$$V(x)=\frac{l_-(l_-+1)}{(x-a_-)^2}\,\theta(x_--x)+\frac{l_+(l_++1)}{(x-a_+)^2}\,\theta(x-x_+)+\bar{u}(x) \tag{1.6}$$

where $l_\pm$ are non-negative integers, $a_\pm$ and $x_\pm$ are real numbers such that $x_-<a_-$ and $x_+>a_+$, $\theta(x)$ is the Heaviside function and $\bar{u}(x)$ is a real function such that

$$\int_{-\infty}^{\infty}(1+x^2)|\bar{u}(x)|\mathrm{d}x<\infty .$$

## 2. Exact solutions

For each positive integer $N$, let

$$P_N(x,\alpha)=(x+1)^{N(N+1)/2}+\alpha(x+1)^{(N-2)(N-1)/2} \tag{2.1}$$

where $\alpha$ is a parameter. The choice of '1' in $x+1$ is arbitrary and this causes no loss of generality in the following; it can actually be replaced by any positive number. Let us assume that the potential has the form

$$V_N(x,\alpha) = -2\left(\frac{P'_N(x,\alpha)}{P_N(x,\alpha)}\right)' \qquad x>0$$

where the prime denotes the $x$ derivative. From (1.5) we obtain

$$m_l(k,x,\alpha) = \sum_{n=0}^{\infty}\left(\frac{\mathrm{i}}{k}\right)^n f_n(x,\alpha) \tag{2.2}$$

where $f_0(x,\alpha)=1$ and

$$f_n(x,\alpha) = \tfrac{1}{2}f'_{n-1}(x,\alpha) - \tfrac{1}{2}\int_{\infty}^{x} \mathrm{d}y\, V_N(y,\alpha) f_{n-1}(y,\alpha) \qquad n\geqslant 1. \tag{2.3}$$

When $P_N(x,\alpha)$ has the form given in (2.1), using (2.3) we obtain

$$f_n(x,\alpha) = \frac{1}{P_N(x,\alpha)}\left[c_n(x+1)^{N(N+1)/2-n} + \alpha d_n(x+1)^{(N-2)(N-1)/2-n}\right] \tag{2.4}$$

where

$$c_n \equiv \frac{(N+n)!}{2^n n!(N-n)!} \qquad 0\leqslant n\leqslant N$$

$$d_n \equiv \frac{(N+n-2)!}{2^n n!(N-n-2)!} \qquad 0\leqslant n\leqslant N-2$$

and

$$c_n = 0 \qquad n\geqslant N+1$$

$$d_n = 0 \qquad n\geqslant N-1.$$

If we replace $x$ by $-x$ and $k$ by $-k$ in the argument given above for $m_l(k,x)$, we obtain the solution $m_r(k,x)$ for $x<0$ to (1.2).

## 3. Family of solutions on the line

Let the potential on the whole axis be given by

$$V_N(x,\alpha,\beta) = c\delta(x) + 2\theta(-x)\frac{1}{(-x+1+\beta)^2} - 2\theta(x)\left(\frac{P'_N(x,\alpha)}{P_N(x,\alpha)}\right)' \tag{3.1}$$

where $\delta(x)$ is the Dirac delta function, $\theta(x)$ is the Heaviside function, $c$ is a constant and $P_N(x,\alpha)$ is as in (2.1); $\alpha$ and $\beta$ are parameters. To avoid potentials with double poles, we assume $1+\alpha>0$ and $1+\beta>0$.

The corresponding physical solutions from the 'left' and 'right' are given respectively by

$$\psi_l(k,x,\alpha,\beta)=\theta(x)T\exp(\mathrm{i}kx)m_l(k,x,\alpha)+\theta(-x)[\exp(\mathrm{i}kx)m_\mathrm{r}(-k,x,\beta) + L\exp(-\mathrm{i}kx)m_\mathrm{r}(k,x,\beta)]$$

and

$$\psi_\mathrm{r}(k,x,\alpha,\beta)=\theta(x)[\exp(-\mathrm{i}kx)m_l(-k,x,\alpha)+R\exp(\mathrm{i}kx)m_l(k,x,\alpha)] + \theta(-x)T\exp(-\mathrm{i}kx)m_\mathrm{r}(k,x,\beta)$$

where

$$S(k)\equiv\begin{bmatrix} T & R \\ L & T\end{bmatrix}$$

is the scattering matrix and $T$, $R$ and $L$, the transmission and reflection coefficients, are to be computed from the boundary conditions

$$\lim_{x\to0^+}\begin{bmatrix}\psi_l(k,x,\alpha,\beta)\\ \psi_\mathrm{r}(k,x,\alpha,\beta)\end{bmatrix}=\lim_{x\to0^-}\begin{bmatrix}\psi_l(k,x,\alpha,\beta)\\ \psi_\mathrm{r}(k,x,\alpha,\beta)\end{bmatrix}$$

$$\lim_{x\to0^+}\begin{bmatrix}\psi'_l(k,x,\alpha,\beta)\\ \psi'_\mathrm{r}(k,x,\alpha,\beta)\end{bmatrix}-\lim_{x\to0^-}\begin{bmatrix}\psi'_l(k,x,\alpha,\beta)\\ \psi'_\mathrm{r}(k,x,\alpha,\beta)\end{bmatrix}=c\lim_{x\to0}\begin{bmatrix}\psi_l(k,x,\alpha,\beta)\\ \psi_\mathrm{r}(k,x,\alpha,\beta)\end{bmatrix}$$

and where

$$m_\mathrm{r}(k,x,\beta)=1-\frac{\mathrm{i}}{k}\frac{1}{x-1-\beta}$$

and $m_l(k, x, \alpha)$ is as in (2.2). The above boundary conditions are equivalent to

$$T=\frac{2\mathrm{i}k}{D(k,\alpha,\beta)}\qquad L=\frac{E(k,\alpha,\beta)}{D(k,\alpha,\beta)}\qquad R=\frac{E(-k,\alpha,\beta)}{D(k,\alpha,\beta)} \tag{3.2}$$

where

$$D(k,\alpha,\beta)\equiv(2\mathrm{i}k-c)m_l(k,0,\alpha)m_\mathrm{r}(k,0,\beta) + m'_l(k,0,\alpha)m_\mathrm{r}(k,0,\beta)-m_l(k,0,\alpha)m'_\mathrm{r}(k,0,\beta) \tag{3.3}$$

$$E(k,\alpha,\beta)\equiv cm_l(k,0,\alpha)m_\mathrm{r}(-k,0,\beta)+m_l(k,0,\alpha)m'_\mathrm{r}(-k,0,\beta) - m'_l(k,0,\alpha)m_\mathrm{r}(-k,0,\beta). \tag{3.4}$$

By choosing

$$c+\frac{1}{1+\beta}+\frac{N-(N-1)\alpha}{1+\alpha}=0 \tag{3.5}$$

we can eliminate one of the parameters $\alpha$ and $\beta$, say $\beta$. Thus, if $\beta$ is chosen as in (3.5), then for each $N$ the potentials in (3.1) form a one-parameter family where $\alpha$ is the parameter. If $\beta$ is chosen as in (3.5), $D(k, \alpha, \beta)$ and $E(K, \alpha, \beta)$ become independent of both $\alpha$ and $\beta$; using (3.3) and (3.4) they can be obtained explicitly:

$$D(k,\alpha,\beta)=2\mathrm{i}k+\sum_{n=0}^{N-1}\left(\frac{\mathrm{i}}{k}\right)^n\left(c-\frac{N(N-1)}{n+1}\right)\frac{(N+n-1)!}{2^n n!(N-n-1)!} \tag{3.6}$$

$$E(k, \alpha, \beta) = \sum_{n=0}^{N-1} \left(\frac{\mathrm{i}}{k}\right)^n \left(\frac{c-n}{2^n n!}\right) \frac{(N+n-1)!}{(N-n-1)!}. \tag{3.7}$$

Hence, from (3.2), as $k \to 0$ we obtain

$$T(k) = \frac{(2\mathrm{i})^N (N-1)!}{(c-N+1)!(2N-2)!} k^N + \mathrm{O}(k^{N+1})$$

$$L(k) = 1 + \mathrm{O}(k)$$

$$R(k) = (-1)^{N-1} + \mathrm{O}(k).$$

## 4. Conclusion

The bound states correspond to the zeros of the denominator of the transmission coefficient $T(k)$, and these zeros are all located on the imaginary axis in the upper-half complex $k$-plane [4]. From (3.2) it is seen that if there is a bound state state at $k = \mathrm{i}\kappa$, where $\kappa > 0$, then $D(\mathrm{i}\kappa, \alpha, \beta) = 0$. Hence, from (3.6) we obtain

$$D(\mathrm{i}\kappa, \alpha, \beta) = -2\kappa + \sum_{n=0}^{N-1} \left(\frac{1}{\kappa}\right)^n \left(c - \frac{N(N-1)}{n+1}\right) \frac{(N+n-1)!}{2^n n!(N-n-1)!}$$

or equivalently

$$\kappa^{N-1} D(\mathrm{i}\kappa, \alpha, \beta) = -2\kappa^N + \sum_{n=0}^{N-1} \kappa^{N-1-n} \left(c - \frac{N(N-1)}{n+1}\right) \frac{(N+n-1)!}{2^n n!(N-n-1)!}. \tag{4.1}$$

We can write (3.5) as

$$c + N - 1 + \frac{1}{1+\beta} + \frac{2N-1}{1+\alpha} = 0. \tag{4.2}$$

Hence, for $0 \leqslant n \leqslant N-1$, we have

$$c - \frac{N(N-1)}{n+1} \leqslant c - (N-1) = -\frac{1}{1+\beta} - \frac{2N-1}{1+\alpha} < 0.$$

Therefore, $\kappa^{N-1} D(\mathrm{i}\kappa, \alpha, \beta)$ given in (4.1) is an $N$th order polynomial in $\kappa$ where all the coefficients are negative and $D(\mathrm{i}\kappa, \alpha, \beta)$ cannot become zero for any $\kappa > 0$. Thus the potentials given in (3.1) do not support any bound states.

It is known [2] that the ratio $m_l(k, 0)/m_r(k, 0)$ at $k = 0$ will uniquely specify the parameter $\alpha$, and hence, once this ratio is fixed, the inverse scattering problem becomes uniquely solvable for the potentials considered here.

The non-uniqueness of the scattering matrix for the potentials considered here is due to the double or higher-order zeros of the transmission coefficient at $k = 0$ or due to the unit value of the reflection coefficient at $k = 0$ [6, 7]. The corresponding wavefunction normalisation constant provides the extra parameter which removes the ambiguity.

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