Inverse scattering for vowel articulation with frequency-domain data

Tuncay Aktosun

Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA

Abstract
An inverse-scattering problem is analysed for vowel articulation in the human vocal tract. When a unit-amplitude, monochromatic, sinusoidal volume velocity is sent from the glottis towards the lips, various types of scattering data acquired at the lips are examined to find whether the cross sectional area of the vocal tract can uniquely be determined by each data set. The data sets considered are the absolute value of the impedance at the lips, the pressure at the lips, the transfer function from the glottis to the lips and a Green's function at the lips. In the case of nonuniqueness, it is indicated what additional information may be used for the unique determination.

1. Introduction

The fundamental inverse problem for vowel articulation is concerned [1–5] with the determination of the geometry of the human vocal tract from some data. In this paper, we consider various types of frequency-domain scattering data acquired at the lips resulting from a unit-amplitude, monochromatic, sinusoidal volume velocity sent from the glottis, and analyse whether each data set uniquely determines the shape of the vocal tract, or else, what additional information may be used for the unique recovery.

Let us use x to denote the distance from the glottis and l for the length of the vocal tract. Hence, the lips are located at a distance l from the glottis. Typically, l varies between 14 cm and 20 cm, usually smaller for children than for adults and smaller for females than for males [1, 2, 5]. Even though the vocal tract is not a right cylinder, to a good approximation it can be treated as one [3, 4].

We will let $A(x)$ denote the cross sectional area as a function of the distance from the glottis, and suppose that $A(x)$ is positive on $(0, l)$. Assuming that the propagation is lossless and planar (these assumptions are known [3, 4] to be reasonable), the acoustics in the vocal tract is governed [1–5] by the first-order linear system of partial differential equations

$$\begin{cases} A(x)p_x(x, t) + \mu v_t(x, t) = 0, \\ A(x)p_t(x, t) + c^2\mu v_x(x, t) = 0, \end{cases} \tag{1.1}$$

where t is the time variable, the subscripts x and t denote the respective partial derivatives, μ is the air density, c is the speed of sound, $v(x, t)$ is the volume velocity of the air flow and $p(x, t)$ is the pressure at location x and at time t.

The pressure is the force per unit cross sectional area and is exerted by the moving air molecules. The volume velocity is equal to the product of the cross sectional area with the average velocity of the air molecules crossing that area. The air density at room temperature is $\mu = 1.2 \times 10^{-3}$ gm cm^{-3}. The speed of sound varies slightly with temperature, and $c = 3.43 \times 10^4$ cm s^{-1} in air at room temperature. In our analysis of the inverse problem, we assume that the values of μ and c are already known. There is no loss of generality in starting the time at $t = 0$.

By using $v_{xt} = v_{tx}$, we can eliminate v in (1.1) and obtain Webster's horn equation

$$\frac{1}{A(x)}[A(x)p_x(x, t)]_x - \frac{1}{c^2}p_{tt}(x, t) = 0, \qquad x \in (0, l), \qquad t > 0.$$

Letting

$$\Phi(x, t) := \sqrt{A(x)}\,p(x, t), \tag{1.2}$$

we find that Φ satisfies the plasma-wave equation

$$\Phi_{xx}(x, t) - \frac{1}{c^2}\Phi_{tt}(x, t) = Q(x)\Phi(x, t), \qquad x \in (0, l), \qquad t > 0, \quad (1.3)$$

where we have defined

$$Q(x) := \frac{[\sqrt{A(x)}]''}{\sqrt{A(x)}}, \tag{1.4}$$

with the prime denoting the x-derivative. The quantity Q is called the relative concavity of the vocal tract or the potential. Separating the variables as

$$\Phi(x, t) := \psi(k, x)\,e^{ickt}, \tag{1.5}$$

we find that $\psi(k, x)$ satisfies the Schrödinger equation

$$\psi''(k, x) + k^2\psi(k, x) = Q(x)\psi(k, x), \qquad x \in (0, l). \tag{1.6}$$

The frequency ν is measured in Hertz and related to k as $\nu = \frac{ck}{2\pi}$. Informally, we can refer to k as the frequency even though the proper term for k is the angular wavenumber.

In order to recover A, we will consider various types of data for $k \in \mathbf{R}^+$ resulting from the glottal volume velocity $v(0, t)$ given in (4.1). As our data sets we consider the absolute value of the impedance at the lips, the absolute value of the pressure at the lips, the absolute value of the transfer function from the glottis to the lips and the absolute value of a Green's function for (1.3) measured at the lips.

The inverse problem of recovery of A can be analysed either as an inverse spectral problem or as an inverse scattering problem. In the inverse spectral formulation, in addition to a boundary condition at the glottis such as (2.1), a boundary condition is also imposed at the lips. The imposition of the boundary conditions at both ends of the vocal tract results in standing waves that are related to an infinite sequence of discrete frequencies. It was established by Borg [6] that Q can be recovered by using two such infinite sequences of discrete frequencies corresponding to two sets of boundary conditions. It then follows [3, 4, 7–11] that A can be recovered from two infinite sequences. For example, such sequences can be chosen as the zeros and poles [7, 8] of the input impedance or the poles and residues [9] of the input impedance, where the input impedance is defined as $p(0, t)/v(0, t)$.

In the inverse scattering formulation, a boundary condition is imposed at only one end of the vocal tract—either at the glottis or at the lips. Then, the measurement of the acoustic data

used in the recovery of A is performed at the same end or at the opposite end. If the boundary condition and the measurement occur at the same end of the vocal tract, the corresponding inverse problem is usually known as a reflection problem. On the other hand, if the boundary condition and the measurement occur at different ends, then we have a transmission problem. The methods based on the inverse scattering formulation may be applied either in the time domain or in the frequency domain, where the data set is a function of t in the former case and of k in the latter. We refer the reader to [3, 4, 12–15] for some approaches as time-domain reflection problems and to [16] for an approach as a time-domain transmission problem. Our approach in this paper is a frequency-domain approach with the data coming from a transmission problem.

Our paper is organized as follows. In section 2 we review some preliminary material related to the Schrödinger equation by introducing the self-adjoint boundary condition involving $\cot\alpha$ given in (2.1) and presenting the Jost solution f, the Jost function F_α and the scattering coefficients T, L and R. In section 3 we briefly review the recovery of Q, $\cot\alpha$, F_α, f, T, L and R from the data $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$. In section 4 we obtain some explicit expressions for the pressure and the volume velocity in the vocal tract in terms of A, f and F_α; we also show in (4.8) that $\cot\alpha$ appearing in (2.1) is directly related to the physical parameters $A(0)$ and $A'(0)$. In section 5 we introduce the relative area $[\eta(x)]^2$ and express it in terms of the Jost solution, $\cot\alpha$ and the scattering coefficients. In sections 6–9 we examine the recovery of Q, η and A from various data sets acquired at the lips. The data set used in section 6 is the absolute value of the impedance at the lips. In section 7 it is the absolute value of the pressure measured at the lips. In section 8 the data set includes the absolute value of the transfer function from the glottis to the lips. In section 9 it is the absolute value of an analogue of the Green's function at the lips introduced in [17] for (1.3). Finally, in section 10 we present some examples to illustrate the theoretical results presented in the earlier sections.

2. Preliminaries

In this section we review the scattering data related to the potential Q appearing in the Schrödinger equation on the half line $\mathbf{R}^+$ with the self-adjoint boundary condition [18–21]

$$\sin\alpha \cdot \varphi'(k, 0) + \cos\alpha \cdot \varphi(k, 0) = 0, \tag{2.1}$$

where α is a number in the interval $(0, \pi)$ identifying the boundary condition at $x = 0$. We can relate the half-line Schrödinger equation to (1.6) by assuming that $Q(x) \equiv 0$ for $x > l$. Note that the mapping $\alpha \mapsto \cot\alpha$ is one-to-one and from $(0, \pi)$ onto $\mathbf{R}$.

The Jost solution f to the half-line Schrödinger equation [18–23] is uniquely determined by the asymptotic conditions

$$f(k, x) = e^{ikx}[1 + o(1)], \qquad f'(k, x) = ik\, e^{ikx}[1 + o(1)], \qquad x \to +\infty.$$

Since Q vanishes when $x > l$, we have

$$f(k, l) = e^{ikl}, \qquad f'(k, l) = ik\, e^{ikl}. \tag{2.2}$$

The Jost function F_α associated with the half-line Schrödinger equation with the boundary condition (2.1) is defined [18–21] as

$$F_\alpha(k) := -i[f'(k, 0) + \cot\alpha \cdot f(k, 0)]. \tag{2.3}$$

Let us emphasize that the subscript in F_α identifies the boundary condition at $x = 0$ and it does not indicate any partial derivative. It is known [18–21] that

$$F_\alpha(-k) = -F_\alpha(k)^*, \qquad k \in \mathbf{R}, \tag{2.4}$$

where the asterisk denotes complex conjugation.

We assume that Q is real valued and integrable on $(0, l)$ and that there are no bound states for the half-line Schrödinger equation with the boundary condition (2.1). The absence of bound states for the corresponding problem is equivalent [18–21] to assuming that $F_\alpha(k)$ has no zeros on $\mathbf{I}^+$, where $\mathbf{I}^+ := \mathrm{i}(0, +\infty)$ is the positive imaginary axis in the complex plane. It is known [19–21] that either $F_\alpha(0) \neq 0$ or $F_\alpha(k)$ has a simple zero at $k = 0$; the former is known as the generic case and the latter as the exceptional case. The exceptional case corresponds to the threshold where the number of bound states can be changed by one under a small perturbation of the potential.

By using the extension $Q(x) \equiv 0$ for $x < 0$, we can relate $f(k, 0)$ and $f'(k, 0)$ to the scattering coefficients associated with the full-line Schrödinger equation. We have [19, 20, 22, 24]

$$f(k, 0) = \frac{1 + L(k)}{T(k)}, \qquad f'(k, 0) = \mathrm{i}k\frac{1 - L(k)}{T(k)}, \tag{2.5}$$

where T and L denote the transmission coefficient and the left reflection coefficient, respectively, associated with Q. The right reflection coefficient R is given by

$$R(k) = -\frac{L(-k)T(k)}{T(-k)}.$$

It is known [19, 20, 22, 24] that

$$T(-k) = T(k)^*, \qquad R(-k) = R(k)^*, \qquad L(-k) = L(k)^*, \qquad k \in \mathbf{R}. \tag{2.6}$$

The absence of bound states for the full-line Schrödinger equation is equivalent [19, 20, 22, 24] for $T(k)$ not to have any poles on $\mathbf{I}^+$; when $Q(x) \equiv 0$ for $x < 0$, this is also equivalent [25] for $f'(k, 0)$ not to have any zeros on $\mathbf{I}^+$.

3. Recovery of Q from $|F_\alpha|$

In the absence of bound states, the fundamental inverse-scattering problem for the half-line Schrödinger equation with the self-adjoint boundary condition (2.1) consists of determining Q and $\cot\alpha$ from various types of scattering data. Recall that the absence of bound states in this case is equivalent for $F_\alpha(k)$ to be nonzero when $k \in \mathbf{I}^+$. In this section we review the solution to this inverse problem when the data set is $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$.

Theorem 3.1. *Assume that Q is real valued, measurable and integrable for $x \in (0, l)$, vanishes for $x > l$, and that the corresponding half-line Schrödinger equation with the boundary condition (2.1) has no bound states. Then, the data set $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$ uniquely determines $Q(x)$ for $x \in (0, l)$ and $\cot\alpha$. The same data set also uniquely determines the corresponding Jost solution $f(k, x)$ and the scattering coefficients $T(k)$, $R(k)$ and $L(k)$.*

Below we outline some steps involved in the solution to the inverse problem stated in theorem 3.1. As seen from (2.4), $|F_\alpha(k)|$ is an even function of $k \in \mathbf{R}$, and hence the data sets $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$ and $\{|F_\alpha(k)| : k \in \mathbf{R}\}$ are equivalent. By using the data $\{|F_\alpha(k)| : k \in \mathbf{R}\}$ as input in the Gel'fand–Levitan method [18–21], we can form the kernel function

$$G_\alpha(x, y) := \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k \left[\frac{k^2}{|F_\alpha(k)|^2} - 1 \right] (\cos kx)(\cos ky),$$

and then solve the Gel'fand–Levitan integral equation

$$h_\alpha(x, y) + G_\alpha(x, y) + \int_0^x \mathrm{d}z\, G_\alpha(y, z)h_\alpha(x, z) = 0, \qquad 0 \leqslant y < x < l. \tag{3.1}$$

The solution to (3.1) is known [18–20] to exist and to be unique. Once $h_\alpha(x, y)$ is obtained, we recover the potential as

$$Q(x) = 2\frac{d}{dx}h_\alpha(x, x^-), \qquad x \in (0, l),$$

where x^- indicates that the limit from the left must be used in the evaluation. We also recover the boundary condition as

$$\cot\alpha = -h_\alpha(0, 0).$$

Alternatively, we can proceed [21] as follows. Let

$$\Lambda_\alpha(k) := \frac{kf(k, 0)}{F_\alpha(k)} - 1, \qquad k \in \overline{\mathbf{C}^+}, \tag{3.2}$$

where we use $\mathbf{C}^+$ for the upper half complex plane and $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+ \cup \mathbf{R}$. The real part of $\Lambda_\alpha(k)$ is given by

$$\mathrm{Re}[\Lambda_\alpha(k)] = \frac{k^2}{|F_\alpha(k)|^2} - 1, \qquad k \in \mathbf{R}. \tag{3.3}$$

From the data $\{|F_\alpha(k)| : k \in \mathbf{R}\}$ we first construct the function $\Lambda_\alpha(k)$ via the Schwarz integral formula as

$$\Lambda_\alpha(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{ds}{s - k - i0^+} \left[\frac{s^2}{|F_\alpha(s)|^2} - 1 \right], \qquad k \in \overline{\mathbf{C}^+}, \tag{3.4}$$

where the quantity $i0^+$ indicates that the values for real k should be obtained as a limit from $\mathbf{C}^+$. Next, $F_\alpha(k)$ is obtained from $|F_\alpha(k)|$ by using

$$F_\alpha(k) = k \exp\left(\frac{-1}{\pi i} \int_{-\infty}^{\infty} ds \frac{\log|s/F_\alpha(s)|}{s - k - i0^+} \right), \qquad k \in \overline{\mathbf{C}^+}. \tag{3.5}$$

Then, we have

$$f(k, 0) = \frac{1}{k} F_\alpha(k)[1 + \Lambda_\alpha(k)], \qquad k \in \overline{\mathbf{C}^+}, \tag{3.6}$$

$$f'(k, 0) = i F_\alpha(k) \left[1 + \frac{1 + \Lambda_\alpha(k)}{k} \lim_{k \to \infty} [k\Lambda_\alpha(k)] \right], \qquad k \in \overline{\mathbf{C}^+}, \tag{3.7}$$

$$\cot\alpha = -i \lim_{k \to \infty} [k\Lambda_\alpha(k)], \tag{3.8}$$

where the limit in (3.8) can be evaluated in any manner in $\overline{\mathbf{C}^+}$. Having both $f(k, 0)$ and $f'(k, 0)$ in hand, we can construct all the quantities that are relevant in the scattering theory for the Schrödinger equation. For example, the scattering coefficients for the corresponding full-line Schrödinger equation can be obtained as

$$T(k) = \frac{2ik}{ikf(k, 0) + f'(k, 0)}, \qquad L(k) = \frac{ikf(k, 0) - f'(k, 0)}{ikf(k, 0) + f'(k, 0)}, \tag{3.9}$$

$$R(k) = \frac{-ikf(-k, 0) - f'(-k, 0)}{ikf(k, 0) + f'(k, 0)}. \tag{3.10}$$

Having obtained such quantities, we can construct the potential by using any one of the various inversion methods available [19, 20, 22, 24]. For example, we can use the Faddeev–Marchenko

method [19, 20, 22, 24] and get

$$Q(x) = -2\frac{d}{dx}K(x, x^+), \qquad x \in \mathbf{R},$$

where $K(x, y)$ is obtained by solving the (left) Faddeev–Marchenko integral equation

$$K(x, y) + \hat{R}(x + y) + \int_x^\infty dz\, \hat{R}(y + z)K(x, z) = 0, \qquad -\infty < x < y, \tag{3.11}$$

with the kernel

$$\hat{R}(y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk\, R(k)\, e^{iky}.$$

The Jost solution $f(k, x)$ can be directly obtained from $K(x, y)$ as

$$f(k, x) = e^{ikx} + \int_x^\infty dy\, K(x, y)\, e^{iky}. \tag{3.12}$$

Let us remark that, in order to obtain $\{\Lambda_\alpha(k) : k \in \mathbf{R}\}$ from $\{|F_\alpha(k)| : k \in \mathbf{R}\}$, instead of using (3.4) we can equivalently construct the real and imaginary parts of $\Lambda_\alpha(k)$ separately via (3.3) and

$$\mathrm{Im}[\Lambda_\alpha(k)] = -\frac{1}{\pi}\mathrm{CPV}\int_{-\infty}^\infty \frac{ds}{s - k}\left[\frac{s^2}{|F_\alpha(s)|^2} - 1\right], \qquad k \in \mathbf{R},$$

where CPV indicates that the integral must be evaluated as a Cauchy principal-value integral. Consequently, $\cot\alpha$ can be recovered (cf (3.8)) by using

$$\cot\alpha = \lim_{k \to +\infty} (k\,\mathrm{Im}[\Lambda_\alpha(k)]).$$

4. Pressure and volume velocity

When the vocal-tract area function A is known, via (1.4) we can evaluate the potential Q, solve the corresponding Schrödinger equation and obtain the Jost solution $f(k, x)$. In this section, with the help of $f(k, x)$, we evaluate the pressure $p(x, t)$ and volume velocity $v(x, t)$ in the vocal tract corresponding to the input glottal volume velocity

$$v(0, t) = e^{ickt}, \qquad t > 0. \tag{4.1}$$

It is known [19, 20, 22] that $f(k, \cdot)$ and $f(-k, \cdot)$ are linearly independent for each $k \in \overline{\mathbf{C}^+} \setminus \{0\}$. Hence, the general solution to (1.6) can be written as a linear combination of $f(k, \cdot)$ and $f(-k, \cdot)$. From (1.2), (1.5) and (1.6), we see that the pressure has the form

$$p(x, t) = P(k, x)\, e^{ickt}, \tag{4.2}$$

with

$$P(k, x) = \frac{1}{\sqrt{A(x)}}[a(k)f(-k, x) + b(k)f(k, x)], \tag{4.3}$$

where $a(k)$ and $b(k)$ are coefficients to be determined. When $x > l$, the pressure $p(x, t)$ should be a wave travelling outward from the lips and should not contain the part proportional to $e^{ik(ct+x)}$ travelling into the mouth. Thus, with the help of (2.2) we see that we must have $b(k) \equiv 0$ in (4.3). Hence, (4.3) is reduced to

$$P(k, x) = \frac{1}{\sqrt{A(x)}}a(k)f(-k, x), \qquad x \in (0, l). \tag{4.4}$$

Next, we need to determine the value of $a(k)$ in terms of some quantities relevant to the acoustics in the vocal tract. From (4.1), (4.2) and the first line of (1.1), for the x-derivative of the pressure at the glottis we get

$$P'(k, 0) = -\frac{ic\mu k}{A(0)}. \tag{4.5}$$

Note that from (4.4) through differentiation we obtain

$$P'(k, 0) = \frac{1}{\sqrt{A(0)}} a(k) \left[f'(-k, 0) - \frac{A'(0)}{2A(0)} f(-k, 0) \right], \tag{4.6}$$

where we have used

$$\frac{[\sqrt{A(x)}]'}{\sqrt{A(x)}} = \frac{A'(x)}{2A(x)}. \tag{4.7}$$

A comparison of (4.6) and (4.7) with (2.3) shows that, by choosing

$$\cot \alpha = -\frac{A'(0)}{2A(0)} = -\frac{[\sqrt{A(x)}]'|_{x=0}}{\sqrt{A(0)}}, \tag{4.8}$$

we can write (4.6) as

$$P'(k, 0) = \frac{i}{\sqrt{A(0)}} a(k) F_\alpha(-k). \tag{4.9}$$

Comparing (4.5) and (4.9) we get

$$a(k) = -\frac{c\mu k}{\sqrt{A(0)} F_\alpha(-k)},$$

and hence we can write (4.4) in the equivalent form

$$P(k, x) = -\frac{c\mu k f(-k, x)}{\sqrt{A(x)} \sqrt{A(0)} F_\alpha(-k)}, \qquad x \in (0, l), \tag{4.10}$$

and obtain $p(x, t)$ by using (4.10) in (4.2)

Having determined the pressure $p(x, t)$ in the vocal tract, from the first line of (1.1), we get

$$v_t(x, t) = \frac{ck A(x) e^{ickt}}{\sqrt{A(0)} F_\alpha(-k)} \frac{d}{dx} \left[\frac{f(-k, x)}{\sqrt{A(x)}} \right], \qquad x \in (0, l), \qquad t > 0, \tag{4.11}$$

and, finally, with the help of (4.1) and (4.11), we obtain the volume velocity for $x \in (0, l)$ and $t > 0$ as

$$v(x, t) = -\frac{i\sqrt{A(x)} e^{ickt}}{\sqrt{A(0)} F_\alpha(-k)} \left[f'(-k, x) - \frac{A'(x)}{2A(x)} f(-k, x) \right]. \tag{4.12}$$

5. Area and relative area

In this section we relate the vocal-tract area function $A(x)$ to various particular solutions of the Schrödinger equation.

Let us view (1.4) as the zero-energy Schrödinger equation, and consider the initial-value problem

$$y'' = Q(x)y, \qquad x \in (0, l), \tag{5.1}$$

with the initial conditions

$$y(0) = \sqrt{A(0)}, \qquad y'(0) = -\sqrt{A(0)} \cot \alpha, \tag{5.2}$$

where $\cot\alpha$ is the quantity in (4.8). It is easy to verify that $\sqrt{A}$ is the unique solution to the system given in (5.1) and (5.2). Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions to (5.1) on the interval $(0, l)$. Then, the solution to the initial-value problem in (5.1) and (5.2) can be written with the help of a determinant as

$$y(x) = \frac{\sqrt{A(0)}}{[y_1(x); y_2(x)]} \begin{vmatrix} 0 & y_1(x) & y_2(x) \\ -1 & y_1(0) & y_2(0) \\ \cot\alpha & y_1'(0) & y_2'(0) \end{vmatrix}, \tag{5.3}$$

where $[F; G] := FG' - F'G$ denotes the Wronskian. Define

$$\eta(x) := \frac{\sqrt{A(x)}}{\sqrt{A(0)}}. \tag{5.4}$$

We see that η satisfies (5.1) with the initial conditions

$$\eta(0) = 1, \qquad \eta'(0) = -\cot\alpha, \tag{5.5}$$

and it is closely related to the regular solution $\varphi_\alpha(k, x)$ to the half-line Schrödinger equation satisfying the initial conditions

$$\varphi_\alpha(k, 0) = 1, \qquad \varphi_\alpha'(k, 0) = -\cot\alpha. \tag{5.6}$$

In fact, we have

$$\eta(x) = \varphi_\alpha(0, x), \qquad x \in (0, l). \tag{5.7}$$

We can write (5.4) in the equivalent form

$$A(x) = A(0)[\eta(x)]^2, \qquad x \in (0, l). \tag{5.8}$$

We will refer to $[\eta(x)]^2$ as the relative area of the vocal tract.

Recall that the Wronskian of any two solutions to the Schrödinger equation is independent of x, and $[y_1(x); y_2(x)] \neq 0$ if and only if y_1 and y_2 are linearly independent on $(0, l)$. For example, we can choose y_1 and y_2 as the zero-energy Jost solutions $g_l(0, x)$ and $g_r(0, x)$, from the left and right respectively, to the full-line Schrödinger equation where the potential agrees with $Q(x)$ on the interval $(0, l)$, is zero when $x < 0$, and is some chosen real-valued, measurable, integrable function without bound states and with a finite first moment when $x > l$. Let $\tau(k)$, $\ell(k)$ and $\rho(k)$ be the corresponding transmission coefficient, the left reflection coefficient and the right reflection coefficient, respectively. Generically, we have $\tau(0) = 0$ or equivalently $[g_l(0, x); g_r(0, x)] \neq 0$. In the exceptional case, we have $\tau(0) \neq 0$ or equivalently $[g_l(0, x); g_r(0, x)] = 0$.

In the generic case, using [19, 20, 22, 24]

$$[g_r(k, x); g_l(k, x)] = \frac{2ik}{\tau(k)}, \tag{5.9}$$

$$g_r(k, x) = \frac{g_l(-k, x) + \rho(k)g_l(k, x)}{\tau(k)},$$

$$g_r(0, 0) = 1, \quad g_r'(0, 0) = 0,$$

with the help of (5.3) and (5.4) we obtain

$$\eta(x) = \begin{vmatrix} 0 & -\frac{i}{2}\dot{\tau}(0)g_l(0, x) & ig_l(0, x) - \frac{i}{2}\dot{\rho}(0)g_l(0, x) \\ 1 & g_l(0, 0) & 1 \\ -\cot\alpha & g_l'(0, 0) & 0 \end{vmatrix}, \qquad x \in (0, l), \tag{5.10}$$

where the overdot denotes the k-derivative.

In the exceptional case, we can choose

$$y_1(x) = g_1(0, x), \qquad y_2(x) = g_1(0, x) \int_0^x \frac{dz}{[g_1(0, z)]^2}.$$

In this case, we have

$$y_1(0) = \frac{1 + \ell(0)}{\tau(0)}, \qquad y_1'(0) = 0, \qquad y_2(0) = 0, \qquad y_2'(0) = \frac{1}{y_1(0)} = \frac{\tau(0)}{1 + \ell(0)},$$

with $y_1(0) \neq 0$ because [19, 20, 22, 24] we have $-1 < \ell(0) < 1$ and $\tau(0) > 0$. Hence, with the help of (5.3) and (5.4) we get

$$\eta(x) = g_1(0, x) \begin{vmatrix} 0 & 1 & \int_0^x \frac{dz}{[g_1(0,z)]^2} \\ -1 & g_1(0, 0) & 0 \\ \cot\alpha & 0 & \frac{1}{g_1(0,0)} \end{vmatrix}, \qquad x \in (0, l). \tag{5.11}$$

Theorem 5.1.   *The relative area $[\eta(x)]^2$ for $x \in (0, l)$ is uniquely determined by the data $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$. Equivalently, $\eta(x)$ for $x \in (0, l)$ is uniquely determined from $\{Q(x) : x \in (0, l), \cot\alpha\}$, where $\cot\alpha$ is the constant in (4.8).*

Proof.   From theorem 3.1 we know that $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$ uniquely determines the potential $Q(x)$ for $x \in (0, l)$ and the constant $\cot\alpha$. From (5.5)–(5.7) we see that $\eta(x)$ is uniquely determined by the regular solution $\varphi_\alpha(k, x)$, which in turn is uniquely determined by $\{Q(x) : x \in (0, l), \cot\alpha\}$. $\square$

It is possible to construct $\eta(x)$ from $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$ as follows. With the help of (3.4)–(3.12), we can construct the corresponding right reflection coefficient $R(k)$, the transmission coefficient $T(k)$ and the Jost solution $f(k, x)$. Hence, in the generic case every term appearing on the right-hand side of (5.10) can be constructed from $\{|F_\alpha(k)| : k \in \mathbf{R}^+\}$, and we get

$$\eta(x) = \begin{vmatrix} 0 & -\frac{i}{2}\dot{T}(0)f(0, x) & if(0, x) - \frac{i}{2}\dot{R}(0)f(0, x) \\ 1 & f(0, 0) & 1 \\ -\cot\alpha & f'(0, 0) & 0 \end{vmatrix}, \qquad x \in (0, l).$$

In the exceptional case, from (5.11) we get

$$\eta(x) = f(0, x) \begin{vmatrix} 0 & 1 & \int_0^x \frac{dz}{[f(0,z)]^2} \\ -1 & f(0, 0) & 0 \\ \cot\alpha & 0 & \frac{1}{f(0,0)} \end{vmatrix}, \qquad x \in (0, l).$$

Note that we can write the Jost solution $f(k, x)$ as a linear combination of $g_1(k, x)$ and $g_1(-k, x)$, where $g_1(k, x)$ is the quantity appearing in (5.9). With the help of (2.2) we obtain

$$f(k, x) = \frac{e^{ikl}}{2ik} \begin{vmatrix} 0 & g_1(k, x) & g_1(-k, x) \\ 1 & g_1(k, l) & g_1(-k, l) \\ ik & g_1'(k, l) & g_1'(-k, l) \end{vmatrix}, \qquad x \in (0, l). \tag{5.12}$$

We can also express $F_\alpha(k)$ with the help of $g_1(k, x)$. To do so, we can obtain $f(k, 0)$ and $f'(k, 0)$ from (5.12) and use (2.3) to get $F_\alpha(k)$. Alternatively, by using (cf (2.5))

$$g_1(k, 0) = \frac{1 + \ell(k)}{\tau(k)}, \qquad g_1'(k, 0) = ik\frac{1 - \ell(k)}{\tau(k)},$$

we can write $F_\alpha(k)$ with the help of the transmission and left reflection coefficients associated with $g_1(k, x)$.

6. Recovery from the impedance at the lips

The impedance at the lips is defined as

$$Z(k, l) := \frac{p(l, t)}{v(l, t)}, \tag{6.1}$$

where $p(l, t)$ and $v(l, t)$ are the pressure and the volume velocity at the lips. From (4.2) and (4.12) we see that $Z(k, l)$ does not depend on t, which justifies our notation for Z not containing t as one of the arguments. In this section we analyse whether or not $\{|Z(k, l)| : k \in \mathbf{R}^+\}$ determines the vocal-tract area $A(x)$ for $x \in (0, l)$.

Using (4.2), (4.10) and (4.12) in (6.1), we get

$$Z(k, l) = \frac{2ic\mu k}{2ikA(l) + A'(l)}. \tag{6.2}$$

Thus, we can only hope to get $A(l)$ and $A'(l)$ from $Z(k, l)$. We can refer to $Z(k, l)$ also as the output impedance by visualizing the input occurring at the glottis and the output at the lips.

Note that from (6.2) we get

$$|Z(k, l)|^2 = \frac{4c^2\mu^2 k^2}{4k^2[A(l)]^2 + [A'(l)]^2}, \qquad k \in \mathbf{R}. \tag{6.3}$$

By using (6.2) at two distinct real k values, say k_1 and k_2, we can recover $A(l)$ and $A'(l)$ by solving a linear algebraic system and get

$$A(l) = \frac{c\mu}{k_1 - k_2}\left[\frac{k_1}{Z(k_1, l)} - \frac{k_2}{Z(k_2, l)}\right],$$

$$A'(l) = \frac{2ic\mu k_1 k_2}{k_1 - k_2}\left[\frac{1}{Z(k_2, l)} - \frac{1}{Z(k_1, l)}\right].$$

On the other hand, if we only know $|Z(k_1, l)|$ and $|Z(k_2, l)|$ without knowing their phases, then from (6.3) we get $A(l)$ and $|A'(l)|$ via

$$A(l) = \sqrt{\frac{c^2\mu^2}{k_1^2 - k_2^2}\left[\frac{k_1^2}{|Z(k_1, l)|^2} - \frac{k_2^2}{|Z(k_2, l)|^2}\right]}, \tag{6.4}$$

$$[A'(l)]^2 = \frac{4c^2\mu^2 k_1^2 k_2^2}{k_1^2 - k_2^2}\left[\frac{1}{|Z(k_2, l)|^2} - \frac{1}{|Z(k_1, l)|^2}\right]. \tag{6.5}$$

As seen from (6.3), $\{|Z(k, l)| : k \in \mathbf{R}^+\}$ by itself contains no other information related to Q, η or A.

7. Recovery from the pressure at the lips

Let us consider the recovery of $A(x)$ for $x \in (0, l)$ from the absolute value of the pressure at the lips resulting from the glottal volume velocity in (4.1). From (4.2) we see that our data set is equivalent to $\{|P(k, l)| : k \in \mathbf{R}^+\}$. With the help of (2.4)–(2.6) and (4.10), we note that $|P(k, l)|$ is an even function of $k \in \mathbf{R}$, and hence we have our data actually available for $k \in \mathbf{R}$. In this section we show that this data set uniquely determines each of $Q(x)$, $\eta(x)$ and $A(x)$ for $x \in (0, l)$, and outline an explicit procedure to recover these quantities.

Theorem 7.1. *The data set* $\{|P(k, l)| : k \in \mathbf{R}^+\}$ *uniquely determines each of* $Q(x)$, $\eta(x)$ *and* $A(x)$ *for* $x \in (0, l)$.

Proof. From (2.2), (2.4) and (4.10) we get

$$|P(k,l)| = \frac{c\mu|k|}{\sqrt{A(l)}\sqrt{A(0)}|F_\alpha(k)|}, \qquad k \in \mathbf{R}. \tag{7.1}$$

It is known (cf (3.9) of [21]) that for any fixed $\alpha \in (0, \pi)$ we have

$$F_\alpha(k) = k + O(1), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{7.2}$$

Using (7.2) in (7.1), we obtain

$$\sqrt{A(0)A(l)} = \frac{c\mu}{\lim_{k\to+\infty}|P(k,l)|}, \tag{7.3}$$

and hence

$$|F_\alpha(k)| = \frac{|k|}{|P(k,l)|}\left(\lim_{k\to+\infty}|P(k,l)|\right), \qquad k \in \mathbf{R}. \tag{7.4}$$

As seen from (7.4) and the evenness of $|P(k,l)|$ in $k \in \mathbf{R}$, by measuring the absolute value of the pressure at the lips for $k \in \mathbf{R}^+$, we get $|F_\alpha(k)|$ for $k \in \mathbf{R}$. Then, by proceeding as in section 3, we can recover $Q(x)$ for $x \in (0, l)$ and the constant $\cot\alpha$ appearing in (4.8). Next, by proceeding as in section 5, we determine $\eta(x)$ for $x \in (0, l)$. Note that $\sqrt{A(0)A(l)}$ is uniquely determined from our data set via (7.3). Furthermore, as seen from (5.4), we have $\eta(l) = \sqrt{A(l)/A(0)}$. Thus, we obtain

$$A(0) = \frac{1}{\eta(l)}\frac{c\mu}{\lim_{k\to+\infty}|P(k,l)|}.$$

Hence, having $A(0)$ and $\eta(x)$ for $x \in (0, l)$ in hand, we get the area function uniquely via (5.8). $\qquad\square$

8. Recovery from the transfer function

The transfer function $\mathbf{T}(k, l)$ from the glottis to the lips is defined as

$$\mathbf{T}(k,l) := \frac{v(l,t)}{v(0,t)}, \tag{8.1}$$

where we recall that $v(x, t)$ is the volume velocity in the vocal tract. In this section we show that the data set $\{|\mathbf{T}(k,l)| : k \in \mathbf{R}^+\}$ by itself does not uniquely determine any of $Q(x), \eta(x)$ or $A(x)$, and we show how additional data may be used for the unique determination.

From (4.1), (4.12) and (8.1), we have

$$\mathbf{T}(k,l) = \frac{\sqrt{A(l)}\,\mathrm{e}^{-ikl}}{\sqrt{A(0)}F_\alpha(-k)}\left[-k + \frac{\mathrm{i}}{2}\frac{A'(l)}{A(l)}\right], \qquad k \in \overline{\mathbf{C}^+}. \tag{8.2}$$

Hence, using (2.4) we get

$$|\mathbf{T}(k,l)|^2 = \frac{A(l)}{A(0)|F_\alpha(k)|^2}\left[k^2 + \frac{[A'(l)]^2}{4[A(l)]^2}\right], \qquad k \in \mathbf{R}. \tag{8.3}$$

With the help of (2.4) and (8.3) we see that $|\mathbf{T}(k,l)|$ is an even function of $k \in \mathbf{R}$, and hence the data sets $\{|\mathbf{T}(k,l)| : k \in \mathbf{R}^+\}$ and $\{|\mathbf{T}(k,l)| : k \in \mathbf{R}\}$ are equivalent.

Theorem 8.1. *The data set $\{|\mathbf{T}(k,l)| : k \in \mathbf{R}^+, A(l), |A'(l)|\}$ uniquely determines each of $Q(x), \eta(x)$ and $A(x)$ for $x \in (0, l)$.*

Proof. Using (7.2) in (8.2) we obtain

$$|\mathbf{T}(k, l)| = \frac{\sqrt{A(l)}}{\sqrt{A(0)}} [1 + O(1/k)], \qquad k \to +\infty,$$

and as a result we can recover $A(0)$ as

$$A(0) = \frac{A(l)}{\lim\limits_{k \to +\infty} |\mathbf{T}(k, l)|^2}. \tag{8.4}$$

Thus, from (8.2) and (8.4), with the help of (2.4), we have

$$|F_\alpha(k)|^2 = \frac{\lim\limits_{k \to +\infty} |\mathbf{T}(k, l)|^2}{|\mathbf{T}(k, l)|^2} \left[k^2 + \frac{1}{4} \frac{[A'(l)]^2}{[A(l)]^2} \right], \qquad k \in \mathbf{R}. \tag{8.5}$$

Since $|\mathbf{T}(k, l)|$ is even in $k \in \mathbf{R}$, from (8.5) we see that our data set uniquely determines $|F_\alpha(k)|$ for $k \in \mathbf{R}$, and hence, as indicated in theorem 3.1, $Q(x)$ is uniquely determined for $x \in (0, l)$. Next, from theorem 5.1 it follows that $\eta(x)$ is also uniquely determined for $x \in (0, l)$. Finally, from (8.4) we see that $A(0)$ is also determined by our data set, and thus we recover $A(x)$ for $x \in (0, l)$ uniquely by using (5.8). $\qquad \square$

Note that we assume that $A(l)$ and $|A'(l)|$ do not change with k, and hence they are constants. As indicated in section 6 they can be obtained via (6.4) and (6.5) by measuring the absolute value of the impedance at the lips at two different frequencies.

Theorem 8.2. *The data set* $\{|\mathbf{T}(k, l)| : k \in \mathbf{R}^+, |A'(l)|/A(l)\}$ *uniquely determines each of* $Q(x)$ *and* $\eta(x)$ *for* $x \in (0, l)$, *and it determines* $A(x)$ *for* $x \in (0, l)$ *up to a multiplicative constant.*

Proof. From (8.4) and the evenness of $|\mathbf{T}(k, l)|$ in $k \in \mathbf{R}$, we see that $|F_\alpha(k)|$ for $k \in \mathbf{R}$ is uniquely determined by our data set. Hence $Q(x)$ and $\eta(x)$ are uniquely determined for $x \in (0, l)$ with the help of theorems 3.1 and 5.1, respectively. Furthermore, from (3.5), (8.2) and (8.3) we see that if we multiply each of $A(0)$, $A(l)$ and $|A'(l)|$ by the same constant, we do not change $|\mathbf{T}(k, l)|$ for $k \in \mathbf{R}$. Thus, our data set corresponds to the one-parameter family for $A(x)$, where the parameter $A(0)$ appears as a multiplicative parameter in (5.8). $\qquad \square$

With the help of (8.5) and theorem 8.2 we have the following conclusions.

Corollary 8.3. *Corresponding to the data set* $\{|\mathbf{T}(k, l)| : k \in \mathbf{R}^+, A(l)\}$, *in general there exists a one-parameter family for each of* $Q(x)$, $\eta(x)$ *and* $A(x)$, *where* $|A'(l)|$ *can be chosen as the parameter.*

Corollary 8.4. *Corresponding to the data set* $\{|\mathbf{T}(k, l)| : k \in \mathbf{R}^+\}$, *in general there exists a two-parameter family for each of* $Q(x)$, $\eta(x)$ *and* $A(x)$, *where* $A(l)$ *and* $|A'(l)|$ *can be chosen as the parameters.*

9. Recovery from the Green's function at the lips

In this section we show that the absolute value of a Green's function for (1.3) at the lips measured for $k \in \mathbf{R}^+$ enables us to uniquely construct each of $Q(x)$, $\eta(x)$ and $A(x)$ for $x \in (0, l)$.

The Green's function at the lips can be defined [17] as the solution $\Phi(l, t)$ given in (1.2) when the glottal volume velocity is as in (4.1). Thus, from (1.2), (4.2) and (4.10), we get the Green's function at the lips as

$$\mathbf{G}(k, l; t) = \frac{-c\mu k \, \mathrm{e}^{\mathrm{i}k(ct - l)}}{\sqrt{A(0)} F_\alpha(-k)}.$$

Hence, with the help of (2.4) we obtain

$$|\mathbf{G}(k, l; t)| = \frac{c\mu|k|}{\sqrt{A(0)}|F_\alpha(k)|}, \qquad k \in \mathbf{R}. \tag{9.1}$$

Note that the expression in (9.1) is closely related to that in (7.1). The two data sets differ from each other by the yet unknown multiplicative factor $\sqrt{A(l)}$.

Theorem 9.1. *The data set $\{|\mathbf{G}(k, l; t)| : k \in \mathbf{R}^+\}$ uniquely determines each of $Q(x), \eta(x)$ and $A(x)$ for $x \in (0, l)$.*

Proof. From (2.4) and (9.1) it follows that $|\mathbf{G}(k, l; t)|$ is independent of t and is an even function of k on $\mathbf{R}$, and hence our data can be extended from $k \in \mathbf{R}^+$ to $k \in \mathbf{R}$. Using (7.2) in (9.1) we get

$$\sqrt{A(0)} = \frac{c\mu}{\lim_{k \to +\infty} |\mathbf{G}(k, l; t)|}, \tag{9.2}$$

and hence

$$\frac{|k|}{|F_\alpha(k)|} = \frac{|\mathbf{G}(k, l; t)|}{\lim_{k \to +\infty} |\mathbf{G}(k, l; t)|}, \qquad k \in \mathbf{R}.$$

Thus, we get $|F_\alpha(k)|$ for $k \in \mathbf{R}$ when $|\mathbf{G}(k, l; t)|$ is available for $k \in \mathbf{R}^+$. Then, as in section 3, we construct $Q(x)$ for $x \in (0, l)$ and $\cot \alpha$. Next, as in section 5, we construct $\eta(x)$ for $x \in (0, l)$. Finally, with the help of (5.8) and (9.2) we obtain

$$A(x) = \frac{c^2 \mu^2 [\eta(x)]^2}{\left(\lim_{k \to +\infty} |\mathbf{G}(k, l; t)| \right)^2}.$$

Thus, the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

10. Examples

In this section we illustrate the theoretical results presented in the previous sections with some examples.

Let us use $l = 17.5$ cm, $c = 3.43 \times 10^4$ cm s^{-1}, $\mu = 1.2 \times 10^{-3}$ g cm^{-3}, $A(0) = 5$ cm^2, $A'(0) = -0.52$ cm and
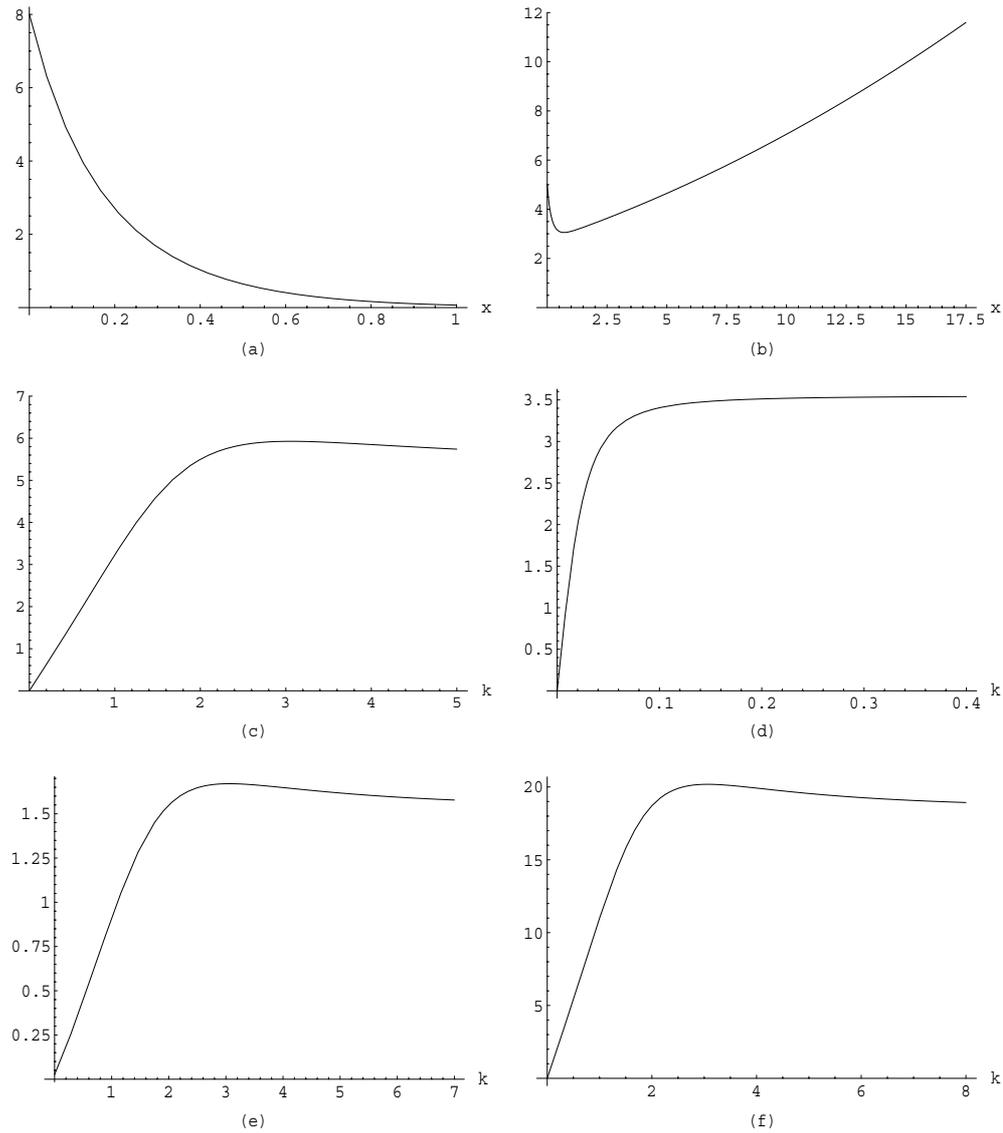
$$Q(x) = \frac{80(7 + 3\sqrt{5}) e^{2\sqrt{5}x}}{[(7 + 3\sqrt{5}) e^{2\sqrt{5}x} - 2]^2}.$$

When Q is viewed as a potential of the full-line Schrödinger equation with support on $\mathbf{R}^+$, the corresponding scattering coefficients $\tau(k), \rho(k), \ell(k)$ and the left Jost solution $g_1(k, x)$ introduced in section 5 are rational functions of k, and it can be verified that

$$g_1(k, x) = e^{ikx} \left[1 + \frac{i}{k + i\sqrt{5}} \frac{4\sqrt{5}}{(7 + 3\sqrt{5}) e^{2\sqrt{5}x} - 2} \right], \qquad x \geqslant 0,$$

$$\tau(k) = \frac{k(k + i\sqrt{5})}{(k + i)(k + 2i)}, \qquad \ell(k) = \frac{2}{(k + i)(k + 2i)},$$

$$\rho(k) = \frac{-2(k + i\sqrt{5})}{(k + i)(k + 2i)(k - i\sqrt{5})}.$$

Figure 1. (a) $Q(x)$, (b) $A(x)$, (c) $|P(k, l)|$, (d) $|Z(k, l)|$, (e) $|\mathbf{T}(k, l)|$ and (f) $|\mathbf{G}(k, l; t)|$ of section 10.

All the quantities related to (1.1), (1.3) and (1.6) can now be explicitly evaluated. For example, the left Jost solution $g_1(k, x)$ for $x \leqslant 0$ for the full-line Schrödinger equation can be obtained as

$$g_1(k, x) = \frac{e^{ikx}}{\tau(k)} + \frac{\ell(k)\, e^{-ikx}}{\tau(k)}, \qquad x \leqslant 0.$$

Via (4.8) we get $\cot \alpha = 0.052$, $\eta(x)$ can be obtained via (5.10), $A(x)$ via (5.8), $f(k, x)$ via (5.12), $F_\alpha(k)$ via (2.3), the scattering coefficients $T(k)$, $R(k)$ and $L(k)$ via (3.9) and (3.10), $P(k, x)$ via (4.10), $p(x, t)$ via (4.2) and (4.10), $v(x, t)$ via (4.12), $|P(k, l)|$ via (7.1), $|Z(k, l)|$ via (6.3), $|\mathbf{T}(k, l)|$ via (8.3), $|\mathbf{G}(k, l; t)|$ via (9.1), $\Lambda_\alpha(k)$ via (3.2). Having obtained $A(x)$, we also compute $A(l) = 11.596$ cm^2 and $A'(l) = 0.681$ cm. Even though all these quantities

can be explicitly written in terms of elementary functions in closed forms, the corresponding expressions are too long to be displayed here, and instead we only show some of their graphs. In figures 1(c)–(g), note that the asymptotics as $k \to +\infty$ are all constants that can be read from the corresponding graphs.

As far as the inverse problem is concerned, it is known from section 6 that the graph in figure 1(d) cannot determine either of the graphs of figures 1(a) and (b). We know from section 7 that the graph in figure 1(c) uniquely determines the graphs of figures 1(a) and (b). From section 9 we know that the graph in figure 1(f) also uniquely determines the graphs of figures 1(a) and (b). We know from section 8 that the information contained in the graph of figure 1(e) is not sufficient to determine uniquely either of the graphs in figures 1(a) and (b); however, the graphs in figures 1(d) and (e) together uniquely determine the graphs of figures 1(a) and (b).

Acknowledgments

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