Inverse Problems 1 (1985) 291–300. Printed in Great Britain

Non-uniqueness in the one-dimensional inverse scattering problem

Tuncay Aktosun and Roger G Newton
Department of Physics, Indiana University, Bloomington, IN 47405, USA

Received 3 June 1985

Abstract. The Schrödinger equation in one dimension is considered for the case when at least one reflection amplitude at zero energy is unity. It is shown that if there exists a corresponding potential that causes no negative-energy bound states then there is a one-parameter family of potentials that causes the same scattering at all energies. Two explicit examples are given.

1. Introduction

If the unitary S matrix of the Schrödinger equation in one dimension is continuous at $k=0$ (as it is when $|x|^2 V \in L^1(\mathbb{R})$) then it must be real there. For the transmission amplitude T and the reflection amplitudes R_l and R_r from the left and right, in terms of which

$$S=\begin{pmatrix} T & R_r \\ R_l & T \end{pmatrix},$$

there are then the following possibilities (taking into account that $\det S = T/\bar{T}$, where $\bar{T}$ is the complex conjugate of T, $S(-k)=\bar{S}(k)$, and listing only the simplest cases, when the leading terms go as integral powers of k):

(1) $T(0)\neq 0$, $\det S(0)=1$

(2) $T(k)=\mathrm{i}ck+\mathrm{o}(k)$, $c\neq 0$, $\det S(0)=-1$

$(a)\ R_l=R_r=-1 \qquad (b)\ R_l=R_r=1$

(3) $T(k)=ck^2+\mathrm{o}(k^2)$, $c\neq 0$, $\det S(0)=1$

$R_l=-R_r=\pm 1$

(4) $T(k)=\mathrm{O}(k^3)$.

If $|x|V\in L^1$, case (2a) is generic, while case (1) is exceptional. In case (1) one says that there is a 'half-bound' state, because it implies the existence of a bounded solution of the Schrödinger equation at $k=0$. (Generically both solutions grow linearly in at least one direction.) None of the other possibilities can occur if $(|x|+|x|^2)V\in L^1$. As is well known, the inverse scattering problem for such potentials has a unique solution (Faddeev 1964, Deift and Trubowitz 1979).

On the other hand, if $|x|V\notin L^1$ then the other listed possibilities may occur, and explicit examples of (2b) (Abraham *et al* 1981, Brownstein 1982), (3) (Moses 1983) and (4)

0266-5611/85/040291 + 10\$02.25

(Brownstein 1982) are known. They are also all known to be associated with more than one potential. In this paper we shall study cases (2*b*) and (3), and we shall show that if a potential leads to an S matrix in class (2*b*) or class (3) then there is always a one-parameter family of potentials that is associated with the same S matrix. In § 4 we show that the previously known examples of ambiguity in the inverse problem are special cases of such families. Our results are closely related to those of Sabatier (1984), who came to the same conclusion in case (2*b*) by a different procedure.

In the matrix inversion method (Newton 1980, 1983) the unique potential for $S(k)$ with $R_l(0)=R_r(0)=-1$ and the family of potentials for

$$S'=ISI=\begin{pmatrix} T & -R_r \\ -R_l & T \end{pmatrix} \qquad I=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are both constructed together, as we shall see in § 2. Section 3 deals with case (3).

2. The general ambiguity in case (2*b*)

This can be split into three parts.

(*a*) The solution of the inverse scattering problem given by Newton (1980, 1983) is based on the solution of the following Riemann–Hilbert problem.

Problem $H_A^0(S)$. Let a continuous 2×2 matrix-valued function $S(k)$ on $\mathbb{R}$ with the following properties be given:

(i) $\quad S(-k)=\bar{S}(k)$;

(ii) $\quad \tilde{S}=QSQ \qquad Q=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

where $\tilde{S}$ is the transpose of S;

(iii) $\quad SS^\dagger=\mathbb{1}$

where $S^\dagger$ is the adjoint of S;

(iv) $\quad (S(k)-\mathbb{1})\in L^2(\mathbb{R})$;

(v) $\quad kR_{l,r}(k)\in L^1(\mathbb{R})$;

(vi) $\quad S(0)=-Q$;

(vii) $\quad \sigma(0)-\sigma(\infty)=-\frac{1}{2}\pi$

where $\sigma(k)=-\frac{1}{2}\mathrm{i}\ln\det S(k)$.

Find a 2×2 matrix-valued function $F(k)=\bar{F}(-k)$ that is the boundary value of an analytic function holomorphic in $\mathbb{C}^+$, such that

$$\lim_{|k|\to\infty} F(k)=A \qquad A=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$(F(k)-A)\in L^2(\mathbb{R})$ and for real k

$$F(-k)=QS(-k)F(k)I. \tag{1}$$

Note that (vi) and (vii) imply that we are restricting ourselves to the case that is generic when $|x|V \in L^1$, with no bound or half-bound states. (Our example in § 4 will show that for potentials outside that class (vii) need not guarantee that.) Comparison with equation (3.30) of Newton (1983) shows that the Jost matrix J corresponding to S is related to F by $J = AF^{-1}$.

It is worth noting explicitly that equation (1) is such that each column of F satisfies an autonomous equation. If $f^{(1)}$ and $f^{(2)}$ are the first and second columns of F, respectively, then as $|k| \to \infty$, $f^{(1)} \to \binom{1}{1}$, $f^{(2)} \to \binom{1}{-1}$ and

$$f^{(1)}(-k) = QS(-k)f^{(1)}(k) \tag{2}$$

$$f^{(2)}(-k) = -QS(-k)f^{(2)}(k). \tag{3}$$

If we define $f^{(3)} := If^{(2)}$, then $f^{(3)} \to \binom{1}{1}$ as $|k| \to \infty$ and

$$f^{(3)}(-k) = QIS(-k)If^{(3)}(k). \tag{4}$$

Thus $f^{(3)}$ corresponds to $S' = ISI$ as $f^{(1)}$ does to S.

If we denote by F_x the solution of $H_A^0(S_x)$, where

$$S_x(k) = \exp(\mathrm{i}kIx)S(k)\exp(-\mathrm{i}kIx),$$

then the first column of F_x is the scattering solution $\Psi^{(1)}$ of the Schrödinger equation with a potential V_1 whose S matrix is $S(k)$ (see Newton (1983)) and the second column is $I\Psi^{(2)}$, where $\Psi^{(2)}$ is the solution of the Schrödinger equation with a potential V_2 whose S matrix is ISI, if it exists. If the first corresponds to T, R_r and R_l, then the second corresponds to T, $-R_r$, $-R_l$. Condition (v) ensures that the solution of $H_A^0(S_x)$ is 'miraculous' and that V_1 and V_2 exist if F_x does (Newton 1984).

Taking determinants of both sides of (1) and defining $\xi(-k) := -\frac{1}{2}\det F(k)$ we get

$$\xi(-k) = \det S(-k)\xi(k). \tag{5}$$

The unique solution of this equation which is the boundary value of a function holomorphic in $\mathbb{C}^+$ and approaches 1 as $|k| \to \infty$ is given by

$$\xi(k) = \exp\left(\mathrm{i}\sigma(k) + \frac{2}{\pi} P \int_0^{\infty} \mathrm{d}k' \, \frac{k'\sigma(k')}{k'^2 - k^2}\right) \tag{6}$$

(here P denotes Cauchy's principal value). As $k \to 0$ this function vanishes linearly. In the usual case, when T vanishes linearly at $k = 0$ and is holomorphic in $\mathbb{C}^+$, $\xi(k) = T(k)$.

If we write near $k = 0$

$$F(k) = M + \mathrm{o}(1)$$

then we obtain from (1) and (vi) that $M(\mathbb{1} + I) = 0$ and hence

$$M = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \tag{7}$$

where a and b are real, so that $F(-k) = \bar{F}(k)$. Now, if $H_A^0(S)$ has a solution F then $-\frac{1}{2}\det F$ must satisfy (5) and hence is given by (6). Thus $\det F$ cannot vanish in $\mathbb{C}^+$ and F^{-1} is holomorphic too.

Lemma 1. If F_1 and F_2 are two solutions of $H_A^0(S)$ with the same ratio of a/b, then $F_1 = F_2$.

Proof. It follows from (1) that $F_1(-k)IF_1(k)^{-1}=F_2(-k)IF_2(k)^{-1}$ and therefore $F_2(-k)^{-1}F_1(-k)=IF_2(k)^{-1}F_1(k)I$. The left-hand side is holomorphic in $\mathbb{C}^-$ and the right-hand side in $\mathbb{C}^+$. Near the origin

$$F_2^{-1}=k^{-1}c\begin{pmatrix} b_2 & -a_2 \\ 0 & 0 \end{pmatrix}+O(1)$$

$$F_1=\begin{pmatrix} 0 & a_1 \\ 0 & b_1 \end{pmatrix}+o(1)$$

where $b_2a_1-a_2b_1=0$. Therefore $F_2^{-1}F_1$ is bounded at $k=0$, and $F_2^{-1}F_1$ is in $L^2(\mathbb{R})$. Thus, since $F_2\approx A$, $F_1\approx A$ at infinity, by Liouville's theorem, $F_2\equiv F_1$.

QED

In other words, the specification of the ratio a/b makes the solution of $H_A^0(S)$ unique if it exists. We shall see below that this ratio can be given arbitrarily, so that we arrive at a one-parameter family of solutions of $H_A^0(S)$. However, this lack of uniqueness of $H_A^0(S_x)$ affects the second column of F_x only. The first column is uniquely determined and leads, as is known, to a unique potential V_1 (if S satisfies the Faddeev–Deift–Trubowitz conditions) whose S matrix is S. The lack of uniqueness of the solution of $H_A^0(S_x)$ then leads to a one-parameter family of potentials V_2 whose S matrix is ISI, because the second column of F depends on the ratio of a/b.

(*b*) In order to solve $H_A^0(S)$ we proceed to remove the pole of F^{-1} at $k=0$ by the same method by which the bound-state poles were removed by Newton (1980, 1983). Let P be the orthogonal projection, $P^2=P=P^\dagger$, such that $PM=0$:

$$P=\frac{1}{a^2+b^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix}.$$

Define

$$\Pi(k):=(k+\mathrm{i})P/k+\mathbb{1}-P \tag{7}$$

and

$$F^{\mathrm{r}}(k):=\Pi(k)F(k). \tag{8}$$

Then the equation

$$F^{\mathrm{r}}(-k)=QS^{\mathrm{r}}(-k)F^{\mathrm{r}}(k)I \tag{9}$$

for F^{r} is equivalent to (1), where

$$S^{\mathrm{r}}(k)=Q\Pi(k)QS(k)\Pi(-k)^{-1}. \tag{10}$$

Since

$$\det S^{\mathrm{r}}=(\det S)\frac{k+\mathrm{i}}{k-\mathrm{i}},$$

(vii) implies that $\sigma^{\mathrm{r}}(0)-\sigma^{\mathrm{r}}(\infty)=0$ if $\sigma^{\mathrm{r}}(k)=-\frac{1}{2}\mathrm{i}\ln\det S^{\mathrm{r}}(k)$. The Riemann–Hilbert problem $H_A^0(S^{\mathrm{r}})$ for F^{r} is of the normal kind, with $\det F^{\mathrm{r}}(0)\neq 0$. If it has a solution then it is unique. Equation (8) then leads to a one-parameter family of solutions of $H_A^0(S)$. It should be noted, however, that although $S^{\mathrm{r}}(k)^{-1}=QS^{\mathrm{r}}(-k)Q$, S^{r} does not satisfy (ii) and (iii).

(*c*) Let us now convert $H_A^0(S)$ into an integral equation by Fourier transformation. Defining

$$\Gamma(\alpha)=(1/2\pi)\int_{-\infty}^{\infty} \mathrm{d}k(F(-k)-A)\exp(\mathrm{i}k\alpha) \tag{11}$$

$$G(\alpha)=(1/2\pi)\int_{-\infty}^{\infty} \mathrm{d}k\, Q(S(-k)-\mathbb{1})\exp(\mathrm{i}k\alpha) \tag{12}$$

we find from (1) for $\alpha>0$

$$\Gamma(\alpha)=G(\alpha)QA+\int_0^{\infty} \mathrm{d}\beta\, G(\alpha+\beta)\Gamma(\beta)I \tag{13}$$

because for $\alpha<0$, $\Gamma(\alpha)=0$ owing to the analyticity of $F(k)$ in $\mathbb{C}^+$. The operator $\mathscr{G}$ whose kernel is $G(\alpha+\beta)$, $\alpha,\beta\in\mathbb{R}_+$, is known to be compact, self-adjoint, and its spectrum lies in $[-1, 1]$. The first column $\eta^{(1)}$ of Γ satisfies the equation for $\alpha>0$,

$$\eta^{(1)}(\alpha)=G(\alpha)\hat{1}+\int_0^{\infty} \mathrm{d}\beta\, G(\alpha+\beta)\eta^{(1)}(\beta) \tag{14}$$

and the second column $\eta^{(2)}$ for $\alpha>0$,

$$\eta^{(2)}(\alpha)=-G(\alpha)I\hat{1}-\int_0^{\infty} \mathrm{d}\beta\, G(\alpha+\beta)\eta^{(2)}(\beta) \tag{15}$$

where $\hat{1}=\binom{1}{1}$. A solution of (2) is connected to a solution of (14), and a solution of (3) to one of (15).

It is known (Newton 1985) that, though a solution of (14) (or (15)) does not necessarily lead to a solution of (2) (or (3)), if one does then so do all. But (2) has a unique solution (if the Faddeev–Deift–Trubowitz conditions are satisfied). Therefore (14) implies that $\mathscr{G}$ does not have the eigenvalue 1.

A sufficient condition for a solution of (14) (or (15)) to yield a solution of (2) (or (3)) is that $\mathscr{G}^{\#}$ does not have the eigenvalue -1 (or $+1$), where $\mathscr{G}^{\#}$ is the operator whose kernel is $G(-\alpha-\beta)$, $\alpha,\beta\in\mathbb{R}_+$ (Newton 1980, 1983). But $\mathscr{G}^{\#}$ is related to $\bar{S}$ as $\mathscr{G}$ is to S. Therefore the absence of 1 in the spectrum of $\mathscr{G}$ ensures that any solution of

$$\xi^{(2)}(\alpha)=-G(-\alpha)I\hat{1}-\int_0^{\infty} \mathrm{d}\beta\, G(-\alpha-\beta)\xi^{(2)}(\beta) \qquad \alpha>0 \tag{15a}$$

leads to a solution of

$$g^{(2)}(-k)=-QS(k)g^{(2)}(k). \tag{3a}$$

However, (15*a*) always has a solution (Newton 1985); hence so does (3*a*). Consequently, the equation

$$g^{(1)}(-k)=QS(k)g^{(1)}(k) \tag{2a}$$

cannot have a solution without poles because (2*a*) and (3*a*) together give

$$F'(-k)=QS(k)F'(k)I \tag{1a}$$

if F' is formed out of the columns of $g^{(1)}$ and $g^{(2)}$. (The Levinson theorem for $\bar{S}$ implies that no solution of (1*a*) is possible without poles. That the poles occur in (2*a*) and not in (3*a*) is in agreement with the fact that (5) implies that near $k=0$ the second column of $\tilde{F}^{-1}$ is bounded.)

We thus conclude that $\mathscr{G}$ must have the eigenvalue -1, which prevents the solution of

$$\xi^{(1)}(\alpha)=G(-\alpha)\hat{1}+\int_0^\infty \mathrm{d}\beta\, G(-\alpha-\beta)\xi^{(1)}(\beta) \qquad \alpha>0 \tag{14a}$$

from leading to a solution of (2*a*).

The presence of -1 in the spectrum of $\mathscr{G}$ makes the solution of (15) non-unique and accounts for the non-uniqueness of the solution of (3) if it exists. However, we do not have any assurance that (3) has a solution (with the appropriate properties).

3. Class (3)

When $T(k)$ vanishes quadratically at $k=0$ then $\det S(0)=1$. If there are no negative-energy bound states then the Levinson theorem says, instead of (vii), that $\sigma(0)-\sigma(\infty)=-\pi$. We must then be looking for a solution of a modified problem $H_A^{0\prime}(S)$ such that $\det F(k)=\alpha k^2+o(k^2)$, where $\alpha\neq 0$. Assuming that S is continuous and $F(k)=M_0+kM_1+O(k^2)$ we find from (1) that $M_0=IM_0I$ if $S(0)=QI$. It follows that either

$$F(k)=\begin{pmatrix}0 & \mathrm{i}ak\\ \mathrm{i}bk & c+\mathrm{i}dk\end{pmatrix}+O(k^2) \tag{16}$$

or

$$F(k)=\begin{pmatrix}c+\mathrm{i}dk & \mathrm{i}bk\\ \mathrm{i}ak & 0\end{pmatrix}+O(k^2) \tag{17}$$

where a, b, c and d are real. We shall assume the first form; the arguments are similar if it is the second. If $S(0)=-QI$ then either

$$F(k)=\begin{pmatrix}\mathrm{i}ak & 0\\ c+\mathrm{i}dk & \mathrm{i}bk\end{pmatrix}+O(k^2) \tag{18}$$

or

$$F(k)=\begin{pmatrix}\mathrm{i}bk & c+\mathrm{i}dk\\ 0 & \mathrm{i}ak\end{pmatrix}+O(k^2) \tag{19}$$

and again the arguments are similar.

Assuming the form of (16) for F, we have

$$F^{-1}(k)=\frac{1}{abk^2}\begin{pmatrix}c+\mathrm{i}dk & -\mathrm{i}ak\\ -\mathrm{i}bk & 0\end{pmatrix}+O(1)$$

Lemma 2. If F_1 and F_2 are two solutions of the same form (16) or (17) of $H_A^{0\prime}(S)$ with $S(0)=QI$ with the same ratio of a/c, then they are identical.

Proof. The proof proceeds like that of lemma 1 and depends on the boundedness of $F_2^{-1}F_1$. But

$$F_2^{-1}F_1=\frac{\mathrm{i}(a_1c_2-a_2c_1)}{a_2b_2k}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+O(1).$$

Therefore $a_1c_2 - a_2c_1 = 0$ implies the required boundedness. Similarly if both are of form (17). If $S(0) = -QI$ then an analogue of lemma 2 holds if both F_2 and F_1 are of form (18) or both of form (19).

Thus the result is that in each of the two cases of class (3) we must specify the form of F, either (16) or (17) in the first case and either (18) or (19) in the second, and in all cases the ratio a/c must be given in order for $H_A^{0\prime}(S)$ to have at most one solution.

To remove the singularity from F^{-1} we define

$$\Pi(k) := \frac{\lambda^2}{k^2}\begin{pmatrix} -1 & \mathrm{i}ak/c\lambda^2 \\ -c/a\lambda & \mathrm{i}k/\lambda \end{pmatrix}$$

where $\lambda = 1 - \mathrm{i}k$ and F^{r} and S^{r} are as in (8) and (10). Then

$$\det S^{\mathrm{r}} = (\det S)\frac{(1-\mathrm{i}k)(2-\mathrm{i}k)}{(1+\mathrm{i}k)(2+\mathrm{i}k)}$$

and $\sigma^{\mathrm{r}}(0) - \sigma^{\mathrm{r}}(\infty) = 0$.

4. Two examples

As our first example we take

$$S = \frac{1}{k+\mathrm{i}}\begin{pmatrix} k & -\mathrm{i} \\ -\mathrm{i} & k \end{pmatrix}$$

which is in class (2*a*). Its unique potential is given by $V(x) = 2\delta(x)$. The associated

$$S' = ISI = \frac{1}{k+\mathrm{i}}\begin{pmatrix} k & \mathrm{i} \\ \mathrm{i} & k \end{pmatrix}$$

is in class (2*b*); this S matrix was considered by Abraham *et al* (1981) and by Brownstein (1982).

The family of potentials corresponding to S' is given by

$$V(a, x) = -2\delta(x) + \frac{8\Theta(x)}{(1+a+2x)^2} + \frac{8\Theta(-x)}{(1+b-2x)^2}$$

where $a > 0$ and $b = 1/a$; $\Theta(x)$ is the Heaviside function. Note that $V(a, x) = V(b, -x)$. The two scattering solutions are

$$\Psi^{(1)}(a, k, x) = \Theta(x)\exp(\mathrm{i}kx)\frac{k}{k+\mathrm{i}}\left(1 + \frac{2\mathrm{i}/k}{1+a+2x}\right)$$
$$+ \Theta(-x)\left[\exp(\mathrm{i}kx)\left(1 - \frac{2\mathrm{i}/k}{1+b-2x}\right) + \frac{\mathrm{i}}{k+\mathrm{i}}\exp(-\mathrm{i}kx)\left(1 + \frac{2\mathrm{i}/k}{1+b-2x}\right)\right]$$

and $\Psi^{(2)}(a, k, x) = \Psi^{(1)}(b, k, -x)$. (We interpret $\delta(x)\Theta(x) = \frac{1}{2}\delta(x)$.)

For $a \to 0$ and $a \to \infty$ we obtain the two potentials of Abraham *et al* and for $a = 1$ that of Brownstein.

The $k = 0$ limit of $\Psi^{(1)}$ is given by

$$\Psi^{(1)}(a, 0, x) = \Theta(x)\frac{2}{1+a+2x} + \Theta(-x)\frac{2b}{1+b-2x}$$

which is in $L^2(\mathbb{R})$. Therefore, unless $a=0$ or $a=\infty$ there is a bound state at $k=0$. (Brownstein noticed this in his example.) For $a=0$, $k=0$ two linearly independent solutions are

$$\Psi_1=\Theta(x)(1+2x)^{-1}+\Theta(-x)$$

$$\Psi_2=\Theta(x)\frac{4x^3+6x^2+3x}{1+2x}+\Theta(-x)3x$$

and those for $a\to\infty$, $k=0$ are obtained by $x\to -x$. Thus in the cases of Abraham *et al* there is a 'half-bound' state, i.e. one solution is bounded but not square integrable (Brownstein noticed this also). It is remarkable that in this example the existence of a bound state or half-bound state cannot be recognised from the S matrix but depends on a free parameter. It thus violates Levinson's theorem. (Note, however, that this simple example is not in the class discussed generally in this paper.)

As a second example we take

$$S=\frac{1}{k^2+\mathrm{i}\sqrt{2}k-1}\begin{pmatrix} k^2 & 1 \\ -1 & k^2\end{pmatrix}$$

which is in class (3). This example was considered by Moses (1983). The family of potentials associated with this S matrix is

$$V(c,x)=4\Theta(x)\frac{(\sqrt{2}x+1)(\sigma(x)-2c+1)}{(\sigma(x)+c)^2}+4\Theta(-x)\frac{1}{(b-\sqrt{2}x)^2}$$

where $c\geqslant 1$, $1/b+1/c=1$ and $\sigma(x)=(2\sqrt{2}/3)x^3+2x^2+\sqrt{2}x$. The two scattering solutions are

$$\Psi^{(1)}(c,k,x)=\Theta(-x)\left[\exp(\mathrm{i}kx)\left(1-\frac{\mathrm{i}}{k}P_1\right)+R_1\exp(-\mathrm{i}kx)\left(1+\frac{\mathrm{i}}{k}P_1\right)\right]$$
$$+\Theta(x)T\exp(\mathrm{i}kx)\left(1+\frac{\mathrm{i}}{k}P_2-\frac{1}{k^2}P_3\right)$$

and

$$\Psi^{(2)}(c,k,x)=\Theta(-x)\,T\exp(-\mathrm{i}kx)\left(1+\frac{\mathrm{i}}{k}P_1\right)+\Theta(x)\left[\exp(-\mathrm{i}kx)\left(1-\frac{\mathrm{i}}{k}P_2-\frac{\mathrm{i}}{k^2}P_3\right)\right.$$
$$\left.+R_\mathrm{r}\exp(\mathrm{i}kx)\left(1+\frac{\mathrm{i}}{k}P_2-\frac{1}{k^2}P_3\right)\right]$$

where

$$T=\frac{k^2}{k^2+\mathrm{i}\sqrt{2}k-1}\qquad R_\mathrm{r}=-R_1=\frac{1}{k^2+\mathrm{i}\sqrt{2}k-1}$$

$$P_1(c,x)=\frac{\sqrt{2}}{b-\sqrt{2}x}$$

$$P_2(c,x)=\frac{\sqrt{2}(1+\sqrt{2}x)^2}{\sigma(x)+c}$$

$$P_3(c,x)=\frac{2(1+\sqrt{2}x)}{\sigma(x)+c}.$$

The $k \to 0$ limit of $\Psi^{(1)}$ is given by

$$\lim_{k\to 0} \Psi^{(1)} = \Theta(x)P_3 + \sqrt{2}(b-1)\Theta(-x)P_1$$

and a linearly independent solution is

$$\Psi(c, 0, x) = \Theta(x)\left(\sigma(x) + 4c - 1 - \frac{(3c-1)^2}{\sigma(x)+c}\right)$$

$$+ \Theta(-x)\tfrac{1}{3}(c-1)\left(\frac{5b^3 - 3b^2 - 9b - 3}{b - \sqrt{2}x} - 5(b - \sqrt{2}x)^2\right).$$

The first of these is square integrable; thus there is a zero-energy bound state. The two potentials given by Moses are obtained in the limits as $c \to \infty$ and $c \to 1$ (though there is an error in the first):

$$V(\infty, x) = 4\Theta(-x)(1 - \sqrt{2}x)^{-2}$$

$$V(1, x) = 4\Theta(x)\frac{(\sqrt{2}x + 1)(\sigma(x) - 1)}{(\sigma(x)+1)^2}.$$

For $c \to \infty$ we find

$$\lim_{k\to 0}\lim_{c\to\infty} \tfrac{1}{2}c\Psi^{(1)} = \Theta(x)(1 + \sqrt{2}x) + \frac{\Theta(-x)}{1 - \sqrt{2}x}$$

and a linearly independent solution

$$\Psi = 3\Theta(x) + \Theta(-x)\left(\frac{2}{1 - \sqrt{2}x} + (1 - \sqrt{2}x)^2\right).$$

On the other hand, for $c \to 1$

$$\lim_{k\to 0}\lim_{c\to 1} \tfrac{1}{2}\Psi^{(1)} = \Theta(x)\frac{1 + \sqrt{2}x}{\sigma(x)+1} + \Theta(-x)$$

and a linearly independent solution

$$\Psi = \Theta(x)\left(\sigma(x) + 3 - \frac{4}{\sigma(x)+1}\right) + \Theta(-x)(5\sqrt{2}x - 1).$$

For $c \to 1$ one solution is bounded but not square integrable, while for $c \to \infty$ neither solution is bounded. Thus there is a $k = 0$ bound state for all values of $c > 1$, a half-bound state for $c \to 1$ and neither for $c \to \infty$.

We note that in both of these examples of a continuous ambiguity there are zero-energy bound states. The half-bound states are associated with a discrete two-fold non-uniqueness.

Acknowledgment

This work was supported in part by the US National Science Foundation.

References

Abraham P B, DeFacio B and Moses H E 1981 *Phys. Rev. Lett.* **46** 1657
Brownstein K R 1982 *Phys. Rev.* D **25** 2704
Deift P and Trubowitz E 1979 *Commun. Pure Appl. Math.* **32** 121
Faddeev L D 1964 *Trudi Mat. Inst. Steklov* **73** 314 (Engl. transl. *Am. Math. Soc.* **2** 139)
Moses H E 1983 *Phys. Rev.* A **27** 2220
Newton R G 1980 *J. Math. Phys.* **21** 493
—— 1983 *Conf. on Inverse Scattering: Theory and Application* ed. J B Bednar *et al* (Philadelphia: SIAM) p 1
—— 1984 *J. Math. Phys.* **25** 2991
—— 1985 *Inverse Problems* **1** 371–80
Sabatier P C 1984 *Inverse Problems of Acoustic and Elastic Waves* ed. F Santosa *et al* (Philadelphia: SIAM) p 82