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Inverse spectral-scattering problem with two sets of discrete spectra for the radial Schrödinger equation

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Abstract
The Schrödinger equation on the half-line is considered with a real-valued, integrable potential having a finite first moment. It is shown that the potential and the boundary conditions are uniquely determined by the data containing the discrete eigenvalues for a boundary condition at the origin, the continuous part of the spectral measure for that boundary condition and a subset of the discrete eigenvalues for a different boundary condition. This result extends the celebrated two-spectrum uniqueness theorem of Borg and Marchenko to the case where there is also a continuous spectrum.

1. Introduction

The inverse spectral theory deals with the determination of a differential operator from an appropriate set of spectral data. Its origin goes back to Ambartsumyan [1] who considered the Sturm–Liouville problem

$$-\psi'' + V(x)\psi = \lambda\psi, \qquad x \in (0, \pi), \qquad (1.1)$$
$$\psi'(0) = \psi'(\pi) = 0,$$

where the prime denotes the spatial x-derivative and the potential V is continuous and real valued. Ambartsumyan indicated that if $\{\lambda_j\}_{j=0}^{\infty}$ is the eigenvalue set for this Sturm–Liouville problem and if $\lambda_j = j^2$ for $j = 0, 1, 2, \ldots$, then $V \equiv 0$. Next, Borg showed [2] that one spectrum in general does not uniquely determine the corresponding Sturm–Liouville operator and that Ambartsumyan's result is really a special case. In particular, Borg gave the proof of the following result: let $\{\lambda_j\}_{j=0}^{\infty}$ be the eigenvalue set for (1.1) with the boundary conditions

$$\cos\alpha \cdot \psi'(0) + \sin\alpha \cdot \psi(0) = 0, \qquad \cos\beta \cdot \psi'(\pi) + \sin\beta \cdot \psi(\pi) = 0,$$

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and let $\{\mu_j\}_{j=0}^{\infty}$ be the eigenvalue set with the boundary condition

$$\cos\gamma \cdot \psi'(0) + \sin\gamma \cdot \psi(0) = 0, \qquad \cos\beta \cdot \psi'(\pi) + \sin\beta \cdot \psi(\pi) = 0,$$

where $\gamma \neq \alpha$. Then, the two sets $\{\lambda_j\}_{j=0}^{\infty}$ and $\{\mu_j\}_{j=0}^{\infty}$ uniquely determine α, β, γ and V.

Borg [3] and Marchenko [4] studied the Sturm–Liouville operator on the half-line $\mathbf{R}^+ := (0, +\infty)$ with a boundary condition at the origin when there is no continuous spectrum. They showed that two sets of discrete spectra associated with distinct boundary conditions at $x = 0$ (with a fixed boundary condition, if any, at $x = +\infty$) uniquely determine the potential and the boundary conditions at the origin.

A continuous spectrum often arises in applications. It comes into play in a natural way in the analysis of potentials vanishing at infinity. In this paper, we generalize the celebrated Borg–Marchenko result to the case where there is also a continuous spectrum; namely, we prove that the potential and boundary conditions are uniquely determined by an appropriate data set containing the discrete eigenvalues and continuous part of the spectral measure corresponding to one boundary condition at the origin and a subset of the discrete eigenvalues for a different boundary condition. Another extension of the Borg–Marchenko theorem to the case with a continuous spectrum is given by Gesztesy and Simon [5], where a uniqueness result is presented when Krein's spectral shift function is known. In our generalization of the Borg–Marchenko theorem, our conditions are directly stated in terms of a subset of the spectral measure, namely, the amplitude of the Jost function and the eigenvalues. There is an extensive literature on the inverse spectral problem; for other important contributions to the field and a more detailed historical account, we refer the reader to [5–8].

Consider the radial Schrödinger equation, related to (1.1) with $\lambda = k^2$,

$$-\psi'' + V(x)\psi = k^2\psi, \qquad x \in \mathbf{R}^+, \tag{1.2}$$

with the boundary condition

$$\sin\alpha \cdot \psi'(k, 0) + \cos\alpha \cdot \psi(k, 0) = 0, \tag{1.3}$$

for some $\alpha \in (0, \pi]$. The condition (1.3) is also written as

$$\begin{cases} \psi'(k, 0) + \cot\alpha \cdot \psi(k, 0) = 0, & \alpha \in (0, \pi), \\ \psi(k, 0) = 0, & \alpha = \pi. \end{cases} \tag{1.4}$$

In (1.3) or (1.4) we get the Dirichlet condition if $\alpha = \pi$, the Neumann condition if $\alpha = \pi/2$ and otherwise the mixed condition. We assume that the potential V in (1.2) belongs to the Faddeev class; i.e., it is real valued and belongs to $L_1^1(\mathbf{R}^+)$, where $L_n^1(J)$ denotes the Lebesgue-measurable functions V defined on a Lebesgue-measurable set J for which $\int_J \mathrm{d}x\,(1 + |x|)^n |V(x)|$ is finite.

Let H_α for $\alpha \in (0, \pi]$ denote the unique self-adjoint realization [9] of $-\mathrm{d}^2/\mathrm{d}x^2 + V$ in $L^2(0, +\infty)$ with the boundary condition (1.3). It is known [8, 9] that H_α has no positive or zero eigenvalues, it has no singular continuous spectrum and its absolutely continuous spectrum consists of $[0, +\infty)$. It has a finite number of simple negative eigenvalues, and we use $\sigma_{\mathrm{d}}(H_\alpha) := \{-\kappa_{\alpha j}^2\}_{j=1}^{N_\alpha}$ to denote the eigenvalue set. The Jost function of (1.2) associated with the boundary condition (1.4) is defined as [8]

$$F_\alpha(k) := \begin{cases} -\mathrm{i}[f'(k, 0) + \cot\alpha \cdot f(k, 0)], & \alpha \in (0, \pi), \\ f(k, 0), & \alpha = \pi, \end{cases} \tag{1.5}$$

where $f(k, x)$ denotes the Jost solution to (1.2) satisfying the asymptotics

$$f(k, x) = \mathrm{e}^{\mathrm{i}kx}[1 + o(1)], \qquad f'(k, x) = \mathrm{i}k\,\mathrm{e}^{\mathrm{i}kx}[1 + o(1)], \qquad x \to +\infty. \tag{1.6}$$

It is known [7, 8] that the set $\{\mathrm{i}\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$ corresponds to the zeros of F_α in $\mathbf{C}^+$. We use $\mathbf{C}^+$ for the upper half complex plane and $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$ for its closure.

There are two main methods to solve the inverse spectral and scattering problems for the radial Schrödinger equation, namely, the Gel'fand–Levitan method and the Marchenko method. The former [7, 8, 10, 11] solves the inverse spectral problem, and the potential and boundary conditions are uniquely reconstructed by solving the Gel'fand–Levitan integral equation (5.4) with the input data (5.5) or (5.6) obtained from the spectral measure $\rho_\alpha(\lambda)$ given in (3.23). The part of the spectral measure associated with the continuous spectrum is absolutely continuous, and as seen from (3.23) its derivative at energy k^2 is determined by $|F_\alpha|$. The part associated with the discrete spectrum is determined by the set of eigenvalues $\{-\kappa_{\alpha j}^2\}_{j=1}^{N_\alpha}$ and the norming constants $\{g_{\alpha j}\}_{j=1}^{N_\alpha}$. On the other hand, the Marchenko method [7, 8, 10, 12] is an inverse scattering procedure, and the potential and boundary conditions are uniquely reconstructed by solving the Marchenko integral equation (5.10) in terms of the scattering data consisting (cf (5.7) and (5.11)) of the scattering matrix S_α, the bound state energies $\{-\kappa_{\alpha j}^2\}_{j=1}^{N_\alpha}$ and the norming constants $\{m_{\alpha j}\}_{j=1}^{N_\alpha}$, where the scattering matrix is defined as

$$S_\alpha(k) := \begin{cases} -\dfrac{F_\alpha(-k)}{F_\alpha(k)}, & \alpha \in (0, \pi), \\ \dfrac{F_\pi(-k)}{F_\pi(k)}, & \alpha = \pi. \end{cases} \tag{1.7}$$

Our generalized Borg–Marchenko problem is stated as follows. Let $\beta \in (0, \pi)$ with $\beta < \alpha \leqslant \pi$ correspond to the boundary condition obtained from (1.3) by replacing α there with β. This leads to, via (1.5), the Jost function F_β with zeros at $k = \mathrm{i}\kappa_{\beta j}$ in $\mathbf{C}^+$, where $j = 1, \ldots, N_\beta$. Assume that we are given some data set $\mathcal{D}$, which contains $|F_\alpha|$ for $k \in \mathbf{R}$, the whole set $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$, and a subset of $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$ consisting of N_α elements. Alternatively, our data $\mathcal{D}$ may include $|F_\beta(k)|$ for $k \in \mathbf{R}$ and the sets $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$ and $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$. Does $\mathcal{D}$ uniquely determine the set $\mathcal{E}$, where $\mathcal{E} := \{V, \alpha, \beta\}$? If not, what additional information do we need besides $\mathcal{D}$ in order to determine $\mathcal{E}$ uniquely? Can we present a constructive method to recover $\mathcal{E}$ from $\mathcal{D}$ or from a data set obtained by some augmentation of $\mathcal{D}$?

This generalized Borg–Marchenko problem can be considered as an inverse scattering problem because both the Faddeev class of potentials and the Jost function are natural elements in scattering theory. On the other hand, this problem is also an inverse spectral problem because in our data we use $|F_\alpha|$ and $\{-\kappa_{\alpha j}^2\}_{j=1}^{N_\alpha}$, which are both contained in the relevant spectral measure. In fact, from this point of view, we replace the N_α norming constants appearing in the discrete portion of the spectral measure by N_α of the eigenvalues for a different boundary condition. This constitutes a natural mathematical problem which is actually an inverse problem with two discrete spectra in the presence of a continuous spectrum. Replacing the norming constants in the Gel'fand–Levitan or the Marchenko method by a set of eigenvalues from a second boundary condition is also interesting from the viewpoint of physical applications. This is because eigenvalues have a direct physical interpretation as energies of the stationary states of a quantum mechanical system, whereas, *a priori*, norming constants do not have such a clear physical interpretation.

Our problem can also be considered as an inverse scattering problem on the line with a potential supported on a half-line. As we show in section 5, from our data, we can uniquely construct the data set $\mathcal{F}$ given in (5.14), which contains enough information [13–18] to reconstruct the potential by using any one of the full-line inversion methods [7, 10, 19–22].

Our motivation for this paper came from a question by Roy Pike [23] as to whether $f'(k, 0) := \mathrm{i}F_{\pi/2}(k)$, the spatial derivative of the Jost solution to the one-dimensional Schrödinger equation evaluated at $x = 0$, can uniquely determine the corresponding potential if that potential is known to be zero on the negative half-line. This question arises in the acoustical analysis of the human vocal tract. When the vocal tract is stimulated by a sinusoidal input volume velocity at the glottis, the impulse response at the lips is (cf (70) in [24]) essentially given by $f'(k,0)$. Such an inverse problem is equivalent to determining a scaled curvature of the duct of the vocal tract when a constant-frequency sound is uttered, and it has important applications in speech recognition [24].

The method we use is a generalization of that used in [23] in the case of a potential that has no bound states for either the Dirichlet or Neumann boundary conditions and that is perturbed by a Dirac delta distribution at $x=0$. The basic idea is to relate our data to the real part of a function that is in the Hardy class $H^2(\mathbf{R})$ [25, 26] of functions analytic in $\mathbf{C}^+$. It turns out that the real part of such a function is determined for $k\in\mathbf{R}$ by our data. Then, the function itself is uniquely constructed in $\overline{\mathbf{C}^+}$ with the help of the Schwarz integral formula [27–29]. Our proofs also present a method for the reconstruction of the potential and boundary conditions.

Our paper is organized as follows. We list our main results as theorems 2.1–2.8 in section 2. Then, in section 3 we present the results needed in order to prove these theorems. In section 4 the proof of each theorem is given by a constructive method; from the appropriate scattering-spectral data sets $\mathcal{D}_1,\ldots,\mathcal{D}_8$ given in (2.4)–(2.11), respectively, it is shown how the boundary conditions are uniquely reconstructed and how appropriate information can be assembled in order to uniquely reconstruct the potential. In section 5 we outline several methods to uniquely reconstruct the potentials. Finally, in section 6, we illustrate the uniqueness and reconstruction by an explicit example.

2. Main theorems

In theorems 2.1–2.8, we generalize the celebrated two-spectra uniqueness theorem proved by Borg [3] and Marchenko [4] from the case of purely discrete spectra to the case where there is also a continuous spectrum. We take into consideration all possibilities with $N_\alpha=N_\beta$ or $N_\alpha=N_\beta-1$, with $\alpha\in(0,\pi)$ or $\alpha=\pi$ and by using $|F_\alpha|$ or $|F_\beta|$ in our data.

In order to state our results in a precise way, we introduce some notation. Define

$$h_{\beta\alpha}:=\cot\beta-\cot\alpha,\qquad \alpha,\beta\in(0,\pi). \tag{2.1}$$

From (1.5) and (2.1), for $\alpha\neq\beta$ we get

$$f(k,0)=\begin{cases}\dfrac{\mathrm{i}}{h_{\beta\alpha}}[F_\beta(k)-F_\alpha(k)], & \alpha,\beta\in(0,\pi),\\ F_\pi(k), & \end{cases} \tag{2.2}$$

$$f'(k,0)=\begin{cases}\dfrac{\mathrm{i}}{h_{\beta\alpha}}[\cot\beta\cdot F_\alpha(k)-\cot\alpha\cdot F_\beta(k)], & \alpha,\beta\in(0,\pi),\\ \mathrm{i}F_\beta(k)-\cot\beta\cdot F_\pi(k), & \beta\in(0,\pi).\end{cases} \tag{2.3}$$

Note that $h_{\beta\alpha}>0$ if $0<\beta<\alpha<\pi$ because the cotangent function is monotone decreasing on $(0,\pi)$. Let $\tilde V$ be another potential in the Faddeev class, $\tilde H_\gamma$ be the corresponding realization of $-\mathrm{d}^2/\mathrm{d}x^2+\tilde V$ in $L^2(0,+\infty)$ with the boundary condition (1.3) in which α is replaced by γ, and $\sigma_{\mathrm{d}}(\tilde H_\gamma)$ denote the corresponding eigenvalue set $\{-\tilde\kappa_{\gamma j}^2\}_{j=1}^{\tilde N_\gamma}$.

Let us introduce the appropriate data sets $\mathcal{D}_1,\ldots,\mathcal{D}_8$ used as inputs in theorems 2.1–2.8, respectively, as follows:

$$\mathcal{D}_1:=\left\{h_{\beta\alpha},|F_\alpha(k)|\text{ for }k\in\mathbf{R},\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha},\{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.4}$$

$$\mathcal{D}_2:=\left\{\beta,|F_\pi(k)|\text{ for }k\in\mathbf{R},\{\kappa_{\pi j}\}_{j=1}^{N_\pi},\{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.5}$$

$$\mathcal{D}_3 := \left\{h_{\beta\alpha}, |F_\alpha(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \text{an } N_\alpha\text{-element subset of } \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.6}$$

$$\mathcal{D}_4 := \left\{\beta, |F_\pi(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \text{an } N_\pi\text{-element subset of } \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.7}$$

$$\mathcal{D}_5 := \left\{h_{\beta\alpha}, |F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.8}$$

$$\mathcal{D}_6 := \left\{|F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.9}$$

$$\mathcal{D}_7 := \left\{\beta, h_{\beta\alpha}, |F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}, \tag{2.10}$$

$$\mathcal{D}_8 := \left\{\beta, |F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\right\}. \tag{2.11}$$

Theorem 2.1. *Let the realizations H_α and H_β correspond to a potential V in the Faddeev class with the boundary conditions identified by α and β, respectively. Similarly, let $\tilde{H}_\gamma$ and $\tilde{H}_\epsilon$ correspond to $\tilde{V}$ in the Faddeev class with the boundary conditions identified with γ and ϵ, respectively. Denote the corresponding Jost functions by F_α, F_β, $\tilde{F}_\gamma$ and $\tilde{F}_\epsilon$, respectively. Suppose that*

(i) $0 < \beta < \alpha < \pi$.
(ii) $N_\alpha = N_\beta \geqslant 0$.
(iii) $h_{\beta\alpha} = h_{\epsilon\gamma}$.
(iv) $\sigma_{\mathrm{d}}(H_\alpha) = \sigma_{\mathrm{d}}(\tilde{H}_\gamma)$.
(v) $\sigma_{\mathrm{d}}(H_\beta) = \sigma_{\mathrm{d}}(\tilde{H}_\epsilon)$.
(vi) $|F_\alpha(k)| = |\tilde{F}_\gamma(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\alpha = \gamma$, $\beta = \epsilon$ and $V = \tilde{V}$. This is equivalent to saying that if $N_\alpha = N_\beta \geqslant 0$ and $0 < \beta < \alpha < \pi$, then the data set $\mathcal{D}_1$ given in (2.4) uniquely determines $\{V, \alpha, \beta\}$.

Next, we consider the analogue of theorem 2.1 when $\alpha = \pi$.

Theorem 2.2. *With the same notation as in theorem 2.1, assume that*

(i) $0 < \beta < \alpha = \pi$.
(ii) $N_\alpha = N_\beta \geqslant 0$.
(iii) $\beta = \epsilon$.
(iv) $\sigma_{\mathrm{d}}(H_\alpha) = \sigma_{\mathrm{d}}(\tilde{H}_\gamma)$.
(v) $\sigma_{\mathrm{d}}(H_\beta) = \sigma_{\mathrm{d}}(\tilde{H}_\epsilon)$.
(vi) $|F_\alpha(k)| = |\tilde{F}_\gamma(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\alpha = \gamma$ and $V = \tilde{V}$. Equivalently, if $N_\alpha = N_\beta \geqslant 0$ and $0 < \beta < \alpha = \pi$, then the data set $\mathcal{D}_2$ given in (2.5) uniquely determines V.

In the next result, the analogue of theorem 2.1 is considered when $N_\alpha = N_\beta - 1$.

Theorem 2.3. *With the same notation as in theorem 2.1, suppose that*

(i) $0 < \beta < \alpha < \pi$.
(ii) $N_\alpha = N_\beta - 1 \geqslant 0$.
(iii) $h_{\beta\alpha} = h_{\epsilon\gamma}$.
(iv) $\sigma_{\mathrm{d}}(H_\alpha) = \sigma_{\mathrm{d}}(\tilde{H}_\gamma)$.
(v) The intersection of $\sigma_{\mathrm{d}}(H_\beta)$ and $\sigma_{\mathrm{d}}(\tilde{H}_\epsilon)$ contains at least N_α common elements.
(vi) $|F_\alpha(k)| = |\tilde{F}_\gamma(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\alpha = \gamma$, $\beta = \epsilon$ *and* $V = \tilde{V}$. *This is equivalent to saying that if* $N_\alpha = N_\beta - 1 \geqslant 0$ *and* $0 < \beta < \alpha < \pi$, *then* $\{V, \alpha, \beta\}$ *is uniquely determined by the data* $\mathcal{D}_3$ *defined in (2.6).*

In the next theorem we consider the analogue of theorem 2.3 when $\alpha = \pi$, or equivalently, the analogue of theorem 2.2 when $N_\alpha = N_\beta - 1$.

Theorem 2.4. *With the same notation as in theorem 2.1, assume that*

(i) $0 < \beta < \alpha = \pi$.
(ii) $N_\alpha = N_\beta - 1 \geqslant 0$.
(iii) $\beta = \epsilon$.
(iv) $\sigma_{\mathrm{d}}(H_\alpha) = \sigma_{\mathrm{d}}(\tilde{H}_\gamma)$.
(v) The intersection of $\sigma_{\mathrm{d}}(H_\beta)$ *and* $\sigma_{\mathrm{d}}(\tilde{H}_\epsilon)$ *contains at least* N_α *common elements.*
(vi) $|F_\alpha(k)| = |\tilde{F}_\gamma(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\alpha = \gamma$ *and* $V = \tilde{V}$. *Equivalently, if* $N_\alpha = N_\beta - 1 \geqslant 0$ *and* $0 < \beta < \alpha = \pi$, *then the data set* $\mathcal{D}_4$ *given in (2.7) determines* V *uniquely.*

We note that if $N_\alpha = 0$ in theorems 2.1–2.4, then V itself is reconstructed uniquely (cf (5.1)–(5.11)) from $|F_\alpha|$ without needing $\beta, h_{\beta\alpha}$, or any possible eigenvalue of H_β. The next result is the analogue of theorem 2.1 but when $|F_\beta|$ is known instead of $|F_\alpha|$.

Theorem 2.5. *With the same notation as in theorem 2.1, suppose that*

(i) $0 < \beta < \alpha < \pi$.
(ii) $N_\alpha = N_\beta \geqslant 0$.
(iii) $h_{\beta\alpha} = h_{\epsilon\gamma}$.
(iv) $\sigma_{\mathrm{d}}(H_\alpha) = \sigma_{\mathrm{d}}(\tilde{H}_\gamma)$.
(v) $\sigma_{\mathrm{d}}(H_\beta) = \sigma_{\mathrm{d}}(\tilde{H}_\epsilon)$.
(vi) $|F_\beta(k)| = |\tilde{F}_\epsilon(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\alpha = \gamma$, $\beta = \epsilon$ *and* $V = \tilde{V}$. *Equivalently, if* $N_\alpha = N_\beta \geqslant 0$ *and* $0 < \beta < \alpha < \pi$, *then the data set* $\mathcal{D}_5$ *given in (2.8) uniquely determines* $\{V, \alpha, \beta\}$.

We note that if $N_\beta = 0$ in theorem 2.5, then V itself is uniquely determined by $|F_\beta|$ without needing $h_{\beta\alpha}$. The analogue of theorem 2.2 is given next when $|F_\beta|$ is known instead of $|F_\alpha|$; it is also the analogue of theorem 2.5 when $\alpha = \pi$.

Theorem 2.6. *With the same notation as in theorem 2.1, assume that*

(i) $0 < \beta < \alpha = \pi$.
(ii) $N_\alpha = N_\beta \geqslant 0$.
(iii) $\alpha = \gamma$.
(iv) $\sigma_{\mathrm{d}}(H_\alpha) = \sigma_{\mathrm{d}}(\tilde{H}_\gamma)$.
(v) $\sigma_{\mathrm{d}}(H_\beta) = \sigma_{\mathrm{d}}(\tilde{H}_\epsilon)$.
(vi) $|F_\beta(k)| = |\tilde{F}_\epsilon(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\beta = \epsilon$ *and* $V = \tilde{V}$. *This is equivalent to saying that if* $N_\alpha = N_\beta \geqslant 0$ *and* $0 < \beta < \alpha = \pi$, *then the data set* $\mathcal{D}_6$ *defined in (2.9) uniquely determines* $\{V, \beta\}$.

In the next theorem we present the analogue of theorem 2.3 when $|F_\beta|$ is known instead of $|F_\alpha|$; equivalently, it is the analogue of theorem 2.5 when $N_\alpha = N_\beta - 1$.

Theorem 2.7. *With the same notation as in theorem 2.1, suppose that*

(i) $0 < \beta < \alpha < \pi$.

(ii) $N_\alpha = N_\beta - 1 \geqslant 0$.
(iii) $\beta = \epsilon$.
(iv) $h_{\beta\alpha} = h_{\epsilon\gamma}$.
(v) $\sigma_{\rm d}(H_\alpha) = \sigma_{\rm d}(\tilde{H}_\gamma)$.
(vi) $\sigma_{\rm d}(H_\beta) = \sigma_{\rm d}(\tilde{H}_\epsilon)$.
(vii) $|F_\beta(k)| = |\tilde{F}_\epsilon(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $\alpha = \gamma$ *and* $V = \tilde{V}$. *Equivalently, if* $N_\alpha = N_\beta - 1 \geqslant 0$ *and* $0 < \beta < \alpha < \pi$, *then* $\{V, \alpha\}$ *is uniquely determined by the data* $\mathcal{D}_7$ *given in (2.10).*

Finally, we state the analogue of theorem 2.4 when $|F_\beta|$ is known instead of $|F_\alpha|$; it is also the analogue of theorem 2.7 when $\alpha = \pi$.

Theorem 2.8. *With the same notation as in theorem 2.1, suppose that*

(i) $0 < \beta < \alpha = \pi$.
(ii) $N_\alpha = N_\beta - 1 \geqslant 0$.
(iii) $\beta = \epsilon$.
(iv) $\alpha = \gamma$.
(v) $\sigma_{\rm d}(H_\alpha) = \sigma_{\rm d}(\tilde{H}_\gamma)$.
(vi) $\sigma_{\rm d}(H_\beta) = \sigma_{\rm d}(\tilde{H}_\epsilon)$.
(vii) $|F_\beta(k)| = |\tilde{F}_\epsilon(k)|$ *for* $k \in \mathbf{R}$.

Then, we have $V = \tilde{V}$. *This is equivalent to saying that if* $N_\alpha = N_\beta - 1 \geqslant 0$ *and* $0 < \beta < \alpha = \pi$, *then* V *is uniquely determined by the data* $\mathcal{D}_8$ *given in (2.11).*

3. Preliminaries

We first state some known results [7, 8, 10–12, 19–22, 30–33] that we need for the proofs of theorems 2.1–2.8. Consider the Jost solution $f(k,x)$ to (1.2) with the asymptotics in (1.6). The properties of $f(k,x)$ are well understood. For each fixed $x \in [0,+\infty)$, it is known that $f(\cdot,x)$ and $f'(\cdot,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Also, $f(k,0)$ and $f'(k,0)$ are real valued if $k \in \mathbf{I}^+ \cup \{0\}$, where $\mathbf{I}^+ := \mathrm{i}(0,+\infty)$ is the positive imaginary axis in $\mathbf{C}^+$. Moreover, as $k \to \infty$ in $\overline{\mathbf{C}^+}$ we have

$$f(k,0) = 1 - \frac{1}{2\mathrm{i}k}\int_0^\infty \mathrm{d}x\, V(x) + \frac{1}{2\mathrm{i}k}\int_0^\infty \mathrm{d}x\, V(x)\,\mathrm{e}^{2\mathrm{i}kx} + O(1/k^2), \tag{3.1}$$

$$f'(k,0) = \mathrm{i}k - \frac{1}{2}\int_0^\infty \mathrm{d}x\, V(x) - \frac{1}{2}\int_0^\infty \mathrm{d}x\, V(x)\,\mathrm{e}^{2\mathrm{i}kx} + O(1/k), \tag{3.2}$$

$$\frac{f'(k,0)}{f(k,0)} = \mathrm{i}k - \int_0^\infty \mathrm{d}x\, V(x)\,\mathrm{e}^{2\mathrm{i}kx} + o(1/k). \tag{3.3}$$

Furthermore [34], as $k \to 0$ in $\overline{\mathbf{C}^+}$ we have

$$\frac{f'(k,0)}{f(k,0)} = \frac{f'(0,0)}{f(0,0)} + \frac{\mathrm{i}k}{f(0,0)^2} + o(k), \qquad f(0,0) \neq 0, \tag{3.4}$$

$$\frac{f(k,0)}{f'(k,0)} = \frac{f(0,0)}{f'(0,0)} - \frac{\mathrm{i}k}{f'(0,0)^2} + o(k), \qquad f'(0,0) \neq 0. \tag{3.5}$$

It is known that $f(\cdot,0)$ has a finite number of simple zeros in $\mathbf{C}^+$, which correspond to the eigenvalues of H_π. The only real zero of $f(\cdot,0)$ may occur as a simple zero at $k = 0$.

The properties of the Jost function F_α defined in (1.5) are also well understood [8] and are summarized in the following proposition.

Proposition 3.1. *For $\alpha \in (0, \pi]$, let F_α be the Jost function associated with a potential in the Faddeev class and related to the boundary condition (1.3). Then, F_α is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Further, F_α has a finite number of zeros in $\mathbf{C}^+$ and they are all located on $\mathbf{I}^+$. The zeros of F_α in $\overline{\mathbf{C}^+}$ are simple, and the only real zero of F_α may occur as a simple zero at $k = 0$.*

As stated below (1.6), we use $\mathrm{i}\kappa_{\alpha j}$ to denote the zeros of F_α in $\mathbf{C}^+$, and we order them as $0 < \kappa_{\alpha 1} < \cdots < \kappa_{\alpha N_\alpha}$.

In the proof of the following proposition, we use $H^2(\mathbf{R})$ to denote the Hardy class of analytic functions in $\mathbf{C}^+$, i.e., the class of functions p that are analytic in $\mathbf{C}^+$ with $\sup_{y>0} \int_{-\infty}^{\infty} \mathrm{d}x\, |p(x+\mathrm{i}y)|^2 < +\infty$.

Proposition 3.2. *Assume that V is in the Faddeev class and $\alpha \in (0, \pi]$. Then, the corresponding Jost function F_α can be uniquely reconstructed from its amplitude given on $\mathbf{R}$ and its zeros in $\mathbf{C}^+$. For $\alpha \in (0, \pi)$ we have*

$$F_\alpha(k) = k \left(\prod_{j=1}^{N_\alpha} \frac{k - \mathrm{i}\kappa_{\alpha j}}{k + \mathrm{i}\kappa_{\alpha j}} \right) \exp\left(\frac{-1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\log |t/F_\alpha(t)|}{t - k - \mathrm{i}0^+} \right), \qquad k \in \overline{\mathbf{C}^+}, \tag{3.6}$$

and for $\alpha = \pi$ we get

$$F_\pi(k) = \left(\prod_{j=1}^{N_\pi} \frac{k - \mathrm{i}\kappa_{\pi j}}{k + \mathrm{i}\kappa_{\pi j}} \right) \exp\left(\frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\log |F_\pi(t)|}{t - k - \mathrm{i}0^+} \right), \qquad k \in \overline{\mathbf{C}^+}, \tag{3.7}$$

where $\mathrm{i}0^+$ indicates that the value for $k \in \mathbf{R}$ must be obtained as a limit from $\mathbf{C}^+$.

Proof. Let

$$G_\alpha(k) := \begin{cases} \dfrac{k}{F_\alpha(k)} \left(\prod_{j=1}^{N_\alpha} \dfrac{k - \mathrm{i}\kappa_{\alpha j}}{k + \mathrm{i}\kappa_{\alpha j}} \right), & \alpha \in (0, \pi), \\ F_\pi(k) \left(\prod_{j=1}^{N_\pi} \dfrac{k + \mathrm{i}\kappa_{\pi j}}{k - \mathrm{i}\kappa_{\pi j}} \right), & \alpha = \pi. \end{cases}$$

With the help of (1.5) and proposition 3.1, we see that G_π has no zeros in $\overline{\mathbf{C}^+} \setminus \{0\}$ and $\log G_\pi$ belongs to the Hardy class $H^2(\mathbf{R})$. From (3.1) we get

$$\log G_\pi(k) = O(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$$

Note that $f(0,0)$ and $f'(0,0)$ cannot simultaneously be zero because this would imply $f(0,x) = 0$ for all $x \geqslant 0$, contradicting (1.6). Thus, when $f(0,0) = 0$, with the help of (3.5) we get

$$\log G_\pi(k) = \log f'(k,0) + \log(-\mathrm{i}k/f'(0,0)^2) + O(1), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}.$$

Consequently,

$$\int_{-\infty}^{\infty} \mathrm{d}t\, |\log G_\pi(t + \mathrm{i}z)|^2 \leqslant C, \qquad z \geqslant 0,$$

for some constant C. Since $\log G_\pi$ is analytic for $k \in \mathbf{C}^+$ and

$$\mathrm{Re}[\log G_\pi(k)] = \log |G_\pi(k)|, \qquad k \in \mathbf{R},$$

it follows from the Schwarz integral formula (see, e.g., theorem 93 on p 125 of [27]) that

$$\log G_\pi(k) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{\log |G_\pi(t)|}{t-k}, \qquad k \in \mathbf{C}^+. \tag{3.8}$$

Moreover, $\log(G_\pi(t+\mathrm{i}z)) \to \log(G_\pi(t))$ as $z \to 0^+$ in the L^2-sense and a.e. in t. Consequently, (3.7) follows from (3.8). We prove (3.6) in a similar way by using the analyticity of $\log G_\alpha$ in $\mathbf{C}^+$, (3.1)–(3.5), proposition 3.1 and

$$\int_{-\infty}^{\infty} \mathrm{d}t \, |\log G_\alpha(t+\mathrm{i}z)|^2 \leqslant C, \qquad z \geqslant 0,$$

for an appropriate constant C. □

The large-k asymptotics of the Jost functions are treated in the next proposition.

Proposition 3.3. *If $\alpha, \beta \in (0, \pi)$, then, as $k \to \infty$ in $\overline{\mathbf{C}^+}$, we have*

$$F_\alpha(k) = k - \mathrm{i}\left[\cot\alpha - \frac{1}{2}\int_0^\infty \mathrm{d}x \, V(x)\right] + o(1), \tag{3.9}$$

$$F_\pi(k) = 1 - \frac{1}{2\mathrm{i}k}\int_0^\infty \mathrm{d}x \, V(x) + o(1/k), \tag{3.10}$$

$$F_\alpha(k) - F_\beta(k) = \mathrm{i}h_{\beta\alpha} - \frac{h_{\beta\alpha}}{2k}\int_0^\infty \mathrm{d}x \, V(x) + o(1/k), \tag{3.11}$$

$$\frac{F_\beta(k)}{F_\pi(k)} = k - \mathrm{i}\cot\beta + \mathrm{i}\int_0^\infty \mathrm{d}x \, V(x)\, \mathrm{e}^{2\mathrm{i}kx} + o(1/k), \tag{3.12}$$

$$\frac{F_\pi(k)}{F_\beta(k)} = \frac{1}{k} + \frac{\mathrm{i}}{k^2}\left[\cot\beta - \int_0^\infty \mathrm{d}x \, V(x)\, \mathrm{e}^{2\mathrm{i}kx}\right] - \frac{\cot^2\beta}{k^3} + o(1/k^3), \tag{3.13}$$

$$\frac{F_\alpha(k)}{F_\beta(k)} = 1 + \frac{\mathrm{i}h_{\beta\alpha}}{k} - \frac{h_{\beta\alpha}\cot\beta}{k^2} + o(1/k^2), \tag{3.14}$$

where $h_{\beta\alpha}$ is the constant defined in (2.1).

Proof. We obtain (3.9)–(3.14) directly by using (3.1)–(3.3) in (1.5). □

Note that $F_\alpha(k)$ is purely imaginary for $k \in \mathbf{I}^+$ if $\alpha \in (0, \pi)$ and that $F_\pi(k)$ is real for $k \in \mathbf{I}^+$. Next, we analyse the small-k asymptotics of the Jost function. Since $f(0, 0)$ and $f'(0, 0)$ cannot be zero at the same time, with the help of (1.5) we see that if $F_\alpha(0) = 0$ for any value of $\alpha \in (0, \pi)\backslash\{\pi/2\}$, then we must necessarily have $f(0, 0) \neq 0$ and $f'(0, 0) \neq 0$. Clearly, (1.5) also implies that $F_\pi(0) = 0$ if and only if $f(0, 0) = 0$ and $f'(0, 0) \neq 0$. Furthermore, from (1.5) we can conclude that for $\alpha, \beta \in (0, \pi]$, if $\alpha \neq \beta$ then we cannot have $F_\alpha(0) = F_\beta(0) = 0$. Hence, in propositions 3.4 and 3.5, we do not need to consider the trivial case with $F_\alpha(0) = F_\beta(0) = 0$.

Proposition 3.4. *Assume $\alpha, \beta \in (0, \pi)$. As $k \to 0$ in $\overline{\mathbf{C}^+}$, we have*

$$\frac{F_\alpha(k)}{F_\beta(k)} = \begin{cases} \dfrac{F_\alpha(0)}{F_\beta(0)} - \dfrac{\mathrm{i}kh_{\beta\alpha}}{F_\beta(0)^2} + o(k), & F_\beta(0) \neq 0, \\ -\dfrac{\mathrm{i}}{k}\dfrac{F_\alpha(0)^2}{h_{\beta\alpha}}[1 + o(1)], & F_\beta(0) = 0, \quad \alpha \neq \beta. \end{cases} \tag{3.15}$$

Proof. Using (1.5), (3.4) and (3.5), we expand F_α/F_β as $k \to 0$ in $\overline{\mathbf{C}^+}$ and use (2.1) to simplify the result. Note that if $F_\beta(0) = 0$, then with the help of (1.5) and (2.1) we obtain $F_\alpha(0) = \mathrm{i}h_{\beta\alpha} f(0,0)$, which enables us to get the asymptotics in the second line of (3.15). □

Proposition 3.5. *Assume $\beta \in (0,\pi)$. As $k \to 0$ in $\overline{\mathbf{C}^+}$, we have*

$$\frac{F_\pi(k)}{F_\beta(k)} = \begin{cases} \dfrac{F_\pi(0)}{F_\beta(0)} - \dfrac{k}{F_\beta(0)^2} + o(k), & F_\beta(0) \neq 0, \\[2mm] \dfrac{F_\pi(0)^2}{k}[1+o(1)], & F_\beta(0) = 0. \end{cases} \tag{3.16}$$

$$\frac{F_\beta(k)}{F_\pi(k)} = \begin{cases} \dfrac{F_\beta(0)}{F_\pi(0)} + \dfrac{k}{F_\pi(0)^2} + o(k), & F_\pi(0) \neq 0, \\[2mm] -\dfrac{F_\beta(0)^2}{k}[1+o(1)], & F_\pi(0) = 0. \end{cases} \tag{3.17}$$

Proof. Using (1.5), (3.4) and (3.5), we get the expansion in the first line of (3.16). Note that, if $F_\beta(0) = 0$, we must have $F_\pi(0) \neq 0$ and hence we get the expansion in the second line of (3.16). In a similar way, the first line of (3.17) is obtained from (1.5) and (3.4), and the second line is obtained from (1.5) and (3.5) by noting that $F_\beta(0) = -\mathrm{i}f'(0,0)$ when $F_\pi(0) = 0$. □

Proposition 3.6. *If $\alpha, \beta \in (0,\pi)$, then for $k \in \mathbf{R}$ we have*

$$\mathrm{Re}\left[\frac{F_\pi(k)}{F_\beta(k)}\right] = \frac{k}{|F_\beta(k)|^2}, \qquad \mathrm{Re}\left[\frac{F_\beta(k)}{F_\pi(k)}\right] = \frac{k}{|F_\pi(k)|^2}, \qquad \mathrm{Re}\left[\frac{\mathrm{i}F_\beta(k)}{F_\alpha(k)}\right] = \frac{kh_{\beta\alpha}}{|F_\alpha(k)|^2}. \tag{3.18}$$

Proof. The first two identities in (3.18) are obtained directly from (1.5) and the well-known Wronskian identity [7, 8, 10, 19–22, 30–33],

$$f(k,0)\overline{f'(k,0)} - \overline{f(k,0)}f'(k,0) = -2\mathrm{i}k, \qquad k \in \mathbf{R}, \tag{3.19}$$

where an overbar denotes complex conjugation. To get the third identity, we use (1.5), (2.1) and (3.19). □

Proposition 3.7. *Let H_α and H_β be two realizations of the Schrödinger operator for the potential V in the Faddeev class with respective boundary conditions α and β, and respective eigenvalues $\{-\kappa_{\alpha j}^2\}_{j=1}^{N_\alpha}$ and $\{-\kappa_{\beta j}^2\}_{j=1}^{N_\beta}$. Assume that $0 < \beta < \alpha \leqslant \pi$. Then, $\sigma_{\mathrm{d}}(H_\alpha)$ and $\sigma_{\mathrm{d}}(H_\beta)$ are disjoint, and either $N_\beta = N_\alpha$ or $N_\beta = N_\alpha + 1$. In the former case we have*

$$0 < \kappa_{\alpha 1} < \kappa_{\beta 1} < \kappa_{\alpha 2} < \kappa_{\beta 2} < \cdots < \kappa_{\alpha N_\alpha} < \kappa_{\beta N_\beta}, \tag{3.20}$$

and in the latter case we have

$$0 < \kappa_{\beta 1} < \kappa_{\alpha 1} < \kappa_{\beta 2} < \kappa_{\alpha 2} < \cdots < \kappa_{\alpha N_\alpha} < \kappa_{\beta N_\beta}. \tag{3.21}$$

Proof. First, let us prove that the eigenvalues of H_α and H_β cannot overlap. Recall that the eigenvalues of H_α correspond to zeros of the Jost function F_α in $\mathbf{C}^+$. If $-\kappa^2$ were a common eigenvalue, then we would have $F_\alpha(\mathrm{i}\kappa) = F_\beta(\mathrm{i}\kappa) = 0$. By (1.5), this would imply $f(\mathrm{i}\kappa,0) = f'(\mathrm{i}\kappa,0) = 0$ because we assume $\alpha > \beta$. This, however, would force $f(\mathrm{i}\kappa,x) = 0$ for all $x \geqslant 0$, which is incompatible with (1.6). Next, let us prove that either $N_\beta = N_\alpha$ or $N_\beta = N_\alpha + 1$, and that either (3.20) or (3.21) holds. The quadratic form [35, 36] Q_α associated with H_α is given by

$$Q_\alpha(\phi,\psi) = \langle\phi',\psi'\rangle + \langle V\phi,\psi\rangle - \cot\alpha\cdot\phi(0)\cdot\overline{\psi(0)}, \qquad \alpha \in (0,\pi),$$

with domain $W_{1,2}(\mathbf{R}^+)$, and

$$Q_\pi(\phi, \psi) = \langle \phi', \psi' \rangle + \langle V\phi, \psi \rangle,$$

with domain $W_{1,2}^{(0)}(\mathbf{R}^+)$. Here, we use $\langle \cdot, \cdot \rangle$ for the standard scalar product in $L^2(\mathbf{R}^+)$, $W_{1,2}(\mathbf{R}^+)$ for the standard Sobolev space [37], and $W_{1,2}^{(0)}(\mathbf{R}^+)$ for that Sobolev space with the Dirichlet boundary condition $\phi(0) = 0$. Note that $W_{1,2}^{(0)}(\mathbf{R}^+) \subset W_{1,2}(\mathbf{R}^+)$. Since the difference of the resolvents of H_α for different values of α is a rank-one operator, it follows from the min–max principle and the spectral mapping theorem [22] that the eigenvalues of H_α and H_β must interlace. Further, we get $-\kappa_{\beta N_\beta}^2 < -\kappa_{\alpha N_\alpha}^2$ because $\beta < \alpha$ and $\sigma_{\rm d}(H_\alpha)$ and $\sigma_{\rm d}(H_\beta)$ are disjoint. Thus, we must have either $N_\beta = N_\alpha$ or $N_\beta = N_\alpha + 1$, and in the former case (3.20) must hold and in the latter case (3.21) must hold. □

Proposition 3.8. *Assume* $0 < \beta < \alpha \leqslant \pi$, *and let* F_α *and* F_β *be the Jost functions associated with a potential in the Faddeev class with respective boundary conditions* α *and* β. *We have the following:*

(i) If $F_\alpha(0) = 0$, *then* $N_\beta = N_\alpha + 1$.
(ii) If $F_\beta(0) = 0$, *then* $N_\beta = N_\alpha$.

Proof. From propositions 3.1 and 3.7 we know that the zeros of F_α and F_β are simple and interlace on $\mathbf{I}^+$ and that either $N_\beta = N_\alpha$ or $N_\beta = N_\alpha + 1$. The asymptotics of F_α and F_β as $k \to \infty$ on $\mathbf{I}^+$ are already known from proposition 3.3; also by analysing the signs of F_α and F_β as $k \to 0$ on $\mathbf{I}^+$, we can tell whether $N_\beta = N_\alpha$ or $N_\beta = N_\alpha + 1$. When $0 < \beta < \alpha < \pi$, we have $N_\beta = N_\alpha$ if F_α/F_β remains positive (or approaches 0^+ or $+\infty$) as $k \to 0$ on $\mathbf{I}^+$, and we have $N_\beta = N_\alpha + 1$ if that sign remains negative (or approaches 0^- or $-\infty$). When $0 < \beta < \alpha = \pi$, in the light of the asymptotics in (3.13) as $k \to \infty$ on $\mathbf{I}^+$, we have $N_\beta = N_\alpha$ if $\mathrm{i}F_\pi/F_\beta$ remains positive (or approaches 0^+ or $+\infty$) as $k \to 0$ on $\mathbf{I}^+$, and we have $N_\beta = N_\alpha + 1$ if that sign remains negative (or approaches 0^- or $-\infty$). In the former case of $\alpha \neq \pi$, using the first line of (3.15) with $F_\alpha(0) = 0$, we see that the sign of F_α/F_β as $k \to 0$ on $\mathbf{I}^+$ coincides with the sign of $h_{\beta\alpha}/F_\beta(0)^2$, which is negative due to the facts that $h_{\beta\alpha} > 0$ and $F_\beta(0)$ is purely imaginary. Thus, (i) holds if $\alpha \in (0, \pi)$. On the other hand, if $\alpha = \pi$, by putting $F_\pi(0) = 0$ in the first line of (3.16) and noting that $F_\beta(0)$ is purely imaginary, we see that the sign of $\mathrm{i}F_\pi/F_\beta$ remains negative as $k \to 0$ on $\mathbf{I}^+$. Thus, (i) is valid also when $\alpha = \pi$. Let us now turn to (ii). If $\alpha \in (0, \pi)$, by first interchanging α and β in the first line of (3.15) and then by setting $F_\beta(0) = 0$ there, we see that the sign of F_β/F_α as $k \to 0$ on $\mathbf{I}^+$ coincides with the sign of $h_{\alpha\beta}/F_\alpha(0)^2$, which is negative due to the facts that $h_{\alpha\beta} = -h_{\beta\alpha} < 0$ and $F_\alpha(0)$ is purely imaginary. Thus, (ii) is proved when $\alpha \in (0, \pi)$. When $\alpha = \pi$, from the second line of (3.16) we see that $\mathrm{i}F_\pi/F_\beta$ remains positive as $k \to 0$ on $\mathbf{I}^+$, and hence $N_\beta = N_\alpha$ if $F_\beta(0) = 0$, as stated in (ii). □

Next, we review certain known results [7, 8, 33, 38–40] related to the spectral function associated with H_α. Let $\varphi_\alpha(k, x)$ be the regular solution to (1.2) satisfying the boundary conditions

$$\begin{cases} \varphi_\alpha(k, 0) = 1, & \varphi_\alpha'(k, 0) = -\cot\alpha, \quad \alpha \in (0, \pi), \\ \varphi_\pi(k, 0) = 0, & \varphi_\pi'(k, 0) = 1. \end{cases} \tag{3.22}$$

There is a monotone increasing function $\rho_\alpha(\lambda)$ with $\lambda \in \mathbf{R}$, known as the spectral function, such that for any $g \in L^2(\mathbf{R}^+)$,

$$(U_\alpha g)(\lambda) := \lim_{n \to +\infty} \int_0^n \mathrm{d}x\, \varphi_\alpha(\sqrt{\lambda}, x)\, g(x)$$

exists as a strong limit in $L^2(\mathbf{R}, d\rho_\alpha)$, and moreover the following Parseval identity holds:

$$\langle g, h\rangle = \langle U_\alpha g, U_\alpha h\rangle,$$

where we recall that $\langle\cdot,\cdot\rangle$ is the standard scalar product in $L^2(\mathbf{R}^+)$. The map U_α allows a spectral representation of H_α. It follows from [7, 8] that

$$\mathrm{d}\rho_\alpha(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}}{\pi}\dfrac{1}{|F_\alpha(\sqrt{\lambda})|^2}\,\mathrm{d}\lambda, & \lambda > 0,\\ \sum_{j=1}^{N_\alpha} g_{\alpha j}^2\,\delta\big(\lambda + \kappa_{\alpha j}^2\big)\,\mathrm{d}\lambda, & \lambda < 0, \end{cases} \tag{3.23}$$

where $\delta(\cdot)$ is the Dirac delta distribution and the constants $g_{\alpha j}$ are given (cf [7, 8]) by

$$g_{\alpha j} := \begin{cases} \dfrac{|f(\mathrm{i}\kappa_{\alpha j}, 0)|}{\|f(\mathrm{i}\kappa_{\alpha j}, \cdot)\|}, & \alpha \in (0,\pi),\\ \dfrac{|f'(\mathrm{i}\kappa_{\pi j}, 0)|}{\|f(\mathrm{i}\kappa_{\pi j}, \cdot)\|}, & \alpha = \pi, \end{cases}$$

with $\|\cdot\|$ denoting the norm in $L^2(\mathbf{R}^+)$ and $f(k,x)$ being the Jost solution to (1.2). Note that the Marchenko norming constants $m_{\alpha j}$ associated with the eigenvalues $-\kappa_{\alpha j}^2$ are defined as

$$m_{\alpha j} := \frac{1}{\|f(\mathrm{i}\kappa_{\alpha j}, \cdot)\|}, \qquad j = 1, \ldots, N_\alpha.$$

With the help of (4.2.19) of [8] and (1.5), one can show that

$$\|f(\mathrm{i}\kappa_{\alpha j}, \cdot)\|^2 = \begin{cases} \dfrac{1}{2\kappa_{\alpha j}}\dot{F}_\alpha(\mathrm{i}\kappa_{\alpha j}) f(\mathrm{i}\kappa_{\alpha j}, 0), & \alpha \in (0,\pi),\\ \dfrac{\mathrm{i}}{2\kappa_{\pi j}}\dot{F}_\pi(\mathrm{i}\kappa_{\pi j}) f'(\mathrm{i}\kappa_{\pi j}, 0), & \alpha = \pi, \end{cases} \tag{3.24}$$

with the overdot denoting the k-derivative. Thus, if $\alpha \in (0,\pi)$, then both $\{g_{\alpha j}\}_{j=1}^{N_\alpha}$ and $\{m_{\alpha j}\}_{j=1}^{N_\alpha}$ can be constructed once F_α and $f(\mathrm{i}\kappa_{\alpha j}, 0)$ are known. On the other hand, if $\alpha = \pi$, then we can construct those norming constants when we know F_π and $f'(\mathrm{i}\kappa_{\pi j}, 0)$. If $0 < \beta < \alpha < \pi$, as seen from (2.2), once we know F_α, F_β and $h_{\beta\alpha}$, we can evaluate $f(k, 0)$ and hence $f(\mathrm{i}\kappa_{\alpha j}, 0)$; in particular, we get $F_\beta(\mathrm{i}\kappa_{\alpha j}) = -\mathrm{i}h_{\beta\alpha} f(\mathrm{i}\kappa_{\alpha j}, 0)$. If $0 < \beta < \alpha = \pi$, from (1.5) it follows that $f'(\mathrm{i}\kappa_{\pi j}, 0) = \mathrm{i}F_\beta(\mathrm{i}\kappa_{\pi j})$, and hence knowledge of F_β and F_π allows us to construct both the Gel'fand–Levitan and Marchenko norming constants. We have

$$g_{\alpha j} = \begin{cases} \sqrt{\dfrac{2\mathrm{i}\kappa_{\alpha j} F_\beta(\mathrm{i}\kappa_{\alpha j})}{h_{\beta\alpha}\dot{F}_\alpha(\mathrm{i}\kappa_{\alpha j})}}, & 0 < \beta < \alpha < \pi,\\ \sqrt{\dfrac{2\kappa_{\pi j} F_\beta(\mathrm{i}\kappa_{\pi j})}{\dot{F}_\pi(\mathrm{i}\kappa_{\pi j})}}, & 0 < \beta < \alpha = \pi, \end{cases} \tag{3.25}$$

$$m_{\alpha j} = \begin{cases} \sqrt{\dfrac{-2\mathrm{i}\kappa_{\alpha j} h_{\beta\alpha}}{F_\beta(\mathrm{i}\kappa_{\alpha j})\dot{F}_\alpha(\mathrm{i}\kappa_{\alpha j})}}, & 0 < \beta < \alpha < \pi,\\ \sqrt{\dfrac{-2\kappa_{\pi j}}{F_\beta(\mathrm{i}\kappa_{\pi j})\dot{F}_\pi(\mathrm{i}\kappa_{\pi j})}}, & 0 < \beta < \alpha = \pi. \end{cases} \tag{3.26}$$

4. Proofs of the main theorems

In this section, we present the proofs of theorems 2.1–2.8. In each proof, we describe how the boundary conditions are uniquely reconstructed and how enough information can be assembled for the unique recovery of the potential via the methods of section 5.

Proof of theorem 2.1. In this case, we have $N_\beta = N_\alpha$ and $0 < \beta < \alpha < \pi$. Since $|\tilde{F}_\gamma(k)| = |F_\alpha(k)|$ for $k \in \mathbf{R}$, it follows from (3.9) and (3.10) that we have $\gamma < \pi$; moreover, we get $\epsilon < \gamma$ because $h_{\epsilon\gamma} = h_{\beta\alpha} > 0$. We would like to show that our data set $\mathcal{D}_1$ given in (2.4) uniquely reconstructs V, α and β. Note that by proposition 3.8(i), we must have $F_\alpha(0) \neq 0$. Define

$$\Lambda_1(k) := -\mathrm{i} + \mathrm{i}\frac{F_\beta(k)}{F_\alpha(k)} \prod_{j=1}^{N_\alpha} \frac{k^2 + \kappa_{\alpha j}^2}{k^2 + \kappa_{\beta j}^2}. \tag{4.1}$$

From the third formula in (3.18) it follows that

$$\mathrm{Re}[\Lambda_1(k)] = \frac{k h_{\beta\alpha}}{|F_\alpha(k)|^2} \prod_{j=1}^{N_\alpha} \frac{k^2 + \kappa_{\alpha j}^2}{k^2 + \kappa_{\beta j}^2}, \qquad k \in \mathbf{R}. \tag{4.2}$$

The properties of F_α and F_β stated in proposition 3.1 indicate that Λ_1 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. Using (3.14) with α and β interchanged, from (4.1) we get

$$\Lambda_1(k) = \frac{h_{\beta\alpha}}{k} + \frac{\mathrm{i}}{k^2}\left[h_{\beta\alpha}\cot\alpha + \sum_{j=1}^{N_\alpha}\left(\kappa_{\alpha j}^2 - \kappa_{\beta j}^2\right)\right] + o(1/k^2), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{4.3}$$

As $k \to 0$ in $\overline{\mathbf{C}^+}$, noting that $F_\alpha(0) \neq 0$ and using the first line in (3.15) with α and β switched, from (4.1) we see that $\Lambda_1(k) = O(1)$ and hence Λ_1 is continuous at $k = 0$. In terms of the data $\mathcal{D}_1$, we construct the right-hand side of (4.2) and use it as input to the Schwarz formula

$$\Lambda_1(k) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t - k - \mathrm{i}0^+} \mathrm{Re}[\Lambda_1(t)], \qquad k \in \overline{\mathbf{C}^+}. \tag{4.4}$$

Thus, Λ_1 is uniquely constructed. Note that using $\mathcal{D}_1$ and (4.3), we can recover $\cot\alpha$ and hence α as well. Then, $\cot\beta$ and hence β can be recovered by using (2.1). Our data set also allows the construction of F_α in $\overline{\mathbf{C}^+}$ via (3.6). Then, having F_α and Λ_1 in hand, we obtain F_β from (4.1) as

$$F_\beta(k) = F_\alpha(k)[1 - \mathrm{i}\Lambda_1(k)] \prod_{j=1}^{N_\alpha} \frac{k^2 + \kappa_{\beta j}^2}{k^2 + \kappa_{\alpha j}^2}.$$

Having F_α, F_β and $h_{\beta\alpha}$, we can reconstruct V uniquely by using any one of the methods described in section 5. Analogous to (2.4), let us define the data set $\tilde{\mathcal{D}}_1$ as

$$\tilde{\mathcal{D}}_1 := \left\{h_{\epsilon\gamma}, |\tilde{F}_\gamma(k)| \text{ for } k \in \mathbf{R}, \{\tilde{\kappa}_{\gamma j}\}_{j=1}^{\tilde{N}_\gamma}, \{\tilde{\kappa}_{\epsilon j}\}_{j=1}^{\tilde{N}_\epsilon}\right\}.$$

Then, the uniqueness for $\mathcal{D}_1 \mapsto \{V, \alpha, \beta\}$ follows from the fact that $\tilde{\mathcal{D}}_1 = \mathcal{D}_1$. □

Proof of theorem 2.2. We have $0 < \beta < \alpha = \pi$ and $N_\pi = N_\beta$. As in the proof of theorem 2.1, we prove that $\epsilon < \gamma = \pi$. We cannot have $F_\pi(0) = 0$ as implied by proposition 3.8(i). We would like to show that our data set $\mathcal{D}_2$ given in (2.5) uniquely reconstructs V. Proceeding as in the proof of theorem 2.1, let us define

$$\Lambda_2(k) := -1 - \frac{1}{k}\frac{F_\beta(0)}{F_\pi(0)} \prod_{j=1}^{N_\pi} \frac{\kappa_{\pi j}^2}{\kappa_{\beta j}^2} + \frac{1}{k}\frac{F_\beta(k)}{F_\pi(k)} \prod_{j=1}^{N_\pi} \frac{k^2 + \kappa_{\pi j}^2}{k^2 + \kappa_{\beta j}^2}. \tag{4.5}$$

Using the second identity of (3.18) in (4.5) and noting that $F_\beta(0)$ is purely imaginary and $F_\pi(0)$ is real, we see that

$$\mathrm{Re}[\Lambda_2(k)] = -1 + \frac{1}{|F_\pi(k)|^2} \prod_{j=1}^{N_\pi} \frac{k^2+\kappa_{\pi j}^2}{k^2+\kappa_{\beta j}^2}, \qquad k \in \mathbf{R}. \tag{4.6}$$

Proposition 3.1 implies that Λ_2 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. Using (3.12) in (4.5) we get

$$\Lambda_2(k) = -\frac{1}{k}\left[\mathrm{i}\cot\beta + \frac{F_\beta(0)}{F_\pi(0)} \prod_{j=1}^{N_\pi} \frac{\kappa_{\pi j}^2}{\kappa_{\beta j}^2}\right] + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{4.7}$$

As $k \to 0$ in $\overline{\mathbf{C}^+}$, since $F_\pi(0) \neq 0$, with the help of the first line in (3.17), from (4.5) we see that $\Lambda_2(k) = O(1)$ and hence Λ_2 is continuous at $k = 0$. Our data set $\mathcal{D}_2$ allows us to construct Λ_2 by using the right-hand side of (4.6) as input to the appropriate Schwarz formula similar to (4.4). Having constructed Λ_2, using (4.7) we obtain

$$\frac{F_\beta(0)}{F_\pi(0)} \prod_{j=1}^{N_\pi} \frac{\kappa_{\pi j}^2}{\kappa_{\beta j}^2} = -\mathrm{i}\cot\beta - \lim_{k\to\infty}[k\Lambda_2(k)], \tag{4.8}$$

where the limit can be evaluated in any way in $\overline{\mathbf{C}^+}$. Next, using (3.7) we construct F_π. Then, using (4.5) and (4.8) we get

$$F_\beta(k) = kF_\pi(k)\left[\Lambda_2(k) + 1 - \frac{\mathrm{i}\cot\beta}{k} - \frac{1}{k}\left(\lim_{k\to\infty}[k\Lambda_2(k)]\right)\right] \prod_{j=1}^{N_\pi} \frac{k^2+\kappa_{\beta j}^2}{k^2+\kappa_{\pi j}^2}.$$

Finally, having both F_π and F_β in hand, V can be reconstructed uniquely as indicated in section 5. □

Proof of theorem 2.3. In this case we have $N_\beta = N_\alpha + 1$ and $0 < \beta < \alpha < \pi$. Arguing as in the proof of theorem 2.1, we get $\epsilon < \gamma < \pi$. We would like to show that our data set $\mathcal{D}_3$ defined in (2.6) uniquely reconstructs V, α and β. Note that exactly one of the $\kappa_{\beta j}$ is missing from our data. Without loss of any generality, we can assume that the missing element in $\mathcal{D}_3$ is $\kappa_{\beta N_\beta}$ and use

$$\mathcal{D}_3 = \left\{h_{\beta\alpha}, |F_\alpha(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{\kappa_{\beta j}\}_{j=1}^{N_\alpha}\right\}. \tag{4.9}$$

Our data set allows us to construct F_α via (3.6). By proposition 3.8(ii), we see that $F_\beta(0) \neq 0$. Define

$$\Lambda_3(k) := \mathrm{i}k\frac{F_\beta(k)}{F_\alpha(k)} \frac{\prod_{j=1}^{N_\alpha}\left(k^2+\kappa_{\alpha j}^2\right)}{\prod_{j=1}^{N_\beta}\left(k^2+\kappa_{\beta j}^2\right)}. \tag{4.10}$$

Proposition 3.1 indicates that Λ_3 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Using (3.14) with α and β switched, we obtain

$$\Lambda_3(k) = \frac{\mathrm{i}}{k} + \frac{h_{\beta\alpha}}{k^2} + \frac{\mathrm{i}}{k^3}\left[h_{\beta\alpha}\cot\alpha + \sum_{j=1}^{N_\alpha}\kappa_{\alpha j}^2 - \sum_{j=1}^{N_\beta}\kappa_{\beta j}^2\right] + o(1/k^3), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{4.11}$$

From the third formula in (3.18) we get

$$\mathrm{Re}[\Lambda_3(k)] = \frac{k^2 h_{\beta\alpha}}{|F_\alpha(k)|^2} \frac{\prod_{j=1}^{N_\alpha}\left(k^2+\kappa_{\alpha j}^2\right)}{\prod_{j=1}^{N_\beta}\left(k^2+\kappa_{\beta j}^2\right)}, \qquad k \in \mathbf{R}. \tag{4.12}$$

If we had $\kappa_{\beta N_\beta}$ in $\mathcal{D}_3$, we would be able to construct Λ_3 by using the right-hand side of (4.12) as input into the appropriate Schwarz formula similar to (4.4) and obtain

$$\Lambda_3(k) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-k-\mathrm{i}0^+} \frac{t^2}{t^2+\kappa_{\beta N_\beta}^2} \frac{h_{\beta\alpha}}{|F_\alpha(t)|^2} \prod_{j=1}^{N_\alpha} \frac{t^2+\kappa_{\alpha j}^2}{t^2+\kappa_{\beta j}^2}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.13}$$

However, since $\kappa_{\beta N_\beta}$ is missing in our data, we proceed in a slightly different manner. By replacing $\kappa_{\beta N_\beta}$ with an arbitrary positive parameter κ on the right-hand side of (4.13), we obtain the one-parameter family of functions

$$\mathcal{H}(k,\kappa) := \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-k-\mathrm{i}0^+} \frac{t^2}{t^2+\kappa^2} \frac{h_{\beta\alpha}}{|F_\alpha(t)|^2} \prod_{j=1}^{N_\alpha} \frac{t^2+\kappa_{\alpha j}^2}{t^2+\kappa_{\beta j}^2}, \qquad k \in \overline{\mathbf{C}^+}, \tag{4.14}$$

that are analytic for $k \in \mathbf{C}^+$ and continuous for $k \in \overline{\mathbf{C}^+}$. Note that $\mathcal{H}(k, \kappa_{\beta N_\beta}) = \Lambda_3(k)$. Having constructed $\mathcal{H}(k,\kappa)$ containing κ as a parameter, we impose the restriction

$$\lim_{k\to\infty} [k\,\mathcal{H}(k,\kappa)] = i, \tag{4.15}$$

so that, as seen from (4.11), the leading terms in the large-k asymptotics in $\mathcal{H}(\cdot,\kappa)$ and Λ_3 agree. Provided we interpret the limit as a nontangential limit in $\overline{\mathbf{C}^+}$, we show in proposition 4.1 that (4.15) has the unique positive solution $\kappa = \kappa_{\beta N_\beta}$. Having constructed $\mathcal{H}(k,\kappa)$ and $\kappa_{\beta N_\beta}$, we obtain $\Lambda_3(k)$ as $\mathcal{H}(k, \kappa_{\beta N_\beta})$. Then, we construct F_β via (4.10) as

$$F_\beta(k) = \frac{1}{\mathrm{i}k} F_\alpha(k)\, \mathcal{H}(k, \kappa_{\beta N_\beta}) \frac{\prod_{j=1}^{N_\beta} (k^2+\kappa_{\beta j}^2)}{\prod_{j=1}^{N_\alpha} (k^2+\kappa_{\alpha j}^2)}.$$

Note that the value of $\cot\alpha$ can now be obtained from (4.11), and then $\cot\beta$ can be computed via (2.1). Thus, our data set allows us to construct α and β. Having F_α, F_β and $h_{\beta\alpha}$ in hand, V can be reconstructed uniquely via a method given in section 5. Alternatively, after obtaining V, we can evaluate α and β with the help of (3.9) and then (2.1). □

Proof of theorem 2.4. We have $N_\beta = N_\alpha + 1$ and $0 < \beta < \alpha = \pi$. As in the proof of theorem 2.1 we prove that $\epsilon < \gamma = \pi$. We will show that $\mathcal{D}_4$ given in (2.7) uniquely reconstructs V. As in the proof of theorem 2.3, without loss of any generality we can assume that the missing element in $\mathcal{D}_4$ is $\kappa_{\beta N_\beta}$ and use

$$\mathcal{D}_4 = \left\{\beta, |F_\pi(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \{\kappa_{\beta j}\}_{j=1}^{N_\pi}\right\}. \tag{4.16}$$

We construct F_π via (3.7). From proposition 3.8(ii), we conclude that $F_\beta(0) \neq 0$. Letting

$$\Lambda_4(k) := -1 + k \frac{F_\beta(k)}{F_\pi(k)} \frac{\prod_{j=1}^{N_\pi} (k^2+\kappa_{\pi j}^2)}{\prod_{j=1}^{N_\beta} (k^2+\kappa_{\beta j}^2)}, \tag{4.17}$$

by proposition 3.1 we observe that Λ_4 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. From (3.12) we obtain

$$\Lambda_4(k) = -\frac{\mathrm{i}\cot\beta}{k} + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}, \tag{4.18}$$

and from the second identity in (3.18) we get

$$\mathrm{Re}[\Lambda_4(k)] = -1 + \frac{k^2}{|F_\pi(k)|^2} \frac{\prod_{j=1}^{N_\pi} (k^2+\kappa_{\pi j}^2)}{\prod_{j=1}^{N_\beta} (k^2+\kappa_{\beta j}^2)}, \qquad k \in \mathbf{R}. \tag{4.19}$$

If we had $\kappa_{\beta N_\beta}$ in $\mathcal{D}_4$, we could construct Λ_4 by using (4.19) as input into the analogue of (4.4) and obtain

$$\Lambda_4(k) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-k-\mathrm{i}0^+} \left[\frac{t^2}{t^2+\kappa_{\beta N_\beta}^2} \frac{1}{|F_\pi(t)|^2} \prod_{j=1}^{N_\pi} \frac{t^2+\kappa_{\pi j}^2}{t^2+\kappa_{\beta j}^2} - 1 \right], \qquad k \in \overline{\mathbf{C}^+}. \quad (4.20)$$

Since $\kappa_{\beta N_\beta}$ is missing in our data, we proceed as in the proof of theorem 2.3. By replacing $\kappa_{\beta N_\beta}$ with an arbitrary positive parameter κ on the right-hand side of (4.20), we obtain the one-parameter family of functions

$$\mathcal{H}_\pi(k,\kappa) := \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-k-\mathrm{i}0^+} \left[\frac{t^2}{t^2+\kappa^2} \frac{1}{|F_\pi(t)|^2} \prod_{j=1}^{N_\pi} \frac{t^2+\kappa_{\pi j}^2}{t^2+\kappa_{\beta j}^2} - 1 \right], \qquad k \in \overline{\mathbf{C}^+},$$

that are analytic for $k \in \mathbf{C}^+$ and continuous for $k \in \overline{\mathbf{C}^+}$. Note that $\mathcal{H}_\pi(k,\kappa_{\beta N_\beta}) = \Lambda_4(k)$. Having constructed $\mathcal{H}_\pi(\cdot,\kappa)$ containing κ as a parameter, we impose the restriction

$$\lim_{k\to\infty} [k\mathcal{H}_\pi(k,\kappa)] = -\mathrm{i}\cot\beta, \qquad (4.21)$$

so that, as seen from (4.18), the leading terms in the large-k asymptotics in $\mathcal{H}_\pi(\cdot,\kappa)$ and Λ_4 agree. We prove in proposition 4.1 that (4.21) has the unique positive solution $\kappa = \kappa_{\beta N_\beta}$ provided the limit in (4.21) is a nontangential limit in $\overline{\mathbf{C}^+}$. Having $\mathcal{H}_\pi(\cdot,\kappa)$ and $\kappa_{\beta N_\beta}$ in hand, we obtain $\Lambda_4(k)$ as $\mathcal{H}_\pi(k,\kappa_{\beta N_\beta})$. Then, F_β is obtained via (4.17) as

$$F_\beta(k) = \frac{1}{k} F_\pi(k)[\mathcal{H}_\pi(k,\kappa_{\beta N_\beta}) + 1] \frac{\prod_{j=1}^{N_\beta}(k^2+\kappa_{\beta j}^2)}{\prod_{j=1}^{N_\pi}(k^2+\kappa_{\pi j}^2)}.$$

Having found F_π and F_β, V can be reconstructed uniquely as explained in section 5. □

Proposition 4.1. *Assume that each of the data sets $\mathcal{D}_3$ and $\mathcal{D}_4$ given in (4.9) and (4.16), respectively, is associated with a potential in the Faddeev class. If the limits in (4.15) and (4.21) are interpreted as nontangential limits in $\overline{\mathbf{C}^+}$, then (4.15) and (4.21) each have a unique positive solution, and that solution is given by $\kappa = \kappa_{\beta N_\beta}$.*

Proof. For the part of the proof related to (4.15), we proceed as follows. Define

$$I_1(k) := \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{k}{t-k-\mathrm{i}0^+} \frac{\mathrm{Re}[\Lambda_3(t)]}{t^2+\kappa^2},$$

$$I_2(k) := \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{t}{t-k-\mathrm{i}0^+} \frac{\mathrm{Re}[\Lambda_3(t)]}{t^2+\kappa^2}.$$

With the help of (4.11) and (4.15) we see that the latter is equivalent to

$$\lim_{k\to\infty} [k\mathcal{H}(k,\kappa) - k\Lambda_3(k)] = 0,$$

and that (4.12)–(4.14) imply

$$k\mathcal{H}(k,\kappa) - k\Lambda_3(k) = \frac{\kappa_{\beta N_\beta}^2 - \kappa^2}{\pi \mathrm{i}} I_1(k),$$

and hence our proof will be completed by showing that the nontangential limit of $I_1(k)$ exists and is nonzero. We note that

$$I_1(k) - I_2(k) = -\int_{-\infty}^{\infty} \mathrm{d}t \, \frac{\mathrm{Re}[\Lambda_3(t)]}{t^2+\kappa^2}. \qquad (4.22)$$

Writing k in terms of its real and imaginary parts as $k := k_{\rm R} + {\rm i}k_{\rm I}$, from (3.9) and (4.12) we obtain

$$|I_2(k)| \leqslant C \int_{-\infty}^{\infty} {\rm d}t\, \frac{|t|}{\sqrt{(t-k_{\rm R})^2 + k_{\rm I}^2}} \frac{1}{(t^2+\kappa^2)\left(t^2+\kappa_{\beta N_\beta}^2\right)},$$

for an appropriate constant C. With the help of the estimate

$$\frac{1}{\sqrt{(t-k_{\rm R})^2+k_{\rm I}^2}} \leqslant \begin{cases} \dfrac{1}{k_{\rm I}}, & |t| \geqslant |k_{\rm R}|/2, \\ \dfrac{2}{\sqrt{k_{\rm R}^2+4k_{\rm I}^2}}, & |t| \leqslant |k_{\rm R}|/2, \end{cases}$$

we get $I_2(k) = o(1)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$ provided $k_{\rm I} \geqslant \delta_1$ for some positive δ_1. Using the facts (cf (4.12)) that $\mathrm{Re}[\Lambda_3(t)]$ is bounded on $\mathbf{R}$ and is positive when $t \neq 0$, we conclude from (4.22) that the nontangential limit $\lim_{k\to\infty} I_1(k)$ exists and is negative.

Arguing as above, we prove that (4.21) has the unique positive solution $\kappa = \kappa_{\beta N_\beta}$ provided that the nontangential limit $\lim_{k\to\infty} I(k)$ in $\overline{\mathbf{C}^+}$ is zero, where we have defined

$$I(k) := \int_{-\infty}^{\infty} {\rm d}t\, \frac{t}{t-k-{\rm i}0^+} \frac{\mathrm{Re}[\Lambda_4(t)+1]}{t^2+\kappa^2}.$$

For any $\Upsilon > 0$, let us write $I(k) = I_3(k) + I_4(k)$ with

$$I_3(k) := \int_{|t|\geqslant\Upsilon} {\rm d}t\, \frac{t}{t-k-{\rm i}0^+} \frac{\mathrm{Re}[\Lambda_4(t)+1]}{t^2+\kappa^2},$$

$$I_4(k) := \int_{|t|\leqslant\Upsilon} {\rm d}t\, \frac{t}{t-k-{\rm i}0^+} \frac{\mathrm{Re}[\Lambda_4(t)+1]}{t^2+\kappa^2}.$$

By the Schwarz inequality, we have

$$|I_3(k)|^2 \leqslant C \left(\int_{-\infty}^{\infty} \frac{{\rm d}t}{t^2+k_{\rm I}^2}\right)\left(\int_{|t|\geqslant\Upsilon} {\rm d}t\, \frac{t^2}{(t^2+\kappa^2)^2}\right),$$

where C is an appropriate constant (cf (4.19)). Thus, given $\delta_2, \delta_3 > 0$ we can take Υ large enough so that $|I_3(k)| \leqslant \delta_2$ for all $k \in \mathbf{C}^+$ with $k_{\rm I} \geqslant \delta_3$. Moreover, with Υ fixed as above, for $|k_{\rm R}| > 2\Upsilon$ we get $|I_4(k)| \leqslant C\Upsilon/(|k_{\rm R}| + k_{\rm I})$ for an appropriate constant C. Hence the nontangential limit $\lim_{k\to\infty} I(k)$ is zero. □

Proof of theorem 2.5. In this case we have $N_\beta = N_\alpha$ and $0 < \beta < \alpha < \pi$. As in the proof of theorem 2.1, we prove that $\epsilon < \gamma < \pi$. We would like to show that $\mathcal{D}_5$ given in (2.8) uniquely reconstructs V, α and β. Let

$$\Lambda_5(k) := {\rm i}k - h_{\beta\alpha} - {\rm i}k \frac{F_\alpha(k)}{F_\beta(k)} \prod_{j=1}^{N_\beta} \frac{k^2+\kappa_{\beta j}^2}{k^2+\kappa_{\alpha j}^2}. \tag{4.23}$$

Using the third identity in (3.18) with α and β switched, from (4.23) it follows that

$$\mathrm{Re}[\Lambda_5(k)] = -h_{\beta\alpha} + \frac{k^2 h_{\beta\alpha}}{|F_\beta(k)|^2} \prod_{j=1}^{N_\beta} \frac{k^2+\kappa_{\beta j}^2}{k^2+\kappa_{\alpha j}^2}, \qquad k \in \mathbf{R}, \tag{4.24}$$

where we have also used $h_{\beta\alpha} = -h_{\alpha\beta}$. The properties of F_α and F_β stated in proposition 3.1 allow us to conclude that Λ_5 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\backslash\{0\}$. With the help of (3.14), from (4.23) we get

$$\Lambda_5(k) = \frac{\rm i}{k}\left[h_{\beta\alpha}\cot\beta + \sum_{j=1}^{N_\beta}\left(\kappa_{\alpha j}^2 - \kappa_{\beta j}^2\right)\right] + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{4.25}$$

As $k \to 0$ in $\overline{\mathbf{C}^+}$, using (3.15) in (4.23) we see that $\Lambda_5(k) = O(1)$ regardless of whether $F_\beta(0) = 0$ or not, and hence Λ_5 is continuous at $k = 0$. Then, in terms of $\mathcal{D}_5$, we construct Λ_5 with the right-hand side of (4.24) as input to the appropriate Schwarz formula analogous to (4.4). Using (4.25) we get the value of $\cot\beta$ and hence β. Then, with the help of (2.1) we get the value of α. Next, using (3.6) our data set allows us to construct F_β in $\overline{\mathbf{C}^+}$. Then, having F_β and Λ_5 in hand, we obtain F_α via (4.23) as

$$F_\alpha(k) = \frac{\mathrm{i}}{k} F_\beta(k)[h_{\beta\alpha} - \mathrm{i}k + \Lambda_5(k)] \prod_{j=1}^{N_\beta} \frac{k^2 + \kappa_{\alpha j}^2}{k^2 + \kappa_{\beta j}^2}.$$

Finally, having F_α, F_β and $h_{\beta\alpha}$ in hand, V is reconstructed uniquely as indicated in section 5. □

Proof of theorem 2.6. We are in the case $0 < \beta < \alpha = \pi$ and $N_\pi = N_\beta$. As in the proof of theorem 2.1 we establish $\epsilon < \gamma = \pi$, and we note that we cannot have $F_\pi(0) = 0$ due to the assumption $N_\pi = N_\beta$. We will show that $\mathcal{D}_6$ defined in (2.9) uniquely reconstructs V and β. Proceeding as in the proof of theorem 2.1, let us define

$$\Lambda_6(k) := -1 + k \frac{F_\pi(k)}{F_\beta(k)} \prod_{j=1}^{N_\beta} \frac{k^2 + \kappa_{\beta j}^2}{k^2 + \kappa_{\pi j}^2}. \tag{4.26}$$

Using the first identity of (3.18) in (4.26), we obtain

$$\mathrm{Re}[\Lambda_6(k)] = -1 + \frac{k^2}{|F_\beta(k)|^2} \prod_{j=1}^{N_\beta} \frac{k^2 + \kappa_{\beta j}^2}{k^2 + \kappa_{\pi j}^2}, \qquad k \in \mathbf{R}. \tag{4.27}$$

Proposition 3.1 implies that Λ_6 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$. Using (3.13) in (4.26) we get

$$\Lambda_6(k) = \frac{\mathrm{i}\cot\beta}{k} + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{4.28}$$

As $k \to 0$ in $\overline{\mathbf{C}^+}$, using (3.16) in (4.26) we see that $\Lambda_6(k) = O(1)$ regardless of whether $F_\beta(0) = 0$ or not, and hence Λ_6 remains continuous at $k = 0$. Our data set $\mathcal{D}_6$ allows us to construct Λ_6 with the right-hand side of (4.27) used as input to the appropriate Schwarz formula, which is the analogue of (4.4). Having constructed Λ_6, we recover β with the help of (4.28). Next, using (3.6) we construct F_β in $\overline{\mathbf{C}^+}$, and from (4.26) we get

$$F_\pi(k) = \frac{1}{k} F_\beta(k)[\Lambda_6(k) + 1] \prod_{j=1}^{N_\beta} \frac{k^2 + \kappa_{\pi j}^2}{k^2 + \kappa_{\beta j}^2}.$$

Then, having both F_π and F_β in hand, V can be reconstructed uniquely as shown in section 5. □

Proof of theorem 2.7. This is the case $N_\beta = N_\alpha + 1$ and $0 < \beta < \alpha < \pi$. We prove $\epsilon < \gamma < \pi$ as in the proof of theorem 2.1. We would like to show that our data set $\mathcal{D}_7$ given in (2.10) uniquely reconstructs V and α. Since $\mathcal{D}_7$ contains β and $h_{\beta\alpha}$, we get α from (2.1). In this case proposition 3.8(ii) implies that $F_\beta(0) \neq 0$. Define

$$\Lambda_7(k) := -\mathrm{i}k + h_{\beta\alpha} - \frac{\mathrm{i}}{k} \frac{F_\alpha(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta} \kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2} + \frac{\mathrm{i}}{k} \frac{F_\alpha(k)}{F_\beta(k)} \frac{\prod_{j=1}^{N_\beta} \left(k^2 + \kappa_{\beta j}^2\right)}{\prod_{j=1}^{N_\beta - 1} \left(k^2 + \kappa_{\alpha j}^2\right)}. \tag{4.29}$$

Using the third identity in (3.18) with α and β switched, from (4.29) we get

$$\mathrm{Re}[\Lambda_7(k)] = h_{\beta\alpha} - \frac{h_{\beta\alpha}}{|F_\beta(k)|^2} \frac{\prod_{j=1}^{N_\beta} \left(k^2 + \kappa_{\beta j}^2\right)}{\prod_{j=1}^{N_\beta - 1} \left(k^2 + \kappa_{\alpha j}^2\right)}, \qquad k \in \mathbf{R}, \tag{4.30}$$

where we have also used $h_{\beta\alpha} = -h_{\alpha\beta}$. Proposition 3.1 indicates that Λ_7 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. With the help of (3.14), as $k \to \infty$ in $\overline{\mathbf{C}^+}$ from (4.29) we get

$$\Lambda_7(k) = \frac{\mathrm{i}}{k} \left[-h_{\beta\alpha} \cot \beta + \sum_{j=1}^{N_\beta} \kappa_{\beta j}^2 - \sum_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2 - \frac{F_\alpha(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta} \kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2} \right] + o(1/k).$$

Setting

$$P(k) := \mathrm{i}k\Lambda_7(k) - h_{\beta\alpha} \cot \beta + \sum_{j=1}^{N_\beta} \kappa_{\beta j}^2 - \sum_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2, \tag{4.31}$$

we see that

$$\frac{F_\alpha(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta} \kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2} = \lim_{k\to\infty} P(k), \tag{4.32}$$

where the limit can be obtained in any manner in $\overline{\mathbf{C}^+}$. As $k \to 0$ in $\overline{\mathbf{C}^+}$, using the first line of (3.15) in (4.29) we see that $\Lambda_7(k) = O(1)$ regardless of $F_\alpha(0) = 0$ or $F_\alpha(0) \neq 0$, and hence Λ_7 is continuous at $k = 0$. Then, the data set $\mathcal{D}_7$ allows us to construct Λ_7 with the right-hand side of (4.30) used as input to the appropriate Schwarz formula, which is the analogue of (4.4). Next, using (3.6) we construct F_β in $\overline{\mathbf{C}^+}$. Consequently, using (4.32) in (4.29) we are able to obtain F_α as

$$F_\alpha(k) = \frac{k}{\mathrm{i}} F_\beta(k) \left[\mathrm{i}k - h_{\beta\alpha} + \Lambda_7(k) + \frac{\mathrm{i}}{k} \left(\lim_{k\to\infty} P(k) \right) \right] \frac{\prod_{j=1}^{N_\beta - 1} \left(k^2 + \kappa_{\alpha j}^2\right)}{\prod_{j=1}^{N_\beta} \left(k^2 + \kappa_{\beta j}^2\right)},$$

where P is as given in (4.31). Finally, having F_α, F_β and $h_{\beta\alpha}$ in hand, V can be reconstructed uniquely as outlined in section 5. □

Proof of theorem 2.8. We have $N_\beta = N_\alpha + 1$ with $0 < \beta < \alpha = \pi$. From (i), (iii) and (iv) we conclude that $\epsilon < \gamma = \pi$. We will show that $\mathcal{D}_8$ given in (2.11) uniquely reconstructs V. Define

$$\Lambda_8(k) := -1 - \frac{1}{k} \frac{F_\pi(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta} \kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2} + \frac{1}{k} \frac{F_\pi(k)}{F_\beta(k)} \frac{\prod_{j=1}^{N_\beta} \left(k^2 + \kappa_{\beta j}^2\right)}{\prod_{j=1}^{N_\beta - 1} \left(k^2 + \kappa_{\alpha j}^2\right)}. \tag{4.33}$$

Via (3.6) we construct F_β in $\overline{\mathbf{C}^+}$. Using the first identity of (3.18) in (4.33) and noting that $F_\beta(0)$ is purely imaginary and $F_\pi(0)$ is real, it follows that

$$\mathrm{Re}[\Lambda_8(k)] = -1 + \frac{1}{|F_\beta(k)|^2} \frac{\prod_{j=1}^{N_\beta} \left(k^2 + \kappa_{\beta j}^2\right)}{\prod_{j=1}^{N_\beta - 1} \left(k^2 + \kappa_{\alpha j}^2\right)}, \qquad k \in \mathbf{R}. \tag{4.34}$$

Proposition 3.1 implies that Λ_8 is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$. With the help of (3.13), from (4.33) we get

$$\Lambda_8(k) = \frac{1}{k} \left[\mathrm{i} \cot \beta - \frac{F_\pi(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta} \kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta - 1} \kappa_{\alpha j}^2} \right] + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{4.35}$$

Again we have $F_\beta(0) \neq 0$ because of proposition 3.8(ii). As $k \to 0$ in $\overline{\mathbf{C}^+}$, using the first line of (3.16) in (4.33) we see that $\Lambda_8(k) = O(1)$ and hence Λ_8 is continuous at $k = 0$. Now from the data $\mathcal{D}_8$, we construct Λ_8 with the right-hand side of (4.34) used as input to the appropriate Schwarz formula similar to (4.4). Then, with the help of (4.33) and (4.35), we construct F_π via

$$F_\pi(k) = kF_\beta(k)\left[1 + \Lambda_8(k) + \frac{\mathrm{i}\cot\beta}{k} - \frac{1}{k}\left(\lim_{k\to\infty}[k\Lambda_8(k)]\right)\right]\frac{\prod_{j=1}^{N_\beta-1}\left(k^2+\kappa_{\alpha j}^2\right)}{\prod_{j=1}^{N_\beta}\left(k^2+\kappa_{\beta j}^2\right)}.$$

Finally, having both F_π and F_β in hand, V can be reconstructed uniquely as outlined in section 5. □

5. Reconstruction of the potential

In this section we outline several methods via which the potential can be uniquely reconstructed from each of the data sets $\mathcal{D}_1, \ldots, \mathcal{D}_8$ given in (2.4)–(2.11). These methods include the Gel'fand–Levitan method [7, 8, 10, 11, 33] and the Marchenko method [7, 8, 10, 12, 33] for the half-line inverse scattering problem, the Faddeev–Marchenko [7, 10, 19–22] method and several other methods [10, 21] used to solve the full-line inverse scattering problem. We will show that each of $\mathcal{D}_1, \ldots, \mathcal{D}_8$ constructs $\mathcal{G}_\alpha$, $\mathcal{M}_\alpha$ and $\mathcal{F}$ defined in (5.1), (5.7) and (5.14), respectively. If we have F_α, F_β and $h_{\beta\alpha}$ in hand, the norming constants $g_{\alpha j}$ and $m_{\alpha j}$ are constructed via the first line of (3.25) and of (3.26), respectively. Thus, each of $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5$ and $\mathcal{D}_7$ yields $\mathcal{G}_\alpha$ and $\mathcal{M}_\alpha$. On the other hand, if we have F_π and F_β in hand, the norming constants $g_{\pi j}$ and $m_{\pi j}$ are constructed as in the second line of (3.25) and of (3.26), respectively. Thus, each of $\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6$, and $\mathcal{D}_8$ yields $\mathcal{G}_\pi$ and $\mathcal{M}_\pi$. The construction of $\mathcal{F}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_8$ is achieved by using (5.18)–(5.23).

The data set $\mathcal{G}_\alpha$ used as input to the Gel'fand–Levitan method is given by

$$\mathcal{G}_\alpha := \left\{|F_\alpha(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{g_{\alpha j}\}_{j=1}^{N_\alpha}\right\}, \qquad \alpha \in (0, \pi]. \tag{5.1}$$

It allows us to reconstruct the corresponding regular solution $\varphi_\alpha(k, x)$ uniquely as (cf (3.22))

$$\varphi_\alpha(k,x) = \begin{cases} \cos kx + \displaystyle\int_0^x \mathrm{d}y\, A_\alpha(x,y)\cos ky, & \alpha \in (0,\pi), \\ \dfrac{\sin kx}{k} + \displaystyle\int_0^x \mathrm{d}y\, A_\pi(x,y)\frac{\sin ky}{k}, & \alpha = \pi, \end{cases} \tag{5.2}$$

and the corresponding potential V uniquely as

$$V(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}A_\alpha(x, x^-), \qquad \alpha \in (0, \pi], \tag{5.3}$$

where $A_\alpha(x, y)$ is obtained by solving the Gel'fand–Levitan integral equation [7, 8, 10, 11]

$$A_\alpha(x,y) + G_\alpha(x,y) + \int_0^x \mathrm{d}z\, G_\alpha(y,z)A_\alpha(x,z) = 0, \qquad 0 < y < x, \tag{5.4}$$

with the kernel $G_\alpha(x, y)$ for $\alpha \in (0, \pi)$ given by

$$G_\alpha(x,y) := \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}k\left[\frac{k^2}{|F_\alpha(k)|^2} - 1\right](\cos kx)(\cos ky) + \sum_{j=1}^{N_\alpha} g_{\alpha j}^2(\cosh\kappa_{\alpha j}x)(\cosh\kappa_{\alpha j}y), \tag{5.5}$$

and the kernel $G_\pi(x, y)$ given by

$$G_\pi(x, y) := \frac{1}{\pi}\int_{-\infty}^{\infty} dk \left[\frac{1}{|F_\pi(k)|^2} - 1\right](\sin kx)(\sin ky) + \sum_{j=1}^{N_\pi} \frac{g_{\pi j}^2}{\kappa_{\pi j}^2}(\sinh \kappa_{\pi j} x)(\sinh \kappa_{\pi j} y). \tag{5.6}$$

We note that, with the help of (3.9) and (3.10), it is possible to tell whether we have $\alpha < \pi$ or $\alpha = \pi$. When $\alpha < \pi$, we observe that α is readily obtained from the solution to (5.4) because (3.22) and (5.2) imply that $\cot\alpha = -A_\alpha(0, 0)$.

The data set $\mathcal{M}_\alpha$ used as input to the Marchenko method is given by

$$\mathcal{M}_\alpha := \left\{S_\alpha(k) \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{m_{\alpha j}\}_{j=1}^{N_\alpha}\right\}, \qquad \alpha \in (0, \pi], \tag{5.7}$$

where S_α is the scattering matrix defined in (1.7). Given $\mathcal{M}_\alpha$, we are able to reconstruct the corresponding Jost solution $f(k, x)$ uniquely as (cf (1.6))

$$f(k, x) = e^{ikx} + \int_x^\infty dy\, K(x, y)\, e^{iky}, \tag{5.8}$$

and the potential V uniquely as

$$V(x) = -2\frac{d}{dx}K(x, x^+), \tag{5.9}$$

where $K(x, y)$ is obtained by solving the Marchenko integral equation [7, 8, 10, 12]

$$K(x, y) + M_\alpha(x + y) + \int_x^\infty dz\, M_\alpha(y + z)K(x, z) = 0, \qquad 0 < x < y, \tag{5.10}$$

with the kernel

$$M_\alpha(y) := \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} dk\,[S_\alpha(k) - 1]\, e^{iky} + \sum_{j=1}^{N_\alpha} m_{\alpha j}^2\, e^{-\kappa_{\alpha j} y}, & \alpha \in (0, \pi), \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} dk\,[1 - S_\pi(k)]\, e^{iky} + \sum_{j=1}^{N_\pi} m_{\pi j}^2\, e^{-\kappa_{\pi j} y}, & \alpha = \pi. \end{cases} \tag{5.11}$$

Note that the solution $K(x, y)$ to (5.10) is the same for all $\alpha \in (0, \pi]$, whereas the solution $A_\alpha(x, y)$ to (5.4) depends on α. This is not surprising because $K(x, y)$ is related (cf (5.8)) to the Fourier transform of the Jost solution $f(k, x)$, which is independent of α, whereas $A_\alpha(x, y)$ is related (cf (5.2)) to the Fourier transform of the regular solution $\varphi_\alpha(k, x)$, which depends on α. Let us also remark on the limiting values $A_\alpha(x, x^-)$ and $K(x, x^+)$ appearing in (5.3) and (5.9), respectively. If we invert the Fourier transforms given in (5.2) and (5.8), we obtain $A_\alpha(x, y) = 0$ for $y > x$ and $K(x, y) = 0$ for $y < x$. To emphasize the jump discontinuities in these functions when $y = x$, we use the appropriate limiting values in (5.3) and (5.9), even though those limits are not always explicitly indicated in the literature (cf [7, 8, 10]).

The potential V can alternatively be reconstructed by using the Gel'fand–Levitan method or the Marchenko method in the Dirichlet case. This can be done as follows. If we have F_α, F_β and $h_{\beta\alpha}$ for some $\alpha, \beta \in (0, \pi)$ with $\alpha \neq \beta$, then by using (2.2) we can construct $F_\pi(k) := f(k, 0)$. Having F_π in hand, we also have $\kappa_{\pi j}$ for $j = 1, \ldots, N$. Finally, the Gel'fand–Levitan norming constants $g_{\pi j}$ and the Marchenko norming constants $m_{\pi j}$ can be constructed by using the second line of (3.25) and of (3.26), respectively.

One can also reconstruct V by viewing it as the potential in the full-line Schrödinger equation with $V \equiv 0$ for $x < 0$. Recall that the left Jost solution $f_l(k, x)$ and the right Jost solution $f_{\rm r}(k,x)$ are the solutions to the full-line Schrödinger equation with the respective asymptotic conditions

$$f_{\rm l}(k,x)=\mathrm{e}^{\mathrm{i}kx}[1+o(1)],\qquad f_{\rm l}'(k,x)=\mathrm{i}k\,\mathrm{e}^{\mathrm{i}kx}[1+o(1)],\qquad x\to+\infty,$$

$$f_{\rm r}(k,x)=\mathrm{e}^{-\mathrm{i}kx}[1+o(1)],\qquad f_{\rm r}'(k,x)=-\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}kx}[1+o(1)],\qquad x\to-\infty.$$

In this case, $f_{\rm l}(k,x)$ satisfies

$$f_{\rm l}(k,x)=\frac{\mathrm{e}^{\mathrm{i}kx}}{T(k)}+\frac{L(k)\,\mathrm{e}^{-\mathrm{i}kx}}{T(k)},\qquad x\leqslant 0, \tag{5.12}$$

and it agrees with (cf (1.6)) the Jost solution $f(k,x)$ when $x\geqslant 0$. Here, L is the left reflection coefficient and T is the transmission coefficient. The right reflection coefficient R is given by

$$R(k)=-\frac{L(-k)T(k)}{T(-k)},\qquad k\in\mathbf{R}. \tag{5.13}$$

The potential can be uniquely reconstructed by using any one of the full-line inversion methods [7, 10, 19–22] provided we can construct the data $\mathcal{F}$ defined as

$$\mathcal{F}:=\left\{L(k),T(k),R(k),\{\tau_j\}_{j=1}^N,\{c_{{\rm l}j}\}_{j=1}^N,\{c_{{\rm r}j}\}_{j=1}^N,\{\gamma_j\}_{j=1}^N\right\}, \tag{5.14}$$

where $-\tau_j^2$ correspond to the full-line bound-state energies. Note that T has poles at $k=\mathrm{i}\tau_j$ in $\mathbf{C}^+$ for $j=1,\ldots,N$, the constants $c_{{\rm l}j}$ are the norming constants defined as (cf (3.24))

$$c_{{\rm l}j}:=\frac{1}{\sqrt{\int_{-\infty}^{\infty}\mathrm{d}x\,f_{\rm l}(\mathrm{i}\tau_j,x)^2}},\qquad j=1,\ldots,N, \tag{5.15}$$

$c_{{\rm r}j}$ are the norming constants defined as in (5.15) by replacing $f_{\rm l}(k,x)$ with $f_{\rm r}(k,x)$, and γ_j are the bound-state dependence constants defined as

$$\gamma_j:=\frac{f_{\rm l}(\mathrm{i}\tau_j,x)}{f_{\rm r}(\mathrm{i}\tau_j,x)},\qquad j=1,\ldots,N. \tag{5.16}$$

For example, in the Faddeev–Marchenko method [7, 10, 19–22] the potential V and $f_{\rm l}(k,x)$ can be uniquely reconstructed as

$$V(x)=-2\frac{\mathrm{d}B_{\rm l}(x,0^+)}{\mathrm{d}x},\qquad f_{\rm l}(k,x)=\mathrm{e}^{\mathrm{i}kx}\left[1+\int_0^\infty\mathrm{d}y\,B_{\rm l}(x,y)\,\mathrm{e}^{\mathrm{i}ky}\right],$$

where $B_{\rm l}(x,y)$ is obtained by solving the left Faddeev–Marchenko integral equation

$$B_{\rm l}(x,y)+\Omega_{\rm l}(2x+y)+\int_0^\infty\mathrm{d}y\,\Omega_{\rm l}(2x+y+z)B_{\rm l}(x,z)=0,\qquad y>0,$$

with the input data

$$\Omega_{\rm l}(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,R(k)\,\mathrm{e}^{\mathrm{i}ky}+\sum_{j=1}^N c_{{\rm l}j}^2\,\mathrm{e}^{-\tau_j y}.$$

Equivalently, the potential V and $f_{\rm r}(k,x)$ can be uniquely reconstructed as

$$V(x)=2\frac{\mathrm{d}B_{\rm r}(x,0^+)}{\mathrm{d}x},\qquad f_{\rm r}(k,x)=\mathrm{e}^{-\mathrm{i}kx}\left[1+\int_0^\infty\mathrm{d}y\,B_{\rm r}(x,y)\,\mathrm{e}^{\mathrm{i}ky}\right],$$

where $B_{\rm r}(x,y)$ is obtained by solving the right Faddeev–Marchenko integral equation

$$B_{\rm r}(x,y)+\Omega_{\rm r}(-2x+y)+\int_0^\infty\mathrm{d}y\,\Omega_{\rm r}(-2x+y+z)B_{\rm r}(x,z)=0,\qquad y>0,$$

with the input data

$$\Omega_{\mathrm{r}}(y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, L(k)\,\mathrm{e}^{\mathrm{i}ky} + \sum_{j=1}^{N} c_{\mathrm{r}j}^{2}\,\mathrm{e}^{-\tau_j y}.$$

Let us now describe the construction of $\mathcal{F}$ given in (5.14) from $\{F_\alpha, F_\beta, \alpha, \beta\}$ with $\alpha \neq \beta$ or from $\{F_\pi, F_\beta, \beta\}$ with $\beta \neq \pi$, enabling us to use any of the full-line inversion methods to reconstruct V. Using (5.12) and its x-derivative evaluated at $x = 0$, we get

$$L(k) = \frac{\mathrm{i}kf(k,0) - f'(k,0)}{\mathrm{i}kf(k,0) + f'(k,0)}, \qquad T(k) = \frac{2\mathrm{i}k}{\mathrm{i}kf(k,0) + f'(k,0)}. \tag{5.17}$$

If $\alpha \neq \beta$, with the help of (2.2), (2.3) and (5.17), for $k \in \overline{\mathbf{C}^+}$ we obtain

$$L(k) = \begin{cases} \dfrac{(k - \mathrm{i}\cot\beta)F_\alpha(k) - (k - \mathrm{i}\cot\alpha)F_\beta(k)}{(k + \mathrm{i}\cot\beta)F_\alpha(k) - (k + \mathrm{i}\cot\alpha)F_\beta(k)}, & \alpha, \beta \in (0, \pi), \\[2ex] \dfrac{(k - \mathrm{i}\cot\beta)F_\pi(k) - F_\beta(k)}{(k + \mathrm{i}\cot\beta)F_\pi(k) + F_\beta(k)}, & \beta \in (0, \pi), \end{cases} \tag{5.18}$$

$$T(k) = \begin{cases} \dfrac{2\mathrm{i}kh_{\beta\alpha}}{(k + \mathrm{i}\cot\beta)F_\alpha(k) - (k + \mathrm{i}\cot\alpha)F_\beta(k)}, & \alpha, \beta \in (0, \pi), \\[2ex] \dfrac{2k}{(k + \mathrm{i}\cot\beta)F_\pi(k) + F_\beta(k)}, & \beta \in (0, \pi), \end{cases} \tag{5.19}$$

and using (5.13), for $k \in \mathbf{R}$ we get

$$R(k) = \begin{cases} \dfrac{-(k + \mathrm{i}\cot\beta)F_\alpha(-k) + (k + \mathrm{i}\cot\alpha)F_\beta(-k)}{(k + \mathrm{i}\cot\beta)F_\alpha(k) - (k + \mathrm{i}\cot\alpha)F_\beta(k)}, & \alpha, \beta \in (0, \pi), \\[2ex] -\dfrac{(k + \mathrm{i}\cot\beta)F_\pi(-k) + F_\beta(-k)}{(k + \mathrm{i}\cot\beta)F_\pi(k) + F_\beta(k)}, & \beta \in (0, \pi). \end{cases} \tag{5.20}$$

Since $V \equiv 0$ for $x < 0$, it is already known that the norming constants $c_{\mathrm{r}j}$ are related [13, 21] to the residues of L at the poles $k = \mathrm{i}\tau_j$ as

$$c_{\mathrm{r}j} = \sqrt{-\mathrm{i}\,\mathrm{Res}(L, \mathrm{i}\tau_j)}, \qquad j = 1, \ldots, N. \tag{5.21}$$

Using (5.12) and the fact that $f_{\mathrm{r}}(k, x) = \mathrm{e}^{-\mathrm{i}kx}$ for $x \leqslant 0$, we have

$$\gamma_j = f_{\mathrm{l}}(\mathrm{i}\tau_j, 0) = f(\mathrm{i}\tau_j, 0) = \frac{L}{T}(\mathrm{i}\tau_j) = \frac{\mathrm{Res}(L, \mathrm{i}\tau_j)}{\mathrm{Res}(T, \mathrm{i}\tau_j)}, \tag{5.22}$$

and then via (5.15) and (5.16) we get

$$c_{\mathrm{l}j} = \frac{c_{\mathrm{r}j}}{|\gamma_j|} = \frac{(-1)^{N-j}c_{\mathrm{r}j}}{\gamma_j} = \frac{\mathrm{i}(-1)^{N-j+1}\mathrm{Res}(T, \mathrm{i}\tau_j)}{\sqrt{-\mathrm{i}\,\mathrm{Res}(L, \mathrm{i}\tau_j)}}, \tag{5.23}$$

where we have used the fact [21] that the sign of γ_j is the same as that of $(-1)^{N-j}$.

6. An example

In this section, we illustrate the uniqueness and recovery described in theorems 2.1–2.8 with a concrete example. The existence of a potential in the Faddeev class corresponding to the scattering data in our example is assured by verifying that the corresponding left reflection coefficient L satisfies the characterization conditions given in theorem 3.3 of [41]. In this example, the Jost function and scattering coefficients are rational functions of k; consequently, the integral equations of Gel'fand–Levitan, Marchenko and Faddeev–Marchenko have degenerate kernels, enabling us to solve them explicitly and to recover the related potentials in closed forms. Such potentials are known as Bargmann potentials and they decay exponentially as $x \to +\infty$.

Example 6.1. In the data $\mathcal{D}_3$ of theorem 2.3, let us specify

$$N_\alpha = 1, \quad N_\beta = 2, \quad \kappa_{\alpha 1} = 2, \quad \kappa_{\beta 2} = 4, \quad |F_\alpha(k)|^2 = k^2 + 4 \qquad \text{for } k \in \mathbf{R},$$

but let us leave the value of $h_{\beta\alpha}$ as yet an unspecified parameter. We get $F_\alpha(k) = k - 2\mathrm{i}$. From the interlacing property stated in proposition 3.7, we have the restriction $\kappa_{\beta 1} \in (0, 2)$. Proceeding as in the proof of theorem 2.3, we find

$$\mathcal{H}(k, \kappa) = \frac{\mathrm{i}kh_{\beta\alpha}}{(\kappa + 4)(k + \mathrm{i}\kappa)(k + 4\mathrm{i})}, \qquad \lim_{k\to\infty} [k\mathcal{H}(k, \kappa)] = \frac{\mathrm{i}h_{\beta\alpha}}{\kappa + 4}.$$

The value of $\kappa_{\beta 1}$ is then obtained via (4.15) as $\kappa_{\beta 1} = h_{\beta\alpha} - 4$. Thus, the restriction $\kappa_{\beta 1} \in (0, 2)$ indicates that $h_{\beta\alpha} \in (4, 6)$. We also get

$$\Lambda_3(k) = \frac{\mathrm{i}k}{(k + 4\mathrm{i})[k + \mathrm{i}(h_{\beta\alpha} - 4)]}, \qquad F_\beta(k) = \frac{(k - 4\mathrm{i})[k - \mathrm{i}(h_{\beta\alpha} - 4)]}{k + 2\mathrm{i}}.$$

Using (4.11) and then (2.1) we obtain

$$\cot\alpha = \frac{12}{h_{\beta\alpha}} - 4, \qquad \cot\beta = h_{\beta\alpha} + \frac{12}{h_{\beta\alpha}} - 4.$$

Via (2.2) we get

$$f(k, 0) = \frac{k - \mathrm{i}(4 - 12/h_{\beta\alpha})}{k + 2\mathrm{i}},$$

and we find that $f(k, 0)$ has exactly one zero in $\mathbf{C}^+$ when $h_{\beta\alpha} \in (4, 6)$. With the help of (5.18) and (5.19), we have

$$L(k) = \frac{6(h_{\beta\alpha} - 6)(h_{\beta\alpha} - 2)/h_{\beta\alpha}^2}{\eta(k, h_{\beta\alpha})}, \qquad T(k) = \frac{k(k + 2\mathrm{i})}{\eta(k, h_{\beta\alpha})},$$

where

$$\eta(k, h_{\beta\alpha}) := k^2 + (-4 + 12/h_{\beta\alpha})\mathrm{i}k + \left(-6 + 48/h_{\beta\alpha} - 72/h_{\beta\alpha}^2\right).$$

Note that unless the value of $h_{\beta\alpha}$ is specified in $\mathcal{D}_3$, we find a one-parameter family for each of V, α and β. The corresponding one-parameter family of potentials can be obtained by using any of the methods outlined in section 5.

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