Inverse Problems 3 (1987) 555–563. Printed in the UK

Stability of the Marchenko inversion

Tuncay Aktosun
Department of Mathematics, Duke University, Durham, NC 27706, USA

Received 16 September 1986, in final form 2 February 1987

Abstract. The stability of the Marchenko inversion in the one-dimensional non-relativistic quantum mechanics is studied. First, two bounds on the integral of the potential difference are given in terms of the integral of the difference in the scattering matrix. Second, a pointwise bound on the potential difference is obtained. Finally, the estimates are refined for the case of two scattering matrices coinciding on the finite energy interval $k \in [0, N]$.

1. Introduction

In an earlier paper (Aktosun 1987), the Marchenko inversion method was generalised to find the change in the potential that corresponds to a finite change in the scattering matrix. In this paper, using the results obtained in Aktosun (1987), we study the stability of the Marchenko inversion.

The problems associated with the Gel'fand–Levitan and Marchenko inversion methods can be summarised as follows. In both methods we need to know the scattering matrix at all energies. First, it is impractical to determine the scattering matrix at all energies, especially at higher energies. The other problem is that the non-relativistic Schrödinger equation is no longer a good description of the physical reality at high energies, and one must use the Dirac equation of the relativistic quantum mechanics instead. Another problem is that there is always some error in our measurements, either intrinsic in nature or coming from the imperfections in our measuring devices: hence, it is impossible to determine the exact scattering matrix at all energies. Therefore, the following questions arise. Can one use partial data or data with some errors in the Gel'fand–Levitan and Marchenko methods and still obtain an approximate potential? Can one estimate the error in the potential if the error in the scattering matrix is known? If the error in the scattering matrix is small in some sense, is the corresponding error in the potential also small; that is, are the Gel'fand–Levitan and Marchenko methods stable? This paper addresses these questions in the case of Marchenko inversion. For the stability of the Gel'fand–Levitan inversion, the interested reader is referred to the classic paper of Newton (1982).

The stability of the Marchenko inversion was first studied by Marchenko himself. In the radial case, when the two scattering matrices $S(k)$ and $S_0(k)$ coincide for $k \in [0, N]$ and differ for $k \in (N, \infty)$, he gave bounds on the potential difference $|V(x) - V_0(x)|$ and $|\int_x^\infty \mathrm{d}t(V(t) - V_0(t))|$ (Marchenko 1968). His bounds, however, are not uniform and they are functions of the space variable x. His bound on the potential difference $|V(x) - V_0(x)|$ was later improved, but it was still a function of x (Lundina and Marchenko 1969). Another relevant study has been done by Prosser (1984). In the one-dimensional case, he gave a bound on $\max_{x \in \mathbb{R}} |V(x) - V_0(x)|$ in terms of the change in the reflection coefficient, but his result is restricted to the reflection coefficients in the class

$$\left\{ R(k): \int_{-\infty}^{\infty} \mathrm{d}x |\hat{R}(x)| < 1 \qquad \text{where} \qquad \hat{R}(x) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} R(k) \exp(\mathrm{i}kx) \right\}.$$

The stability of the Marchenko inversion also follows from the convergence of the Neumann-series solution to the Marchenko equation. An incomplete proof of the convergence of the Neumann series for the Marchenko equation was given by Segur (1973) and was later completed by Sabatier (1985). A complete proof is also given by Xu Bang Qing and Shao Chang Qui (1985). In these three references, the authors consider the one-dimensional case and they use one of the reflection coefficients as the scattering data.

This paper is organised as follows. In § 2, some estimates are given on the wavefunctions and these estimates will be used later. In § 3, the relation between the potential difference and the difference in the scattering matrix is obtained. Section 4 contains estimates of the integral of the potential difference. In § 5, the estimate of the potential difference is given. In § 6, such estimates are refined for scattering matrices $S(k)$ and $S_0(k)$ which coincide for $k \in [0, N]$ and which differ for $k \in (N, \infty)$.

As in Aktosun (1987), we assume that both $S(k)$ and $S_0(k)$ are free of bound states, and we let

$$S(k) \equiv \begin{pmatrix} T & R \\ L & T \end{pmatrix} \qquad \text{and} \qquad S_0(k) \equiv \begin{pmatrix} T_0 & R_0 \\ L_0 & T_0 \end{pmatrix}.$$

Note that the subscript will be used for the quantities on which the perturbation is built.

2. Bounds on wavefunctions

Let ψ_l and ψ_{r} be the solutions to the one-dimensional Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(k, x) + k^2\psi(k, x) = V(x)\psi(k, x)$$

which satisfy

$$\psi_l(k, x) = T(k) \exp(\mathrm{i}kx) + \mathrm{o}(1) \qquad \text{as } x \to \infty$$

$$\psi_{\mathrm{r}}(k, x) = \exp(-\mathrm{i}kx) + R(k) \exp(\mathrm{i}kx) + \mathrm{o}(1) \qquad \text{as } x \to \infty$$

$$\psi_l(k, x) = \exp(\mathrm{i}kx) + L(k) \exp(-\mathrm{i}kx) + \mathrm{o}(1) \qquad \text{as } x \to -\infty$$

$$\psi_{\mathrm{r}}(k, x) = T(k) \exp(-\mathrm{i}kx) + \mathrm{o}(1) \qquad \text{as } x \to -\infty$$

where $V(x)$ is the potential and $V(x) \sim 0$ as $x \to \pm\infty$. Let us define

$$m_l(k, x) \equiv \frac{1}{T} \exp(-\mathrm{i}kx)\psi_l(k, x) \qquad \text{and} \qquad m_{\mathrm{r}}(k, x) \equiv \frac{1}{T} \exp(\mathrm{i}kx)\psi_{\mathrm{r}}(k, x).$$

The subscripts l and r are used because ψ_l and ψ_{r} are usually called waves travelling from the 'left' and from the 'right' respectively. Let ψ_{0l}, $\psi_{0\mathrm{r}}$, m_{0l} and $m_{0\mathrm{r}}$ be the corresponding quantities for the potential $V_0(x)$ and the scattering matrix $S_0(k)$. As in Aktosun (1987), we define the absolute value of a matrix $A = (A_{ij})$ as

$$|A| \equiv \max_i \sum_j |A_{ij}|. \tag{2.1}$$

Let us also define the constants

$$\gamma \equiv \int_{-\infty}^{\infty} \mathrm{d}x |xV(x)| \qquad \text{and} \qquad \gamma_0 \equiv \int_{-\infty}^{\infty} \mathrm{d}x |xV_0(x)| \tag{2.2}$$

$$\sigma \equiv \int_{-\infty}^{\infty} \mathrm{d}x |V(x)| \qquad \text{and} \qquad \sigma_0 \equiv \int_{-\infty}^{\infty} \mathrm{d}x |V_0(x)|. \tag{2.3}$$

Then, using the results in Deift and Trubowitz (1979), we have the following bounds on the wavefunctions:

$$|\psi_l| \leqslant |T|\, \mathrm{e}^{\gamma} \quad \text{for} \quad x \geqslant 0; \qquad |\psi_l| \leqslant 2\, \mathrm{e}^{\gamma} \quad \text{for} \quad x \leqslant 0 \tag{2.4}$$

$$|\psi_\mathrm{r}| \leqslant 2\, \mathrm{e}^{\gamma} \quad \text{for} \quad x \geqslant 0; \qquad |\psi_\mathrm{r}| \leqslant |T|\, \mathrm{e}^{\gamma} \quad \text{for} \quad x \leqslant 0 \tag{2.5}$$

$$|m_l| \leqslant \exp(\sigma/|k|) \quad \text{and} \quad |m_\mathrm{r}| \leqslant \exp(\sigma/|k|) \quad \text{for} \quad x \in \mathbb{R} \tag{2.6}$$

$$\left|\frac{\mathrm{d}m_l}{\mathrm{d}x}\right| \leqslant \sigma \exp(\sigma/|k|) \quad \text{and} \quad \left|\frac{\mathrm{d}m_\mathrm{r}}{\mathrm{d}x}\right| \leqslant \sigma \exp(\sigma/|k|) \quad \text{for} \quad x \in \mathbb{R} \tag{2.7}$$

The bounds on ψ_{0l}, $\psi_{0\mathrm{r}}$, m_{0l}, $m_{0\mathrm{r}}$, $\mathrm{d}m_{0l}/\mathrm{d}x$ and $\mathrm{d}m_{0\mathrm{r}}/\mathrm{d}x$ are obtained by replacing γ and σ in the above inequalities by γ_0 and σ_0 respectively.

3. Potential difference

From equation (8.17) in Aktosun (1987), we have

$$V(x) - V_0(x) = \frac{\mathrm{d}}{\mathrm{d}x} \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} A(k, x) \exp(-\mathrm{i}ky) \tag{3.1}$$

where we have defined, as in equation (8.16) of the same paper

$$A(k, x) \equiv \frac{1}{T_0} (\psi_{0l}, \psi_{0\mathrm{r}}) I (S - S_0) \begin{pmatrix} \psi_l^* \\ \psi_\mathrm{r}^* \end{pmatrix} \tag{3.2}$$

where the matrix I is given as

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and * denotes the complex conjugation. Note that $(\psi_{0l}, \psi_{0\mathrm{r}})$ and $\binom{\psi_l}{\psi_\mathrm{r}}$ denote the row and column vectors, respectively.

Assuming that $A(k, x)$ and $\partial A(k, x)/\partial x$ are in $L^1(k)$ for each x, from (3.1), we have

$$|V(x) - V_0(x)| \leqslant \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \left| \frac{\partial A}{\partial x}(k, x) \right| \tag{3.3}$$

and for any $x_1, x_2 \in \mathbb{R}$

$$\left| \int_{x_1}^{x_2} \mathrm{d}x (V(x) - V_0(x)) \right| \leqslant \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\pi} \max_{x \in \mathbb{R}} |A(k, x)|. \tag{3.4}$$

Note that (3.1) also implies a stronger inequality than (3.4), namely

$$\left| \int_{x_1}^{x_2} \mathrm{d}x (V(x) - V_0(x)) \right| \leqslant \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\pi} \max_{x \in [x_1, x_2]} |A(k, x)|.$$

4. Estimates on integral of potential difference

The theorem below gives an estimate of the integral of the potential difference.

Theorem 4.1.

$$\max_{x_1, x_2 \in \mathbb{R}} \left| \int_{x_1}^{x_2} \mathrm{d}x (V(x) - V_0(x)) \right| \leqslant \frac{6}{\pi} \exp(\gamma_0 + \gamma) \int_{-\infty}^{\infty} \mathrm{d}k \frac{|S - S_0|}{|T_0|}$$

where γ and γ_0 are the constants defined in (2.2) and where the absolute value of the matrix $S - S_0$ is defined as in (2.1).

Proof. Using the matrix norm defined in (2.1) on $A(k, x)$, which is given in (3.2), we obtain

$$|A| \leqslant |(\psi_{0l}, \psi_{0r})| \frac{1}{|T_0|} |S - S_0| \left| \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix} \right| \tag{4.1}$$

where we have used $|I| = 1$. From (2.1) we also have

$$|(\psi_{0l}, \psi_{0r})| \leqslant \max(|\psi_{0l}|, |\psi_{0r}|) \quad \text{and} \quad \left| \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix} \right| = \max(|\psi_l| + |\psi_r|).$$

Hence, using the estimates given in § 2, we obtain for $x \in \mathbb{R}$

$$|(\psi_{0l}, \psi_{0r})| \leqslant \max(|T_0|\, \mathrm{e}^{\gamma_0}, 2\, \mathrm{e}^{\gamma_0}) = 2\, \mathrm{e}^{\gamma_0} \qquad \text{for } x \in \mathbb{R}$$

$$\left| \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix} \right| \leqslant |T|\, \mathrm{e}^{\gamma} + 2\, \mathrm{e}^{\gamma} \leqslant 3\, \mathrm{e}^{\gamma} \qquad \text{for } x \in \mathbb{R}.$$

Thus, from (4.1) we have $|A| = 6 \exp(\gamma_0 + \gamma)(1/|T_0|)|S - S_0|$ and from (3.4) we obtain the inequality stated in the theorem. QED.

In the sense of the inequality given in the above theorem, the Marchenko inversion is stable for potentials in class $\mathscr{D}_\alpha$ where $\mathscr{D}_\alpha \equiv \{V(x): \int_{-\infty}^{\infty} \mathrm{d}x |xV(x)| < \alpha\}$; i.e., the class of potentials whose first moment is bounded by a positive number α. The inequality given in the preceding theorem can be improved at the expense of simplicity as in the following.

Theorem 4.2.

$$\max_{x_1, x_2 \in \mathbb{R}} \left| \int_{x_2}^{x_1} \mathrm{d}x (V(x) - V_0(x)) \right| \leqslant \exp(\gamma_0 + \gamma) \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\pi} B(k)$$

where

$$B(k) \equiv 5 \left| \frac{T - T_0}{T_0} \right| + 2 \left(1 + \left| \frac{T}{T_0} \right| \right) (|R - R_0| + |L - L_0|)$$

and where γ_0 and γ are the constants defined in (2.2).

Proof. We can write (3.2) explicitly as

$$A(k, x) = \frac{1}{T_0} (\psi_{0l}, \psi_{0r}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} T - T_0 & R - R_0 \\ L - L_0 & T - T_0 \end{pmatrix} \begin{pmatrix} \psi_l^* \\ \psi_r^* \end{pmatrix}$$

$$= \frac{T - T_0}{T_0} (\psi_{0l} \psi_l^* - \psi_{0r} \psi_r^*) + \frac{R - R_0}{T_0} \psi_{0l} \psi_r^* - \frac{L - l_0}{T_0} \psi_{0r} \psi_l^*. \tag{4.2}$$

Hence we have

$$|A| \leqslant \left|\frac{T-T_0}{T_0}\right| (|\psi_{0l}\psi_l| + |\psi_{0r}\psi_r|) + \left|\frac{R-R_0}{T_0}\right| |\psi_{0l}\psi_r| + \left|\frac{L-L_0}{T_0}\right| |\psi_{0r}\psi_l|.$$

Using the bounds in (2.4) and (2.5), for $x \geqslant 0$, we obtain

$$|A| \leqslant \left|\frac{T-T_0}{T_0}\right| [|T_0 T| \exp(\gamma_0+\gamma) + 4\exp(\gamma_0+\gamma)] + \left|\frac{R-R_0}{T_0}\right| 2|T_0| \exp(\gamma_0+\gamma)$$
$$+ \left|\frac{L-L_0}{T_0}\right| 2|T| \exp(\gamma_0+\gamma)$$
$$\leqslant \left(5\left|\frac{T-T_0}{T_0}\right| + 2|R-R_0| + 2\left|\frac{T}{T_0}\right| |L-L_0|\right) \exp(\gamma_0+\gamma)$$

and similarly, for $x \leqslant 0$

$$|A| \leqslant \left(5\left|\frac{T-T_0}{T_0}\right| + 2\left|\frac{T}{T_0}\right| |R-R_0| + 2|L-L_0|\right) \exp(\gamma_0+\gamma).$$

Combining the bounds for $x \geqslant 0$ and for $x \leqslant 0$, we obtain for $x \in \mathbb{R}$,

$$|A| \leqslant \exp(\gamma_0+\gamma)\left[5\left|\frac{T-T_0}{T_0}\right| + 2\left(1+\left|\frac{T}{T_0}\right|\right)(|R-R_0| + |L-L_0|)\right]$$

and hence from (3.4), we have the inequality stated in the theorem. QED.

We can again interpret the inequality given in the above theorem as a stability of the Marchenko inversion for potential in class $\mathscr{D}_\alpha$.

5. Estimate on potential difference

To obtain an estimate on $|V(x) - V_0(x)|$, we can use the inequality given in (3.3). For this, we need a bound on $|\partial A(k,x)/\partial x|$. Let us write (4.2) as

$$A = \frac{T-T_0}{T_0} A_1 + (R-R_0)A_2 - (L-L_0)A_3 \tag{5.1}$$

where we have defined

$$A_1 \equiv \psi_{0l}\psi_l^* - \psi_{0r}\psi_r^* \qquad A_2 \equiv \frac{1}{T_0}\psi_{0l}\psi_r^* \qquad A_3 \equiv \frac{1}{T_0}\psi_{0r}\psi_l^*.$$

It is known (Chadan and Sabatier 1977) that

$$\psi_l^* = (\psi_r - R\psi_r^*)/T \tag{5.2}$$

$$\psi_r^* = (\psi_l - L\psi_l^*)/T. \tag{5.3}$$

Hence we can write A_1, A_2 and A_3 in terms of the solutions from the left and from the right only as follows.

$$A_1 = \psi_{0l}\psi_l^* - \frac{\psi_{0l}^* - L_0^*\psi_{0l}}{T_0^*}\,\frac{\psi_l - L\psi_l^*}{T} = \frac{\psi_{0r}^* - R_0^*\psi_{0r}}{T_0^*}\,\frac{\psi_r - R\psi_r^*}{T} - \psi_{0r}\psi_r^*$$

and

$$A_2=\frac{\psi_{0l}}{T_0}\,\frac{\psi_l-L\psi_l^*}{T}=\frac{1}{T_0}\,\frac{\psi_{0\mathrm{r}}^*-R_0^*\psi_{0\mathrm{r}}}{T_0^*}\,\psi_{\mathrm{r}}^*$$

and

$$A_3=\frac{1}{T_0}\,\frac{\psi_{0l}^*-L_0^*\psi_{0l}}{T_0^*}\,\psi_l^*=\frac{\psi_{0\mathrm{r}}}{T_0}\,\frac{\psi_{\mathrm{r}}-R\psi_{\mathrm{r}}^*}{T}.$$

In terms of $m_{0l}\equiv(1/T_0)\exp(-\mathrm{i}kx)\psi_{0l}$ and $m_l\equiv(1/T)\exp(-\mathrm{i}kx)\psi_l$, we have

$$A_1=T_0T^*m_{0l}m_l^*-m_{0l}^*m_l-R^*\exp(-2\mathrm{i}kx)m_{0l}^*m_l^*-R_0\exp(2\mathrm{i}kx)m_{0l}m_l-R_0R^*m_{0l}m_l^*$$

$$A_2=\exp(2\mathrm{i}kx)m_{0l}m_l+R^*m_{0l}m_l^*$$

$$A_3=\exp(-2\mathrm{i}kx)\frac{T^*}{T_0}\,m_{0l}^*m_l^*-\frac{L_0^*T^*}{T_0^*}\,m_{0l}m_l^*$$

and in terms of $m_{0\mathrm{r}}\equiv(1/T_0)\exp(\mathrm{i}kx)\psi_{0\mathrm{r}}$ and $m_{\mathrm{r}}\equiv(1/T)\exp(\mathrm{i}kx)\psi_{\mathrm{r}}$, we have

$$A_1=m_{0\mathrm{r}}^*m_{\mathrm{r}}+\exp(2\mathrm{i}kx)L^*m_{0\mathrm{r}}^*m_{\mathrm{r}}^*+\exp(-2\mathrm{i}kx)L_0m_{0\mathrm{r}}m_{\mathrm{r}}+L_0L^*m_{0\mathrm{r}}m_{\mathrm{r}}^*-T_0T^*m_{0\mathrm{r}}m_{\mathrm{r}}^*$$

$$A_2=\exp(2\mathrm{i}kx)\frac{T^*}{T_0}\,m_{0\mathrm{r}}^*m_{\mathrm{r}}^*-\frac{R_0^*T^*}{T_0^*}\,m_{0\mathrm{r}}m_{\mathrm{r}}^*$$

$$A_3=\exp(-2\mathrm{i}kx)m_{0\mathrm{r}}m_{\mathrm{r}}+L^*m_{0\mathrm{r}}m_{\mathrm{r}}^*.$$

Letting a prime denote the x derivative, we obtain

$$\begin{aligned}A_1'={}&2\mathrm{i}k\exp(-2\mathrm{i}kx)R^*m_{0l}^*m_l^*-2\mathrm{i}k\exp(2\mathrm{i}kx)R_0m_{0l}m_l+T_0T^*(m_{0l}'m_l^*+m_{0l}m_l^{*\prime})\\&-(m_{0l}^{*\prime}m_l+m_{0l}^*m_l')-\exp(-2\mathrm{i}kx)R^*(m_{0l}^{*\prime}m_l^*+m_{0l}^*m_l^{*\prime})\\&-\exp(2\mathrm{i}kx)R_0(m_{0l}'m_l+m_{0l}m_l')-R_0R^*(m_{0l}'m_l^*+m_{0l}m_l^{*\prime})\end{aligned}$$

$$A_2'=2\mathrm{i}k\exp(2\mathrm{i}kx)m_{0l}m_l+\exp(2\mathrm{i}kx)(m_{0l}'m_l+m_{0l}m_l')+R^*(m_{0l}'m_l^*+m_{0l}m_l^{*\prime})$$

$$\begin{aligned}A_3'={}&-2\mathrm{i}k\exp(-2\mathrm{i}kx)(T^*/T_0)m_{0l}^*m_l^*+\exp(-2\mathrm{i}kx)(T^*/T_0)(m_{0l}^{*\prime}m_l^*+m_{0l}^*m_l^{*\prime})\\&-(L_0^*T^*/T_0^*)(m_{0l}'m_l^*+m_{0l}m_l^{*\prime})\end{aligned}$$

and in terms of the solutions from the right, we have

$$\begin{aligned}A_1'={}&2\mathrm{i}k\exp(2\mathrm{i}kx)L^*m_{0\mathrm{r}}^*m_{\mathrm{r}}^*-2\mathrm{i}k\exp(-2\mathrm{i}kx)L_0m_{0\mathrm{r}}m_{\mathrm{r}}-T_0T^*(m_{0\mathrm{r}}'m_{\mathrm{r}}^*+m_{0\mathrm{r}}m_{\mathrm{r}}^{*\prime})\\&+(m_{0\mathrm{r}}^{*\prime}m_{\mathrm{r}}+m_{0\mathrm{r}}^*m_{\mathrm{r}}')+\exp(2\mathrm{i}kx)L^*(m_{0\mathrm{r}}^{*\prime}m_{\mathrm{r}}^*+m_{0\mathrm{r}}^*m_{\mathrm{r}}^{*\prime})\\&+\exp(-2\mathrm{i}kx)L_0(m_{0\mathrm{r}}'m_{\mathrm{r}}+m_{0\mathrm{r}}m_{\mathrm{r}}')+L_0L^*(m_{0\mathrm{r}}'m_{\mathrm{r}}^*+m_{0\mathrm{r}}m_{\mathrm{r}}^{*\prime})\end{aligned}$$

$$\begin{aligned}A_2'={}&2\mathrm{i}k\exp(2\mathrm{i}kx)(T^*/T_0)m_{0\mathrm{r}}^*m_{\mathrm{r}}^*+\exp(2\mathrm{i}kx)(T^*/T_0)(m_{0\mathrm{r}}^{*\prime}m_{\mathrm{r}}^*+m_{0\mathrm{r}}^*m_{\mathrm{r}}^{*\prime})\\&-(R_0^*T^*/T_0^*)(m_{0\mathrm{r}}'m_{\mathrm{r}}^*+m_{0\mathrm{r}}m_{\mathrm{r}}^{*\prime})\end{aligned}$$

$$A_3'=-2\mathrm{i}k\exp(-2\mathrm{i}kx)m_{0\mathrm{r}}m_{\mathrm{r}}+\exp(-2\mathrm{i}kx)(m_{0\mathrm{r}}'m_{\mathrm{r}}+m_{0\mathrm{r}}m_{\mathrm{r}}')+L^*(m_{0\mathrm{r}}'m_{\mathrm{r}}^*+m_{0\mathrm{r}}m_{\mathrm{r}}^{*\prime}).$$

Using the bounds given in (2.6) and (2.7) and the fact that the transmission and reflection coefficients are bounded by one in absolute value, we obtain for $x\geqslant 0$

$$|A_1'|\leqslant\exp(\gamma_0+\gamma)[5(\sigma_0+\sigma)+2|k|(|R_0|+|R|)]$$

$$|A_2'|\leqslant\exp(\gamma_0+\gamma)[2(|k|+\sigma_0+\sigma)]$$

$$|A_3'|\leqslant\exp(\gamma_0+\gamma)[2|T/T_0|(|k|+\sigma_0+\sigma)]$$

and similarly, for $x \leqslant 0$

$$|A_1'| \leqslant \exp(\gamma_0 + \gamma)[5(\sigma_0 + \sigma) + 2|k|(|L_0| + |L|)]$$

$$|A_2'| \leqslant \exp(\gamma_0 + \gamma)[2|T/T_0|(|k| + \sigma_0 + \sigma)]$$

$$|A_3'| \leqslant \exp(\gamma_0 + \gamma)[2(|k| + \sigma_0 + \sigma)]$$

and hence, after using $|R_0| = |L_0|$ and $|R| = |L|$, we have for $x \in \mathbb{R}$

$$|A_1'| \leqslant [5(\sigma_0 + \sigma) + 2|k|(|R_0| + |R|)] \exp(\gamma_0 + \gamma)$$

$$|A_2'| \leqslant 2(1 + |T/T_0|)(|k| + \sigma_0 + \sigma) \exp(\gamma_0 + \gamma)$$

$$|A_3'| \leqslant 2(1 + |T/T_0|)(|k| + \sigma_0 + \sigma) \exp(\gamma_0 + \gamma).$$

From (5.1), we obtain

$$\left|\frac{\partial A}{\partial x}(k, x)\right| \leqslant \left|\frac{T - T_0}{T_0}\right| |A_1'| + |R - R_0||A_2'| + |L - L_0||A_3'|$$

and hence from (3.3), it is seen that we have proven the following theorem.

Theorem 5.1.

$$\max_{x \in \mathbb{R}} |V(x) - V_0(x)| \leqslant \exp(\gamma_0 + \gamma) \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\pi} C(k)$$

where

$$C(k) \equiv \left|\frac{T - T_0}{T_0}\right| [\tfrac{5}{2}(\sigma_0 + \sigma) + |k|(|R_0| + |R|)]$$
$$+ (1 + |T/T_0|)(|k| + \sigma + \sigma_0)(|R - R_0| + |L - L_0|)$$

and where σ_0, σ, γ_0 and γ are the constants defined in (2.2) and (2.3).

Note that we can interpret the inequality in the above theorem as a stability property of the Marchenko inversion for potentials in class $\mathscr{C}_\alpha \equiv \{V(x) \colon \int_{-\infty}^{\infty} \mathrm{d}x(1 + |x|)|V(x)| < \alpha\}$ where α is a positive number.

6. Stability in a special case

In this section we refine the inequalities obtained in §§ 4 and 5 in the case of two scattering matrices coinciding for $k \in [0, N]$. A similar problem has been studied by Marchenko in the radial case (Marchenko 1968, Lundina and Marchenko 1969). The physical importance of this case comes from the fact that in practice we can only obtain the scattering data in a finite energy interval, and for very high energies the scattering process can no longer be described by the Schrödinger equation of the non-relativistic quantum mechanics. However, we have

$$S(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{O}(1/k) \qquad \text{as} \qquad |k| \to \infty,$$

from Deift and Trubowitz (1979) and Chadan and Sabatier (1977), and hence we can expect our estimates to improve in the special case of two scattering matrices coinciding for $k \in [0, N]$.

It is known from Aktosun (1987) that $A(-k,x)=A(k,x)^*$ and hence $A(k,x)=0$ for $k\in[-N,N]$ whenever $S(k)=S_0(k)$ for $k\in[0,N]$. Using the bounds given in (2.6), for $|k|\geqslant N$, from (5.1) we obtain

$$|A_1|\leqslant|T_0T|(|m_{0l}m_l|+|m_{0r}m_r|)\leqslant 2|T_0T|\exp[(\sigma_0+\sigma)/N]$$

$$|A_2|\leqslant|T||m_{0l}m_r|\leqslant|T|\exp[(\sigma_0+\sigma)/N]$$

$$|A_3|\leqslant|T||m_{0r}m_l|\leqslant|T|\exp[(\sigma_0+\sigma)/N]$$

and hence we have for $|k|\geqslant N$

$$\begin{aligned}|A|&\leqslant|T|(2|T-T_0|+|R-R_0|+|L-L_0|)\exp[(\sigma_0+\sigma)/N]\\&\leqslant(2|T-T_0|+|R-R_0|+|L-L_0|)\exp[(\sigma_0+\sigma)/N]\end{aligned}$$

and thus from (3.4), we obtain

$$\begin{aligned}&\max_{m_x,x_2\in\mathbb{R}}\left|\int_{x_1}^{x_2}\mathrm{d}x(V(x)-V_0(x))\right|\\&\qquad\leqslant\exp[(\sigma_0+\sigma)/N]\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\pi}(2|T-T_0|+|R-R_0|+|L-L_0|).\end{aligned}$$

To get a bound on $|V(x)-V_0(x)|$, we can write (5.1) in terms of m_{0l}, m_{0r}, m_l and m_r as

$$\begin{aligned}A=&(T-T_0)T^*(m_{0l}m_l^*-m_{0r}m_r^*)+(R-R_0)T^*m_{0l}m_r^*\exp(2\mathrm{i}kx)\\&-(L-L_0)T^*\exp(-2\mathrm{i}kx)m_{0r}m_l^*.\end{aligned}$$

After taking the x derivative and using the inequalities in (2.6) and (2.7), we obtain

$$\begin{aligned}|A'|\leqslant&|T|(2|T-T_0|+|R-R_0|+|L-L_0|)(\sigma+\sigma_0)\exp[(\sigma_0+\sigma)/N]\\&+|T|(2|k|)(|R-R_0|+|L-L_0|)\exp[(\sigma_0+\sigma)/N]\\\leqslant&[2|T-T_0|(\sigma_0+\sigma)+(|R-R_0|+|L-L_0|)(2|k|+\sigma_0+\sigma)]\exp[(\sigma_0+\sigma)/N]\end{aligned}$$

and hence from (3.3), we have

$$\begin{aligned}&\max_{x\in\mathbb{R}}|V(x)-V_0(x)|\\&\qquad\leqslant\exp[(\sigma_0+\sigma)/N]\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}[2|T-T_0|(\sigma_0+\sigma)+(|R-R_0|\\&\qquad\quad+|L-L_0|)(2|k|+\sigma_0+\sigma)].\end{aligned}$$

7. Conclusion

In this paper we have given the stability of the Marchenko inversion by giving some estimates of the integral of the potential difference and some estimates of the potential difference in terms of the difference in the scattering matrix by using the results obtained in Aktosun (1987). First, we have obtained two bounds on the integral of the potential difference. Second, a pointwise bound on the potential difference is given. Finally, the estimates are refined for the case of two scattering matrices coinciding on the finite energy interval $k\in[0,N]$.

Acknowledgment

The results in this paper, with the exception of those in § 6, are summaries of parts of my PhD thesis. I am grateful to my advisor, Roger G Newton, for his help and encouragement throughout. I also thank Margaret Cheney for reading the manuscript and for her comments. The adjudicator's comments on the convergence of the Neumann-series solution to the Marchenko equation are also acknowledged.

References

Aktosun T 1987 *Inverse Problems* **3** 523
Chadan K and Sabatier P C 1977 *Inverse Problems in Quantum Scattering Theory* (New York: Springer)
Deift P and Trubowitz E 1979 *Commun. Pure Appl. Math.* **32** 121
Lundina D S and Marchenko V A 1969 *Mat. Sbornik* **78** 120 (Engl. transl. 1969 *Math. USSR Sbornik* **7** 467)
Marchenko V A 1968 *Mat. Sbornik* **77** 119 (Engl. transl. 1968 *Math. USSR Sbornik* **6** 125)
Newton R G 1982 *Scattering Theory of Waves and Particles* 2nd edn (New York: Springer)
Prosser R T 1984 *J. Math. Phys.* **25** 1924
Sabatier P C 1985 *Distributed Parameter Systems (Lecture Notes in Control and Information Sciences, No. 75)* ed. F Kappel *et al* (Berlin: Springer) p 312
Segur H 1973 *J. Fluid Mech.* **59** 721
Xu Bang Qing and Shao Chang Qui 1985 *Lett. Math. Phys.* **9** 183