

LETTER TO THE EDITOR

Examples of non-uniqueness in one-dimensional inverse scattering for which $T(k) = O(k^3)$ and $O(k^4)$ as $k \rightarrow 0$

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Abstract. An example of a one-parameter family of potentials in one-dimensional non-relativistic quantum mechanics is given such that all the potentials in the family have the same scattering matrix at all energies and the transmission coefficient behaves like $T(k) = O(k^3)$ as $k \rightarrow 0$. An example where $T(k) = O(k^4)$ as $k \rightarrow 0$ is also given.

It is already known [1-4] that when one of the reflection coefficients at zero energy is unity, the inverse problem in one-dimensional non-relativistic quantum mechanics has a non-unique solution, and in fact a family of potentials can cause the same scattering at all energies [3, 4]. Examples of such families are given [4] when the transmission coefficient behaves like $T(k) = O(k)$ and $O(k^2)$ as $k \rightarrow 0$. Below we give examples of such families when $T(k) = O(k^3)$ and $O(k^4)$ as $k \rightarrow 0$.

In the matrix inversion method given by Newton [5, 6], the potentials which correspond to the scattering matrices $S(k)$ and $S'(k)$ are constructed together where

$$S(k) = \begin{bmatrix} T & R \\ L & T \end{bmatrix} \quad S'(k) = \begin{bmatrix} T & -R \\ -L & T \end{bmatrix}.$$

Newton's method is used to obtain the examples below.

For our example of the case where $T(k) = O(k^3)$ as $k \rightarrow 0$, we will consider a scattering matrix that has already been studied elsewhere: Brownstein [2] has demonstrated that the potentials

$$V_1(x) = -\delta(x) + \frac{6\theta(x)}{(x+1)^2} \quad V_2(x) = -\delta(x) + \frac{6\theta(-x)}{(x-1)^2}$$

both correspond to the scattering matrix

$$S'(k) = \begin{bmatrix} -2ik^3 & k^2 + 3 \\ k^2 + 3 & -2ik^3 \end{bmatrix} \frac{1}{(ik-1)(-2k^2-3ik+3)}$$

where $\delta(x)$ is the Dirac delta function and $\theta(x)$ is the Heaviside function. Consider the scattering matrix obtained from the above matrix by multiplying the reflection coefficients by -1 , namely

$$S(k) = \begin{bmatrix} T & R \\ L & T \end{bmatrix} = \begin{bmatrix} -2ik^3 & -k^2 - 3 \\ -k^2 - 3 & -2ik^3 \end{bmatrix} \frac{1}{(ik-1)(-2k^2-3ik+3)}$$

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Note that as $k \rightarrow 0$

$$S(k) = \begin{bmatrix} O(k^3) & 1 + O(k) \\ 1 + O(k) & O(k^3) \end{bmatrix}$$

and hence this matrix violates the L_2^1 characterisation conditions of Faddeev [7] and Deift and Trubowitz [8].

The family of potentials corresponding to $S(k)$ is given by

$$V(a, x) = \delta(x) + \frac{2\theta(x)}{(x+1+a)^2} - 2\theta(-x) \left(\frac{P'(a, x)}{P(a, x)} \right)'$$

where $a \geq 0$ is the parameter, the prime denotes the derivative with respect to the space variable x , and

$$P(a, x) = (a+1)(1-x) + a(-x + 3x^2 - 4x^3 + 3x^4 - \frac{5}{3}x^5 + \frac{1}{3}x^6).$$

The corresponding physical solutions from the left and right respectively are given by

$$\psi_l(k, a, x) = T e^{ikx} m_l(a, k, x) \theta(x) + [e^{ikx} m_r(a, -k, x) + L e^{-ikx} m_r(a, k, x)] \theta(-x)$$

$$\psi_r(k, a, x) = [e^{-ikx} m_l(a, -k, x) + R e^{ikx} m_l(a, k, x)] \theta(x) + T e^{-ikx} m_r(a, k, x) \theta(-x)$$

where

$$m_l(a, k, x) = 1 + \frac{i}{k} \frac{1}{x+1+a}$$

$$m_r(a, k, x) = 1 - \frac{i}{k} \frac{P'(a, x)}{P(a, x)} - \frac{1}{2k^2} \frac{P''(a, x)}{P(a, x)} + \frac{i}{8k^3} \frac{P'''(a, x)}{P(a, x)}$$

(we interpret $\delta(x)\theta(x) = \frac{1}{2}\delta(x)$). Since this matrix is symmetric, we can replace x by $-x$ and interchange 'left' and 'right' to obtain another family of potentials and solutions for the same scattering matrix.

As an example of the case where the transmission coefficient behaves like $T(k) = O(k^4)$ as $k \rightarrow 0$, we consider the S matrix

$$S(k) = \begin{bmatrix} T & R \\ L & T \end{bmatrix} = \frac{1}{D(k)} \begin{bmatrix} 2ik & \frac{6i}{k} + \frac{30}{k^2} - \frac{45i}{k^3} \\ -\frac{6i}{k} + \frac{30}{k^2} + \frac{45i}{k^3} & 2ik \end{bmatrix}$$

where

$$D(k) = 2ik - 4 - \frac{36i}{k} + \frac{60}{k^2} + \frac{45i}{k^3}.$$

Note that as $k \rightarrow 0$

$$S(k) = \begin{bmatrix} O(k^4) & -1 + O(k) \\ 1 + O(k) & O(k^4) \end{bmatrix}$$

and hence the L_2^1 characterisation conditions are again violated. The corresponding one-parameter family of potentials is given by

$$V(a, x) = -2\theta(x) \left(\frac{G'(a, x)}{G(a, x)} \right)' - 2\theta(-x) \left(\frac{1}{Q(a, x)} \right)'$$

where $a \geq 0$ is the parameter, the prime denotes the x derivative, and

$$G(a, x) = (x+1)^{10} + a(x+1)^3$$

$$Q(a, x) = x + \frac{1+a}{4-3a}.$$

The corresponding solutions from the left and right respectively are given by

$$\psi_l(k, a, x) = T e^{ikx} m_l(a, k, x) \theta(x) + [e^{ikx} m_r(a, -k, x) + L e^{-ikx} m_r(a, k, x)] \theta(-x)$$

$$\psi_r(k, a, x) = [e^{-ikx} m_l(a, -k, x) + R e^{ikx} m_l(a, k, x)] \theta(x) + T e^{-ikx} m_r(a, k, x) \theta(-x)$$

where

$$m_r(a, k, x) = 1 - \frac{i}{k} \frac{1}{Q(a, x)}$$

$$m_l(a, k, x) = 1 + \frac{i}{k} \frac{G'(a, x)}{G(a, x)} - \frac{1}{2k^2} \frac{G''(a, x)}{G(a, x)} - \frac{105i}{k^3} \frac{(x+1)^7}{G(a, x)} + \frac{1}{48k^3} \frac{G'''(a, x)}{G(a, x)}.$$

Note that in the above S matrix, the reflection coefficients from the left and right are negatives of each other. Hence the one-parameter family of potentials and physical solutions which correspond to the scattering matrix

$$S'(k) = \begin{bmatrix} T & -R \\ -L & T \end{bmatrix}$$

can be obtained by replacing x by $-x$ and interchanging 'left' and 'right' in the above family and solution set.

References

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