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Reconstruction of the wave speed from transmission eigenvalues for the spherically symmetric variable-speed wave equation

Tuncay Aktosun[1] **and Vassilis G Papanicolaou**[2]

[1] Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019-0408, USA
[2] Department of Mathematics, National Technical University of Athens, Zografou Campus, 157 80, Athens, Greece

E-mail: aktosun@uta.edu and papanico@math.ntua.gr

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Abstract

The unique reconstruction of a spherically symmetric wave speed v is considered in a bounded spherical region of radius b from the set of corresponding transmission eigenvalues for which the corresponding eigenfunctions are also spherically symmetric. If the integral of $1/v$ on the interval $[0, b]$ is less than b, assuming that there exists at least one v corresponding to the data, then v is uniquely reconstructed from the data consisting of such transmission eigenvalues and their 'multiplicities', where the multiplicity is defined as the multiplicity of the transmission eigenvalue as a zero of a key quantity. When that integral is equal to b, the unique reconstruction is presented when the data set contains one additional piece of information. Some similar results are presented for the unique reconstruction of the potential from the transmission eigenvalues with multiplicities for a related Schrödinger equation.

1. Introduction

Let us consider the mathematical problem [8–10]

$$\begin{cases} \Delta\Psi + \lambda\rho(\mathbf{x})\Psi = 0, & \mathbf{x} \in \Omega, \\ \Delta\Psi_0 + \lambda\Psi_0 = 0, & \mathbf{x} \in \Omega, \\ \Psi = \Psi_0, \quad \dfrac{\partial\Psi}{\partial\mathbf{n}} = \dfrac{\partial\Psi_0}{\partial\mathbf{n}}, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{1.1}$$

where Δ denotes the Laplacian, λ is the spectral parameter, Ω is a bounded and simply connected domain in $\mathbf{R}^n$ for any positive integer n with the sufficiently smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$ and the positive quantity $\rho(\mathbf{x})$ is assumed to be 1 outside Ω. Those λ-values for which there are nontrivial solutions Ψ and Ψ_0 to (1.1) are known as the transmission eigenvalues for (1.1). Obviously, $\lambda = 0$ is always a transmission eigenvalue, which we view as the trivial one. The problem described in (1.1) arises in acoustic scattering from a bounded region with $1/\sqrt{\rho(\mathbf{x})}$ denoting the wave speed v and also in electromagnetic scattering from a bounded nonhomogeneity with refractive index $\sqrt{\rho(\mathbf{x})}$ as a function of location.

The relevant direct problem involves the determination of the transmission eigenvalues when the nonhomogeneity ρ is known. The relevant inverse problem is to determine the nonhomogeneity everywhere in the given domain Ω by using an appropriate set of λ-values related to the transmission eigenvalues of (1.1). It is already known that the transmission eigenvalues for (1.1) can be determined from some far-field measurements [4, 5]. Letting $\lambda = k^2$, we see that, corresponding to each transmission eigenvalue λ, we have two k-values, namely k and $-k$.

The research field of direct and inverse problems involving transmission eigenvalues is now very active, and the related literature is growing rapidly, and hence it is impossible to provide a complete bibliography on the general topic of transmission eigenvalues. We refer the reader to [10] and the references therein to trace the important developments in the field and to [10, 15–17] and the references therein for the related inverse problem of recovery of a bounded nonhomogeneity from an appropriate set of transmission eigenvalues.

We are interested in (1.1) in the special case where Ω is a sphere of radius b centered at the origin in $\mathbf{R}^3$ and $\rho(\mathbf{x})$ is spherically symmetric, which we write as $\rho(x)$ with $x := |\mathbf{x}|$. We further consider a subset of the transmission eigenvalues for which the corresponding eigenfunctions are also spherically symmetric. As in [2], we refer to such λ-eigenvalues as the *special transmission eigenvalues* of (1.1).

Let us assume that ρ belongs to the *admissible class* $\mathcal{A}$, by which we mean that $\rho(x)$ for $x \in [0, +\infty)$ is positive, continuously differentiable and equal to 1 for $x \geqslant b$, and that $\rho''(x)$ exists almost everywhere, where the prime denotes the x-derivative.

Let us introduce our key quantity

$$D(k) := \frac{\sin(kb)}{k}\,\phi'(b;k) - \cos(kb)\,\phi(b;k), \tag{1.2}$$

where $\phi(x;k)$ is the unique solution to

$$\begin{cases} \phi'' + k^2\rho(x)\phi = 0, & 0 < x < b, \\ \phi(0) = 0, \quad \phi'(0) = 1. \end{cases} \tag{1.3}$$

Note that $D(-k) = D(k)$, and hence $D(k)$ is actually a function of k^2.

The following result is known [2], and here we restate it in terms of k rather than λ for later use.

Theorem 1.1. *Consider the special case of (1.1), with Ω being the three-dimensional ball of radius b centered at the origin, where only spherically symmetric wavefunctions are allowed and it is assumed that such wavefunctions are continuous in the closure of Ω. Suppose that ρ belongs to the admissible class $\mathcal{A}$. Then, the corresponding special transmission eigenvalues of (1.1) coincide with the k^2-values related to the zeros of the quantity $D(k)$ defined in (1.2).*

As in [2], with each nonzero special transmission eigenvalue k_n^2 of (1.1), we associate a multiplicity, which is the same as the multiplicity of k_n as a zero of $D(k)$. When ρ belongs to the admissible class $\mathcal{A}$, it is known that $D(k)$ is entire in k^2 and has the representation [2]

$$D(k) = \gamma E(k), \quad E(k) := k^{2d} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_n^2}\right), \tag{1.4}$$

where γ is a real constant, k_n^2 for $n \in \mathbf{N}$ correspond to the nonzero transmission eigenvalues and d is the multiplicity of the trivial zero transmission eigenvalue. Here we use $\mathbf{N} := \{1, 2, 3, \ldots\}$.

It is known that some k_n^2 may be repeated [2], d is at least 1 and the actual value of d is determined by ρ. In the trivial case $\rho(x) \equiv 1$, we have $\gamma = 0$ and hence $D(k) \equiv 0$.

In analyzing a typical transmission eigenvalue problem, one has to deal with a nonself-adjoint eigenvalue problem [2, 8–10], and hence in general we cannot expect that all eigenvalues will be real. The nonself-adjointness in a natural way forces us to consider complex transmission eigenvalues and also transmission eigenvalues with multiplicities. The mathematical necessity of including complex and nonsimple transmission eigenvalues certainly complicates the analysis. The experimental determination of such transmission eigenvalues, especially in the presence of nonsimple eigenvalues, presents a challenge and it is an interesting and important open question how to measure them and how to determine their multiplicities.

The constant a defined as

$$a := \int_0^b \mathrm{d}x\, \sqrt{\rho(x)} \tag{1.5}$$

has the physical interpretation as the travel time for the wave to move from $x = 0$ to $x = b$. In [2], we have presented the following uniqueness results regarding the determination of ρ corresponding to the special transmission eigenvalues of (1.1) in the spherically symmetric case. Note that it is assumed that we know the value of b, which is a reasonable assumption as far as the applications are concerned.

Theorem 1.2. *Consider the special case of (1.1), with Ω being the three-dimensional ball of radius b centered at the origin, where only spherically symmetric wavefunctions are allowed and it is assumed that such wavefunctions are continuous in the closure of Ω. Suppose that our data set consists of the corresponding special transmission eigenvalues with their multiplicities, and assume that there exists at least one corresponding ρ in the admissible class $\mathcal{A}$. Let a be the constant defined in (1.5). We have the following.*

(a) If $a < b$, then our data set uniquely determines ρ; in other words, if both ρ_1 and ρ_2 correspond to our data, then we must have $\rho_1 \equiv \rho_2$.

(b) If $a = b$, then our data set along with the value of γ appearing in (1.4) uniquely determines ρ; in other words, if both ρ_1 and ρ_2 correspond to our data and the same γ, then we must have $\rho_1 \equiv \rho_2$.

In this paper, we give an alternate proof of theorem 1.2, a proof different from that given in [2], by providing an algorithm to reconstruct ρ from the relevant data set. Our paper is organized as follows. In section 2, we present some preliminary results that are needed later on; the key result in theorem 2.3 is crucial for the unique reconstruction of ρ given in sections 3 and 4. In section 3, we present the alternate proof of theorem 1.2(a) and the reconstruction when $a < b$. In section 4, we present the alternate proof of theorem 1.2(b) and the reconstruction when $a = b$. In section 5, we consider the analogous problem for the Schrödinger equation. We present theorem 5.2, which is the analogue of theorem 2.3 and which plays a key role in the reconstruction of the potential in the Schrödinger equation. We then give an alternate proof of the uniqueness result of theorem 5.4, a proof different from that given in [2], by providing a reconstruction procedure for the potential in terms of the data set consisting of the corresponding transmission eigenvalues with their multiplicities and the parameter $\tilde{\gamma}$ appearing in (5.15). In section 5, we also provide an illustrative example showing that we cannot have uniqueness if the data set does not include $\tilde{\gamma}$; let us mention, though, that the reconstructed potential is outside the admissible class of potentials $\tilde{A}$ considered in section 5 of our paper. Finally, in section 6, we present some explicit examples where we display $\rho(x)$ and the corresponding quantities γ and $E(k)$ appearing in (1.4); however, in each of those examples, $\rho'(x)$ has a jump discontinuity and hence is outside the admissible class $\mathcal{A}$. In one of the examples presented in section 6, it is shown that the same $E(k)$ yields two distinct $\rho(x)$ quantities, for one of which we have $a < b$ and for the other we have $b > a$.

In the recovery algorithms given in sections 3–5, we solve some basic Riemann–Hilbert problems and use basic facts related to their unique solutions. For the benefit of the readers who are unfamiliar with the theory of Riemann–Hilbert problems, we summarize below their formulation and unique solutions relevant to our paper. Let us use $\mathbf{C}$ for the complex plane, $\mathbf{C}^+$ for the open upper-half-complex plane, $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+ \cup \mathbf{R}$, $\mathbf{C}^-$ for the open lower-half-complex plane and $\overline{\mathbf{C}^-}$ for $\mathbf{C}^- \cup \mathbf{R}$.

The idea behind solving a basic Riemann–Hilbert problem is to determine a sectionally analytic function on $\mathbf{C}$ by determining its sections on $\mathbf{C}^+$ and on $\mathbf{C}^-$, respectively, from its jump value on the real axis $\mathbf{R}$. Mathematically, we need to solve the functional equation

$$F(k) - F(-k) = G(k), \qquad k \in \mathbf{R}, \tag{1.6}$$

where $G(k)$ is relevant only for real values of k and it indicates the jump. In other words, given $G(k)$ for $k \in \mathbf{R}$, we need to determine $F(k)$ for $k \in \mathbf{C}^+$ in such a way that $F(k)$ is analytic in $k \in \mathbf{C}^+$, continuous in $k \in \overline{\mathbf{C}^+}$ and $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Certainly, we then have $F(-k)$ analytic in $k \in \mathbf{C}^-$, continuous in $k \in \overline{\mathbf{C}^-}$ and $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^-}$. In general, $G(k)$ may not have any extension off the real axis, but even if it does only the values of $G(k)$ for $k \in \mathbf{R}$ are relevant and needed in solving (1.6). For the unique solvability of (1.6), it is sufficient to assume that $G(k)$ behaves like $O(1/k)$ as $k \to \pm\infty$ on the real axis $\mathbf{R}$ and that $G(k)$ is Hölder continuous on $\mathbf{R}$ with a positive index α. The latter condition is expressed as

$$|G(k_1) - G(k_2)| \leqslant C\, |k_1 - k_2|^{\alpha}, \qquad k_1, k_2 \in \mathbf{R}, \tag{1.7}$$

for some positive constant C independent of k. In the special case $\alpha = 1$, the condition given in (1.7) is known as the Lipschitz continuity of $G(k)$ on $\mathbf{R}$. In fact, in our paper the relevant $G(k)$ is Lipschitz continuous on $\mathbf{R}$ due to the fact that $G(k)$ has an analytic continuation from $k \in \mathbf{R}$ to $k \in \mathbf{C}$; even though the relevant $G(k)$ is bounded on $\mathbf{R}$ and decays as $O(1/k)$ as $k \to \pm\infty$ on $\mathbf{R}$, in general, $G(k)$ grows exponentially as $k \to \infty$ in $\mathbf{C}^+$ and in $\mathbf{C}^-$, but as already stated that unboundedness off the real axis is irrelevant in the analysis of the Riemann–Hilbert problem given in (1.6). Under the two aforementioned sufficiency conditions on $G(k)$ for $k \in \mathbf{R}$, the Riemann–Hilbert problem given in (1.6) is uniquely solvable and the analytic section $F(k)$ defined on $k \in \overline{\mathbf{C}^+}$ is explicitly expressed in terms of the values of $G(k)$ known for $k \in \mathbf{R}$ as

$$F(k) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{G(t)}{t - k - \mathrm{i}0^+}, \qquad k \in \overline{\mathbf{C}^+}, \tag{1.8}$$

where one can ignore $\mathrm{i}0^+$ in (1.7) if $k \in \mathbf{C}^+$ and one can interpret the presence of $\mathrm{i}0^+$ by stating that $F(k)$ for $k \in \mathbf{R}$ must be obtained by first evaluating the integral in (1.8) when $k \in \mathbf{C}^+$ and then letting k approach its real value from $\mathbf{C}^+$. When $G(k)$ satisfies the two relevant conditions on $\mathbf{R}$, it may sometimes be possible to solve (1.6) readily by finding a familiar function $F(k)$ satisfying (1.6) with the appropriate properties on $\overline{\mathbf{C}^+}$. Since the existence and uniqueness are ensured, we can then conclude that the function $F(k)$ must satisfy (1.8). We refer the reader to [11, 13] for further information on Riemann–Hilbert problems and their solutions.

In our paper, the Riemann–Hilbert problem stated in (1.6) arises in (3.4), (4.5) and (5.14). For example, the basic idea behind solving the Riemann–Hilbert problem in (4.5) is to split $2\mathrm{i}kD(k)$ into the two pieces $Q(k)$ and $Q(-k)$, where we know that $2\mathrm{i}kD(k)$ has the behavior $O(1/k)$ as $k \to \pm\infty$ on $\mathbf{R}$ and is Lipschitz continuous for $k \in \mathbf{R}$. In our specific case, it turns out that $2\mathrm{i}kD(k)$ has an analytic extension in k to the entire complex plane $\mathbf{C}$ with exponential growth as $k \to \infty$ in $\mathbf{C}$ except when $k \to \pm\infty$ on the real axis. The splitting is such that $Q(k)$ is analytic in $k \in \mathbf{C}^+$ and bounded in $\overline{\mathbf{C}^+}$ and in fact $Q(k) = O(1/k)$ as $k \to \infty$ in

$\overline{\mathbf{C}^+}$. Similarly, the other piece $Q(-k)$ is analytic in $k \in \mathbf{C}^-$ and bounded in $\overline{\mathbf{C}^-}$ and in fact $Q(-k) = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^-}$.

In sections 3 and 4, the cases $a < b$ and $a = b$, respectively, are analyzed, where a and b are the constants appearing in (1.5). In both cases, we show that the relevant $G(k)$ in (1.6) can be split into the appropriate functions $F(k)$ and $F(-k)$ in such a way that we are able to recognize $F(k)$ and express it explicitly in terms of a familiar spectral function, and hence we are able to reconstruct the bounded nonhomogeneity from the explicitly constructed $F(k)$. On the other hand, when $a > b$, even though we know that the relevant Riemann–Hilbert given in (1.6) is uniquely solvable, we are unable to express the corresponding $F(k)$ explicitly in terms of a familiar spectral function yielding the nonhomogeneity. Thus, our reconstruction method exploiting the relevant Riemann–Hilbert given in (1.6) does not seem to yield the nonhomogeneity when $a > b$. In other words, the case $a > b$ is an open problem, and it is not known in that case whether the nonhomogeneity can be recovered by a method similar to that used in sections 3 and 4.

2. Preliminaries

We consider the extension of the differential equation in (1.3) to the half-line $\mathbf{R}^+ := (0, +\infty)$, namely

$$\psi'' + k^2 \rho(x)\, \psi = 0, \qquad x \in \mathbf{R}^+, \tag{2.1}$$

where $\rho(x)$ belongs to the admissible class $\mathcal{A}$ and hence $\rho(x) \equiv 1$ for $x \geqslant b$. Let us define the travel-time coordinate y as

$$y(x) := \int_0^x \mathrm{d}s\, \sqrt{\rho(s)}, \qquad x \in [0, +\infty), \tag{2.2}$$

and note that (1.5) and (2.2) imply that $a = y(b)$. We remark that (2.2) yields

$$y(x) = x + a - b, \qquad x \geqslant b.$$

Let $f(x; k)$ denote the Jost solution to (2.1), i.e. $f(x; k)$ satisfies (2.1) and

$$f(x; k) = \mathrm{e}^{\mathrm{i}kx}, \quad f'(x; k) = \mathrm{i}k\,\mathrm{e}^{\mathrm{i}kx}, \qquad x \geqslant b. \tag{2.3}$$

We also note that, when $k = 0$, (2.1) reduces to $\psi''(x; 0) = 0$, and hence with the help of (2.3) we obtain

$$f(x; 0) = 1, \qquad x \in [0, +\infty). \tag{2.4}$$

Via a Liouville transformation, (2.1) can be transformed into a Schrödinger equation. In other words, if we let

$$\tilde{f}(y; k) := [\rho(x)]^{1/4}\, \mathrm{e}^{-\mathrm{i}k(b-a)} f(x; k), \tag{2.5}$$

where y is related to x as in (2.2), then $\tilde{f}(y; k)$ becomes the Jost solution to the Schrödinger equation

$$\tilde{f}''(y; k) + k^2 \tilde{f}(y; k) = V(y)\, \tilde{f}(y; k), \qquad y \in \mathbf{R}^+, \tag{2.6}$$

where the prime now denotes the y-derivative and we have

$$\tilde{f}(y; k) = \mathrm{e}^{\mathrm{i}ky}, \quad \tilde{f}'(y; k) = \mathrm{i}k\,\mathrm{e}^{\mathrm{i}ky}, \qquad y \geqslant a, \tag{2.7}$$

$$V(y) := V(y(x)) = \frac{\rho''(x)}{4[\rho(x)]^2} - \frac{5[\rho'(x)]^2}{16[\rho(x)]^3}. \tag{2.8}$$

With the help of (2.2) and multiple use of the chain rule in taking the derivatives on the right-hand side of (2.8), we write (2.8) as the second-order linear differential equation

$$\frac{\mathrm{d}^2([\rho(x(y))]^{1/4})}{\mathrm{d}y^2} = V(y)[\rho(x(y))]^{1/4}, \qquad y \in \mathbf{R}^+. \tag{2.9}$$

Note that $\rho(x(y))$ satisfies

$$\rho(x(y))|_{y=a} = 1, \quad \left.\frac{\mathrm{d}[\rho(x(y))]}{\mathrm{d}y}\right|_{y=a} = 0,$$

and hence

$$[\rho(x(y))]^{1/4}|_{y=a} = 1, \quad \left.\frac{\mathrm{d}[\rho(x(y))]^{1/4}}{\mathrm{d}y}\right|_{y=a} = 0.$$

As we shall see, (2.9) will be useful in constructing explicit illustrative examples of $\rho(x)$ and $V(y)$ from some appropriate sets of data.

From (2.4) and (2.5) we conclude the following result.

Corollary 2.1. *Assume that $\rho(x)$ belongs to the admissible class $\mathcal{A}$. Let $\tilde{f}(y;k)$ be the corresponding Jost solution to (2.6) with $V(y)$ as in (2.8). Then, we have*

$$\tilde{f}(y;0) = [\rho(x(y))]^{1/4}, \qquad y \in [0, +\infty). \tag{2.10}$$

Let us remark that, when $\rho(x)$ is in the admissible class $\mathcal{A}$, we have [6, 12, 14, 18]

$$f(x;-k) = f(x;k)^*, \qquad k \in \mathbf{R}, \tag{2.11}$$

where the asterisk denotes complex conjugation. The result in (2.11) will be useful in establishing (2.18) and in the reconstruction of $\rho(x)$ from a data set containing the associated transmission eigenvalues.

The results in the following proposition are already known, but we state them with a brief proof for the convenience of the reader.

Proposition 2.2. *Assume that $\rho(x)$ belongs to the admissible class $\mathcal{A}$. Let $V(y)$ be the potential obtained from $\rho(x)$ as in (2.8) and $\tilde{f}(y;k)$ be the corresponding Jost solution to (2.6) satisfying (2.7). Then, we have the following.*

(a) The potential $V(y)$ belongs to the admissible class $\tilde{A}$ described in section 5. Consequently, the Jost solution $\tilde{f}(y;k)$ has the properties outlined in proposition 5.1.

(b) The differential equation (2.1) with the Dirichlet boundary condition $\psi(0) = 0$ cannot have, for any negative value of k^2, any solutions that are square integrable in $x \in \mathbf{R}^+$.

(c) The corresponding half-line Schrödinger equation (2.6) with the Dirichlet boundary condition at $y = 0$ cannot have any nontrivial solutions that are square integrable in $y \in \mathbf{R}^+$. Hence, the corresponding Schrödinger operator has no bound states, and therefore $\tilde{f}(0;k)$ cannot vanish for $k \in \overline{\mathbf{C}^+} \setminus \{0\}$.

Proof. When ρ is in the admissible class $\mathcal{A}$, the corresponding $V(y)$, as seen from (2.2) and (2.8), is real valued, compactly supported and integrable. Thus, $V(y)$ belongs to the admissible class $\tilde{A}$ described in section 5, and (a) is proved. Note that (c) directly follows from (b) because of the Liouville transformation given in (2.5), proposition 5.1, and the fact that the bound states correspond to the square-integrable solutions of the relevant differential equations. Hence, we only need to prove (b). Because of the self-adjointness of the corresponding Schrödinger operator, any existing bound states may occur only at negative values of k^2. If $\psi(x)$ were a nontrivial square-integrable solution to (2.1) at some negative value of k^2, i.e. if $\psi \in L^2(\mathbf{R}^+)$, then from (2.1) we would obtain $\psi'' \in L^2(\mathbf{R}^+)$, and hence, e.g. via Fourier transforms, we would have $\psi' \in L^2(\mathbf{R}^+)$. By the Cauchy–Schwarz inequality, we would then have $\psi\psi' \in L^1(\mathbf{R}^+)$. However, that would imply the existence of a sequence x_n converging to $+\infty$ such that

$$|\psi'(x_n)\,\psi(x_n)| \to 0, \qquad x_n \to +\infty.$$

From (2.1) through integration, we would then obtain

$$\int_0^{x_n} dx\,[\psi(x)]^*\psi''(x) + k^2\int_0^{x_n} dx\,\rho(x)\,|\psi(x)|^2 = 0. \tag{2.12}$$

Using the Dirichlet condition $\psi(0) = 0$ and an integration by parts on the first integral in (2.12), we would obtain

$$[\psi(x_n)]^*\psi'(x_n) - [\psi(0)]^*\psi'(0^+) - \int_0^{x_n} dx\,|\psi'(x)|^2 + k^2\int_0^{x_n} dx\,\rho(x)\,|\psi(x)|^2 = 0. \tag{2.13}$$

Since $\psi(0) = 0$, from (2.1) it follows that $\psi''(0^+) = 0$ and hence $\psi'(0^+)$ is finite. Thus, letting $x_n \to +\infty$, from (2.13) we would obtain

$$-\int_0^{\infty} dx\,|\psi'(x)|^2 + k^2\int_0^{\infty} dx\,\rho(x)\,|\psi(x)|^2 = 0, \tag{2.14}$$

which is a contradiction because the left-hand side of (2.14) would be strictly negative due to the fact that $k^2 < 0$, $\psi(x)$ is assumed to be a nontrivial solution, and $\rho(x) > 0$ for $x \in \mathbf{R}^+$. □

Let $\phi(x;k)$ and $\tilde\phi(y;k)$ denote the solutions to the initial-value problems on the half-line that are respectively given by

$$\begin{cases} \phi''(x;k) + k^2\rho(x)\,\phi(x;k) = 0, & x \in \mathbf{R}^+,\\ \phi(0;k) = 0, \quad \phi'(0;k) = 1, \end{cases} \tag{2.15}$$

$$\begin{cases} \tilde\phi''(y;k) + k^2\tilde\phi(y;k) = V(y)\,\tilde\phi(y;k), & y \in \mathbf{R}^+,\\ \tilde\phi(0;k) = 0, \quad \tilde\phi'(0;k) = 1, \end{cases} \tag{2.16}$$

where $V(y)$ is related to $\rho(x)$ as in (2.8) and x and y are related to each other as in (2.2). Note that (2.15) and (2.16) are uniquely solvable [7] and that the corresponding solutions are entire in k^2. We remark that (2.15) is actually the extension of (1.3) from the interval $(0, b)$ to $\mathbf{R}^+$ and hence we use $\phi(x;k)$ to denote the unique solution to both (1.3) and (2.15).

The result in the following theorem is crucial for the reconstruction of $\rho(x)$ from the data containing transmission eigenvalues, and it will be used in sections 3 and 4.

Theorem 2.3. *Let $\rho(x)$ belong to the admissible class $\mathcal{A}$. Then, the quantity $D(k)$ defined in (1.2) is related to the Jost solution $f(x;k)$ to (2.1) as*

$$D(k) = \frac{f(0;k) - f(0;-k)}{2ik}, \qquad k \in \mathbf{C}. \tag{2.17}$$

For real k-values, we have

$$\mathrm{Im}[f(0;k)] = kD(k), \qquad k \in \mathbf{R}, \tag{2.18}$$

where Im *denotes the imaginary part.*

Proof. Let us express the solution $\phi(x;k)$ to (2.15) as a linear combination of the linearly independent solutions $f(x;k)$ and $f(x;-k)$ to (2.1), where $f(x;k)$ is the Jost solution satisfying (2.3). We have

$$\begin{bmatrix} \phi(x;k)\\ \phi'(x;k) \end{bmatrix} = \begin{bmatrix} f(x;k) & f(x;-k)\\ f'(x;k) & f'(x;-k) \end{bmatrix} \begin{bmatrix} c_1(k)\\ c_2(k) \end{bmatrix}, \tag{2.19}$$

where the coefficients $c_1(k)$ and $c_2(k)$ are independent of x and are yet to be determined. With the help of (2.3) and the second line of (2.15), we evaluate (2.19) at $x = b$ and $x = 0$, respectively, and we obtain

$$\begin{bmatrix} \phi(b;k) \\ \phi'(b;k) \end{bmatrix} = \begin{bmatrix} e^{ikb} & e^{-ikb} \\ ik\, e^{ikb} & -ik\, e^{-ikb} \end{bmatrix} \begin{bmatrix} c_1(k) \\ c_2(k) \end{bmatrix}, \tag{2.20}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f(0;k) & f(0;-k) \\ f'(0;k) & f'(0;-k) \end{bmatrix} \begin{bmatrix} c_1(k) \\ c_2(k) \end{bmatrix}. \tag{2.21}$$

From (2.20) and (2.21), by eliminating $c_1(k)$ and $c_2(k)$, we obtain

$$\begin{bmatrix} \phi(b;k) \\ \phi'(b;k) \end{bmatrix} = \begin{bmatrix} e^{ikb} & e^{-ikb} \\ ik\, e^{ikb} & -ik\, e^{-ikb} \end{bmatrix} \begin{bmatrix} f(0;k) & f(0;-k) \\ f'(0;k) & f'(0;-k) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.22}$$

Let $[g;h] := gh' - g'h$ denote the Wronskian of any two functions g and h. It is known and can also directly be verified that the Wronskian of any two solutions to (2.1) is independent of x [7]. With the help of (2.1) and (2.3), we obtain

$$[f(x;k); f(x;-k)] = -2ik,$$

and hence

$$\begin{bmatrix} f(0;k) & f(0;-k) \\ f'(0;k) & f'(0;-k) \end{bmatrix}^{-1} = \frac{-1}{2ik} \begin{bmatrix} f'(0;-k) & -f(0;-k) \\ -f'(0;k) & f(0;k) \end{bmatrix}. \tag{2.23}$$

Using (2.23) in (2.22), we obtain

$$\begin{bmatrix} \phi(b;k) \\ \phi'(b;k) \end{bmatrix} = \frac{-1}{2ik} \begin{bmatrix} e^{ikb} & e^{-ikb} \\ ik\, e^{ikb} & -ik\, e^{-ikb} \end{bmatrix} \begin{bmatrix} -f(0;-k) \\ f(0;k) \end{bmatrix}. \tag{2.24}$$

Writing (1.2) as the matrix product

$$D(k) = \begin{bmatrix} -\cos(kb) & \dfrac{\sin(kb)}{k} \end{bmatrix} \begin{bmatrix} \phi(b;k) \\ \phi'(b;k) \end{bmatrix}, \tag{2.25}$$

and using (2.24) in (2.25), after some simplification we obtain (2.17). Finally, using (2.11) in (2.17), we obtain (2.18). □

3. Reconstruction of $\rho(x)$ when $a < b$

In this section, we give a proof of theorem 1.2(a) by providing a reconstruction algorithm for the unique recovery of $\rho(x)$ in terms of the data consisting of the corresponding special transmission eigenvalues with their multiplicities. Thus, our data set is equivalent to the set of zeros (including the multiplicities of those zeros) of the quantity $E(k)$ defined in (1.4). Equivalently, the knowledge of our data is equivalent to the knowledge of $E(k)$. Using (2.5) in (2.17), with the help of (1.4), we obtain

$$E(k) = \frac{e^{ik(b-a)}\tilde{f}(0;k) - e^{-ik(b-a)}\tilde{f}(0;-k)}{2ik\gamma[\rho(0)]^{1/4}}. \tag{3.1}$$

Note that we assume $a < b$, where a is the constant defined in (1.5). For the reconstruction, we assume that the existence problem is solved, i.e. we assume the existence of at least one ρ in the admissible class $\mathcal{A}$ corresponding to our data. The uniqueness aspect in the recovery of $\rho(x)$ follows from the uniqueness in each of the reconstruction steps outlined below.

(a) When $\rho(x)$ is in the admissible class $\mathcal{A}$, as stated in theorem 2.2(a), the corresponding Jost solution $\tilde{f}(y;k)$ given in (2.5) satisfies the properties listed in proposition 5.1, and in particular, (5.3) holds. Thus, using (5.3) in (3.1) we obtain the large-k asymptotics of $E(k)$ for $k \in \mathbf{R}$ as

$$E(k) = \frac{\sin k(b-a)}{k\gamma[\rho(0)]^{1/4}} + O\left(\frac{1}{k^2}\right), \qquad k \to \pm\infty. \tag{3.2}$$

Since $(b-a)$ is assumed to be positive, from (3.2) we can determine the value of $(b-a)$ and hence a and also the value of $\gamma\,[\rho(0)]^{1/4}$. Note that we do not have the values of γ and $[\rho(0)]^{1/4}$ separately, but as we will see this does not create an obstacle for the reconstruction of $\rho(x)$.

(b) Letting

$$P(k) := \tilde{f}(0;k) - 1, \tag{3.3}$$

we write (3.1) as

$$e^{ik(b-a)}P(k) - e^{-ik(b-a)}P(-k) = \varphi(k), \qquad k \in \mathbf{R}, \tag{3.4}$$

where we have defined

$$\varphi(k) := 2i[kE(k)\gamma[\rho(0)]^{1/4} - \sin k(b-a)]. \tag{3.5}$$

By the previous step given in (a) above, we know that our data set uniquely determines the value of $\varphi(k)$. Furthermore, using (3.2) in (3.5), we obtain $\varphi(k) = O(1/k)$ as $k \to \pm\infty$. When ρ is in the admissible class $\mathcal{A}$, it is known that $E(k)$ is entire in k^2 and hence also in k. The Lipschitz continuity of $\varphi(k)$ for $k \in \mathbf{R}$ follows from the fact that the right-hand side in (3.5) has an analytic extension to the entire complex plane and that $\varphi(k) = O(1/k)$ as $k \to \pm\infty$.

(c) Note that (3.4) constitutes a Riemann–Hilbert problem on the complex plane where the function $\varphi(k)$ is specified for $k \in \mathbf{R}$ and it satisfies the Lipschitz continuity on $\mathbf{R}$ and behaves as $O(1/k)$ as $k \to \pm\infty$ on $\mathbf{R}$. The goal is to obtain $e^{ik(b-a)}P(k)$ and $e^{-ik(b-a)}P(-k)$ in such a way that $e^{ik(b-a)}P(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$ and $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Since $b > a$, those properties of $e^{ik(b-a)}P(k)$ follow if and only if $P(k)$ satisfies those properties. The unique solvability of (3.4) follows from the Lipschitz continuity of $\varphi(k)$ on $\mathbf{R}$ and the fact that $\varphi(k) = O(1/k)$ as $k \to \pm\infty$. When ρ in the admissible class, we already know from proposition 5.1 that $\tilde{f}(0;k)$ is analytic in $k \in \mathbf{C}^+$, is continuous in $k \in \overline{\mathbf{C}^+}$ and satisfies (5.3). Thus, the function $P(k)$ given in (3.3) helps us to obtain the unique solution to (3.4). As we have indicated in section 1, the unique solution to the Riemann–Hilbert problem given in (3.4) is then expressed with the help of (1.8) as

$$e^{ik(b-a)}P(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \frac{\varphi(t)}{t-k-i0^+}, \qquad k \in \overline{\mathbf{C}^+}. \tag{3.6}$$

We see from (3.3) and (3.6) that

$$\tilde{f}(0;k) = 1 + \frac{e^{-ik(b-a)}}{2\pi i}\int_{-\infty}^{\infty} dt\, \frac{\varphi(t)}{t-k-i0^+}, \qquad k \in \overline{\mathbf{C}^+}.$$

(d) Having constructed $\tilde{f}(0;k)$ from $E(k)$, we remark that, as stated in theorem 2.2(c), $\tilde{f}(0;k)$ does not have any zeros on the positive imaginary axis in the complex k-plane and hence the corresponding half-line Schrödinger equation with the Dirichlet boundary condition has no bound states. Thus, we can use the Marchenko procedure described in proposition 5.1 to uniquely reconstruct $V(y)$ of (2.8) and the Jost solution $\tilde{f}(y;k)$ from the already reconstructed $\tilde{f}(0;k)$. This is done by first constructing the scattering matrix $\tilde{S}(k)$ from $\tilde{f}(0;k)$ as in (5.5). Next, the Marchenko kernel $M(\xi)$ is constructed as in (5.9), but without the summation term there due to the fact that there are no bound states. Then, the Marchenko integral equation (5.8) is uniquely solved for $K(y, \xi)$, and the quantities $V(y)$ and $\tilde{f}(y; k)$ are uniquely recovered as in (5.10) and (5.11), respectively.

(e) Having recovered $\tilde{f}(y; k)$, we obtain $\rho(x(y))$ from $\tilde{f}(y; 0)$ with the help of (2.10), namely

$$\rho(x(y)) = [\tilde{f}(y; 0)]^4, \qquad y \in [0, +\infty). \tag{3.7}$$

Our next task is to obtain ρ in terms of x by establishing the relationship between x and y. From (2.2), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\rho(x(y))}. \tag{3.8}$$

Hence, from (2.2) and (3.8), we obtain the first-order, separable ordinary differential equation

$$\frac{\mathrm{d}y}{[\tilde{f}(y; 0)]^2} = \mathrm{d}x, \qquad x \in \mathbf{R}^+, \tag{3.9}$$

with the initial condition $y(0) = 0$. By integrating (3.9), the relationship between x and y is obtained as

$$x = \int_0^y \frac{\mathrm{d}s}{[\tilde{f}(s; 0)]^2}, \qquad y \in [0, +\infty). \tag{3.10}$$

Since $\rho(x)$ is assumed to be positive, from (3.10) it follows that the mapping $x \mapsto y$ is one-to-one and onto on $\mathbf{R}^+$. Having x as a function of y in (3.10), we can invert it to obtain y as a function of x. Thus, by using (3.10) in (3.7), we recover $\rho(x)$ in terms of x as

$$\rho(x) = [\tilde{f}(y(x), 0)]^4, \qquad x \in [0, +\infty).$$

Thus, the reconstruction of $\rho(x)$ for $x \in [0, +\infty)$ from $E(k)$ for $k \in \mathbf{R}$ is accomplished.

Finally, let us note that our procedure yields the value of the constant γ from the knowledge of $E(k)$. This is because we already have the value of $\gamma\, [\rho(0)]^{1/4}$ from (3.2) and the value of $\rho(0)$ from (3.7) evaluated at $y = 0$.

4. Reconstruction of $\rho(x)$ when $a = b$

In this section, we consider the case $a = b$, where a and b are the quantities appearing in (1.5). We give an independent proof of theorem 1.2(b) by providing a reconstruction algorithm for $\rho(x)$ from the data consisting of the corresponding special transmission eigenvalues with their multiplicities and the constant γ appearing in (1.4). By theorem 1.1, the knowledge of our data is equivalent to knowing the zeros (with multiplicities) of the quantity $D(k)$ given in (1.4) as well as the value of γ there. Hence, our data set is equivalent to the knowledge of $D(k)$.

As seen from (1.4) and (3.2), if $D(k) = O(1/k^2)$ as $k \to \pm\infty$, we can deduce that $a = b$. When $a = b$, let us outline the unique recovery of $\rho(x)$ from $D(k)$ given in (1.4).

(a) Since $a = b$, from (1.4) and (3.1) we see that

$$2\mathrm{i}kD(k) = \frac{\tilde{f}(0; k) - \tilde{f}(0; -k)}{[\rho(0)]^{1/4}}, \qquad k \in \mathbf{C}. \tag{4.1}$$

On the other hand, from (2.10) we know that

$$\tilde{f}(0; 0) = [\rho(0)]^{1/4}, \tag{4.2}$$

and hence we rewrite (4.1) as

$$2\mathrm{i}kD(k) = \frac{\tilde{f}(0; k) - \tilde{f}(0; -k)}{\tilde{f}(0; 0)}, \qquad k \in \mathbf{C}. \tag{4.3}$$

We know from (4.2) that $\tilde{f}(0; 0)$ is real and in fact positive because $\rho(0) > 0$. When $\rho(x)$ is in the admissible class $\mathcal{A}$, by using (5.3) and (5.4) in (4.1), we conclude that $2ikD(k) = O(1/k)$ as $k \to \pm\infty$ on $\mathbf{R}$. Furthermore, we know that $D(k)$ has an analytic extension to the entire complex plane $\mathbf{C}$ and hence $2ikD(k)$ is Lipschitz continuous on $\mathbf{R}$. As seen from (5.3) and (5.4), $2ikD(k)$ is unbounded as $k \to \infty$ in $\mathbf{C}^+$ and in $\mathbf{C}^-$, but for the analysis of the Riemann–Hilbert problem to be studied we need the large-k asymptotics of $2ikD(k)$ only on the real axis.

(b) Letting

$$Q(k) := \frac{\tilde{f}(0; k) - 1}{\tilde{f}(0; 0)}, \tag{4.4}$$

we write (4.3) for real k-values as

$$Q(k) - Q(-k) = 2ikD(k), \qquad k \in \mathbf{R}. \tag{4.5}$$

Note that (4.5) constitutes a Riemann–Hilbert problem on the complex plane where the function $2ikD(k)$ is specified for $k \in \mathbf{R}$ and it is Lipschitz continuous on $\mathbf{R}$ and behaves as $O(1/k)$ as $k \to \pm\infty$ on $\mathbf{R}$. The goal is to obtain $Q(k)$ and $Q(-k)$ in such a way that $Q(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$ and $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Since the unique solvability of (4.5) is ensured by the two relevant properties of $2ikD(k)$ stated in (a), we know that the function given in (4.4) must be that unique solution. The corresponding properties of $Q(k)$ are deduced from (4.4) by using the relevant properties of $\tilde{f}(0; k)$ given in proposition 5.1. In particular, using proposition 5.1(a) we establish the analyticity of $Q(k)$ in $\mathbf{C}^+$; using in (4.4) proposition 5.1(a) and the fact that $\tilde{f}(0; 0) > 0$, we conclude the continuity of $Q(k)$ in $\overline{\mathbf{C}^+}$; using (5.3) in (4.4), we obtain $Q(k) = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Therefore, as indicated in (1.8), the unique solution to the Riemann–Hilbert problem given in (4.5) satisfies

$$Q(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{2itD(t)}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.6}$$

We see from (4.4) and (4.6) that

$$\tilde{f}(0; k) = 1 + \frac{\tilde{f}(0; 0)}{\pi} \int_{-\infty}^{\infty} dt \frac{tD(t)}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.7}$$

(c) Let us remark that (4.7) also follows from the Schwarz integral formula for the half-plane [1] by using the following argument. As stated earlier, $Q(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$ and $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. For real k-values, from (4.5) and (5.2), we obtain

$$kD(k) = \text{Im}[Q(k)], \qquad k \in \mathbf{R}. \tag{4.8}$$

Thus, we can construct $Q(k)$ for $k \in \overline{\mathbf{C}^+}$ from its imaginary part known for $k \in \mathbf{R}$ by using the Schwarz integral formula [1]

$$Q(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{\text{Im}[Q(t)]}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.9}$$

Hence, (4.8) and (4.9) yield (4.6) and in turn (4.7).

(d) Having constructed $Q(k)$ from $D(k)$, we can evaluate the value of $\tilde{f}(0; 0)$ by setting $k = 0$ in (4.4), which yields

$$\tilde{f}(0; 0) = \frac{1}{1 - Q(0)}. \tag{4.10}$$

Thus, (4.6), (4.7) and (4.10) imply that $D(k)$ uniquely determines $\tilde{f}(0; k)$.

(e) Having reconstructed $\tilde{f}(0; k)$ from $D(k)$, we can then repeat the reconstruction steps (d) and (e) of section 3 in order to uniquely reconstruct $\rho(x)$. Thus, the reconstruction of $\rho(x)$ for $x \in [0, b]$ with $b = a$ from the data set $D(k)$ is accomplished.

As stated in [2], when $b = a$, it is an open problem whether the value of the constant γ appearing in (1.4) is needed or whether γ can be determined from the data set consisting of $E(k)$ given in (1.4) alone.

5. Reconstruction of the potential in the Schrödinger equation

The transmission eigenvalue problem for (1.1) has an analogue for the Schrödinger equation. The transmission eigenvalues in that case correspond to those values of k^2 for which there exists a nontrivial solution pair Ψ and Ψ_0 to the system

$$\begin{cases} -\Delta\tilde{\Psi} + V(\mathbf{y})\tilde{\Psi} = k^2\tilde{\Psi}, & \mathbf{y} \in \Omega, \\ -\Delta\tilde{\Psi}_0 = k^2\tilde{\Psi}_0, & \mathbf{y} \in \Omega, \\ \tilde{\Psi} = \tilde{\Psi}_0, \quad \dfrac{\partial\tilde{\Psi}}{\partial\mathbf{n}} = \dfrac{\partial\tilde{\Psi}_0}{\partial\mathbf{n}}, & \mathbf{y} \in \partial\Omega, \end{cases} \tag{5.1}$$

where $V(\mathbf{y})$ is a real-valued potential that is square integrable on Ω, and it is assumed that $V(\mathbf{y}) \equiv 0$ outside Ω. In the spherically symmetric case in $\mathbf{R}^3$, using $V(y)$ instead of $V(\mathbf{y})$ with $y := |\mathbf{y}|$, we define the *special transmission eigenvalues* of (5.1) as those transmission eigenvalues for which the corresponding wavefunctions are spherically symmetric in addition to V being spherically symmetric.

We remark that the potential $V(y)$ we use in this section does not necessarily come from any function ρ appearing in (1.1) or (2.1) via the Liouville transformation in (2.8). The only assumption we make on $V(y)$ is that it is real valued, compactly supported within the interval $y \in [0, a]$ and integrable on $(0, a)$. We will say that $V(y)$ belongs to the admissible class $\tilde{A}$ if $V(y)$ satisfies those conditions.

The results given in the following proposition are either known or can easily be proved by using the available results for the half-line Schrödinger equation by exploiting the compact-support property of the potential [3, 6, 12, 14, 18]. We provide a brief proof for the convenience of the reader.

Proposition 5.1. *Assume that the potential $V(y)$ belongs to the admissible class $\tilde{A}$ with support within the interval $[0, a]$. We then have the following.*

(a) The corresponding Jost solution $\tilde{f}(y; k)$ has an analytic extension from $k \in \mathbf{R}$ to the entire complex plane $\mathbf{C}$ for each fixed y. Similar to (2.11), we have

$$\tilde{f}(y; -k) = \tilde{f}(y; k)^*, \qquad k \in \mathbf{R}. \tag{5.2}$$

(b) The quantity $\tilde{f}(0; k)$ is nonzero in $\mathbf{C}^+$ except perhaps at a finite number of points on the positive imaginary axis, say at $k = i\beta_j$ for $j = 1, \ldots, N$, for some non-negative integer N. Such zeros are all simple and they correspond to the bound states of the half-line Schrödinger equation with the Dirichlet boundary condition at the origin.

(c) The large-k asymptotics of $\tilde{f}(0; k)$ in $\mathbf{C}$ are obtained via

$$\tilde{f}(0; k) = 1 + O\left(\frac{1}{k}\right), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{5.3}$$

$$\tilde{f}(0; -k) = 1 + O\left(\frac{1}{k}\right) + e^{-2ika} o\frac{1}{k}, \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{5.4}$$

Hence, $e^{2ikb}\tilde{f}(0; -k)$ is bounded in $\overline{\mathbf{C}^+}$ for any b satisfying $b \geqslant a$.

(d) *The scattering matrix* $\tilde{S}(k)$ *for the half-line Schrödinger equation with the Dirichlet boundary condition is defined as* [3, 6, 12, 14, 18]

$$\tilde{S}(k) := \frac{\tilde{f}(0; -k)}{\tilde{f}(0; k)}, \qquad k \in \mathbf{R}, \tag{5.5}$$

and it has a meromorphic extension from $k \in \mathbf{R}$ *to* $k \in \mathbf{C}^+$ *with simple poles occurring at* $k = \mathrm{i}\beta_j$ *for* $j = 1, \ldots, N$.

(e) *The scattering matrix satisfies*

$$\tilde{S}(k) = 1 + O\left(\frac{1}{k}\right) + \mathrm{e}^{-2\mathrm{i}ka} o \frac{1}{k}, \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{5.6}$$

(f) *Associated with each bound state there is a positive number, known as the corresponding norming constant, defined as*

$$c_j := \frac{1}{\sqrt{\int_0^\infty \mathrm{d}y\, [\tilde{f}(y; \mathrm{i}\beta_j)]^2}}, \qquad j = 1, \ldots, N.$$

Because $V(y)$ *has support confined to the finite interval* $y \in [0, a]$, *the norming constants are uniquely determined by* $\tilde{f}(0; k)$ *alone, or equivalently by the scattering matrix* $\tilde{S}(k)$ *alone, as*

$$c_j = \sqrt{\mathrm{i}\, \mathrm{Res}(\tilde{S}(k), \mathrm{i}\beta_j)}, \qquad j = 1, \ldots, N, \tag{5.7}$$

where $\mathrm{Res}(\tilde{S}(k), \mathrm{i}\beta_j)$ *denotes the residue of* $\tilde{S}(k)$ *at the pole* $k = \mathrm{i}\beta_j$.

(g) *The potential* $V(y)$ *and the Jost solution* $\tilde{f}(y; k)$ *are reconstructed from the solution* $K(y, \xi)$ *to the Marchenko integral equation*

$$K(y, \xi) + M(y + \xi) + \int_y^\infty \mathrm{d}s\, K(y, s)\, M(s + \xi), \qquad 0 < y < \xi < +\infty, \tag{5.8}$$

where

$$M(\xi) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, [1 - \tilde{S}(k)]\, \mathrm{e}^{\mathrm{i}k\xi} + \sum_{j=1}^{N} c_j^2\, \mathrm{e}^{-\beta_j \xi}, \qquad \xi \in \mathbf{R}. \tag{5.9}$$

In fact, we have

$$V(y) = -2 \frac{\mathrm{d}K(y, y)}{\mathrm{d}y}, \qquad y > 0, \tag{5.10}$$

$$\tilde{f}(y; k) = \mathrm{e}^{\mathrm{i}ky} + \int_y^\infty \mathrm{d}s\, K(y, s)\, \mathrm{e}^{\mathrm{i}ks}. \tag{5.11}$$

Proof. The results are mainly known [3, 6, 12, 14, 18]; for example, (5.3) and (5.4) can be derived by using the integral representation [3, 6, 12, 14, 18] for the Jost solution, namely by using

$$\tilde{f}(y; k) = \mathrm{e}^{\mathrm{i}ky} + \frac{1}{k} \int_x^a \mathrm{d}s\, [\sin k(s - y)]\, V(s)\, \tilde{f}(s; k).$$

We then obtain (5.6) by using (5.3) and (5.4) in (5.5). The proof of (5.7) can be outlined as follows. From (2.7) and (5.11), it follows that $K(y, \xi) = 0$ for $a < y < \xi < +\infty$, and hence (5.8) in turn implies that $M(y + \xi) = 0$ for $a < y < \xi < +\infty$. This fact combined with (d) and (e) allows us to evaluate $M(y + \xi)$ for $a < y < \xi < +\infty$ by using (5.9) as a contour integral along a semicircle in $\mathbf{C}^+$ with its center at the origin and its radius becoming infinite in the limit. Using $M(y+\xi)=0$ for $a<y<\xi<+\infty$ in (5.9), we obtain (5.7). □

Analogously to (1.2), let us define

$$\tilde{D}(k):=\frac{\sin(ka)}{k}\,\tilde{\phi}'(a;k)-\cos(ka)\,\tilde{\phi}(a;k), \tag{5.12}$$

where a is the positive constant related to the support of $V(y)$ and $\tilde{\phi}(y;k)$ is the unique solution to (2.16). The following fundamental result is the analogue of theorem 2.3, and its proof is omitted because it is similar to the proof of theorem 2.3.

Theorem 5.2. *Assume that the potential $V(y)$ belongs to the admissible class $\tilde{A}$. Then, the quantity $\tilde{D}(k)$ defined in (5.12) is related to the Jost solution $\tilde{f}(y;k)$ appearing in (2.6) and (2.7) as*

$$\tilde{D}(k)=\frac{\tilde{f}(0;k)-\tilde{f}(0;-k)}{2ik},\qquad k\in\mathbf{C}. \tag{5.13}$$

For real k-values, we then have

$$\mathrm{Im}[\tilde{f}(0;k)]=k\,\tilde{D}(k),\qquad k\in\mathbf{R}. \tag{5.14}$$

We have the following analogue of theorem 1.1 [2].

Theorem 5.3. *Consider the special case of (5.1), with Ω being the three-dimensional ball of radius a centered at the origin, where only spherically symmetric wavefunctions are allowed and it is assumed that such wavefunctions are continuous in the closure of Ω. Then, the corresponding special transmission eigenvalues of (5.1) coincide with the k^2-values related to the zeros of the quantity $\tilde{D}(k)$ defined in (5.12), where $\tilde{\phi}(y;k)$ is the unique solution to (2.16) with the potential $V(y)$ belonging to the admissible class $\tilde{A}$.*

When $V(y)$ belongs to the admissible class $\tilde{A}$, the quantity $\tilde{D}(k)$ defined in (5.12) is known to be entire in k^2 [2] and has a representation analogous to (1.4), namely

$$\tilde{D}(k)=\tilde{\gamma}\,k^{2\tilde{d}}\prod_{n=1}^{\infty}\left(1-\frac{k^2}{\tilde{k}_n^2}\right), \tag{5.15}$$

with $\tilde{k}_n^2$ for $n\in\mathbf{N}$ being the nonzero transmission eigenvalues, some of which may be repeated and $\tilde{d}$ denoting the multiplicity of the zero transmission eigenvalue. As in [2], we refer to the multiplicity of a nonzero zero $\tilde{k}_n$ of $\tilde{D}(k)$ as the multiplicity of the special transmission eigenvalue $\tilde{k}_n^2$.

The following uniqueness result was proved in [2] and is the analogue of theorem 1.2.

Theorem 5.4. *Assume that $V(y)$ belongs to the admissible class $\tilde{A}$. Then, $V(y)$ is uniquely determined by the function $\tilde{D}(k)$ appearing in (5.12) and (5.15) if we assume that there exists at least one $V(y)$ in $\tilde{A}$ corresponding to $\tilde{D}(k)$. Equivalently stated, if the existence is ensured, $V(y)$ is uniquely determined by the knowledge of the special transmission eigenvalues of (5.1) with their multiplicities and the constant $\tilde{\gamma}$ appearing in (5.15).*

Our goal in this section is to give an independent proof of theorem 5.4 and further provide a reconstruction of $V(y)$ from $\tilde{D}(k)$. The reconstruction consists of the following steps and the uniqueness follows as a result of the uniqueness in each reconstruction step.

(a) First, reconstruct $\tilde{f}(0; k)$ from $\tilde{D}(k)$, where $\tilde{f}(y; k)$ is the Jost solution appearing in (2.6) and (2.7). This is done by solving the Riemann–Hilbert problem given by

$$[\tilde{f}(0; k) - 1] - [\tilde{f}(0; -k) - 1] = 2ik\, \tilde{D}(k), \qquad k \in \mathbf{R}, \tag{5.16}$$

which is obtained from (5.13). It follows from (5.3) and (5.16) that $2ik\tilde{D}(k)$ behaves as $O(1/k)$ as $k \to \pm\infty$ on $\mathbf{R}$. Furthermore, because $\tilde{D}(k)$ has an analytic extension to the entire complex plane, it follows that $2ik\tilde{D}(k)$ satisfies the Lipschitz continuity in $\mathbf{R}$. Thus, the Riemann–Hilbert problem in (5.16) has a unique solution that is given by

$$\tilde{f}(0; k) = 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} dt\, \frac{t\tilde{D}(t)}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+}. \tag{5.17}$$

We can write (5.14) as

$$\mathrm{Im}[\tilde{f}(0; k) - 1] = k\tilde{D}(k), \qquad k \in \mathbf{R}. \tag{5.18}$$

The result given in (5.17) also follows by using the Schwarz integral formula (4.9); by replacing $Q(k)$ there with $\tilde{f}(0; k) - 1$, with the help of (5.18), we obtain (5.17).

(b) Having obtained $\tilde{f}(0; k)$ from $\tilde{D}(k)$, we can use (5.5) and (5.7) in (5.9) and obtain the Marchenko kernel $M(\xi)$ from $\tilde{f}(0; k)$.

(c) The potential $V(y)$ is then uniquely reconstructed as in (5.10) by using $M(\xi)$ as input to the Marchenko integral equation (5.8) and by obtaining $K(y, \xi)$ as the unique solution to the Marchenko equation.

Let us mention that it is an open problem whether the value of $\tilde{\gamma}$ appearing in (5.3) can be determined from the zeros of $\tilde{D}(k)$. If the answer is yes, then $\tilde{\gamma}$ is not needed for the unique determination of $V(y)$ and the zeros of $\tilde{D}(k)$ with their multiplicities would be sufficient for the reconstruction of $V(y)$. In the following example, we show that $\tilde{\gamma}$ is needed to construct a potential, which is, however, outside the admissible class $\tilde{A}$.

Example 5.5. Let the potential $V(y)$ be given as

$$V(y) = c\,\delta(y - a),$$

where c is a real nonzero constant, $\delta(y - a)$ denotes the Dirac delta function with argument $y - a$ and a is the positive number related to the interval $[0, a]$ containing the support of $V(y)$. The corresponding Jost solution is obtained by solving (2.6) and (2.7), and we obtain

$$\tilde{f}(y; k) = \begin{cases} \left(1 - \dfrac{c}{2ik}\right) e^{iky} + \dfrac{c}{2ik}\, e^{2ika - iky}, & y \leqslant a, \\ e^{iky}, & y \geqslant a. \end{cases} \tag{5.19}$$

From (5.19), we evaluate $\tilde{f}(0; k)$ and then using (5.13) and (5.15), we obtain the values of $\tilde{\gamma}$ and $\tilde{D}(k)$, yielding

$$\tilde{\gamma} = ca^2, \quad \tilde{E}(k) := \frac{\tilde{D}(k)}{\tilde{\gamma}} = \left(\frac{\sin(ka)}{ka}\right)^2 = \prod_{n=1}^{\infty} \left(1 - \frac{a^2k^2}{n^2\pi^2}\right)^2. \tag{5.20}$$

Hence, in this example the transmission eigenvalues, i.e. the k^2-values corresponding to the zeros of $\tilde{D}(k)$, all have double multiplicities and are given by $k_n^2 = n^2\pi^2/a^2$ for $n \in \mathbf{N}$. However, as seen from (5.20), $\tilde{E}(k)$ alone does not uniquely determine c, and hence c or equivalently $\tilde{\gamma}$ is also needed for the unique determination of V.

6. Examples

In this section, we illustrate the transmission eigenvalue problem corresponding to $\rho(x)$ appearing in (1.3) with some explicit examples. In our first example, with the help of (2.2), (2.5), (2.9) and example 5.5, we present a concrete $\rho(x)$ for which we can explicitly evaluate the relevant quantities $D(k)$ and $E(k)$, given in (1.2) and (1.4), respectively.

Example 6.1. Let ϵ be a positive parameter and c be a real nonzero parameter. Assume (2.2) is given by

$$y(x) = \begin{cases} \dfrac{\epsilon^2 x}{\epsilon c x + 1}, & x \leqslant x_0, \\ x - x_0 + y_0, & x \geqslant x_0, \end{cases}$$

where

$$x_0 := \frac{\epsilon - 1}{\epsilon c}, \quad y_0 := y(x_0) = \frac{\epsilon - 1}{c}. \tag{6.1}$$

In order to have x_0 and y_0 positive, we must have $c > 0$ if $\epsilon > 1$ and we must have $c < 0$ if $\epsilon < 1$. Note that (6.1) implies that

$$b - a = x_0 - y_0 = -\frac{(\epsilon - 1)^2}{\epsilon c}, \tag{6.2}$$

where a and b are the parameters appearing in (1.5). We have $b \geqslant x_0$ and hence from (6.2) we see that $a < b$ if $\epsilon > 1$ and that $a > b$ if $\epsilon < 1$. Using (2.2), we obtain

$$\rho(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \begin{cases} \dfrac{\epsilon^4}{(\epsilon c x + 1)^4}, & x \leqslant x_0, \\ 1, & x \geqslant x_0, \end{cases} \tag{6.3}$$

Because of (6.1), we see from (6.3) that $\rho(x)$ is continuous at x_0, whereas $\rho'(x)$ jumps from $\rho'(x_0^-) = -4c$ to $\rho'(x_0^+) = 0$. One can directly verify that the Jost solution $f(x; k)$ to (2.1) is given by

$$f(x; k) = \begin{cases} \dfrac{\epsilon c x + 1}{2k\epsilon}\, \mathrm{e}^{-\mathrm{i}k(\epsilon-1)^2/(\epsilon c)} Z(x; k, \epsilon, c), & x \leqslant x_0, \\ \mathrm{e}^{\mathrm{i}kx}, & x \geqslant x_0, \end{cases} \tag{6.4}$$

where we have defined

$$Z(x; k, \epsilon, c) := (2k + \mathrm{i}c)\, \mathrm{e}^{\mathrm{i}k\epsilon^2 x/(\epsilon c x + 1)} - \mathrm{i}c\, \mathrm{e}^{2\mathrm{i}k(\epsilon-1)/c - \mathrm{i}k\epsilon^2 x/(\epsilon c x+1)}.$$

One can check that $f(x; k)$ and $f'(x; k)$ are continuous at $x = x_0$. From (6.4), we obtain

$$f(0; k) = \frac{1}{2\epsilon k} \mathrm{e}^{-\mathrm{i}k(\epsilon-1)^2/(\epsilon c)} [(2k + \mathrm{i}c) - \mathrm{i}c\, \mathrm{e}^{2\mathrm{i}k(\epsilon-1)/c}]. \tag{6.5}$$

Using (6.5) in (2.17), we obtain

$$D(k) = \frac{c}{2\epsilon k^2} \left[\cos\left(\frac{k(\epsilon - 1)^2}{\epsilon c}\right) - \cos\left(\frac{k(\epsilon^2 - 1)}{\epsilon c}\right) - \frac{2k}{c} \sin\left(\frac{k(\epsilon - 1)^2}{\epsilon c}\right) \right]. \tag{6.6}$$

By expanding (6.6) in powers of k^2, we obtain

$$D(k) = \gamma k^2 + O(k^4), \qquad k \to 0 \text{ in } \mathbf{C},$$

where a comparison with (1.4) reveals that

$$d = 1, \quad \gamma = -\frac{(\epsilon - 1)^4}{3\epsilon^3 c^3}. \tag{6.7}$$

Using (6.6) and (6.7) in (1.4), we have

$$E(k) = -\frac{3\epsilon^2 c^4}{2(\epsilon-1)^4 k^2}\left[\cos\left(\frac{k(\epsilon-1)^2}{\epsilon c}\right) - \cos\left(\frac{k(\epsilon^2-1)}{\epsilon c}\right) - \frac{2k}{c}\sin\left(\frac{k(\epsilon-1)^2}{\epsilon c}\right)\right]. \tag{6.8}$$

From (6.8), we obtain

$$E(k) = \frac{3\epsilon^2 c^3}{(\epsilon-1)^4 k}\sin\left(\frac{k(\epsilon-1)^2}{\epsilon c}\right) + O\left(\frac{1}{k^2}\right), \qquad k \to \pm\infty, \tag{6.9}$$

and hence a comparison of (6.9) with (3.2) reveals that

$$b - a = -\frac{(\epsilon-1)^2}{\epsilon c}, \qquad \gamma[\rho(0)]^{1/4} = -\frac{(\epsilon-1)^4}{3\epsilon^2 c^3},$$

which is compatible with the value of $(b-a)$ given in (6.2), $\rho(0)$ from (6.3) and γ in (6.7). Note that $\rho(x)$ in this example is outside the admissible class $\mathcal{A}$ because of the jump discontinuity of $\rho'(x)$ at x_0. Let us remark that $D(k)$ given in (6.6) can also be obtained by using (1.2), where the unique solution $\phi(x;k)$ to (2.15) in this case is given by

$$\phi(x;k) = \begin{cases} \dfrac{(\epsilon c x + 1)}{\epsilon^2 k}\sin\left(\dfrac{\epsilon^2 k x}{\epsilon c x + 1}\right), & x \leqslant x_0, \\ c_3(k,\epsilon,c)\sin(kx) + c_4(k,\epsilon,c)\cos(kx), & x \geqslant x_0, \end{cases}$$

with the constants $c_3(k,\epsilon,c)$ and $c_4(k,\epsilon,c)$ specified as

$$c_3(k,\epsilon,c) := \frac{1}{2\epsilon k}\cos\left(\frac{k(1-\epsilon)^2}{\epsilon c}\right) + \frac{c}{4\epsilon k^2}\left[\sin\left(\frac{k(1-\epsilon)^2}{\epsilon c}\right) - \sin\left(\frac{k(1-\epsilon^2)}{\epsilon c}\right)\right],$$

$$c_4(k,\epsilon,c) := \frac{1}{2\epsilon k}\sin\left(\frac{k(1-\epsilon)^2}{\epsilon c}\right) - \frac{c}{4\epsilon k^2}\left[\cos\left(\frac{k(1-\epsilon)^2}{\epsilon c}\right) - \cos\left(\frac{k(1-\epsilon^2)}{\epsilon c}\right)\right].$$

Using the result of example 6.1, in the next example we will produce two distinct profiles $\rho(x)$ corresponding to the same $E(k)$ but to different γ values; in fact, in one case we will have $a > b$ and in the other case we will have $a < b$.

Example 6.2. In example 6.1 above, let us use the following values for the parameters:

$$\epsilon = 2, \qquad c = \frac{1}{b},$$

where b is the constant that appears in (1.2) and is related to the known support of $\rho(x) - 1$. Using (6.2), (6.3), (6.7) and (6.8), we obtain

$$\rho(x) = \begin{cases} \dfrac{16b^4}{(2x+b)^4}, & x \leqslant b/2, \\ 1, & x \geqslant b/2, \end{cases} \tag{6.10}$$

$$\gamma = \frac{-b^3}{24}, \qquad a = \frac{3b}{2}, \tag{6.11}$$

$$E(k) = \frac{-6}{b^4 k^2}\left[\cos\left(\frac{bk}{2}\right) - \cos\left(\frac{3bk}{2}\right) - 2bk\sin\left(\frac{bk}{2}\right)\right]. \tag{6.12}$$

On the other hand, in example 6.1 if we use the parameters

$$\epsilon = \frac{1}{2}, \qquad c = -\frac{1}{b},$$

from (6.2), (6.3), (6.7) and (6.8), then we obtain

$$\rho(x) = \begin{cases} \dfrac{b^4}{(2b-x)^4}, & x \leqslant b, \\ 1, & x \geqslant b, \end{cases} \tag{6.13}$$

$$\gamma = \frac{b^3}{6}, \quad a = \frac{b}{2}, \tag{6.14}$$

$$E(k) = \frac{-6}{b^4k^2}\left[\cos\left(\frac{bk}{2}\right) - \cos\left(\frac{3bk}{2}\right) - 2bk\sin\left(\frac{bk}{2}\right)\right]. \tag{6.15}$$

Thus, as seen from (6.12) and (6.15), we have produced two distinct profiles for $\rho(x)$ given in (6.10) and (6.13), respectively, corresponding to the same $E(k)$, but two different γ values. In fact, as seen from (6.11) and (6.14), the former corresponds to the case $a > b$ and the latter to $a < b$. We can simplify and rewrite (6.12) as

$$E(k) = \frac{12}{b^3k}\sin\left(\frac{bk}{2}\right)\left(1 - \frac{\sin(bk)}{bk}\right). \tag{6.16}$$

As seen from (6.16), corresponding to the two distinct profiles given in (6.10) and (6.13), we have a simple zero transmission eigenvalue, infinitely many simple nonzero real transmission eigenvalues that are given by $k_n^2 = 4n^2\pi^2/b^2$ for $n \in \mathbf{N}$ and infinitely many simple complex transmission eigenvalues that are related to nonzero zeros of $kb - \sin(kb)$. Note that for each complex transmission eigenvalue, its complex conjugate is also a transmission eigenvalue.

We conclude with another explicit example.

Example 6.3. For a positive parameter c, let

$$\rho(x) = \begin{cases} \dfrac{(b+c)^2}{(x+c)^2}, & x \leqslant b, \\ 1, & x \geqslant b, \end{cases}$$

where b is the positive parameter appearing in (1.5). Using (2.2), we obtain

$$y(x) = \begin{cases} (b+c)\log\left(1 + \dfrac{x}{c}\right), & x \leqslant b, \\ x - b + a, & x \geqslant b, \end{cases} \tag{6.17}$$

where a is the parameter appearing in (1.5) and its value is obtained from (6.17) as

$$a := y(b) = (b+c)\log\left(1 + \frac{b}{c}\right). \tag{6.18}$$

In this case, $a > b$ because from (6.18) it follows that

$$\frac{a}{b} = \left(1 + \frac{c}{b}\right)\log\left(1 + \frac{b}{c}\right) > 1,$$

based on the positivity assumption on c. We can solve (1.3) explicitly and obtain

$$\phi(x;k) = \frac{1}{r_+ - r_-}[-c^{r_+}(x+c)^{r_-} + c^{r_-}(x+c)^{r_+}], \qquad x \in [0, b], \tag{6.19}$$

where we have defined

$$r_\pm := \tfrac{1}{2}\left[1 \pm \sqrt{1 - 4(b+c)^2k^2}\right]. \tag{6.20}$$

Using (6.19) in (1.2), we obtain

$$D(k) = \frac{\sin(bk)}{k}\,\phi'(b;k) - \cos(kb)\,\phi(b;k), \tag{6.21}$$

where as we see from (6.19) and (6.20), with the help of $r_+ + r_- = 1$, we have

$$\phi(b;k) = \frac{c}{r_+ - r_-}\left[-\left(1+\frac{b}{c}\right)^{r_-} + \left(1+\frac{b}{c}\right)^{r_+}\right],$$

$$\phi'(b;k) = \frac{c}{r_+ - r_-}\left[-\frac{r_-}{b+c}\left(1+\frac{b}{c}\right)^{r_-} + \frac{r_+}{b+c}\left(1+\frac{b}{c}\right)^{r_+}\right].$$

Letting $k \to 0$ in (6.21), with the help of (1.4), we obtain

$$d = 1, \quad \gamma = c^3\left[-\frac{2}{3}\left(\frac{b}{c}\right)^3 - 3\left(\frac{b}{c}\right)^2 - 2\left(\frac{b}{c}\right) + 2\left(1+\frac{b}{c}\right)^2 \log\left(1+\frac{b}{c}\right)\right], \tag{6.22}$$

where, by using a graphical argument, it can be shown that $\gamma < 0$. We remark that the results in this example are also valid if $c < 0$ but $b > -c$. In that case, from (6.18) we obtain $a < b$ and from (6.22) we obtain $\gamma > 0$.

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