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The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation

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Abstract

The recovery of a spherically symmetric wave speed v is considered in a bounded spherical region of radius b from the set of the corresponding transmission eigenvalues for which the corresponding eigenfunctions are also spherically symmetric. If the integral of $1/v$ on the interval $[0, b]$ is less than b, assuming that there exists at least one v corresponding to the data, it is shown that v is uniquely determined by the data consisting of such transmission eigenvalues and their 'multiplicities', where the 'multiplicity' is defined as the multiplicity of the transmission eigenvalue as a zero of a key quantity. When that integral is equal to b, the unique recovery is obtained when the data contain one additional piece of information. Some similar results are presented for the unique determination of the potential from the transmission eigenvalues with 'multiplicities' for a related Schrödinger equation.

1. Introduction

The interior transmission problem is a nonselfadjoint boundary-value problem for a pair of fields Ψ and Ψ_0 in a bounded and simply connected domain Ω of $\mathbf{R}^n$ with the sufficiently smooth boundary $\partial\Omega$. It was first stated in [13] and can be formulated [12, 13,16] as

$$\begin{cases} \Delta\Psi + \lambda\rho(\mathbf{x})\Psi = 0, & \mathbf{x} \in \Omega, \\ \Delta\Psi_0 + \lambda\Psi_0 = 0, & \mathbf{x} \in \Omega, \\ \Psi = \Psi_0, \quad \dfrac{\partial\Psi}{\partial\mathbf{n}} = \dfrac{\partial\Psi_0}{\partial\mathbf{n}}, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{1.1}$$

where Δ denotes the Laplacian, λ is the spectral parameter, $\mathbf{n}$ represents the outward unit normal to the boundary $\partial\Omega$, and the positive quantity $\rho(\mathbf{x})$ corresponds to the square of the refractive index of the medium at location $\mathbf{x}$ in the electromagnetic case or the reciprocal of the square of the sound speed $v(\mathbf{x})$ in the acoustic case, i.e. $v(\mathbf{x}) := 1/\sqrt{\rho(\mathbf{x})}$. In the acoustic case, $\sqrt{\rho(\mathbf{x})}$ is usually called the slowness. Without loss of generality, we can assume that in the region exterior to Ω, the speed of the electromagnetic wave is 1 or the sound speed is 1 in the acoustic case.

This interior transmission problem arises in the inverse scattering theory in inhomogeneous media, where the goal is to determine the function ρ in Ω from an appropriate set of λ-values related to (1.1). The values of λ for which (1.1) has a pair of nontrivial solutions Ψ and Ψ_0 are called *transmission eigenvalues*. It is already known that those transmission eigenvalues can be determined from some far-field measurements [6, 7] as well as from some near-field measurements [29].

Since there does not exist a standard theory to analyze nonselfadjoint eigenvalue problems, the existence of transmission eigenvalues for (1.1) is a difficult problem. The proof of the existence of transmission eigenvalues was known [14] in the spherically symmetric case, but in the absence of spherical symmetry it was first established in [30] under certain appropriate conditions and was later provided [5, 8–10, 15, 19] under other appropriate conditions. Using some techniques related to the Fredholm theory of integral equations, it has been shown [11, 33] that the transmission eigenvalues for (1.1) form a discrete set with infinity as the only possible accumulation point. In general, we expect [4, 15] transmission eigenvalues to be complex numbers although some of them may be real and some, in fact, may be positive.

A fundamental problem related to (1.1) is the relationship between ρ in Ω and the corresponding transmission eigenvalues. The case $n = 3$ is naturally the most relevant in applications. A key question is whether we can uniquely determine ρ in Ω if all the transmission eigenvalues are known. Another important question is whether the unique recovery is possible if we know only a certain subset of the transmission eigenvalues.

In the case where Ω is the ball of radius b centered at the origin and $\rho(\mathbf{x})$ is spherically symmetric, it has recently been shown [4] that the set of all transmission eigenvalues uniquely determine ρ in Ω. In the spherically symmetric case, let us use $\rho(x)$, instead of $\rho(\mathbf{x})$, with $x := |\mathbf{x}|$. In this case, it is natural to ask whether $\rho(x)$ can be determined from a subset of transmission eigenvalues, such as those transmission eigenvalues for which the corresponding eigenfunctions are also spherically symmetric. We will refer to such eigenvalues as *special transmission eigenvalues*. Another variant of the transmission eigenvalue problem in the spherically symmetric case has been studied in [25–27], where some uniqueness results were established when only the positive special transmission eigenvalues are used in the determination.

In the case $n = 3$, where Ω is the ball of radius $b > 0$ centered at the origin and ρ is spherically symmetric, the boundary-value problem (1.1) becomes equivalent to a nonstandard Sturm–Liouville-type eigenvalue problem, which is formulated in the following proposition. Here, 'nonstandard' refers to the fact that the spectral parameter appears in the boundary condition at the right endpoint. Our assumptions on ρ are that $\rho(x)$ is positive and continuously differentiable and that ρ'' exists almost everywhere and is square integrable, i.e.

$$\rho(x) > 0, \qquad x \in (0, b); \quad \rho \in C^1(0, b); \quad \rho'' \in L^2(0, b), \tag{1.2}$$

where a prime is used to denote the x-derivative.

Proposition 1.1. *Consider the special case of (1.1) with Ω being the three-dimensional ball of radius b centered at the origin, where only spherically symmetric wavefunctions are allowed and it is assumed that such wavefunctions are continuous in the closure of Ω. Then, the corresponding special transmission eigenvalues of (1.1) coincide with the eigenvalues of the nonstandard boundary-value problem*

$$\begin{cases} \Phi'' + \lambda\rho(x)\Phi = 0, & 0 < x < b, \\ \Phi(0) = 0, & \dfrac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}\Phi'(b) - \cos(\sqrt{\lambda}b)\Phi(b) = 0. \end{cases} \tag{1.3}$$

Proof. The Laplacian in $\mathbf{R}^3$ in spherical coordinates (x, θ, φ) is given by

$$\Delta := \frac{1}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x} + \frac{1}{x^2 \sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{x^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}, \tag{1.4}$$

where we recall that $x := |\mathbf{x}|$. If the wavefunctions Φ and Φ_0 are spherically symmetric, i.e. if they do not depend on θ and φ, then with the help of (1.4) we transform (1.1) into

$$\begin{cases} \Psi'' + \dfrac{2\Psi'}{x} + \lambda\rho(x)\Psi = 0, & 0 < x < b, \\ \Psi_0'' + \dfrac{2\Psi_0'}{x} + \lambda\Psi_0 = 0, & 0 < x < b, \\ \Psi(b) = \Psi_0(b), \quad \Psi'(b) = \Psi_0'(b), \end{cases} \tag{1.5}$$

where $\Psi(0)$ and $\Psi_0(0)$ must be finite because of the continuity of Ψ and Ψ_0 in Ω. Letting $\Phi := x\Psi$ and $\Phi_0 := x\Psi_0$, from (1.5) we get

$$\begin{cases} \Phi'' + \lambda\rho(x)\Phi = 0, & 0 < x < b, \\ \Phi_0'' + \lambda\Phi_0 = 0, & 0 < x < b, \\ \Phi(0) = \Phi_0(0) = 0, \quad \Phi(b) = \Phi_0(b), \quad \Phi'(b) = \Phi_0'(b). \end{cases} \tag{1.6}$$

From the second line in (1.6), we see that the solution $\Phi_0(x)$ satisfying $\Phi_0(0) = 0$ must be a constant multiple of $\sin(\sqrt{\lambda}\,x)/\sqrt{\lambda}$. Thus, we see that (1.6) is equivalent to (1.3). □

The eigenvalues of (1.3), namely the λ-values for which (1.3) has a nontrivial solution $\Phi(x)$, are the special transmission eigenvalues mentioned earlier. In other words, the corresponding eigenfunctions are spherically symmetric and hence functions of x only. Note that such eigenfunctions of (1.3) can only be determined up to a multiplicative constant, and it is clear from (1.3) that there exists only one linearly independent eigenfunction for each eigenvalue of (1.3). Nevertheless, for each eigenvalue λ_j of (1.3), we will associate a 'multiplicity' in a special sense, namely the multiplicity of λ_j as a zero of the quantity $D(\lambda)$ defined in (2.10). We will elaborate on the meaning of 'multiplicity' in section 2.

We define the relevant quantity a as

$$a := \int_0^b \sqrt{\rho(x)}\,\mathrm{d}x, \tag{1.7}$$

which has the physical interpretation as the travel time for the wave to move from $x = 0$ to $x = b$. Our main result in this paper is the proof that the knowledge of eigenvalues of (1.3) with their 'multiplicities' uniquely determine $\rho(x)$ for $0 < x < b$ provided $a < b$. If $a = b$, we prove the unique determination of ρ provided we know one additional parameter, namely the value of the constant γ appearing in (2.13). Let us clarify that we do not study the existence aspect of the inverse problem but we only analyze the uniqueness aspect. In other words, corresponding to our data we assume that there exists at least one function ρ satisfying (1.2), and we prove that if ρ_1 and ρ_2 are two such functions then we must have $\rho_1 \equiv \rho_2$.

When $a = b$, it is an open question if the knowledge of γ is necessary or whether γ can be determined from the knowledge of eigenvalues of (1.3) including their 'multiplicities'. In the discrete version of (1.3), assuming the existence aspect of the inverse problem is solved, it is already known [31] that generically, except for one exceptional case, ρ is uniquely determined from the knowledge of the special transmission eigenvalues and their 'multiplicities' and hence γ is in general uniquely determined without needing to know any additional parameter.

If $\rho(x)$ satisfies (1.2) with $a \neq b$, it follows from [25, 27] that (1.3) has a countably infinite sequence of nonzero eigenvalues λ_j such that

$$\lambda_j = \frac{j^2\pi^2}{(a-b)^2} + O(1), \qquad j \to +\infty,$$

with $j \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of positive integers. Furthermore, that countable sequence of eigenvalues contains [25, 27] a countably infinite subsequence of real eigenvalues such that

$$\lambda_{n_j} = \frac{n_j^2\pi^2}{(a-b)^2} + O(1), \qquad n_j \to +\infty,$$

where λ_{n_j} for $j \in \mathbf{N}$ are the real eigenvalues of (1.3) indexed in an increasing order. Hence, the quantity a can be determined if the eigenvalues of (1.3) are known or even if only the real eigenvalues of (1.3) are known, where we assume that we know the value of b already. In other words, if ρ_1 and ρ_2 satisfy (1.2) and they correspond to the same set of special transmission eigenvalues, then we must have $a_1 = a_2$, where

$$a_1 := \int_0^b \sqrt{\rho_1(x)}\,\mathrm{d}x, \quad a_2 := \int_0^b \sqrt{\rho_2(x)}\,\mathrm{d}x. \tag{1.8}$$

Let us elaborate on the eigenvalues of (1.3). As we illustrate with some examples in section 2, besides real eigenvalues, (1.3) has in general nonreal eigenvalues and in fact the number of nonreal eigenvalues may be infinite or it is even possible that there may not be any nonreal eigenvalues. Because $\rho(x)$ is real valued, from (1.3) it is seen that if λ is an eigenvalue, then λ^* is also an eigenvalue of (1.3), where we use an asterisk to denote complex conjugation. In our present work, for the unique recovery of ρ we assume the knowledge of all the eigenvalues (both real and complex nonreal) including their 'multiplicities'. In the previously established uniqueness results [25–27] regarding (1.3), it has been assumed that either $a \leqslant b/3$ or that some partial information on ρ is available. On the other hand, in those results [25–27], it is assumed that only the positive eigenvalues are known and no 'multiplicities' are used in the data.

Our paper is organized as follows. In section 2, we present some preliminary results that are needed to prove the uniqueness theorems of sections 3 and 4. In section 3, we consider the uniqueness in the recovery of ρ from the knowledge of special transmission eigenvalues of (1.1) with 'multiplicities'. When $a < b$, where a is the quantity in (1.7), we establish the uniqueness. When $a = b$, we show that the combined knowledge of special transmission eigenvalues of (1.1) with 'multiplicities' and the constant γ appearing in (2.13) ensures the uniqueness. We also elaborate on the case $a > b$ and indicate why the technique we use does not apply in that case to prove the uniqueness. In section 4, we consider the uniqueness in the recovery of the potential V of the Schrödinger equation from the data consisting of special transmission eigenvalues of (4.1) with 'multiplicities'. We prove the unique recovery if our data contain one additional parameter, namely the constant $\tilde{\gamma}$ appearing in (4.5).

2. Preliminaries

Let us recall [1] that an entire function of order 1/2 grows no faster than $O(e^{c|\lambda|^{(1/2)+\epsilon}})$ as $\lambda \to \infty$ in the complex plane $\mathbf{C}$ for any given positive ϵ, where c is some positive constant. The sums and products of such functions are entire of order not exceeding 1/2.

We first consider a problem closely related to (1.3), namely

$$\begin{cases} \phi'' + \lambda\rho(x)\phi = 0, & 0 < x < b, \\ \phi(0) = 0, \quad \phi'(0) = 1. \end{cases} \tag{2.1}$$

It is known [32] that, for every λ in the complex plane $\mathbf{C}$, (2.1) has a unique solution $\phi(x)$, which we also write as $\phi(x;\lambda)$ to emphasize its dependence on λ. Since $\rho(x)$ is real valued, the solution to (2.1) satisfies

$$\phi(x;\lambda^*) = \phi(x;\lambda)^*, \qquad \lambda \in \mathbf{C}. \tag{2.2}$$

Proposition 2.1. *Assume that ρ satisfies (1.2). Then, (2.1) is uniquely solvable, and for each fixed $x \in (0,b]$, the quantities $\phi(x;\cdot)$ and $\phi'(x;\cdot)$ are entire in λ of order $1/2$. Furthermore, $\phi(x;\lambda)$ and $\phi'(x;\lambda)$ cannot simultaneously vanish at the same x-value.*

Proof. We refer the reader to [32] for the proof that $\phi(x;\cdot)$ and $\phi'(x;\cdot)$ are entire in λ of order $1/2$. If $\phi(x_0;\lambda) = \phi'(x_0;\lambda) = 0$ for some x_0 value in $[0,b]$, then the unique solution to the corresponding initial-value problem would have to be the zero solution, which is incompatible with $\phi'(0) = 1$ in (2.1). □

When ρ satisfies (1.2), it is known (see e.g. [32]) that the variable-speed wave equation in (2.1) can be transformed into a Schrödinger equation via a Liouville transformation. In other words, by using the change of variables

$$y = y(x) := \int_0^x \sqrt{\rho(s)}\,\mathrm{d}s, \quad \varphi(y) = \varphi(y(x)) := \rho(x)^{1/4}\phi(x), \tag{2.3}$$

we can transform (2.1) into the equivalent Sturm–Liouville problem for the Schrödinger equation that is given by

$$\begin{cases} -\varphi''(y) + q(y)\varphi(y) = \lambda\varphi(y), & 0 < y < a, \\ \varphi(0) = 0, \quad \varphi'(0) = \dfrac{1}{\rho(0)^{1/4}}, \end{cases} \tag{2.4}$$

where a is the quantity defined in (1.7) and

$$q(y) = q(y(x)) := \frac{1}{4}\frac{\rho''(x)}{\rho(x)^2} - \frac{5}{16}\frac{\rho'(x)^2}{\rho(x)^3}.$$

Let us use $\mathrm{Im}[\sqrt{\lambda}]$ to denote the imaginary part of $\sqrt{\lambda}$, where the argument of the square-root function is chosen so that $\arg(\sqrt{\lambda}) \in (-\pi/2, \pi/2]$. The proof of the following proposition can be obtained [32] with the help of the Liouville transformation (2.3) and some estimates for the corresponding Schrödinger equation in (2.4), and hence it will not be given here.

Proposition 2.2. *Assume that ρ satisfies (1.2). Then, there exists a positive constant A such that, for all $x \in [0,b]$ and $\lambda \in \mathbf{C}$, the solution $\phi(x;\lambda)$ to (2.1) and its x-derivative, respectively, satisfy*

$$\left|\phi(x;\lambda) - \frac{1}{[\rho(0)\rho(x)]^{1/4}\sqrt{\lambda}}\sin(\sqrt{\lambda}y(x))\right| \leqslant \frac{A}{|\sqrt{\lambda}|}\exp(|\mathrm{Im}[\sqrt{\lambda}]|y(x)), \tag{2.5}$$

$$\left|\phi'(x;\lambda) - \left[\frac{\rho(x)}{\rho(0)}\right]^{1/4}\cos(\sqrt{\lambda}y(x))\right| \leqslant A\exp(|\mathrm{Im}[\sqrt{\lambda}]|y(x)), \tag{2.6}$$

where $y(x)$ is the quantity given in (2.3).

For a positive ε, let $\mathbf{C}_\varepsilon$ denote the sector in the complex plane defined as

$$\mathbf{C}_\varepsilon := \{\lambda \in \mathbf{C} : \ \varepsilon \leqslant \arg(\lambda) \leqslant 2\pi - \varepsilon\}. \tag{2.7}$$

The proof of the following result is already known [28].

Proposition 2.3. *Assume that* ρ *satisfies (1.2). Then, for each fixed* $x \in [0, b]$, *as* $\lambda \to \infty$ *in* $\mathbf{C}_\varepsilon$, *the unique solution* $\phi(x; \lambda)$ *to (2.1) satisfies*

$$\phi(x;\lambda) = \frac{1}{[\rho(0)\rho(x)]^{1/4}\sqrt{\lambda}}[\sin(\sqrt{\lambda}y(x))]\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right], \tag{2.8}$$

$$\phi'(x;\lambda) = \left[\frac{\rho(x)}{\rho(0)}\right]^{1/4}[\cos(\sqrt{\lambda}y(x))]\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right], \tag{2.9}$$

where $y(x)$ *is the quantity given in (2.3).*

Let us now clarify the relationship between (1.3) and (2.1). In general, for a given $\lambda \in \mathbf{C}$, (1.3) may not have a nontrivial solution. Suppose that λ_j is an eigenvalue of (1.3). Then, a solution $\Phi(x; \lambda_j)$ to (1.3) can only be determined up to a multiplicative constant, and in fact any such solution must be a constant multiple of the unique solution $\phi(x; \lambda_j)$ to (2.1) due to the fact that $\Phi(0) = 0$ in (1.3) and $\phi(0) = 0$ in (2.1).

We now introduce the key function $D(\lambda)$ as

$$D(\lambda) := \frac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}\phi'(b;\lambda) - \cos(\sqrt{\lambda}b)\phi(b;\lambda), \tag{2.10}$$

where we recall that $\phi(x; \lambda)$ is the unique solution to (2.1). Let us remark that if $\rho(x) \equiv 1$ in (2.1), then $\phi(x; \lambda) = \sin(\sqrt{\lambda}\, x)/\sqrt{\lambda}$ and hence $D(\lambda) \equiv 0$.

Theorem 2.4. *Assume that* ρ *satisfies (1.2). Then, the quantity* $D(\lambda)$ *defined in (2.10) is entire in* λ *of order not exceeding* $1/2$. *Each zero of* $D(\lambda)$ *in the complex plane* $\mathbf{C}$ *corresponds to an eigenvalue of (1.3) and vice versa. The value* $\lambda = 0$ *is always a zero of* $D(\lambda)$ *of some multiplicity* d *with* $d \geqslant 1$, *and hence*

$$D(0) = 0. \tag{2.11}$$

Furthermore,

$$D(\lambda^*) = D(\lambda)^*, \qquad \lambda \in \mathbf{C}, \tag{2.12}$$

and there exists a real constant γ *such that*

$$D(\lambda) = \gamma\,\Xi(\lambda), \tag{2.13}$$

where the auxiliary quantity $\Xi(\lambda)$ *is uniquely determined from the zeros (including multiplicities) of* $D(\lambda)$ *and has the representation*

$$\Xi(\lambda) = \lambda^d \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right), \tag{2.14}$$

with λ_n *for* $n \in \mathbf{N}$ *being the nonzero zeros of* $D(\lambda)$, *some of which may be repeated.*

Proof. From their representations in terms of exponential functions, we know that $\sin(\sqrt{\lambda}\, b)/\sqrt{\lambda}$ and $\cos(\sqrt{\lambda}\, b)$ are entire in λ of order $1/2$. From proposition 2.1, we know that $\phi(b; \lambda)$ and $\phi'(b; \lambda)$ are entire of order $1/2$, and hence the right side of (2.10) is entire of order not exceeding $1/2$. If λ_j is an eigenvalue of (1.3) with an eigenfunction $\Phi(x; \lambda_j)$, we already know that $\Phi(x; \lambda_j)$ is a constant multiple of the solution $\phi(x; \lambda_j)$ to (2.1), and hence from (2.10) we see that $D(\lambda_j) = 0$. Conversely, if $D(\lambda_j) = 0$ for some λ_j, then a comparison of (1.3) and (2.1) shows that the unique solution $\phi(x; \lambda_j)$ to (2.1) satisfies (1.3) and hence λ_j is an eigenvalue for (1.3) with the eigenfunction $\phi(x; \lambda_j)$. In particular, we note that when $\lambda = 0$, the unique solution to (2.1) is given by

$$\phi(x; 0) = x, \tag{2.15}$$

which indicates that

$$\phi(0;0)=0,\quad \phi'(0;0)=1,\quad \phi(b;0)=b,\quad \phi'(b;0)=1, \tag{2.16}$$

and hence $\phi(x;0)$ indeed satisfies (1.3) when $\lambda=0$. Thus, $\lambda=0$ is always a zero of $D(\lambda)$ with some multiplicity d, which is at least 1. We obtain (2.12) from (2.2) and (2.10). Since $D(\lambda)$ is entire of order not exceeding $1/2$, by the Hadamard factorization theorem, we must have the representation in (2.13), where γ is a complex constant and $\Xi(\lambda)$ is as in (2.14). In fact, γ turns out to be real as a result of (2.12). □

As we have seen in proposition 1.1 and theorem 2.4, the special transmission eigenvalues of (1.1), the eigenvalues of (1.3), and the zeros of $D(\lambda)$ defined in (2.10) all coincide. On the other hand, each zero of $D(\lambda)$ may have a multiplicity greater than 1 even though there exists only one linearly independent eigenfunction for the corresponding eigenvalue of (1.3). We refer to the multiplicity of a zero λ_j of $D(\lambda)$ also as the 'multiplicity' of the special transmission eigenvalue λ_j. Next, we elaborate on the 'multiplicities' with an illustrative example.

Example 2.5. When $\rho(x)$ is constant on $(0,b)$, by using ρ to denote that constant value, the unique solution to (1.3) is obtained as

$$\phi(x;\lambda)=\frac{1}{\sqrt{\lambda\rho}}\sin(\sqrt{\lambda\rho}\,x),\qquad 0<x<b,$$

and hence the corresponding quantity in (2.10) is given by

$$D(\lambda)=\frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}b)\cos(\sqrt{\lambda\rho}b)-\frac{1}{\sqrt{\lambda\rho}}\cos(\sqrt{\lambda}b)\sin(\sqrt{\lambda\rho}b). \tag{2.17}$$

When $\rho(x)\equiv 1/4$, from (2.17) we get

$$D(\lambda)=\frac{2}{\sqrt{\lambda}}\sin^3\left(\frac{\sqrt{\lambda}b}{2}\right),$$

and hence $D(\lambda)$ has a simple zero at $\lambda=0$ and an infinite set of real zeros at the λ-values $4j^2\pi^2/b^2$ for $j\in\mathbf{N}$, each having a multiplicity of 3. On the other hand, when $\rho(x)\equiv 4/9$, from (2.17) we get

$$D(\lambda)=\frac{1}{\sqrt{\lambda}}\sin^3\left(\frac{\sqrt{\lambda}b}{3}\right)\left[3+2\cos\left(\frac{2\sqrt{\lambda}b}{3}\right)\right],$$

and hence $D(\lambda)$ has a simple zero at $\lambda=0$, an infinite set of real zeros of multiplicity 3 at the λ-values $9j^2\pi^2/b^2$ for $j\in\mathbf{N}$, and an infinite set of simple complex zeros at the λ-values that are given by

$$\frac{9(2j-1)^2\pi^2}{4b^2}-\frac{9}{4b^2}\left[\log\left(\frac{3+\sqrt{5}}{2}\right)\right]^2\pm i\frac{9(2j-1)\pi}{2b^2}\left[\log\left(\frac{3+\sqrt{5}}{2}\right)\right],\qquad j\in\mathbf{N}.$$

Let us remark that the knowledge of $\Xi(\lambda)$ given in (2.15) is equivalent to the knowledge of the eigenvalues of (1.3) with their 'multiplicities'. Furthermore, the knowledge of $\Xi(\lambda)$ is equivalent to the knowledge of its zeros including their multiplicities. Hence, in proving our uniqueness results, as our data we can equivalently use $\Xi(\lambda)$, the zeros of $\Xi(\lambda)$ with their multiplicities, the eigenvalues of (1.3) with their 'multiplicities', or the special transmission eigenvalues of (1.1) with their 'multiplicities'.

Since $D(\lambda)$ given in (2.10) is entire, we can obtain its Maclaurin expansion with the help of the Maclaurin expansion of the unique solution $\phi(x;\lambda)$ to (2.1), which we write as

$$\phi(x;\lambda)=\phi_0(x)+\lambda\phi_1(x)+\lambda^2\phi_2(x)+O(\lambda^3),\qquad \lambda\to 0 \text{ in } \mathbf{C}, \tag{2.18}$$

where we have defined

$$\phi_0(x) := x, \quad \phi_1(x) := M_2(x) - xM_1(x), \tag{2.19}$$

$$\phi_2(x) := \frac{1}{2}\int_0^x [M_1(z)]^2\,\mathrm{d}z - x\int_0^x zM_2(z)\,\mathrm{d}z + \int_0^x z\rho(z)M_2(z)\,\mathrm{d}z, \tag{2.20}$$

with

$$M_1(x) := \int_0^x z\rho(z)\,\mathrm{d}z, \quad M_2(x) := \int_0^x z^2\rho(z)\,\mathrm{d}z.$$

Using (2.18)–(2.20) and their x-derivatives in (2.10) we obtain

$$D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + O(\lambda^3), \qquad \lambda \to 0 \text{ in } \mathbf{C}, \tag{2.21}$$

where

$$D_0 := 0, \quad D_1 := \frac{b^3}{3} - M_2(b),$$

$$D_2 := -\frac{b^5}{30} + b[M_1(b)]^2 - M_1(b)M_2(b) - \frac{b^3}{3}M_1(b) + \frac{b^2}{2}M_2(b) - \int_0^b [M_1(z)]^2\,\mathrm{d}z.$$

If $d = 1$ in (2.14), with the help of (2.10) and (2.21) we see that

$$D_1 \neq 0, \quad \gamma = D_1, \quad -\gamma\sum_{j=1}^{\infty}\frac{1}{\lambda_j} = D_2,$$

where γ is the parameter appearing in (2.13) and λ_j for $j \in \mathbf{N}$ are the nonzero zeros of $D(\lambda)$, some of which may be repeated. On the other hand, if $d = 2$ in (2.14), then we must have

$$D_1 = 0, \quad D_2 \neq 0, \quad \gamma = D_2.$$

The results in the following two propositions will be used in the proof of the unique determination of ρ.

Proposition 2.6. *Let f be an entire function of λ such that*

$$\begin{cases} f(\lambda) = \dfrac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|c)}{\sqrt{\lambda}}O(1), & \lambda \to \infty \text{ in } \mathbf{C}, \\ f\left(\dfrac{\pi^2 n^2}{c^2}\right) = 0, & n \in \mathbf{N}, \end{cases} \tag{2.22}$$

where c is a positive constant. Then, there is a constant C_1 such that

$$f(\lambda) = C_1\frac{\sin(\sqrt{\lambda}c)}{\sqrt{\lambda}} = C_1 c\prod_{n=1}^{\infty}\left(1 - \frac{c^2\lambda}{\pi^2 n^2}\right). \tag{2.23}$$

Similarly, if g is an entire function of λ such that

$$\begin{cases} g(\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|c)O(1), & \lambda \to \infty \text{ in } \mathbf{C}, \\ g\left(\dfrac{\pi^2(2n-1)^2}{4c^2}\right) = 0, & n \in \mathbf{N}, \end{cases} \tag{2.24}$$

then there is a constant C_2 such that

$$g(\lambda) = C_2\cos(\sqrt{\lambda}c) = C_2\prod_{n=1}^{\infty}\left(1 - \frac{4c^2\lambda}{\pi^2(2n-1)^2}\right). \tag{2.25}$$

Proof. The second line of (2.22) implies that $f(\lambda)$ can be written as

$$f(\lambda) = h(\lambda)\frac{\sin(\sqrt{\lambda}c)}{\sqrt{\lambda}}, \tag{2.26}$$

for some entire function $h(\lambda)$. Using (2.26) in the first line of (2.22), we get

$$\left|h(\lambda)\frac{\sin(\sqrt{\lambda}c)}{\sqrt{\lambda}}\right| \leqslant \frac{B\exp(|\mathrm{Im}[\sqrt{\lambda}]|c)}{|\sqrt{\lambda}|}, \qquad \lambda \in \mathbf{C}, \tag{2.27}$$

for some positive constant B. With the help of (2.22) and (2.26), one can prove that the order of h cannot exceed $1/2$. On the other hand, by using the exponential representation of the sine function, as $\lambda \to \infty$ along any ray other than the positive real axis, we have

$$\left|\frac{\sin(\sqrt{\lambda}c)}{\sqrt{\lambda}}\right| = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|c)}{2|\sqrt{\lambda}|}[1 + o(1)]. \tag{2.28}$$

Hence, from (2.27) and (2.28) we see that $h(\lambda)$ must be bounded on any ray other than the positive real axis. By invoking a consequence of the Phragmén–Lindelöf principle (see theorem 18.1.3 of [18]), we conclude that $h(\lambda)$ must be a constant, which establishes (2.23). The proof of (2.25) is obtained in a similar manner. □

Proposition 2.7. *Let f be an entire function of λ satisfying (2.22), and assume that as $\lambda \to \infty$ along some fixed ray in the complex plane, we have*

$$f(\lambda) = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|c)}{\sqrt{\lambda}}o(1). \tag{2.29}$$

Then, $f(\lambda) \equiv 0$. Similarly, let g be an entire function of λ satisfying (2.24), and assume that as $\lambda \to \infty$ along some fixed ray in the complex plane we have

$$g(\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|c)o(1). \tag{2.30}$$

Then, $g(\lambda) \equiv 0$.

Proof. In the proof of proposition 2.6, the further restriction given in (2.29) forces us to have $C_1 = 0$ in (2.23), and hence we get $f(\lambda) \equiv 0$. Similarly, (2.30) forces to have $C_2 = 0$ in (2.25), yielding $g(\lambda) \equiv 0$. □

We state a relevant relationship between (2.1) and two Sturm–Liouville problems in the following corollary.

Corollary 2.8. *Let ρ satisfy (1.2). Then, the eigenvalues of the Sturm–Liouville problem*

$$\begin{cases} \psi'' + \lambda\rho(x)\psi = 0, & 0 < x < b, \\ \psi(0) = \psi(b) = 0, \end{cases} \tag{2.31}$$

exactly correspond to the zeros of $\phi(b;\lambda)$, where $\phi(x;\lambda)$ is the unique solution to (2.1). Similarly, the eigenvalues of the Sturm–Liouville problem

$$\begin{cases} \psi'' + \lambda\rho(x)\psi = 0, & 0 < x < b, \\ \psi(0) = \psi'(b) = 0 \end{cases} \tag{2.32}$$

exactly correspond to the zeros of $\phi'(b;\lambda)$.

The fundamental uniqueness theorem of inverse spectral theory for Sturm–Liouville problems indicates that, assuming the existence problem is solved, the knowledge of two sets of spectra uniquely determines ρ. It is already known [2, 17, 20–24] that the combined knowledge of the eigenvalues of (2.31) and the eigenvalues of (2.32) uniquely determines $\rho(x)$ for $x \in [0, b]$. Thus, with the help of corollary 2.8, we have the following result.

Corollary 2.9. *Let ρ_1 and ρ_2 satisfy (1.2), and let $\phi_1(x;\lambda)$ and $\phi_2(x;\lambda)$ be the corresponding unique solutions to (2.1) with $\rho = \rho_1$ and $\rho = \rho_2$, respectively. If $\phi_1(b;\lambda)$ and $\phi_2(b;\lambda)$ have the same set of zeros and also $\phi_1'(b;\lambda)$ and $\phi_2'(b;\lambda)$ have the same set of zeros, then $\rho_1 \equiv \rho_2$.*

3. The inverse problem

We assume that ρ satisfies (1.2). The relevant direct problem is the determination of the special transmission eigenvalues of (1.1) including their 'multiplicities' when $\rho(x)$ is known for $x \in [0, b]$. Conversely, our relevant inverse problem is the determination of $\rho(x)$ for $x \in [0, b]$ from the knowledge of the special transmission eigenvalues of (1.1) including their 'multiplicities'. From (2.13) and (2.14), we see that the direct problem can be equivalently stated as the determination of the map $\rho \mapsto \Xi$ and the inverse problem as the determination of the map $\Xi \mapsto \rho$, where Ξ is the quantity appearing in (2.14). Recall that we are only concerned with the uniqueness aspect of the inverse problem and not with the existence aspect. In other words, corresponding to our data we assume that there exists at least one function ρ satisfying (1.2) and we show that our data lead to a unique ρ.

The main conclusion in our paper is that once the existence problem is known to be solvable, the function Ξ uniquely determines ρ in the case $a < b$, where a is the quantity defined in (1.7). On the other hand, when $a = b$ it is unclear if Ξ uniquely determines ρ, but we show that Ξ and γ together uniquely determine ρ, where γ is the constant appearing in (2.13). In other words, if $a = b$, then $\rho(x)$ for $x \in [0, b]$ is uniquely determined by $D(\lambda)$ for $\lambda \in \mathbf{C}$. First, we present a special case of the uniqueness result in the following theorem, which also includes the solution to the relevant existence problem.

Theorem 3.1. *Assume that ρ satisfies (1.2), and let the corresponding $D(\lambda)$ be as in (2.10). If $D(\lambda) \equiv 0$ for $\lambda \in \mathbf{C}$, then $\rho(x) \equiv 1$ for $x \in [0, b]$.*

Proof. If $D(\lambda) \equiv 0$, then (2.10) implies that

$$\frac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}\phi'(b;\lambda) = \cos(\sqrt{\lambda}b)\phi(b;\lambda), \qquad \lambda \in \mathbf{C}. \tag{3.1}$$

Note that each of the four functions in (3.1), namely $\sin(\sqrt{\lambda}\,b)/\sqrt{\lambda}$, $\cos(\sqrt{\lambda}\,b)$, $\phi(b;\lambda)$ and $\phi'(b;\lambda)$ are entire of order $1/2$. Furthermore, $\phi(b;\lambda)$ and $\phi'(b;\lambda)$ cannot vanish simultaneously, and $\sin(\sqrt{\lambda}\,b)/\sqrt{\lambda}$ and $\cos(\sqrt{\lambda}\,b)$ cannot vanish simultaneously. Thus, (3.1) implies that $\sin(\sqrt{\lambda}\,b)/\sqrt{\lambda}$ and $\phi(b;\lambda)$ must have the same set of zeros including multiplicities and that $\cos(\sqrt{\lambda}\,b)$ and $\phi'(b;\lambda)$ must have the same set of zeros including multiplicities (note that, in this particular case, the multiplicities must all be one). Hence, by the Hadamard factorization theorem, considering the fact that the order of each of these four functions is $1/2$, we must have

$$\phi(b;\lambda) = c_1\frac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}, \qquad \phi'(b;\lambda) = c_1\cos(\sqrt{\lambda}b) \tag{3.2}$$

for some nonzero constant c_1; in fact, (2.16) implies that $c_1 = 1$. By corollary 2.9 we know that the combined knowledge of the zeros of $\phi(b;\lambda)$ and of $\phi'(b;\lambda)$ uniquely determines ρ. Thus, ρ is uniquely determined by the combined knowledge of the zeros of $\sin(\sqrt{\lambda}\,b)/\sqrt{\lambda}$ and of $\cos(\sqrt{\lambda}\,b)$, and it is already known that those combined zeros correspond to $\rho(x) \equiv 1$ for $x \in [0, b]$. □

In the next theorem, we present our uniqueness result when $a < b$.

Theorem 3.2. *Assume that for the function Ξ appearing in (2.14), there corresponds at least one function ρ satisfying (1.2); assume also that $a < b$, where a is the quantity defined in (1.7). Then, ρ is uniquely determined by Ξ; in other words, the knowledge of special transmission eigenvalues of (1.1) with 'multiplicities' uniquely determines ρ.*

Proof. Let us assume that ρ_1 and ρ_2 correspond to Ξ_1 and Ξ_2, respectively, and let $D_1(\lambda)$ and $D_2(\lambda)$ be the corresponding quantities in (2.13) with γ_1 and γ_2 being the respective constants there. We will show that $\rho_1 \equiv \rho_2$ if $\Xi_1 \equiv \Xi_2$. Let us also use ϕ_1 and ϕ_2 to denote the solutions to (2.1) corresponding to ρ_1 and ρ_2, respectively. From the line above (1.8), we see that if $\Xi_1(\lambda) \equiv \Xi_2(\lambda)$, then $a_1 = a_2$, where a_1 and a_2 are the corresponding quantities for ρ_1 and ρ_2, respectively. Let us use a to denote the common value of a_1 and a_2. Since we assume that $a < b$, by (2.5), (2.6), (2.8) and (2.9) we have

$$\phi_1(b;\lambda) = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|a)}{\sqrt{\lambda}}O(1), \quad \phi_2(b;\lambda) = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|a)}{\sqrt{\lambda}}O(1), \qquad \lambda \to \infty \text{ in } \mathbf{C}, \tag{3.3}$$

$$\phi_1(b;\lambda) = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|b)}{\sqrt{\lambda}}o(1), \quad \phi_2(b;\lambda) = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|b)}{\sqrt{\lambda}}o(1), \qquad \lambda \to \infty \text{ in } \mathbf{C}_\varepsilon, \tag{3.4}$$

$$\phi_1'(b;\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|a)O(1), \quad \phi_2'(b;\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|a)O(1), \qquad \lambda \to \infty \text{ in } \mathbf{C}, \tag{3.5}$$

$$\phi_1'(b;\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|b)o(1), \quad \phi_2'(b;\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|b)o(1), \qquad \lambda \to \infty \text{ in } \mathbf{C}_\varepsilon, \tag{3.6}$$

where $\mathbf{C}_\varepsilon$ is the sector defined in (2.7). Since $\Xi_1(\lambda) \equiv \Xi_2(\lambda)$, from (2.13) and (2.14) it follows that

$$\frac{1}{\gamma_1}D_1\left(\frac{n^2\pi^2}{b^2}\right) = \frac{1}{\gamma_2}D_2\left(\frac{n^2\pi^2}{b^2}\right), \qquad n \in \mathbf{N}, \tag{3.7}$$

and hence from (2.10) we get

$$\frac{1}{\gamma_1}\phi_1\left(b;\frac{n^2\pi^2}{b^2}\right) = \frac{1}{\gamma_2}\phi_2\left(b;\frac{n^2\pi^2}{b^2}\right), \qquad n \in \mathbf{N}. \tag{3.8}$$

In a similar way, with the help of (2.9), (2.10), (2.13) and (2.14), by using

$$\Xi_1\left(b;\frac{(2n-1)^2\pi^2}{4b^2}\right) = \Xi_2\left(b;\frac{(2n-1)^2\pi^2}{4b^2}\right), \qquad n \in \mathbf{N}, \tag{3.9}$$

we obtain

$$\frac{1}{\gamma_1}\phi_1'\left(b;\frac{(2n-1)^2\pi^2}{4b^2}\right) = \frac{1}{\gamma_2}\phi_2'\left(b;\frac{(2n-1)^2\pi^2}{4b^2}\right), \qquad n \in \mathbf{N}. \tag{3.10}$$

In proposition 2.7 by choosing $c = b$ and $f(\lambda) = \phi_1(b;\lambda)/\gamma_1 - \phi_2(b;\lambda)/\gamma_2$, we see that (3.3), (3.4) and (3.8) imply that $f(\lambda) \equiv 0$. Similarly, in proposition 2.7 by choosing $c = b$ and $g(\lambda) = \phi_1'(b;\lambda)/\gamma_1 - \phi_2'(b;\lambda)/\gamma_2$, we see that (3.5), (3.6) and (3.10) imply that $g(\lambda) \equiv 0$. On the other hand, $f(\lambda) \equiv 0$ indicates that $\phi_1(b;\lambda)$ and $\phi_2(b;\lambda)$ have the same set of zeros, and $g(\lambda) \equiv 0$ indicates that $\phi_1'(b;\lambda)$ and $\phi_2'(b;\lambda)$ have the same set of zeros. Thus, using corollary 2.9, we conclude that $\rho_1 \equiv \rho_2$. □

The next uniqueness theorem applies to the case $a = b$.

Theorem 3.3. *Suppose that for the function Ξ appearing in (2.14) there corresponds at least one function ρ satisfying (1.2); assume also that $a = b$, where a is the quantity defined in (1.7). Then, ρ is uniquely determined by the combined knowledge of Ξ and the constant γ appearing in (2.13); in other words, the knowledge of special transmission eigenvalues of (1.1) with 'multiplicities' along with the knowledge of γ uniquely determines ρ.*

Proof. The proof is similar to the proof of theorem 3.2 with appropriate modifications we indicate here. As in the proof of theorem 3.2, we have $\Xi_1(\lambda) \equiv \Xi_2(\lambda)$, but we also have $\gamma_1 = \gamma_2$, and we want to show that $\rho_1 \equiv \rho_2$. By corollary 2.9, it is sufficient to prove that $\phi_1(b;\lambda)$ and $\phi_2(b;\lambda)$ have the same set of zeros and that $\phi_1'(b;\lambda)$ and $\phi_2'(b;\lambda)$ have the same set of zeros. Since $a = b$, this time we have (3.3) and (3.5), but not (3.4) or (3.6). Proceeding as in (3.7)–(3.10) verbatim, and in proposition 2.6 by choosing f and g as in the proof of theorem 3.2, we obtain

$$\frac{1}{\gamma_1}\phi_1(b;\lambda) - \frac{1}{\gamma_2}\phi_2(b;\lambda) = C_1\frac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}, \tag{3.11}$$

$$\frac{1}{\gamma_1}\phi_1'(b;\lambda) - \frac{1}{\gamma_2}\phi_2'(b;\lambda) = C_2\cos(\sqrt{\lambda}b), \tag{3.12}$$

for some constants C_1 and C_2. Evaluating (3.11) and (3.12) at $\lambda = 0$ and using (2.16), we get

$$C_1 = C_2 = \frac{1}{\gamma_1} - \frac{1}{\gamma_2}. \tag{3.13}$$

Since we assume $\gamma_1 = \gamma_2$, we see from (3.13) that $C_1 = C_2 = 0$. Thus, from (3.11) and (3.12) we get $\phi_1(b;\lambda) = \phi_2(b;\lambda)$ and $\phi_1'(b;\lambda) = \phi_2'(b;\lambda)$, indicating that $\phi_1(b;\lambda)$ and $\phi_2(b;\lambda)$ have the same set of zeros and also that $\phi_1'(b;\lambda)$ and $\phi_2'(b;\lambda)$ have the same set of zeros. $\square$

Having considered the inverse problem when $a < b$ and $a = b$ in theorems 3.2 and 3.3, respectively, let us now comment on the case $a > b$. The method we use to prove the uniqueness for $a \leqslant b$ does not apply to the case $a > b$, as the following argument clarifies. The lack of applicability of our technique to the case $a > b$ certainly does not mean that a uniqueness result does not exist when $a > b$. The unique recovery of ρ from $D(\lambda)$ defined in (2.10) is based on our ability to extract each of $\phi(b;\lambda)$ and $\phi'(b;\lambda)$ up to a constant multiplicative factor. When $a > b$ let us the consider the determination of two functions $\phi(b;\lambda)$ and $\phi'(b;\lambda)$ that are entire in λ and of order 1/2 and that satisfy the respective asymptotics related to (2.5) and (2.6), namely, as $\lambda \to \infty$ in $\mathbf{C}$

$$\phi(b;\lambda) = \frac{\exp(|\mathrm{Im}[\sqrt{\lambda}]|a)}{\sqrt{\lambda}}O(1), \quad \phi'(b;\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|a)O(1), \tag{3.14}$$

for which

$$\frac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}\phi'(b;\lambda) - \cos(\sqrt{\lambda}b)\phi(b;\lambda) = D(\lambda). \tag{3.15}$$

Let ζ be any entire function of λ having the asymptotics

$$\zeta(\lambda) = \exp(|\mathrm{Im}[\sqrt{\lambda}]|(a-b))O(1), \qquad \lambda \to \infty \text{ in } \mathbf{C}. \tag{3.16}$$

Letting

$$\check{\phi}(b;\lambda) := \phi(b;\lambda) + \frac{\sin(\sqrt{\lambda}b)}{\sqrt{\lambda}}\zeta(\lambda), \quad \check{\phi}'(b;\lambda) := \phi'(b;\lambda) + \cos(\sqrt{\lambda}b)\zeta(\lambda),$$

we see that $\check{\phi}(b;\lambda)$ and $\check{\phi}'(b;\lambda)$ are entire in λ and that (3.14) and (3.15) are satisfied when we replace in them $\phi(b;\lambda)$ with $\check{\phi}(b;\lambda)$ and replace $\phi'(b;\lambda)$ with $\check{\phi}'(b;\lambda)$. Because of (3.16), $\zeta(\lambda)$ must be a constant when $a = b$ and must be zero when $a < b$, but no such restrictions exist when $a > b$. Thus, our method does not allow us to conclude the unique determination of ρ from $D(\lambda)$ when $a > b$.

4. The inverse problem for the Schrödinger equation

In the case of the Schrödinger operator, the interior transmission eigenvalue problem is analogous to the corresponding problem for the wave equation with variable speed. Instead of (1.1), we have

$$\begin{cases} -\Delta\tilde{\Psi} + V(\mathbf{x})\tilde{\Psi} = \mu\tilde{\Psi}, & \mathbf{x} \in \Omega, \\ -\Delta\tilde{\Psi}_0 = \mu\tilde{\Psi}_0, & \mathbf{x} \in \Omega, \\ \tilde{\Psi} = \tilde{\Psi}_0, \quad \dfrac{\partial\tilde{\Psi}}{\partial\mathbf{n}} = \dfrac{\partial\tilde{\Psi}_0}{\partial\mathbf{n}}, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{4.1}$$

where μ is the spectral parameter, $V(\mathbf{x})$ is a real-valued potential that is square integrable on Ω, and it is assumed that $V(\mathbf{x}) \equiv 0$ outside Ω. Those μ-values yielding nontrivial solutions $\tilde{\Psi}$ and $\tilde{\Psi}_0$ to (4.1) are called *transmission eigenvalues* of (4.1). In the spherically symmetric case, using $V(x)$ instead of $V(\mathbf{x})$ with $x := |\mathbf{x}|$, we have the following analog of proposition 1.1. We omit its proof because it is similar to the proof of proposition 1.1.

Proposition 4.1. *Consider the special case of (4.1) with Ω being the three-dimensional ball of radius b centered at the origin, where only spherically symmetric wavefunctions are allowed and it is assumed that such wavefunctions are continuous in the closure of Ω. Then, the corresponding transmission eigenvalues of (4.1) coincide with the eigenvalues of the boundary-value problem*

$$\begin{cases} -\tilde{\Phi}'' + V(x)\tilde{\Phi} = \mu\tilde{\Phi}, & 0 < x < b, \\ \tilde{\Phi}(0) = 0, \quad \dfrac{\sin(\sqrt{\mu}b)}{\sqrt{\mu}}\tilde{\Phi}'(b) - \cos(\sqrt{\mu}b)\tilde{\Phi}(b) = 0. \end{cases} \tag{4.2}$$

The eigenvalues of (4.2), namely the μ-values for which (4.2) has a nontrivial solution, coincide with the *special transmission eigenvalues* of (4.1), namely those transmission eigenvalues of (4.1) for which the corresponding wavefunctions are spherically symmetric in addition to V being spherically symmetric. Note that the boundary condition at $x = b$ in (4.2) suggests an analog of $D(\lambda)$ appearing in (2.10). We define

$$\tilde{D}(\mu) := \frac{\sin(\sqrt{\mu}b)}{\sqrt{\mu}}\tilde{\phi}'(b;\mu) - \cos(\sqrt{\mu}b)\tilde{\phi}(b;\mu), \tag{4.3}$$

where $\tilde{\phi}(x;\mu)$ is the analog of $\phi(x;\lambda)$ appearing in (2.4) and is the unique solution to the initial-value problem

$$\begin{cases} -\tilde{\phi}'' + V(x)\tilde{\phi} = \mu\tilde{\phi}, & 0 < x < b, \\ \tilde{\phi}(0) = 0, \quad \tilde{\phi}'(0) = 1. \end{cases} \tag{4.4}$$

The uniqueness of $\tilde{\phi}(x;\mu)$ for each $\mu \in \mathbf{C}$ is ensured [32] when V is real valued and square integrable on $[0, b]$.

We note that (4.2), (4.3) and (4.4) are closely related. If μ_j is an eigenvalue of (4.2) with an eigenfunction $\tilde{\Phi}(x;\mu_j)$, then $\tilde{\Phi}(x;\mu_j)$ must be a constant multiple of $\tilde{\phi}(x;\mu_j)$, where $\tilde{\phi}(x;\mu)$ denotes the unique solution to (4.4). Hence, from (4.3) we conclude that $\tilde{D}(\mu_j) = 0$. Thus, with the help of proposition 4.1, we conclude that the special transmission eigenvalues of (4.1), the eigenvalues of (4.2), and the zeros of (4.4) all coincide. On the other hand, there exists only one linearly independent eigenfunction for a given eigenvalue μ_j of (4.2) whereas the multiplicity of μ_j as a zero of $\tilde{D}(\mu)$ may be greater than 1. We will refer to the multiplicity of a zero μ_j of $\tilde{D}(\mu)$ as the 'multiplicity' of the special transmission eigenvalue μ_j and also as the 'multiplicity' of the eigenvalue μ_j of (4.2).

Note that from (4.3) and the second line of (4.4), we obtain

$$\tilde{D}(0) = b\tilde{\phi}'(b; 0) - \tilde{\phi}(b; 0),$$

and, contrary to (2.11), generically we have $\tilde{D}(0) \neq 0$, although we may have $\tilde{D}(0) = 0$ for some potentials. For example, if $V(x) \equiv 0$, then we have $\tilde{\phi}(x; \mu) = \sin(\sqrt{\mu}\, x)/\sqrt{\mu}$, yielding $\tilde{D}(0) = 0$. In fact, $\tilde{D}(\mu) \equiv 0$ for $\mu \in \mathbf{C}$ in that special case.

Our goal in this section is to show that $V(x)$ for $0 < x < b$ is uniquely determined by the corresponding $\tilde{D}(\mu)$ known for all $\mu \in \mathbf{C}$. In fact, we will see that, up to the multiplicative constant $\tilde{\gamma}$ appearing in (4.5), $\tilde{D}(\mu)$ is uniquely determined by the knowledge of its zeros including the multiplicities of those zeros. Since those zeros are exactly the eigenvalues of (4.2), we will conclude that the knowledge of the eigenvalues of (4.2) including their 'multiplicities' and the value of $\tilde{\gamma}$ uniquely determines V. Since the eigenvalues of (4.2) are the special transmission eigenvalues of (4.1), we will also conclude that the knowledge of those special transmission eigenvalues including their 'multiplicities' and $\tilde{\gamma}$ uniquely determines V. Since the proofs are similar to those in the case of the variable-speed wave equation studied in the previous sections, we will omit the details of the proofs.

As in the case of the variable-speed wave equation, we only consider the uniqueness aspect of our inverse problem and not the existence aspect. In other words, corresponding to our data $\tilde{D}(\mu)$ or its equivalents, we assume that there exists at least one potential V, where $V(x)$ is real valued and belongs to $L^2(0, b)$. We then prove that if V_1 and V_2 are two such potentials, then we must have $V_1 \equiv V_2$. Let us also clarify that the equality $V_1 \equiv V_2$ is meant to be an equality in the almost-everywhere sense because we deal with potentials in the class $L^2(0, b)$ whereas the corresponding equality $\rho_1 \equiv \rho_2$ obtained in section 3 holds pointwise because ρ_1 and ρ_2 satisfy (1.2).

The following theorem summarizes the properties of $\tilde{D}(\mu)$ defined in (4.3), and it is an analog of theorem 2.4. We omit the proof because it is similar to the proof of theorem 2.4.

Proposition 4.2. *Assume that V is real valued and square integrable on* $[0, b]$. *Then, the quantity* $\tilde{D}(\mu)$ *defined in (4.3) is entire in* μ *and its order does not exceed 1/2. Thus, by the Hadamard factorization theorem,* $\tilde{D}(\mu)$ *is determined, uniquely up to a multiplicative constant, from its zeros as*

$$\tilde{D}(\mu) = \tilde{\gamma}\mu^{\tilde{d}} \prod_{n=1}^{\infty} \left(1 - \frac{\mu}{\mu_n}\right), \tag{4.5}$$

with μ_n *for* $n \in \mathbf{N}$ *being the nonzero zeros of* $\tilde{D}(\mu)$, *some of which may be repeated, and* $\tilde{d}$ *denoting the multiplicity of the zero as a zero of* $\tilde{D}(\mu)$.

The results stated in the following theorem are analogous to those stated in corollaries 2.8 and 2.9.

Theorem 4.3. *Assume that V is real valued and square integrable on* $[0, b]$, *and let* $\tilde{\phi}(x; \mu)$ *denote the unique solution to (4.4). We then have the following:*

(i) *The zeros of* $\tilde{\phi}(b; \mu)$ *coincide with the eigenvalues of the Sturm–Liouville problem*

$$\begin{cases} -\tilde{\psi}'' + V(x)\tilde{\psi} = \mu\tilde{\psi}, & 0 < x < b, \\ \tilde{\psi}(0) = \tilde{\psi}(b) = 0. \end{cases} \tag{4.6}$$

(ii) *The zeros of* $\tilde{\phi}'(b; \mu)$ *coincide with the eigenvalues of the Sturm–Liouville problem*

$$\begin{cases} -\tilde{\psi}'' + V(x)\tilde{\psi} = \mu\tilde{\psi}, & 0 < x < b, \\ \tilde{\psi}(0) = \tilde{\psi}'(b) = 0. \end{cases} \tag{4.7}$$

(iii) *The data consisting of the eigenvalues of (4.6) and (4.7) uniquely determine* V *if the existence is ensured. In other words, assuming that there exists at least one* V *corresponding to the data, if* V_1 *and* V_2 *correspond to the same data, then we must have* $V_1(x) \equiv V_2(x)$ *on* $[0, b]$.

(iv) *The data consisting of the zeros of* $\tilde{\phi}(b; \mu)$ *and* $\tilde{\phi}'(b; \mu)$ *uniquely determine* V *if the existence is ensured. In other words, assuming that there exists at least one* V *corresponding to the data, if* V_1 *and* V_2 *correspond to the same data, then we must have* $V_1(x) \equiv V_2(x)$ *on* $[0, b]$.

Proof. We obtain (i) and (ii) by comparing (4.4) and (4.6) and by noting that $\tilde{\phi}(0) = \tilde{\psi}(0) = 0$. We note that (iii) is a version of the well-known uniqueness result by Borg [3]. Finally, (iv) is a consequence of (i)–(iii). □

The following is the analog of the uniqueness result stated in theorem 3.3.

Theorem 4.4. *Assume that* $V(x)$ *is real valued and square integrable on* $[0, b]$. *Then,* V *is uniquely determined by the function* $\tilde{D}(\mu)$ *appearing in (4.3) if we assume that there exists at least one* V *corresponding to* $\tilde{D}$. *Equivalently stated, if the existence is ensured,* V *is uniquely determined by the knowledge of the special transmission eigenvalues of (4.1) with their 'multiplicities' and the constant* $\tilde{\gamma}$ *appearing in (4.5).*

Proof. If V_1 and V_2 correspond to $\tilde{D}_1(\mu)$ and $\tilde{D}_2(\mu)$, then we need to show that $V_1 \equiv V_2$ when $\tilde{D}_1(\mu) \equiv \tilde{D}_2(\mu)$. Let $\tilde{\phi}_1(x; \mu)$ and $\tilde{\phi}_2(x; \mu)$ be the solutions to (4.4) corresponding to V_1 and V_2, respectively. Because of theorem 4.3 (iv), it is sufficient to show that $\tilde{\phi}_1(b; \mu) = \tilde{\phi}_2(b; \mu)$ and $\tilde{\phi}_1'(b; \mu) = \tilde{\phi}_2'(b; \mu)$, which is proved by proceeding as in the proof of theorem 3.3. □

By proposition 4.2, we know that the knowledge of $\tilde{D}(\mu)$ is equivalent to the knowledge of its zeros with their multiplicities and the constant $\tilde{\gamma}$ appearing in (4.5). We have already seen that the zeros of $\tilde{D}(\mu)$, the eigenvalues of (4.2), and the special transmission eigenvalues of (4.1) all coincide. Thus, from theorem 4.4, we obtain the following corollary.

Corollary 4.5. *Suppose that* $V(x)$ *is real valued and square integrable on* $[0, b]$. *Assuming that there exists at least one* V *corresponding to the data,* V *is uniquely determined by the data consisting of the zeros of* $\tilde{D}(\mu)$ *in (4.5) with their multiplicities and the constant* $\tilde{\gamma}$ *there. Equivalently, assuming the existence,* V *is uniquely determined by the data consisting of the eigenvalues of (4.2) with their 'multiplicities' and the constant* $\tilde{\gamma}$.

One consequence of corollary 4.5 is that if $\tilde{D}(\mu) \equiv 0$, then $V(x) \equiv 0$, which is the analog of theorem 3.1.

Let us mention that it is an open problem whether the value of $\tilde{\gamma}$ appearing in (4.5) can be determined from the zeros of $\tilde{D}(\mu)$. If the answer is yes, then $\tilde{\gamma}$ is not needed for the unique determination of V, and as seen from corollary 4.5 the zeros of $\tilde{D}(\mu)$ with their multiplicities would be sufficient for that purpose. The technique we use to prove the uniqueness assumes the knowledge of $\tilde{\gamma}$, but this does not rule out the possibility that there might be another method to obtain the uniqueness from the data consisting only of the zeros of $\tilde{D}(\mu)$ and their multiplicities. We note that in the discrete version of the inverse transmission problem for the Schrödinger equation, $\tilde{\gamma}$ is determined [31] in the generic case by the zeros of $\tilde{D}(\mu)$ and their multiplicities.

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