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LETTER TO THE EDITOR

An integral expression for the Jost matrix of non-relativistic scattering on the line

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Abstract. An integral expression in terms of the potential and the physical solutions is given for the Jost matrix of one-dimensional non-relativistic quantum mechanics.

The Schrödinger equation on the line,

$$\frac{d^2\psi}{dx^2}(k,x)+k^2\psi(k,x)=V(x)\psi(k,x),$$

assuming the potential $V(x)\sim 0$ in some sense, has two linearly independent solutions. The *physical solutions* from the 'left' and from the 'right' respectively satisfy the boundary conditions

$$\psi_l(k,x)=\begin{cases}T(k)\,e^{ikx}+o(1) & \text{as } x\to\infty\\ e^{ikx}+L(k)\,e^{-ikx}+o(1) & \text{as } x\to-\infty\end{cases}$$

$$\psi_r(k,x)=\begin{cases}e^{-ikx}+R(k)\,e^{ikx}+o(1) & \text{as } x\to\infty\\ T(k)\,e^{-ikx}+o(1) & \text{as } x\to-\infty\end{cases}$$

where

$$S(k)\equiv\begin{bmatrix}T(k) & R(k)\\ L(k) & T(k)\end{bmatrix}$$

is the scattering matrix.

Another pair of linearly independent solutions, $\varphi_l(k,x)$ and $\varphi_r(k,x)$, called *regular solutions* from the 'left' and from the 'right' respectively satisfy the boundary conditions

$$\varphi_l(k,0)=1 \qquad \text{and} \qquad \frac{d\varphi_l}{dx}(k,0)=ik$$

$$\varphi_r(k,0)=1 \qquad \text{and} \qquad \frac{d\varphi_r}{dx}(k,0)=-ik.$$

The Jost matrix $J(k)$ is defined to be the matrix that connects the physical and regular solutions as

$$J(k)\begin{bmatrix}\psi_l(k,x)\\ \psi_r(k,x)\end{bmatrix}=\begin{bmatrix}\varphi_l(k,x)\\ \varphi_r(k,x)\end{bmatrix} \tag{1}$$

and it is used in the inverse scattering method of Gel'fand and Levitan [1]. The scattering matrix $S(k)$ has the canonical decomposition in terms of the Jost matrix as [1]

$$S(k)=qJ^{-1}(k)qJ(-k)$$

where $q\equiv\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $J^{-1}(k)$ is the matrix inverse of $J(k)$. For all these definitions and further properties of $J(k)$, the reader is referred to the papers of Newton [1, 2].

For a given potential $V(x)$, in order to obtain the properties of the Jost matrix $J(k)$ from (1), one needs to know the properties of both the physical and regular solutions. Below we give an integral expression for $J^{-1}(k)$ in terms of the physical solutions and the potential only. In most cases one can obtain the properties of the Jost matrix more directly from such an integral expression than from (1). There are no explicit formulae for $J(k)$ for a given potential [2]. Furthermore, little is known about the analogue of $J(k)$ in higher dimensions, and the expression presented here might also be useful for generalising $J(k)$ to higher dimensions.

From (1) we obtain

$$\begin{bmatrix}\psi_l(k,x)\\ \psi_r(k,x)\end{bmatrix}=J^{-1}(k)\begin{bmatrix}\varphi_l(k,x)\\ \varphi_r(k,x)\end{bmatrix}$$

and

$$\begin{bmatrix}\dfrac{d\psi_l}{dx}(k,x)\\ \dfrac{d\psi_r}{dx}(k,x)\end{bmatrix}=J^{-1}(k)\begin{bmatrix}\dfrac{d\varphi_l}{dx}(k,x)\\ \dfrac{d\varphi_r}{dx}(k,x)\end{bmatrix}.$$

Define the matrices $\Psi(k,x)$ and $\Phi(k,x)$ as

$$\Psi(k,x)\equiv\frac{1}{2}\begin{bmatrix}\psi_l(k,x)+\dfrac{1}{ik}\dfrac{d\psi_l}{dx}(k,x) & \psi_l(k,x)-\dfrac{1}{ik}\dfrac{d\psi_l}{dx}(k,x)\\ \psi_r(k,x)+\dfrac{1}{ik}\dfrac{d\psi_r}{dx}(k,x) & \psi_r(k,x)-\dfrac{1}{ik}\dfrac{d\psi_r}{dx}(k,x)\end{bmatrix}$$

and

$$\Phi(k,x)\equiv\frac{1}{2}\begin{bmatrix}\varphi_l(k,x)+\dfrac{1}{ik}\dfrac{d\varphi_l}{dx}(k,x) & \varphi_l(k,x)-\dfrac{1}{ik}\dfrac{d\varphi_l}{dx}(k,x)\\ \varphi_r(k,x)+\dfrac{1}{ik}\dfrac{d\varphi_r}{dx}(k,x) & \varphi_r(k,x)-\dfrac{1}{ik}\dfrac{d\varphi_r}{dx}(k,x)\end{bmatrix}.$$

Hence $\Psi(k,x)=J^{-1}(k)\Phi(k,x)$ and since $\Phi(k,0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, we obtain

$$J^{-1}(k)=\frac{1}{2}\begin{bmatrix}\psi_l(k,0)+\dfrac{1}{ik}\dfrac{d\psi_l}{dx}(k,0) & \psi_l(k,0)-\dfrac{1}{ik}\dfrac{d\psi_l}{dx}(k,0)\\ \psi_r(k,0)+\dfrac{1}{ik}\dfrac{d\psi_r}{dx}(k,0) & \psi_r(k,0)-\dfrac{1}{ik}\dfrac{d\psi_r}{dx}(k,0)\end{bmatrix}. \qquad (2)$$

The result in (2) can also be obtained in an entirely different way [3].

To obtain the integral expression for $J^{-1}(k)$, we can use the Lippmann–Schwinger equation

$$\begin{bmatrix}\psi_l(k,x)\\ \psi_r(k,x)\end{bmatrix}=\begin{bmatrix}e^{ikx}\\ e^{-ikx}\end{bmatrix}+\frac{1}{2ik}\int_{-\infty}^{\infty}dy\,\exp(ik|x-y|)V(y)\begin{bmatrix}\psi_l(k,y)\\ \psi_r(k,y)\end{bmatrix}$$

in order to obtain

$$\begin{bmatrix}\psi_l(k,0)\\ \psi_r(k,0)\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}+\frac{1}{2ik}\int_{-\infty}^{0}\mathrm{d}y\,\mathrm{e}^{-iky}V(y)\begin{bmatrix}\psi_l(k,y)\\ \psi_r(k,y)\end{bmatrix}+\frac{1}{2ik}\int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{iky}V(y)\begin{bmatrix}\psi_l(k,y)\\ \psi_r(k,y)\end{bmatrix}$$

and

$$\frac{1}{ik}\begin{bmatrix}\dfrac{\mathrm{d}\psi_l}{\mathrm{d}x}(k,0)\\[2ex] \dfrac{\mathrm{d}\psi_r}{\mathrm{d}x}(k,0)\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}+\frac{1}{2ik}\int_{-\infty}^{0}\mathrm{d}y\,\mathrm{e}^{-iky}V(y)\begin{bmatrix}\psi_l(k,y)\\ \psi_r(k,y)\end{bmatrix}-\frac{1}{2ik}\int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{iky}V(y)\begin{bmatrix}\psi_l(k,y)\\ \psi_r(k,y)\end{bmatrix}.$$

Thus we have

$$J^{-1}(k)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\frac{1}{2ik}\begin{bmatrix}\int_{-\infty}^{0}\mathrm{d}y\,\mathrm{e}^{-iky}V(y)\psi_l(k,y) & \int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{iky}V(y)\psi_l(k,y)\\ \int_{-\infty}^{0}\mathrm{d}y\,\mathrm{e}^{-iky}V(y)\psi_r(k,y) & \int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{iky}V(y)\psi_r(k,y)\end{bmatrix}.$$

Note that if $V(x)$ vanishes for $x<0$, this integral expression reduces to equation (3.31) of [2].

It is known [1, 2] that the determinant of $J^{-1}(k)$ is equal to the transmission coefficient $T(k)$. Hence, we have

$$J(k)=\frac{1}{T(k)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\frac{1}{2ikT(k)}\times\begin{bmatrix}\int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{iky}V(y)\psi_r(k,y) & -\int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{iky}V(y)\psi_l(k,y)\\ \int_{-\infty}^{0}\mathrm{d}y\,\mathrm{e}^{-iky}V(y)\psi_r(k,y) & \int_{-\infty}^{0}\mathrm{d}y\,\mathrm{e}^{-iky}V(y)\psi_l(k,y)\end{bmatrix}.$$

References

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