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Marchenko inversion for perturbations. II: Inclusion of bound states

Tuncay Aktosun
Department of Mathematics, Duke University, Durham, NC 27706, USA

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Abstract. The Marchenko inversion method of the one-dimensional non-relativistic quantum mechanics is generalised to perturbations when the bound states are present. The change in the potential that corresponds to a finite change in the scattering matrix is given as a sum of two terms. The first term, not related to the bound states, is obtained from the solution to a Marchenko equation for perturbations and the second term is related to the bound states.

1. Introduction

In an earlier paper (Aktosun 1987), we have given the generalisation of the inversion method of Marchenko perturbations; i.e., we have obtained the potential difference that corresponds to the difference of two scattering matrices. The generalisation in Aktosun (1987) is given in the case where there are no bound states. In this paper, we will present a generalisation of the Marchenko method to perturbations in the case where there are bound states.

This paper is organised in the following way. In § 2, the preliminaries are given and the notation is established. In § 3, Newton's method of removing the bound states from the scattering matrix and hence a reduced Riemann–Hilbert problem are given in terms of 2×2 matrices. Solving the reduced Riemann–Hilbert problem becomes equivalent to solving a reduced matrix Marchenko equation. The potential is given as the sum of two terms: the first term is the part not related to the bound states and it can be obtained from the solution to the reduced Marchenko equation; the second term is due to the bound states. In § 4, Newton's method of removing the bound states is generalised to perturbations. Starting with a reduced Riemann–Hilbert problem, a reduced Marchenko equation for perturbations is obtained. The potential difference is given again as a sum of two terms: the first term is obtained in terms of the solution to the Marchenko equation for perturbations and the second term is related to the bound states.

2. Preliminaries

In order to obtain the potential difference from the difference in the scattering matrix by a Marchenko-like method, in Aktosun (1987) we have started with a Riemann–Hilbert problem formulated in terms of 2×2 matrices. This was done in the following way.

Let $\psi_l^{(v)}$ and $\psi_r^{(v)}$ be the solutions of the one-dimensional Schrödinger equation

$$\frac{d^2}{dx^2}\psi(k,x)+k^2\psi(k,x)=V(x)\psi(k,x)$$

where the potential $V(x)\sim 0$ in some sense, with the boundary conditions

$$\begin{aligned}
\psi_l(k,x)&=T(k)\exp(ikx)+o(1) && \text{as } x\to\infty\\
\psi_l(k,x)&=\exp(ikx)+L(k)\exp(-ikx)+o(1) && \text{as } x\to-\infty\\
\psi_r(k,x)&=\exp(-ikx)+R(k)\exp(ikx)+o(1) && \text{as } x\to\infty\\
\psi_r(k,x)&=T(k)\exp(-ikx)+o(1) && \text{as } x\to-\infty
\end{aligned}$$

where

$$S(k)\equiv\begin{pmatrix}T & R\\ L & T\end{pmatrix}$$

is the scattering matrix, $T(k)$ is the transmission coefficient, and $R(k)$ and $L(k)$ are the reflection coefficients from the 'right' and from the 'left' respectively. The functions $\psi_l^{(v)}$ and $\psi_r^{(v)}$ are usually called physical solutions to the Schrödinger equation from the left and from the right respectively, which is indicated by the subscripts. The superscript refers to the potential.

Let $U(x)$ be the potential that corresponds to the scattering matrix

$$ISI=\begin{pmatrix}T & -R\\ -L & T\end{pmatrix}\qquad\text{where}\qquad I\equiv\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The existence of $U(x)$ is assured if $kL, kR\in L^1$ (Newton 1984). Let us denote the corresponding physical solutions to the Schrödinger equation by $\psi_l^{(u)}$ and $\psi_r^{(u)}$, where the superscript again refers to the potential. It can be shown (Aktosun 1987) that the matrix wavefunction $\Psi(k,x)$, which is defined as

$$\Psi(k,x)\equiv\frac{1}{2}\begin{pmatrix}\psi_l^{(v)}+\psi_l^{(u)} & \psi_l^{(v)}-\psi_l^{(u)}\\ \psi_r^{(v)}-\psi_r^{(u)} & \psi_r^{(v)}+\psi_r^{(u)}\end{pmatrix}, \tag{2.1}$$

satisfies the 2×2 matrix Schrödinger equation

$$\frac{d^2\Psi}{dx^2}+k^2\Psi=\Psi\lambda$$

where $\lambda(x)$ is the potential matrix that is given in terms of the potentials $V(x)$ and $U(x)$ as

$$\lambda(x)\equiv\frac{1}{2}\begin{pmatrix}V+U & V-U\\ V-U & V+U\end{pmatrix}. \tag{2.2}$$

Let us define the matrix $F(k,x)$ as

$$F(k,x)\equiv\exp(-iIkx)\Psi(k,x). \tag{2.3}$$

It is known (Aktosun 1987) that for each x, $F(k,x)$ has a meromorphic extension into the upper-half complex k plane, which is denoted by $\mathbb{C}^+$, $F(k,x)=\mathbb{1}+O(1/k)$ as $|k|\to\infty$ in $\bar{\mathbb{C}}^+$ where

$$\mathbb{1}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and $\bar{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{R}$; $\det F(k, x) = T(k)$ and hence $F(k, x)$ is meromorphic due to the poles $\{i\beta_1, \ldots, i\beta_n\}$ of $T(k)$ that are all located on the imaginary axis in $\mathbb{C}^+$.

3. Reduced problem

The bound states correspond to the poles of $T(k)$ in $\mathbb{C}^+$ and they can be removed from $S(k)$ and $F(k, x)$ by a method from Newton (1980).

Let us assume for simplicity that there is only one bound state located at $k = i\beta$. Since $\det F(k, x) = T(k)$, we have $\det F^{-1}(i\beta, x) = 0$ where F^{-1} is the matrix inverse of F. Hence there is a 2×1 column vector

$$l(x) \equiv \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \qquad \text{such that} \qquad F^{-1}(i\beta, x) l = 0.$$

Since $F^{-1}(i\beta, k)$ is real (Aktosun 1987), l can be chosen as real. Define the 2×2 matrix $D(x)$ as

$$D \equiv \frac{l\tilde{l}}{\tilde{l}l} = \frac{1}{l_1^2 + l_2^2} \begin{pmatrix} l_1^2 & l_1 l_2 \\ l_1 l_2 & l_2^2 \end{pmatrix} \tag{3.1}$$

where the tilde denotes the matrix transpose. Note that $D^2 = l\tilde{l}l\tilde{l}/\tilde{l}l\tilde{l}l = D$ and hence $D(D - \mathbb{1}) = 0$. As a consequence, we have for $\alpha \neq 0$

$$(\mathbb{1} - D + \alpha D)\left(\mathbb{1} - D + \frac{1}{\alpha} D\right) = \mathbb{1} - D + \frac{\alpha}{\alpha} D^2 = \mathbb{1}. \tag{3.2}$$

Let us define the reduced 2×2 matrix $F_{\text{red}}(k, x)$ as

$$F \equiv \Gamma F_{\text{red}} \tag{3.3}$$

where we have defined

$$\begin{aligned} \Gamma(k, x) &\equiv \mathbb{1} - D + \frac{k + i\beta}{k - i\beta} D = \mathbb{1} + \frac{2i\beta}{k - i\beta} D \\ &= \begin{pmatrix} 1 + \dfrac{2i\beta}{k - i\beta} \dfrac{l_1^2}{l_1^2 + l_2^2} & \dfrac{2i\beta}{k - i\beta} \dfrac{l_1 l_2}{l_1^2 + l_2^2} \\ \dfrac{2i\beta}{k - i\beta} \dfrac{l_1 l_2}{l_1^2 + l_2^2} & 1 + \dfrac{2i\beta}{k - i\beta} \dfrac{l_2^2}{l_1^2 + l_2^2} \end{pmatrix}. \end{aligned} \tag{3.4}$$

Since the matrix $D(x)$ is real, we see from (3.4) that

$$\Gamma(-k, x) = \Gamma(k, x)^* \qquad \text{when} \qquad k \in \mathbb{R} \tag{3.5}$$

where * denotes the complex conjugation. Let $A^{\#}(k, x) \equiv A(-k, x)$. Hence, (3.5) can be written as $\Gamma^{\#} = \Gamma^*$ for $k \in \mathbb{R}$. From (3.4) we have $\det \Gamma = (k + i\beta)/(k - i\beta)$, and using (3.2) we obtain $\Gamma\Gamma^{\#} = \mathbb{1}$. Thus

$$\Gamma^{-1} = \Gamma^{\#} \tag{3.6}$$

and we can write (3.3) as

$$F_{\text{red}} = \Gamma^{\#} F. \tag{3.7}$$

Note that when $k \in \mathbb{R}$, (3.7) is equivalent to $F_{\text{red}} = \Gamma^* F$.

The matrix $F_{\rm red}$ is holomorphic in $\mathbb{C}^+$ and its determinant can be computed by using (3.7):

$$\det F_{\rm red} = (\det \Gamma^{\#})(\det F) = \frac{k-\mathrm{i}\beta}{k+\mathrm{i}\beta}\, T(k) \tag{3.8}$$

which does not vanish in $\mathbb{C}^+$.

From (3.4), it is seen that for each x

$$\Gamma(k,x) = \mathbb{1} + \mathrm{O}(1/k) \qquad \text{as} \qquad |k|\to\infty \qquad \text{in} \qquad \bar{\mathbb{C}}^+ \tag{3.9}$$

and hence from (3.7), we obtain for each x

$$F_{\rm red}(k,x) = \mathbb{1} + \mathrm{O}(1/k) \qquad \text{as} \qquad |k|\to\infty \qquad \text{in} \qquad \bar{\mathbb{C}}^+. \tag{3.10}$$

It is known (Aktosun 1987) that $F(k,x)$ satisfies the Riemann–Hilbert problem

$$F^{\#} = \exp(\mathrm{i}Ikx) S^{-1} \exp(-\mathrm{i}Ikx)\, qFq \qquad \text{for } k\in\mathbb{R}. \tag{3.11}$$

Note that $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus using (3.3) in (3.11) we obtain

$$\Gamma^{\#} F_{\rm red}^{\#} = \exp(\mathrm{i}Ikx) S^{-1} \exp(-\mathrm{i}Ikx) q \Gamma F_{\rm red}\, q$$

or equivalently

$$F_{\rm red}^{\#} = (\Gamma^{\#-1} \exp(\mathrm{i}Ikx) S^{-1} \exp(-\mathrm{i}Ikx) q \Gamma q) q F_{\rm red}\, q \tag{3.12}$$

where we have used the fact that $q^2 = \mathbb{1}$. We can write (3.12) in such a way that it will resemble (3.11):

$$F_{\rm red}^{\#} = \exp(\mathrm{i}Ikx) S_{\rm red}^{-1} \exp(-\mathrm{i}Ikx) q F_{\rm red}\, q \tag{3.13}$$

where

$$S_{\rm red}^{-1} = \exp(-\mathrm{i}Ikx) \Gamma \exp(\mathrm{i}Ikx) S^{-1} \exp(-\mathrm{i}Ikx) q \Gamma q \exp(\mathrm{i}Ikx) \tag{3.14}$$

or equivalently the reduced scattering matrix $S_{\rm red}$ is given by

$$S_{\rm red} \equiv \exp(-\mathrm{i}Ikx) q \Gamma^{\#} q \exp(\mathrm{i}Ikx) S \exp(-\mathrm{i}Ikx) \Gamma^{\#} \exp(\mathrm{i}Ikx). \tag{3.15}$$

Lemma 3.1. The reduced scattering matrix $S_{\rm red}$ that is defined in (3.15) is reciprocal and unitary; i.e., $q\tilde{S}_{\rm red}\, q = S_{\rm red}$ and $\tilde{S}_{\rm red}^{*} = S_{\rm red}^{-1}$ respectively.

Proof. The matrix $\Gamma(k,x)$ defined in (3.4) is symmetric because the matrix $D(x)$ defined in (3.1) is symmetric. Using this fact and that $qI = -Iq$, we obtain

$$\begin{aligned} q\tilde{S}_{\rm red}\, q &= q \exp(\mathrm{i}Ikx) \tilde{\Gamma}^{\#} \exp(-\mathrm{i}Ikx) \tilde{S} \exp(\mathrm{i}Ikx) q \tilde{\Gamma}^{\#} \exp(\mathrm{i}Ikx) \\ &= S_{\rm red} \end{aligned}$$

where we have also used $q^2 = \mathbb{1}$ and that $q\tilde{S}q = S$: this last property follows from the fact that $T_l = T_r \equiv T$ in (Newton 1983)

$$S \equiv \begin{pmatrix} T_l & R \\ L & T_r \end{pmatrix}.$$

Hence the reciprocity of $S_{\rm red}$ is established.

To prove the unitarity, note that from (3.5) and (3.15) we obtain

$$\tilde{S}_{\rm red}^{*} = \exp(-\mathrm{i}Ikx) \Gamma \exp(\mathrm{i}Ikx) \tilde{S}^{*} \exp(-\mathrm{i}Ikx) q \Gamma q \exp(\mathrm{i}Ikx)$$

which is equal to $S_{\rm red}^{-1}$ given in (3.14). QED

Let us write (3.13) as

$$F^{\#}_{\rm red} = \Lambda_{\rm red}\, q F_{\rm red}\, q \tag{3.16}$$

where we have defined

$$\Lambda_{\rm red}(k, x) \equiv \exp(\mathrm{i}Ikx) S^{-1}_{\rm red}(k, x) \exp(-\mathrm{i}Ikx). \tag{3.17}$$

Since $S(k) = \mathbb{1} + \mathrm{O}(1/k)$ as $|k| \to \infty$ in $\mathbb{R}$ (Newton 1983), we have for each x, $\Lambda_{\rm red}(k, x) = \mathbb{1} + \mathrm{O}(1/k)$ as $|k| \to \infty$ in $\mathbb{R}$, which can be directly obtained from (3.15) and (3.17). Using the same argument given in § 6 of Aktosun (1987), we subtract the asymptotic values of $F_{\rm red}$ and $\Lambda_{\rm red}$ in order to obtain

$$\begin{aligned} F^{\#}_{\rm red} - \mathbb{1} &= \Lambda_{\rm red}\, q F_{\rm red}\, q - \mathbb{1} \\ &= \Lambda_{\rm red} - \mathbb{1} + (\Lambda_{\rm red} - \mathbb{1}) q (F_{\rm red} - \mathbb{1}) q + q (F_{\rm red} - \mathbb{1}) q. \end{aligned} \tag{3.18}$$

Taking the Fourier transform in $L^2(k)$ of (3.18) by $\int_{-\infty}^{\infty} (\mathrm{d}k/2\pi) \exp(\mathrm{i}ky)$, we obtain

$$\eta_{\rm red}(x, y) = g_{\rm red}(x, y) + \int_{-\infty}^{\infty} \mathrm{d}z\, g_{\rm red}(x, y+z) q \eta_{\rm red}(x, z) q + q \eta_{\rm red}(x, -y) q \tag{3.19}$$

where we have defined

$$\eta_{\rm red}(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} (F_{\rm red}(k, x) - \mathbb{1}) \exp(-\mathrm{i}ky)$$

$$g_{\rm red}(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} (\Lambda_{\rm red}(k, x) - \mathbb{1}) \exp(\mathrm{i}ky).$$

$F_{\rm red}$ possesses all the analyticity properties that F has in the absence of bound states (Aktosun 1987): $F_{\rm red}$ is holomorphic in $\mathbb{C}^+$ for each x and $F_{\rm red} = \mathbb{1} + \mathrm{O}(1/k)$ as $|k| \to \infty$ in $\bar{\mathbb{C}}^+$; we have $\eta_{\rm red}(x, y) = 0$ when $y < 0$ and hence (3.19) becomes

$$\eta_{\rm red}(x, y) = g_{\rm red}(x, y) + \int_0^{\infty} \mathrm{d}z\, g_{\rm red}(x, y+z) q \eta_{\rm red}(x, z) q \qquad y > 0, \tag{3.20}$$

which is the matrix Marchenko equation.

From (3.3) we obtain

$$\begin{aligned} F - \mathbb{1} &= (\Gamma - \mathbb{1} + \mathbb{1})(\mathbb{1} + F_{\rm red} + \mathbb{1}) - \mathbb{1} \\ &= (\Gamma - \mathbb{1}) + (F_{\rm red} - \mathbb{1}) + (\Gamma - \mathbb{1})(F_{\rm red} - \mathbb{1}). \end{aligned} \tag{3.21}$$

The Fourier transform of (3.21) by $\int_{-\infty}^{\infty} (\mathrm{d}k/2\pi) \exp(-\mathrm{i}ky)$ gives us

$$\eta(x, y) = N(x, y) + \eta_{\rm red}(x, y) + \int_{-\infty}^{\infty} \mathrm{d}z\, N(x, y-z) \eta_{\rm red}(x, z) \tag{3.22}$$

where we have defined

$$\eta(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(-\mathrm{i}ky)(F(k, x) - \mathbb{1}) \tag{3.23}$$

$$N(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(-\mathrm{i}ky)(N(k, x) - \mathbb{1}). \tag{3.24}$$

When there is only one bound state at $k=\mathrm{i}\beta$, using (3.4) we obtain

$$N(x,y)=\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{2\mathrm{i}\beta}{k-\mathrm{i}\beta}D(x)\exp(-\mathrm{i}ky)$$
$$=\begin{cases}0 & \text{if } y>0\\ -2\beta D(x)\exp(\beta y) & \text{if } y<0.\end{cases}$$

$N(x,y)$ vanishes for $y>0$ even when the number of bound states is more than one. This is because if there are n bound states located at $\{\mathrm{i}\beta_1,\mathrm{i}\beta_2,\ldots,\mathrm{i}\beta_n\}$, we can define $\Gamma(k,x)$ as

$$\Gamma(k,x)\equiv\prod_{j=1}^{n}\left(1+\frac{2\mathrm{i}\beta_j}{k-\mathrm{i}\beta_j}\right)D_j,$$

which is still holomorphic in the lower half-plane $\mathbb{C}^-$ and $\Gamma(k,x)=\mathbb{1}+O(1/k)$ as $|k|\to\infty$ in $\bar{\mathbb{C}}^-$ for each x. Hence (3.22) becomes

$$\eta(x,y)=\eta_{\mathrm{red}}(x,y)+\int_{y}^{\infty}\mathrm{d}z\,N(x,y-z)\,\eta_{\mathrm{red}}(x,z)\qquad y>0$$

$$\eta(x,y)=N(x,y)+\int_{0}^{\infty}\mathrm{d}z\,N(x,y-z)\,\eta_{\mathrm{red}}(x,z)\qquad y<0.$$

Thus, we obtain

$$\eta(x,0+)-\eta(x,0-)=\eta_{\mathrm{red}}(x,0+)-N(x,0-)\tag{3.25}$$

where we have defined $A(x,0\pm)\equiv\lim_{y\to 0\pm}A(x,y)$.

It is known (Aktosun 1987) that $F(k,x)$ satisfies the matrix Schrödinger equation

$$\frac{\mathrm{d}^2F(k,x)}{\mathrm{d}x^2}+2\mathrm{i}Ik\frac{\mathrm{d}F(k,x)}{\mathrm{d}x}=F(k,x)\lambda(x)\tag{3.26}$$

where $\lambda(x)$ is the matrix potential given in (2.2). Inverting (3.23) we obtain

$$F(k,x)-\mathbb{1}=\int_{-\infty}^{\infty}\mathrm{d}y\;\eta(x,y)\exp(\mathrm{i}ky)\tag{3.27}$$

and hence we can write (3.26) as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_{-\infty}^{\infty}\mathrm{d}y\exp(\mathrm{i}ky)\,\eta(x,y)+2\mathrm{i}Ik\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\mathrm{d}y\exp(\mathrm{i}ky)\eta(x,y)$$
$$=\lambda(x)+\int_{-\infty}^{\infty}\mathrm{d}y\exp(\mathrm{i}ky)\eta(x,y)\lambda(x).\tag{3.28}$$

Using integration by parts, we obtain

$$2\mathrm{i}Ik\int_{-\infty}^{\infty}\mathrm{d}y\;\eta(x,y)\exp(\mathrm{i}ky)=2I(\eta(x,0-)-\eta(x,0+))-2I\int_{-\infty}^{\infty}\mathrm{d}y\exp(\mathrm{i}ky)\frac{\partial}{\partial y}\eta(x,y)$$

and thus (3.28) becomes

$$\int_{-\infty}^{\infty}\mathrm{d}y\exp(\mathrm{i}ky)\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}-2I\frac{\partial}{\partial y}\right)\eta(x,y)-\eta(x,y)\lambda(x)\right]$$
$$=\lambda+2I\frac{\mathrm{d}}{\mathrm{d}x}(\eta(x,0+)-\eta(x,0-)).$$

Assuming that the integrand in the above equation is in $L^2(y)$, we obtain the partial differential equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}-2I\frac{\partial}{\partial y}\right)\eta(x,y)=\eta(x,y)\lambda(x)$$

and the expression for the potential matrix

$$\lambda(x)=-2I\frac{\mathrm{d}}{\mathrm{d}x}(\eta(x,0+)-\eta(x,0-)). \tag{3.29}$$

Using (3.25), we can also write (3.29) as

$$\lambda(x)=-2I\frac{\mathrm{d}}{\mathrm{d}x}\eta_{\mathrm{red}}(x,0+)+2I\frac{\mathrm{d}}{\mathrm{d}x}N(x,0-). \tag{3.30}$$

4. Generalisation of Newton's method to perturbations

In this section we will generalise Newton's method of removing the bound states to perturbations. As in Aktosun (1987), we will use the subscript '0' to denote the quantities upon which the perturbation is built; i.e., we let

$$S_0(k)\equiv\begin{pmatrix}T_0 & R_0\\ L_0 & T_0\end{pmatrix}$$

be the scattering matrix which corresponds to the potential

$$\lambda_0(x)\equiv\frac{1}{2}\begin{pmatrix}V_0+U_0 & V_0-U_0\\ V_0-U_0 & V_0+U_0\end{pmatrix}.$$

Let Ψ_0 and F_0 be the corresponding solutions of the matrix Schrödinger equation, which are the counterparts of (2.1) and (2.3) respectively. We can define $D_0(x)$, $\Gamma_0(k,x)$, $F_{0_{\mathrm{red}}}(k,x)$, and $S_{0_{\mathrm{red}}}(k,x)$ in a similar way as in (3.1), (3.4), (3.3) and (3.15) respectively.

Let us assume for simplicity that the transmission coefficients $T_0(k)$ and $T(k)$ each have one pole at $k=\mathrm{i}\beta_0$ and $k=\mathrm{i}\beta$ respectively. We then have

$$F_0=\Gamma_0F_{0_{\mathrm{red}}}$$

where

$$\Gamma_0=\mathbb{1}-D_0+\frac{k+\mathrm{i}\beta_0}{k-\mathrm{i}\beta_0}D_0.$$

In the case where there is more than one bound state, say at $\{\mathrm{i}\beta_{0_1},\mathrm{i}\beta_{0_2},\ldots,\mathrm{i}\beta_{0_m}\}$, we can define Γ_0 as

$$\Gamma_0\equiv\prod_{j=1}^{m}\left(1+\frac{2\mathrm{i}\beta_{0_j}}{k-\mathrm{i}\beta_{0_j}}D_{0_j}\right).$$

In Aktosun (1987) we have defined the 2×2 matrix $H(k,x)$ as

$$H\equiv F_0^{-1}F. \tag{4.1}$$

In terms of $F_{0_{\rm red}}$ and $F_{\rm red}$, we can write (4.1) as

$$\begin{aligned} H &= F_{0_{\rm red}}^{-1}\,\Gamma_0^{-1}\,\Gamma\,F_{\rm red} \\ &= (F_{0_{\rm red}}^{-1}\,\Gamma_0^{-1}\,\Gamma\,F_{0_{\rm red}})F_{0_{\rm red}}^{-1}\,F_{\rm red} \\ &= YH_{\rm red} \end{aligned} \tag{4.2}$$

where we have defined the matrices $H_{\rm red}(k, x)$ and $Y(k, x)$ as

$$H_{\rm red} \equiv F_{0_{\rm red}}^{-1}\,F_{\rm red} \tag{4.3}$$

$$Y \equiv F_{0_{\rm red}}^{-1}\,\Gamma_0^{-1}\,\Gamma\,F_{0_{\rm red}}. \tag{4.4}$$

The matrix $H_{\rm red}$ is holomorphic in $\mathbb{C}^+$ for the same reasons H is holomorphic in $\mathbb{C}^+$ in the absence of bound states (Aktosun 1987). The asymptotic value of $H_{\rm red}$ as $|k| \to \infty$ in $\mathbb{C}^+$ is given in the following.

Lemma 4.1. The matrix $H_{\rm red}(k, x)$ defined in (4.3) has the property $H_{\rm red}(k, x) = \mathbb{1} + O(1/k)$ as $|k| \to \infty$ in $\bar{\mathbb{C}}^+$ for each x.

Proof. Using (3.8) we can write (4.3) as

$$H_{\rm red} = \frac{k + i\beta_0}{k - i\beta_0}\,\frac{1}{T_0}\,qI\tilde{F}_{0_{\rm red}}\,Iq\,F_{\rm red}. \tag{4.5}$$

As $|k| \to \infty$ in $\bar{\mathbb{C}}^+$ we have $F_{0_{\rm red}}, F_{\rm red} = \mathbb{1} + O(1/k)$ as seen from (3.10) and that $1/T_0 = 1 + O(1/k)$ (Deift and Trubowitz 1979). Since

$$\frac{k + i\beta_0}{k - i\beta_0} = 1 + O(1/k),$$

from (4.5) we obtain $H_{\rm red} = \mathbb{1} + O(1/k)$as $|k| \to \infty$ in $\bar{\mathbb{C}}^+$ for x. QED

Using (3.13) and (4.3) we can formulate a Riemann–Hilbert problem for $H_{\rm red}(k, x)$ as follows:

$$\begin{aligned} H_{\rm red}^{\#} &= F_{0_{\rm red}}^{\#-1}\,F_{\rm red}^{\#} \\ &= qF_{0_{\rm red}}^{-1}\,q\,\exp(iIkx)\,S_{0_{\rm red}}\,\exp(-iIkx)\,\exp(iIkx)\,S_{\rm red}^{-1}\,\exp(-iIkx)qF_{\rm red}q \\ &= (qF_{\rm red}^{-1}\,q\,\exp(iIkx)\,S_{0_{\rm red}}\,S_{\rm red}^{-1}\,\exp(-iIkx)\,qF_{0_{\rm red}}\,q)q\,F_{0_{\rm red}}^{-1}\,F_{\rm red}\,q \end{aligned}$$

and hence

$$H_{\rm red}^{\#} = \mathscr{L}_{\rm red}qH_{\rm red}q \qquad k \in \mathbb{R} \tag{4.6}$$

where we have defined the matrix $\mathscr{L}_{\rm red}(k, x)$ as

$$\mathscr{L}_{\rm red} \equiv qF_{0_{\rm red}}^{-1}\,q\,\exp(iIkx)\,S_{0_{\rm red}}\,S_{\rm red}^{-1}\,\exp(-iIkx)\,qF_{0_{\rm red}}\,q. \tag{4.7}$$

Solving (4.6) for $H_{\rm red}$ constitutes a Riemann–Hilbert problem. We will solve (4.6) by a Marchenko-like method given in Aktosun (1987).

Define the matrices $\Delta_{\rm red}(k, x)$ and $\mathscr{E}_{\rm red}(k, x)$ as

$$\Delta_{\rm red} \equiv S_{\rm red} - S_{0_{\rm red}} \tag{4.8}$$

$$\mathscr{E}_{\rm red} \equiv \mathscr{L}_{\rm red} - \mathbb{1}. \tag{4.9}$$

By lemma 3.1, $S_{0_{\rm red}}$ and $S_{\rm red}$ are still unitary and hence

$$S_{\rm red}^{-1} = \tilde{S}^*_{0_{\rm red}} + \tilde{\Delta}^*_{\rm red} = S_{0_{\rm red}}^{-1} + \tilde{\Delta}^*_{\rm red}.$$

Hence from (4.7) we obtain

$$\begin{aligned}\mathscr{L}_{\rm red} &= qF_{0_{\rm red}}^{-1} q \exp(\mathrm{i}Ikx) S_{0_{\rm red}} (S_{0_{\rm red}}^{-1} + \tilde{\Delta}^*) \exp(-\mathrm{i}Ikx) q F_{0_{\rm red}} q \\ &= \mathbb{1} + qF_{0_{\rm red}}^{-1} q \exp(\mathrm{i}Ikx) S_{0_{red}} \tilde{\Delta}^* \exp(-\mathrm{i}Ikx) qF_{0_{\rm red}} q\end{aligned}$$

and thus from (4.9) we see that

$$\mathscr{E}_{\rm red} = qF_{0_{\rm red}}^{-1} q \exp(\mathrm{i}Ikx) S_{0_{\rm red}} \tilde{\Delta}^* \exp(-\mathrm{i}Ikx) qF_{0_{\rm red}} q. \tag{4.10}$$

As seen in the proof of lemma 4.1, $F_{0_{\rm red}}, F_{0_{\rm red}}^{-1} = \mathbb{1} + O(1/k)$ as $|k| \to \infty$ in $\bar{\mathbb{C}}^+$. Since $S_{0_{\rm red}}, S_{\rm red} = \mathbb{1} + O(1/k)$ as $|k| \to \infty$ in $\mathbb{R}$, from (4.10) we obtain for each x, $\mathscr{E}_{\rm red}(k, x) = O(1/k)$ as $|k| \to \infty$ in $\mathbb{R}$.

To obtain the Marchenko equation for perturbations we can proceed as follows. By subtracting the asymptotic value of $H_{\rm red}$ from both sides of (4.6), we obtain

$$\begin{aligned}H^{\#}_{\rm red} - \mathbb{1} &= (\mathscr{E}_{\rm red} + \mathbb{1}) q H_{\rm red} q - \mathbb{1} \\ &= \mathscr{E}_{\rm red} q H_{\rm red} q + q(H_{\rm red} - \mathbb{1}) q.\end{aligned}$$

The Fourier transform by $\int_{-\infty}^{\infty} (\mathrm{d}k/2\pi) \exp(\mathrm{i}ky)$ of the above equation gives us

$$\xi_{\rm red}(x, y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathscr{E}_{\rm red} q H_{\rm red} q \exp(\mathrm{i}ky) + \xi_{\rm red}(x, -y) \tag{4.11}$$

where we have defined

$$\xi_{\rm red}(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(-\mathrm{i}ky)(H_{\rm red}(k, x) - \mathbb{1}). \tag{4.12}$$

$H_{\rm red}$ is holomorphic in $k \in \mathbb{C}^+$ for each x and by lemma 4.1, $H_{\rm red} = \mathbb{1} + O(1/k)$ as $|k| \to \infty$ in $\mathbb{C}^+$. Hence $\xi_{\rm red}(x, y) = 0$ when $y < 0$ and we can write (4.11) for $y > 0$ as

$$\begin{aligned}\xi_{\rm red}(x, y) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(\mathrm{i}ky) \mathscr{E}_{\rm red} + \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(\mathrm{i}ky) \mathscr{E}_{\rm red} q(H_{\rm red} - \mathbb{1}) q \\ &= w_{\rm red}(x, y) + \int_{-\infty}^{\infty} \mathrm{d}z\, w_{\rm red}(x, y+z) q \xi_{\rm red}(x, z) q \qquad y > 0\end{aligned} \tag{4.13}$$

where we have defined the matrix $w_{\rm red}(x, y)$ as

$$w_{\rm red}(x, y) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(\mathrm{i}ky) \mathscr{E}_{\rm red}(k, x).$$

Using the fact that $\xi_{\rm red}(x, y) = 0$ for $y < 0$, (4.13) becomes the Marchenko equation for perturbations:

$$\xi_{\rm red}(x, y) = w_{\rm red}(x, y) + \int_0^{\infty} \mathrm{d}z\, w_{\rm red}(x, y+z) q \xi_{\rm red}(x, z) q \qquad y > 0.$$

To obtain the change in the matrix potential, $\lambda(x) - \lambda_0(x)$, we can proceed as follows. From (4.2) we have

$$\begin{aligned}H - \mathbb{1} &= (\mathbb{1} + Y - \mathbb{1})(\mathbb{1} + H_{\rm red} - \mathbb{1}) - \mathbb{1} \\ &= (Y - \mathbb{1}) + (H_{\rm red} - \mathbb{1}) + (Y - \mathbb{1})(H_{\rm red} - \mathbb{1}).\end{aligned}$$

The Fourier transform of the above equation by $\int_{-\infty}^{\infty}(\mathrm{d}k/2\pi)\exp(-\mathrm{i}ky)$ gives us

$$\xi(x,y)=Z(x,y)+\xi_{\text{red}}(x,y)+\int_{-\infty}^{\infty}\mathrm{d}z\,Z(x,y-z)\xi_{\text{red}}(x,z) \tag{4.14}$$

where we have defined

$$\begin{aligned}\xi(x,y)&\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\exp(-\mathrm{i}ky)(H(k,x)-\mathbb{1})\\ Z(x,y)&\equiv\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\exp(-\mathrm{i}ky)(Y(k,x)-\mathbb{1}).\end{aligned} \tag{4.15}$$

Since $\xi_{\text{red}}(x,y)=0$ for $y<0$, we can write (4.14) as

$$\xi(x,y)=Z(x,y)+\xi_{\text{red}}(x,y)+\int_{0}^{\infty}\mathrm{d}z\,Z(x,y-z)\xi_{\text{red}}(x,z)\qquad y>0$$

and

$$\xi(x,y)=Z(x,y)+\int_{0}^{\infty}\mathrm{d}z\,Z(x,y-z)\xi_{\text{red}}(x,z)\qquad y<0$$

and hence

$$\xi(x,0+)-\xi(x,0-)=Z(x,0+)-Z(x,0-)+\xi_{\text{red}}(x,0+). \tag{4.16}$$

The following lemma shows that (4.16) is related to the perturbation of the matrix potential.

Lemma 4.2. The perturbation $\lambda(x)-\lambda_0(x)$ of the matrix potential is given by

$$\lambda(x)-\lambda_0(x)=-2I\frac{\mathrm{d}}{\mathrm{d}x}(\xi(x,0+)-\xi(x,0-))$$

where $\xi(x,y)$ is the matrix defined in (4.15).

Proof. Unfolding the Fourier transform in (4.15), we obtain

$$H(k,x)=\mathbb{1}+\int_{-\infty}^{\infty}\mathrm{d}y\,\xi(x,y)\exp(\mathrm{i}ky). \tag{4.17}$$

From equation (9.5) in Aktosun (1987), we know that $H(k,x)$ satisfies the matrix Schrödinger equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}H+2\mathrm{i}Ik\frac{\mathrm{d}H}{\mathrm{d}x}+Kq\frac{\mathrm{d}H}{\mathrm{d}x}+\lambda_0H=H\lambda \tag{4.18}$$

where $K(x)$ is the scalar defined as

$$K(x)\equiv\int_{-\infty}^{x}\mathrm{d}t(V_0(t)-U_0(t)). \tag{4.19}$$

Using (4.17) in (4.18) we obtain

$$\begin{aligned}&\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+K(x)\frac{\mathrm{d}}{\mathrm{d}x}+\lambda_0(x)\right)\int_{-\infty}^{\infty}\mathrm{d}y\,\xi(x,y)\exp(\mathrm{i}ky)-\int_{-\infty}^{\infty}\mathrm{d}y\,\xi(x,y)\exp(\mathrm{i}ky)\lambda(x)\\ &\quad+2I\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\mathrm{d}y\,\xi(x,y)\mathrm{i}k\exp(\mathrm{i}ky)=\lambda(x)-\lambda_0(x).\end{aligned} \tag{4.20}$$

Integration by parts gives us

$$2I\int_{-\infty}^{\infty} dy\, \xi(x,y) ik \exp(iky) = 2I(-\xi(x,0+)+\xi(x,0-)) - 2I\int_{-\infty}^{\infty} dy \exp(iky)\frac{\partial \xi}{\partial y}(x,y)$$

and hence (4.20) becomes

$$\int_{-\infty}^{\infty} dy \left\{ \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - 2I\frac{\partial}{\partial y}\right) + K(x)\frac{\partial}{\partial x} + \lambda_0(x)\right]\xi(x,y) - \xi(x,y)\lambda(x)\right\} \exp(iky)$$

$$= \lambda(x) - \lambda_0(x) + 2I\frac{d}{dx}(\xi(x,0+) - \xi(x,0-)).$$

Assuming that the integrand above is in $L^2(y)$, by unfolding the Fourier transform, we obtain

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - 2I\frac{\partial}{\partial y}\right) + K(x) + \lambda_0(x)\right]\xi(x,y) = \xi(x,y)\lambda(x)$$

and

$$\lambda(x) - \lambda_0(x) = -2I\frac{d}{dx}(\xi(x,0+) - \xi(x,0-)) \qquad \text{QED}$$

Using the result of lemma 4.2 in (4.16), we obtain

$$\lambda(x) - \lambda_0(x) = -2I\frac{d}{dx}\xi_{red}(x,0+) - 2I\frac{d}{dx}(Z(x,0+) - Z(x,0-)).$$

5. Conclusion

In this paper the Marchenko method of obtaining the potential difference is given when the bound states are present. The potential difference is given as a sum of two terms: the first term is the part not related to the bound states and it can be obtained from the solution to the Marchenko equation for perturbations formulated here; the second term is the part that is due to the bound states.

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