Inverse Scattering Transform and the Theory of Solitons

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Glossary

AKNS method A method introduced by Ablowitz, Kaup, Newell, and Segur in 1973 that identifies the nonlinear partial differential equation (NPDE) associated with a given first-order system of linear ordinary differential equations (LODEs) so that the initial value problem (IVP) for that NPDE can be solved by the inverse scattering transform (IST) method.

Direct scattering problem The problem of determining the scattering data corresponding to a given potential in a differential equation.

Integrability A NPDE is said to be integrable if its IVP can be solved via an IST.

Inverse scattering problem The problem of determining the potential that corresponds to a given set of scattering data in a differential equation.

Inverse scattering transform A method introduced in 1967 by Gardner, Greene, Kruskal, and Miura that yields a solution to the IVP for a NPDE with the help of the solutions to the direct and inverse scattering problems for an associated LODE.

Lax method A method introduced by Lax in 1968 that determines the integrable NPDE associated with a given LODE so that the IVP for that NPDE can be solved with the help of an IST.

Scattering data The scattering data associated with a LODE usually consists of a reflection coefficient which is a function of the spectral parameter λ, a finite number of constants λ_j that correspond to the poles of the transmission coefficient in the upper half complex plane, and the bound-state norming constants whose number for each bound-state pole λ_j is the same as the order of that pole. It is desirable that the potential in the LODE is uniquely determined by the corresponding scattering data and vice versa.

Soliton The part of a solution to an integrable NPDE due to a pole of the transmission coefficient in the upper half complex plane. The term soliton was introduced by Zabusky and Kruskal in 1965 to denote a solitary wave pulse with a particle-like behavior in the solution to the Korteweg-de Vries (KdV) equation.

Time evolution of the scattering data The evolvement of the scattering data from its initial value $S(\lambda, 0)$ at $t = 0$ to its value $S(\lambda, t)$ at a later time t.

I. Definition of the Subject and Its Importance

A general theory to solve NPDEs does not seem to exist. However, there are certain NPDEs, usually first order in time, for which the corresponding IVPs can be solved by the IST method. Such NPDEs are sometimes referred to as integrable evolution equations. Some exact solutions to such equations may be available in terms of elementary functions, and such solutions are important to understand nonlinearity better and they may also be useful in testing accuracy of numerical methods to solve such NPDEs.

Certain special solutions to some of such NPDEs exhibit particle-like behaviors. A single-soliton solution is usually a localized disturbance that retains its shape but only changes its location in time. A multi-soliton solution consists of several solitons that interact nonlinearly when they are close to each other but come out of such interactions unchanged in shape except for a phase shift.

Integrable NPDEs have important physical applications. For example, the KdV equation is used to describe [14,23] surface water waves in long, narrow, shallow canals; it also arises [23] in the description of hydromagnetic waves in a cold plasma, and ion-acoustic waves in anharmonic crystals. The nonlinear Schrödinger (NLS) equation arises in modeling [24] electromagnetic waves in optical fibers as well as surface waves in deep waters. The sine-Gordon equation is helpful [1] in analyzing the magnetic field in a Josephson junction (gap between two superconductors).

II. Introduction

The first observation of a soliton was made in 1834 by the Scottish engineer John Scott Russell at the Union Canal between Edinburgh and Glasgow. Russell reported [21] his observation to the British Association of the Advancement of Science in September 1844, but he did not seem to be successful in convincing the scientific community. For example, his contemporary George Airy, the influential mathematician of the time, did not believe in the existence of solitary water waves [1].

The Dutch mathematician Korteweg and his doctoral student de Vries published [14] a paper in 1895 based on de Vries' Ph.D. dissertation, in which surface waves in shallow, narrow canals were modeled by what is now known as the KdV equation. The importance of this paper was not understood until 1965 even though it contained as a special solution what is now known as the one-soliton solution.

Enrico Fermi in his summer visits to the Los Alamos National Laboratory, together with J. Pasta and S. Ulam, used the computer named Maniac I to computationally analyze a one-dimensional dynamical system of 64 particles in which adjacent particles were joined by springs where the forces also included some nonlinear terms. Their main goal was to determine the rate of approach to the equipartition of energy among different modes of the system. Contrary to their expectations there was little tendency towards the equipartition of energy but instead the almost ongoing recurrence to the initial state, which was puzzling. After Fermi died in November 1954, Pasta and Ulam completed their last few computational examples and finished writing a preprint [11], which was never published as a journal article. This preprint appears in Fermi's Collected Papers [10] and is also available on the internet [25].

In 1965 Zabusky and Kruskal explained [23] the Fermi-Pasta-Ulam puzzle in terms of solitary wave solutions to the KdV equation. In their numerical analysis they observed "solitary-wave pulses," named such pulses "solitons" because of their particle-like behavior, and noted that such pulses interact with each other nonlinearly but come out of interactions unaffected in size or shape except for some phase shifts. Such unusual interactions among solitons generated a lot of excitement, but at that time no one knew how to solve the IVP for the KdV equation, except numerically. In 1967 Gardner, Greene, Kruskal, and Miura presented [12] a method, now known as the IST, to solve that IVP, assuming that the initial profile $u(x,0)$ decays to zero sufficiently rapidly as $x \to \pm\infty$. They showed that the integrable NPDE, i.e. the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0, \tag{2.1}$$

is associated with a LODE, i.e. the 1-D Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + u(x,t)\,\psi = k^2\psi, \tag{2.2}$$

and that the solution $u(x,t)$ to (2.1) can be recovered from the initial profile $u(x,0)$ as explained in the diagram given in Section III. They also explained that soliton solutions to the KdV equation correspond to a zero reflection coefficient in the associated scattering data. Note that the subscripts x and t in (2.1) and throughout denote the partial derivatives with respect to those variables.

In 1972 Zakharov and Shabat showed [24] that the IST method is applicable also to the IVP for the NLS equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{2.3}$$

where i denotes the imaginary number $\sqrt{-1}$. They proved that the associated LODE is the first-order linear system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2mm] \dfrac{d\eta}{dx} = i\lambda\eta - \overline{u(x,t)}\,\xi, \end{cases} \tag{2.4}$$

where λ is the spectral parameter and an overline denotes complex conjugation. The system (2.4) is now known as the Zakharov-Shabat system.

Soon afterwards, again in 1972 Wadati showed in a one-page publication [22] that the IVP for the modified Korteweg-de Vries (mKdV) equation

$$u_t + 6u^2u_x + u_{xxx} = 0, \tag{2.5}$$

can be solved with the help of the inverse scattering problem for the linear system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2mm] \dfrac{d\eta}{dx} = i\lambda\eta - u(x,t)\,\xi. \end{cases} \tag{2.6}$$

Next, in 1973 Ablowitz, Kaup, Newell, and Segur showed [2,3] that the IVP for the sine-Gordon equation

$$u_{xt} = \sin u,$$

can be solved in the same way by exploiting the inverse scattering problem associated with the linear system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi - \dfrac{1}{2}\,u_x(x,t)\,\eta, \\[2mm] \dfrac{d\eta}{dx} = i\lambda\eta + \dfrac{1}{2}\,u_x(x,t)\,\xi. \end{cases}$$

Since then, many other NPDEs have been discovered to be solvable by the IST method.

Our review is organized as follows. In the next section we explain the idea behind the IST. Given a LODE known to be associated with an integrable NPDE, there are two primary methods enabling us to determine the corresponding NPDE. We review those two methods, the Lax method and the AKNS method, in Section IV and in Section V, respectively. In Section VI we introduce the scattering data associated with a LODE containing a spectral parameter and a potential, and we illustrate it for the Schrödinger equation and for the Zakharov-Shabat system. In Section VII we explain the time evolution of the scattering data and indicate how the scattering data sets evolve for those two LODEs. In Section VIII we summarize the Marchenko method to solve the inverse scattering problem for the Schrödinger equation and that for the Zakharov-Shabat system, and we outline how the solutions to the IVPs for the KdV equation and the NLS equation are obtained with the help of the IST. In Section IX we present soliton solutions to the KdV and NLS equations. A brief conclusion is provided in Section X.

III. Inverse Scattering Transform

Certain NPDEs are classified as integrable in the sense that their corresponding IVPs can be solved with the help of an IST. The idea behind the IST method is as follows: Each integrable NPDE is associated with a LODE (or a system of LODEs) containing a parameter λ (usually known as the spectral parameter), and the solution $u(x,t)$ to the NPDE appears as a coefficient (usually known as the potential) in the corresponding LODE. In the NPDE the quantities x and t appear as independent variables (usually known as the spatial and temporal coordinates, respectively), and in the LODE x is an independent variable and λ and t appear as parameters. It is usually the case that $u(x,t)$ vanishes at each fixed t as x becomes infinite so that a scattering scenario can be created for the related LODE, in which the potential $u(x,t)$ can uniquely be associated with some scattering data $S(\lambda,t)$. The problem of determining $S(\lambda,t)$ for all λ values from $u(x,t)$ given for all x values is known as the direct scattering problem for the LODE. On the other hand, the problem of determining $u(x,t)$ from $S(\lambda,t)$ is known as the inverse scattering problem for that LODE.

The IST method for an integrable NPDE can be explained with the help of the diagram

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct scattering for LODE at } t=0} & S(\lambda,0) \\ {\scriptstyle\text{solution to NPDE}}\big\downarrow & & \big\downarrow{\scriptstyle\text{time evolution of scattering data}} \\ u(x,t) & \xleftarrow[\text{inverse scattering for LODE at time } t]{} & S(\lambda,t) \end{array}$$

In order to solve the IVP for the NPDE, i.e. in order to determine $u(x,t)$ from $u(x,0)$, one needs to perform the following three steps:

(i) Solve the corresponding direct scattering problem for the associated LODE at $t=0$, i.e. determine the initial scattering data $S(\lambda,0)$ from the initial potential $u(x,0)$.

(ii) Time evolve the scattering data from its initial value $S(\lambda,0)$ to its value $S(\lambda,t)$ at time t. Such an evolution is usually a simple one and is particular to each integrable NPDE.

(iii) Solve the corresponding inverse scattering problem for the associated LODE at fixed t, i.e. determine the potential $u(x,t)$ from the scattering data $S(\lambda,t)$.

It is amazing that the resulting $u(x,t)$ satisfies the integrable NPDE and that the limiting value of $u(x,t)$ as $t\to 0$ agrees with the initial profile $u(x,0)$.

IV. The Lax Method

In 1968 Peter Lax introduced [15] a method yielding an integrable NPDE corresponding to a given LODE. The basic idea behind the Lax method is the following. Given a linear differential operator $\mathcal{L}$ appearing in the spectral problem $\mathcal{L}\psi = \lambda\psi$, find an operator $\mathcal{A}$ (the operators $\mathcal{A}$ and $\mathcal{L}$ are said to form a Lax pair) such that:

(i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.

(ii) The quantity $\psi_t - \mathcal{A}\psi$ remains a solution to the same linear problem $\mathcal{L}\psi = \lambda\psi$.

(iii) The quantity $\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$ is a multiplication operator, i.e. it is not a differential operator.

From condition (ii) we get

$$\mathcal{L}\left(\psi_t - \mathcal{A}\psi\right) = \lambda\left(\psi_t - \mathcal{A}\psi\right), \tag{4.1}$$

and with the help of $\mathcal{L}\psi = \lambda\psi$ and $\lambda_t = 0$, from (4.1) we obtain

$$\mathcal{L}\psi_t - \mathcal{L}\mathcal{A}\psi = \lambda\psi_t - \mathcal{A}\left(\lambda\psi\right) = \partial_t\left(\lambda\psi\right) - \mathcal{A}\mathcal{L}\psi = \partial_t\left(\mathcal{L}\psi\right) - \mathcal{A}\mathcal{L}\psi = \mathcal{L}_t\psi + \mathcal{L}\psi_t - \mathcal{A}\mathcal{L}\psi, \tag{4.2}$$

where ∂_t denotes the partial differential operator with respect to t. After canceling the term $\mathcal{L}\psi_t$ on the left and right hand sides of (4.2), we get

$$\left(\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}\right)\psi = 0,$$

which, because of (iii), yields

$$\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0. \tag{4.3}$$

Note that (4.3) is an evolution equation containing a first-order time derivative, and it is the desired integrable NPDE. The equation (4.3) is often called a compatibility condition.

Having outlined the Lax method, let us now illustrate it to derive the KdV equation (2.1) from the Schrödinger equation (2.2). For this purpose, we write the Schrödinger equation as $\mathcal{L}\psi = \lambda\psi$ with $\lambda := k^2$ and

$$\mathcal{L} := -\partial_x^2 + u(x,t), \tag{4.4}$$

where the notation := is used to indicate a definition so that the quantity on the left should be understood as the quantity on the right hand side. Given the linear differential operator $\mathcal{L}$ defined as in (4.4), let us try to determine the associated operator $\mathcal{A}$ by assuming that it has the form

$$\mathcal{A} = \alpha_3\partial_x^3 + \alpha_2\partial_x^2 + \alpha_1\partial_x + \alpha_0, \tag{4.5}$$

where the coefficients α_j with $j = 0, 1, 2, 3$ may depend on x and t, but not on the spectral parameter λ. Note that $\mathcal{L}_t = u_t$. Using (4.4) and (4.5) in (4.3), we obtain

$$(\ )\partial_x^5 + (\ )\partial_x^4 + (\ )\partial_x^3 + (\ )\partial_x^2 + (\ )\partial_x + (\ ) = 0, \tag{4.6}$$

where, because of (iii), each coefficient denoted by ( ) must vanish. The coefficient of ∂_x^5 vanishes automatically. Setting the coefficients of ∂_x^j to zero for $j = 4, 3, 2, 1$, we obtain

$$\alpha_3 = c_1, \quad \alpha_2 = c_2, \quad \alpha_1 = c_3 - \frac{3}{2}c_1 u, \quad \alpha_0 = c_4 - \frac{3}{4}c_1 u_x - c_2 u,$$

with c_1, c_2, c_3, and c_4 denoting arbitrary constants. Choosing $c_1 = -4$ and $c_3 = 0$ in the last coefficient in (4.6) and setting that coefficient to zero, we get the KdV equation (2.1). Moreover, by letting $c_2 = c_4 = 0$, we obtain the operator $\mathcal{A}$ as

$$\mathcal{A} = -4\partial_x^3 + 6u\partial_x + 3u_x. \tag{4.7}$$

For the Zakharov-Shabat system (2.4), we proceed in a similar way. Let us write it as $\mathcal{L}\psi = \lambda\psi$, where the linear differential operator $\mathcal{L}$ is defined via

$$\mathcal{L} := i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x - i \begin{bmatrix} 0 & u(x,t) \\ \overline{u(x,t)} & 0 \end{bmatrix}.$$

Then, the operator $\mathcal{A}$ is obtained as

$$\mathcal{A} = 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 - 2i \begin{bmatrix} 0 & u \\ \overline{u} & 0 \end{bmatrix} \partial_x - i \begin{bmatrix} -|u|^2 & u_x \\ \overline{u}_x & |u|^2 \end{bmatrix}, \tag{4.8}$$

and the compatibility condition (4.3) gives us the NLS equation (2.3).

For the first-order system (2.6), by writing it as $\mathcal{L}\psi = \lambda\psi$, where the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L} := i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x - i \begin{bmatrix} 0 & u(x,t) \\ u(x,t) & 0 \end{bmatrix},$$

we obtain the corresponding operator $\mathcal{A}$ as

$$\mathcal{A} = -4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \partial_x^3 - 6 \begin{bmatrix} u^2 & -u_x \\ u_x & u^2 \end{bmatrix} \partial_x - \begin{bmatrix} 6uu_x & -3u_{xx} \\ 3u_{xx} & 6uu_x \end{bmatrix},$$

and the compatibility condition (4.3) yields the mKdV equation (2.5).

V. The AKNS Method

In 1973 Ablowitz, Kaup, Newell, and Segur introduced [2,3] another method to determine an integrable NPDE corresponding to a LODE. This method is now known as the AKNS method, and the basic idea behind it is the following. Given a linear operator $\mathcal{X}$ associated with the first-order system $v_x = \mathcal{X}v$, we are interested in finding an operator $\mathcal{T}$ (the operators $\mathcal{T}$ and $\mathcal{X}$ are said to form an AKNS pair) such that:

(i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.

(ii) The quantity $v_t - \mathcal{T}v$ is also a solution to $v_x = \mathcal{X}v$, i.e. we have $(v_t - \mathcal{T}v)_x = \mathcal{X}(v_t - \mathcal{T}v)$.

(iii) The quantity $\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X}$ is a (matrix) multiplication operator, i.e. it is not a differential operator.

From condition (ii) we get

$$v_{tx} - \mathcal{T}_x v - \mathcal{T}v_x = \mathcal{X}v_t - \mathcal{X}\mathcal{T}v = (\mathcal{X}v)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v = (v_x)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v = v_{xt} - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v. \tag{5.1}$$

Using $v_{tx} = v_{xt}$ and replacing $\mathcal{T}v_x$ by $\mathcal{T}\mathcal{X}v$ on the left side and equating the left and right hand sides in (5.1), we obtain

$$(\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X})v = 0,$$

which in turn, because of (iii), implies

$$\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} = 0. \tag{5.2}$$

We can view (5.2) as an integrable NPDE solvable with the help of the solutions to the direct and inverse scattering problems for the linear system $v_x = \mathcal{X}v$. Like (4.3), the compatibility condition (5.2) yields a nonlinear evolution equation containing a first-order time derivative. Note that $\mathcal{X}$ contains the spectral parameter λ, and hence $\mathcal{T}$ also depends on λ as well. This is in contrast with the Lax method in the sense that the operator $\mathcal{A}$ does not contain λ.

Let us illustrate the AKNS method by deriving the KdV equation (2.1) from the Schrödinger equation (2.2). For this purpose we write the Schrödinger equation, by replacing the spectral parameter k^2 with λ, as a first-order linear system $v_x = \mathcal{X}v$, where we have defined

$$v := \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \quad \mathcal{X} := \begin{bmatrix} 0 & u(x,t) - \lambda \\ 1 & 0 \end{bmatrix}.$$

Let us look for $\mathcal{T}$ in the form

$$\mathcal{T} = \begin{bmatrix} \alpha & \beta \\ \rho & \sigma \end{bmatrix},$$

where the entries α, β, ρ, and σ may depend on x, t, and λ. The compatibility condition (5.2) yields

$$\begin{bmatrix} -\alpha_x - \beta + \rho(u-\lambda) & u_t - \beta_x + \sigma(u-\lambda) - \alpha(u-\lambda) \\ -\rho_x + \alpha - \sigma & -\sigma_x + \beta - \rho(u-\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.3}$$

The $(1,1)$, $(2,1)$, and $(2,2)$-entries in the matrix equation (5.3) imply

$$\beta = -\alpha_x + (u-\lambda)\rho, \quad \sigma = \alpha - \rho_x, \quad \sigma_x = -\alpha_x. \tag{5.4}$$

Then from the $(1,2)$-entry in (5.3) we obtain

$$u_t + \frac{1}{2}\rho_{xxx} - u_x\rho - 2\rho_x(u-\lambda) = 0. \tag{5.5}$$

Assuming a linear dependence of ρ on the spectral parameter and hence letting $\rho = \lambda\zeta + \mu$ in (5.5), we get

$$2\zeta_x\lambda^2 + \left(\frac{1}{2}\zeta_{xxx} - 2\zeta_x u + 2\mu_x - u_x\zeta\right)\lambda + \left(u_t + \frac{1}{2}\mu_{xxx} - 2\mu_x u - u_x\mu\right) = 0.$$

Equating the coefficients of each power of λ to zero, we have

$$\zeta = c_1, \quad \mu = \frac{1}{2}c_1 u + c_2, \quad u_t - \frac{3}{2}c_1 u u_x - c_2 u_x + \frac{1}{4}c_1 u_{xxx} = 0, \tag{5.6}$$

with c_1 and c_2 denoting arbitrary constants. Choosing $c_1 = 4$ and $c_2 = 0$, from (5.6) we obtain the KdV equation given in (2.1). Moreover, with the help of (5.4) we get

$$\alpha = u_x + c_3, \quad \beta = -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}, \quad \rho = 4\lambda + 2u, \quad \sigma = c_3 - u_x,$$

where c_3 is an arbitrary constant. Choosing $c_3 = 0$, we find

$$\mathcal{T} = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}.$$

As for the Zakharov-Shabat system (2.4), writing it as $v_x = \mathcal{X}v$, where we have defined

$$\mathcal{X} := \begin{bmatrix} -i\lambda & u(x,t) \\ -\overline{u(x,t)} & i\lambda \end{bmatrix},$$

we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \begin{bmatrix} -2i\lambda^2 + i|u|^2 & 2\lambda u + iu_x \\ -2\lambda\overline{u} + i\overline{u}_x & 2i\lambda^2 - i|u|^2 \end{bmatrix},$$

and the compatibility condition (5.2) yields the NLS equation (2.3).

As for the first-order linear system (2.6), by writing it as $v_x = \mathcal{X}v$, where

$$\mathcal{X} := \begin{bmatrix} -i\lambda & u(x,t) \\ -u(x,t) & i\lambda \end{bmatrix},$$

we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \begin{bmatrix} -4i\lambda^3 + 2i\lambda u^2 & 4\lambda^2 u + 2i\lambda u_x - u_{xx} - 2u^3 \\ -4\lambda^2 u + 2i\lambda u_x + u_{xx} + 2u^3 & 4i\lambda^3 - 2i\lambda u^2 \end{bmatrix},$$

and the compatibility condition (5.2) yields the mKdV equation (2.5).

As for the first-order system $v_x = \mathcal{X}v$, where

$$\mathcal{X} := \begin{bmatrix} -i\lambda & -\frac{1}{2}\,u_x(x,t) \\ \frac{1}{2}\,u_x(x,t) & i\lambda \end{bmatrix},$$

we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \frac{i}{4\lambda}\begin{bmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{bmatrix}.$$

Then, the compatibility condition (5.2) gives us the sine-Gordon equation

$$u_{xt} = \sin u.$$

VI. Direct Scattering Problem

The direct scattering problem consists of determining the scattering data when the potential is known. This problem is usually solved by obtaining certain specific solutions, known as the Jost solutions, to the relevant LODE. The appropriate scattering data can be constructed with the help of spatial asymptotics of the Jost solutions at infinity or from certain Wronskian relations among the Jost solutions. In this section we review the scattering data corresponding to the Schrödinger equation (2.2) and to the Zakharov-Shabat system (2.4). The scattering data sets for other LODEs can similarly be obtained.

Consider (2.2) at fixed t by assuming that the potential $u(x,t)$ belongs to the Faddeev class, i.e. $u(x,t)$ is real valued and $\int_{-\infty}^{\infty} dx\,(1+|x|)\,|u(x,t)|$ is finite. The Schrödinger equation has two types of solutions; namely, scattering solutions and bound-state solutions. The scattering solutions are those that consist of linear combinations of e^{ikx} and e^{-ikx} as $x \to \pm\infty$, and they occur for $k \in \mathbf{R} \setminus \{0\}$, i.e. for real nonzero values of k. Two linearly independent scattering solutions $f_{\rm l}$ and $f_{\rm r}$, known as the Jost solution from the left and from the right, respectively, are those solutions to (2.2) satisfying the respective asymptotic conditions

$$f_{\rm l}(k,x,t) = e^{ikx} + o(1), \quad f_{\rm l}'(k,x,t) = ike^{ikx} + o(1), \qquad x \to +\infty, \tag{6.1}$$

$$f_{\rm r}(k,x,t)=e^{-ikx}+o(1),\quad f'_{\rm r}(k,x,t)=-ike^{-ikx}+o(1),\qquad x\to-\infty,$$

where the notation $o(1)$ indicates the quantities that vanish. Writing their remaining spatial asymptotics in the form

$$f_{\rm l}(k,x,t)=\frac{e^{ikx}}{T(k,t)}+\frac{L(k,t)\,e^{-ikx}}{T(k,t)}+o(1),\qquad x\to-\infty,\tag{6.2}$$

$$f_{\rm r}(k,x,t)=\frac{e^{-ikx}}{T(k,t)}+\frac{R(k,t)\,e^{ikx}}{T(k,t)}+o(1),\qquad x\to+\infty,$$

we obtain the scattering coefficients; namely, the transmission coefficient T and the reflection coefficients L and R, from the left and right, respectively.

Let $\mathbf{C}^+$ denote the upper half complex plane. A bound-state solution to (2.2) is a solution that belongs to $L^2(\mathbf{R})$ in the x variable. Note that $L^2(\mathbf{R})$ denotes the set of complex-valued functions whose absolute squares are integrable on the real line $\mathbf{R}$. When $u(x,t)$ is in the Faddeev class, it is known [5,7-9,16-19] that the number of bound states is finite, the multiplicity of each bound state is one, and the bound-state solutions can occur only at certain k-values on the imaginary axis in $\mathbf{C}^+$. Let us use N to denote the number of bound states, and suppose that the bound states occur at $k=i\kappa_j$ with the ordering $0<\kappa_1<\cdots<\kappa_N$. Each bound state corresponds to a pole of T in $\mathbf{C}^+$. Any bound-state solution at $k=i\kappa_j$ is a constant multiple of $f_{\rm l}(i\kappa_j,x,t)$. The left and right bound-state norming constants $c_{{\rm l}j}(t)$ and $c_{{\rm r}j}(t)$, respectively, can be defined as

$$c_{{\rm l}j}(t):=\left[\int_{-\infty}^{\infty}dx\,f_{\rm l}(i\kappa_j,x,t)^2\right]^{-1/2},\quad c_{{\rm r}j}(t):=\left[\int_{-\infty}^{\infty}dx\,f_{\rm r}(i\kappa_j,x,t)^2\right]^{-1/2},$$

and they are related to each other through the residues of T via

$$\operatorname{Res}(T,i\kappa_j)=i\,c_{{\rm l}j}(t)^2\,\gamma_j(t)=i\,\frac{c_{{\rm r}j}(t)^2}{\gamma_j(t)},\tag{6.3}$$

where the $\gamma_j(t)$ are the dependency constants defined as

$$\gamma_j(t):=\frac{f_{\rm l}(i\kappa_j,x,t)}{f_{\rm r}(i\kappa_j,x,t)}.\tag{6.4}$$

The sign of $\gamma_j(t)$ is the same as that of $(-1)^{N-j}$, and hence $c_{{\rm r}j}(t)=(-1)^{N-j}\gamma_j(t)\,c_{{\rm l}j}(t)$.

The scattering matrix associated with (2.2) consists of the transmission coefficient T and the two reflection coefficients R and L, and it can be constructed from $\{\kappa_j\}_{j=1}^N$ and one of the reflection coefficients. For example, if we start with the right reflection coefficient $R(k,t)$ for $k\in\mathbf{R}$, we get

$$T(k,t)=\left(\prod_{j=1}^N\frac{k+i\kappa_j}{k-i\kappa_j}\right)\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}ds\,\frac{\log(1-|R(s,t)|^2)}{s-k-i0^+}\right),\qquad k\in\mathbf{C}^+\cup\mathbf{R},$$

where the quantity $i0^+$ indicates that the value for $k\in\mathbf{R}$ must be obtained as a limit from $\mathbf{C}^+$. Then, the left reflection coefficient $L(k,t)$ can be constructed via

$$L(k,t)=-\frac{\overline{R(k,t)}\,T(k,t)}{\overline{T(k,t)}},\qquad k\in\mathbf{R}.$$

We will see in the next section that $T(k,t)=T(k,0)$, $|R(k,t)|=|R(k,0)|$, and $|L(k,t)|=|L(k,0)|$.

For a detailed study of the direct scattering problem for the 1-D Schrödinger equation, we refer the reader to [5,7-9,16-19]. It is important to remember that $u(x,t)$ for $x \in \mathbf{R}$ at each fixed t is uniquely determined [5,7-9,16-18] by the scattering data $\{R, \{\kappa_j\}, \{c_{\mathrm{l}j}(t)\}\}$ or one of its equivalents. Letting $c_j(t) := c_{\mathrm{l}j}(t)^2$, we will work with one such data set, namely $\{R, \{\kappa_j\}, \{c_j(t)\}\}$, in Sections VII and VIII.

Having described the scattering data associated with the Schrödinger equation, let us briefly describe the scattering data associated with the Zakharov-Shabat system (2.4). Assuming that $u(x,t)$ for each t is integrable in x on $\mathbf{R}$, the two Jost solutions $\psi(\lambda,x,t)$ and $\phi(\lambda,x,t)$, from the left and from the right, respectively, are those unique solutions to (2.4) satisfying the respective asymptotic conditions

$$\psi(\lambda,x,t) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \to +\infty; \qquad \phi(\lambda,x,t) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty. \tag{6.5}$$

The transmission coefficient T, the left reflection coefficient L, and the right reflection coefficient R are obtained via the asymptotics

$$\psi(\lambda,x,t) = \begin{bmatrix} \dfrac{L(\lambda,t)\, e^{-i\lambda x}}{T(\lambda,t)} \\[2ex] \dfrac{e^{i\lambda x}}{T(\lambda,t)} \end{bmatrix} + o(1), \quad x \to -\infty; \qquad \phi(\lambda,x,t) = \begin{bmatrix} \dfrac{e^{-i\lambda x}}{T(\lambda,t)} \\[2ex] \dfrac{R(\lambda,t)\, e^{i\lambda x}}{T(\lambda,t)} \end{bmatrix} + o(1), \quad x \to +\infty. \tag{6.6}$$

The bound-state solutions to (2.4) occur at those λ values corresponding to the poles of T in $\mathbf{C}^+$. Let us use $\{\lambda_j\}_{j=1}^N$ to denote the set of such poles. It should be noted that such poles are not necessarily located on the positive imaginary axis. Furthermore, unlike the Schrödinger equation, the multiplicities of such poles may be greater than one. Let us assume that the pole λ_j has multiplicity n_j. Corresponding to the pole λ_j, one associates [4,20] n_j bound-state norming constants $c_{js}(t)$ for $s = 0, 1, \dots, n_j - 1$. We assume that, for each fixed t, the potential $u(x,t)$ in the Zakharov-Shabat system is uniquely determined by the scattering data $\{R, \{\lambda_j\}, \{c_{js}(t)\}\}$ and vice versa.

VII. Time Evolution of the Scattering Data

As the initial profile $u(x,0)$ evolves to $u(x,t)$ while satisfying the NPDE, the corresponding initial scattering data $S(\lambda,0)$ evolves to $S(\lambda,t)$. Since the scattering data can be obtained from the Jost solutions to the associated LODE, in order to determine the time evolution of the scattering data, we can analyze the time evolution of the Jost solutions with the help of the Lax method or the AKNS method.

Let us illustrate how to determine the time evolution of the scattering data in the Schrödinger equation with the help of the Lax method. As indicated in Section IV, the spectral parameter k and hence also the values κ_j related to the bound states remain unchanged in time. Let us obtain the time evolution of $f_{\mathrm{l}}(k,x,t)$, the Jost solution from the left. From condition (ii) in Section IV, we see that the quantity $\partial_t f_{\mathrm{l}} - \mathcal{A} f_{\mathrm{l}}$ remains a solution to (2.2) and hence we can write it as a linear combination of the two linearly independent Jost solutions f_{l} and f_{r} as

$$\partial_t f_{\mathrm{l}}(k,x,t) - \left(-4\partial_x^3 + 6u\partial_x + 3u_x\right) f_{\mathrm{l}}(k,x,t) = p(k,t)\, f_{\mathrm{l}}(k,x,t) + q(k,t)\, f_{\mathrm{r}}(k,x,t), \tag{7.1}$$

where the coefficients $p(k,t)$ and $q(k,t)$ are yet to be determined and $\mathcal{A}$ is the operator in (4.7). For each fixed t, assuming $u(x,t) = o(1)$ and $u_x(x,t) = o(1)$ as $x \to +\infty$ and using (6.1) and (6.2) in (7.1) as $x \to +\infty$, we get

$$\partial_t e^{ikx} + 4\partial_x^3 e^{ikx} = p(k,t)\, e^{ikx} + q(k,t) \left[\frac{1}{T(k,t)}\, e^{-ikx} + \frac{R(k,t)}{T(k,t)}\, e^{ikx} \right] + o(1). \tag{7.2}$$

Comparing the coefficients of e^{ikx} and e^{-ikx} on the two sides of (7.2), we obtain

$$q(k,t) = 0, \quad p(k,t) = -4ik^3.$$

Thus, $f_{\mathrm{l}}(k,x,t)$ evolves in time by obeying the linear third-order PDE

$$\partial_t f_{\mathrm{l}} - \mathcal{A} f_{\mathrm{l}} = -4ik^3 f_{\mathrm{l}}. \tag{7.3}$$

Proceeding in a similar manner, we find that $f_{\mathrm{r}}(k,x,t)$ evolves in time according to

$$\partial_t f_{\mathrm{r}} - \mathcal{A} f_{\mathrm{r}} = 4ik^3 f_{\mathrm{r}}. \tag{7.4}$$

Notice that the time evolution of each Jost solution is fairly complicated. We will see, however, that the time evolution of the scattering data is very simple. Letting $x \to -\infty$ in (7.3), using (6.2) and $u(x,t) = o(1)$ and $u_x(x,t) = o(1)$ as $x \to -\infty$, and comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$\partial_t T(k,t) = 0, \quad \partial_t L(k,t) = -8ik^3 L(k,t),$$

yielding

$$T(k,t) = T(k,0), \quad L(k,t) = L(k,0)\, e^{-8ik^3 t}.$$

In a similar way, from (7.4) as $x \to +\infty$, we get

$$R(k,t) = R(k,0)\, e^{8ik^3 t}. \tag{7.5}$$

Thus, the transmission coefficient remains unchanged and only the phases of the reflection coefficients change as time progresses.

Let us also evaluate the time evolution of the dependency constants $\gamma_j(t)$ defined in (6.4). Evaluating (7.3) at $k = i\kappa_j$ and replacing $f_{\mathrm{l}}(i\kappa_j,x,t)$ by $\gamma_j(t) f_{\mathrm{r}}(i\kappa_j,x,t)$, we get

$$f_{\mathrm{r}}(i\kappa_j,x,t)\,\partial_t \gamma_j(t) + \gamma_j(t)\,\partial_t f_{\mathrm{r}}(i\kappa_j,x,t) - \gamma_j(t) \mathcal{A} f_{\mathrm{r}}(i\kappa_j,x,t) = -4\kappa_j^3 \gamma_j(t) f_{\mathrm{r}}(i\kappa_j,x,t). \tag{7.6}$$

On the other hand, evaluating (7.4) at $k = i\kappa_j$, we obtain

$$\gamma_j(t)\,\partial_t f_{\mathrm{r}}(i\kappa_j,x,t) - \gamma_j(t)\,\mathcal{A} f_{\mathrm{r}}(i\kappa_j,x,t) = 4\kappa_j^3 \gamma_j(t)\, f_{\mathrm{r}}(i\kappa_j,x,t). \tag{7.7}$$

Comparing (7.6) and (7.7) we see that $\partial_t \gamma_j(t) = -8\kappa_j^3 \gamma_j(t)$, or equivalently

$$\gamma_j(t) = \gamma_j(0)\, e^{-8\kappa_j^3 t}. \tag{7.8}$$

Then, with the help of (6.3) and (7.8), we determine the time evolutions of the norming constants as

$$c_{\mathrm{l}j}(t) = c_{\mathrm{l}j}(0)\, e^{4\kappa_j^3 t}, \quad c_{\mathrm{r}j}(t) = c_{\mathrm{r}j}(0)\, e^{-4\kappa_j^3 t}.$$

The norming constants $c_j(t)$ appearing in the Marchenko kernel (8.1) are related to $c_{\mathrm{l}j}(t)$ as $c_j(t) := c_{\mathrm{l}j}(t)^2$, and hence their time evolution is described as

$$c_j(t) = c_j(0)\, e^{8\kappa_j^3 t}. \tag{7.9}$$

As for the NLS equation and other integrable NPDEs, the time evolution of the related scattering data sets can be obtained in a similar way. For the former, in terms of the operator $\mathcal{A}$ in (4.8), the Jost solutions $\psi(\lambda, x, t)$ and $\phi(\lambda, x, t)$ appearing in (6.5) evolve according to the respective linear PDEs

$$\psi_t - \mathcal{A}\psi = -2i\lambda^2\psi, \quad \phi_t - \mathcal{A}\phi = 2i\lambda^2\phi.$$

The scattering coefficients appearing in (6.6) evolve according to

$$T(\lambda, t) = T(\lambda, 0), \quad R(\lambda, t) = R(\lambda, 0)\, e^{4i\lambda^2 t}, \quad L(\lambda, t) = L(\lambda, 0)\, e^{-4i\lambda^2 t}. \tag{7.10}$$

Associated with the bound-state pole λ_j of T, we have the bound-state norming constants $c_{js}(t)$ appearing in the Marchenko kernel $\Omega(y, t)$ given in (8.4). Their time evolution is governed [4] by

$$[\, c_{j(n_j-1)}(t) \quad c_{j(n_j-2)}(t) \quad \dots \quad c_{j0}(t)\,] = [\, c_{j(n_j-1)}(0) \quad c_{j(n_j-2)}(0) \quad \dots \quad c_{j0}(0)\,]\, e^{-4iA_j^2 t}, \tag{7.11}$$

where the $n_j \times n_j$ matrix A_j appearing in the exponent is defined as

$$A_j := \begin{bmatrix} -i\lambda_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -i\lambda_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -i\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -i\lambda_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -i\lambda_j \end{bmatrix}.$$

VIII. Inverse Scattering Problem

In Section VI we have seen how the initial scattering data $S(\lambda, 0)$ can be constructed from the initial profile $u(x, 0)$ of the potential by solving the direct scattering problem for the relevant LODE. Then, in Section VII we have seen how to obtain the time-evolved scattering data $S(\lambda, t)$ from the initial scattering data $S(\lambda, 0)$. As the final step in the IST, in this section we outline how to obtain $u(x, t)$ from $S(\lambda, t)$ by solving the relevant inverse scattering problem. Such an inverse scattering problem may be solved by the Marchenko method [5,7-9,16-19]. Unfortunately, in the literature many researchers refer to this method as the Gel'fand-Levitan method or the Gel'fand-Levitan-Marchenko method, both of which are misnomers. The Gel'fand-Levitan method [5,7,16,17,19] is a different method to solve the inverse scattering problem, and the corresponding Gel'fand-Levitan integral equation involves an integration on the finite interval $(0, x)$ and its kernel is related to the Fourier transform of the spectral measure associated with the LODE. On the other hand, the Marchenko integral equation involves an integration on the semi-infinite interval $(x, +\infty)$, and its kernel is related to the Fourier transform of the scattering data.

In this section we first outline the recovery of the solution $u(x, t)$ to the KdV equation from the corresponding time-evolved scattering data $\{R, \{\kappa_j\}, \{c_j(t)\}\}$ appearing in (7.5) and (7.9). Later, we will also outline the recovery of the solution $u(x, t)$ to the NLS equation from the corresponding time-evolved scattering data $\{R, \{\lambda_j\}, \{c_{js}(t)\}\}$ appearing in (7.10) and (7.11).

The solution $u(x, t)$ to the KdV equation (2.1) can be obtained from the time-evolved scattering data by using the Marchenko method as follows:

(a) From the scattering data $\{R(k, t), \{\kappa_j\}, \{c_j(t)\}\}$ appearing in (7.5) and (7.9), form the Marchenko kernel Ω defined via

$$\Omega(y, t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, R(k, t)\, e^{iky} + \sum_{j=1}^{N} c_j(t)\, e^{-\kappa_j y}. \tag{8.1}$$

(b) Solve the corresponding Marchenko integral equation

$$K(x,y,t)+\Omega(x+y,t)+\int_x^\infty dz\,K(x,z,t)\,\Omega(z+y,t)=0,\qquad x<y<+\infty, \tag{8.2}$$

and obtain its solution $K(x,y,t)$.

(c) Recover $u(x,t)$ by using

$$u(x,t)=-2\,\frac{\partial K(x,x,t)}{\partial x}. \tag{8.3}$$

The solution $u(x,t)$ to the NLS equation (2.3) can be obtained from the time-evolved scattering data by using the Marchenko method as follows:

(i) From the scattering data $\{R(\lambda,t),\{\lambda_j\},\{c_{js}(t)\}\}$ appearing in (7.10) and (7.11), form the Marchenko kernel Ω as

$$\Omega(y,t):=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,R(\lambda,t)\,e^{i\lambda y}+\sum_{j=1}^{N}\sum_{s=0}^{n_j-1} c_{js}(t)\,\frac{y^s}{s!}\,e^{i\lambda_j y}. \tag{8.4}$$

(ii) Solve the Marchenko integral equation

$$K(x,y,t)-\overline{\Omega(x+y,t)}+\int_x^\infty dz\int_x^\infty ds\,K(x,s,t)\,\Omega(s+z,t)\,\overline{\Omega(z+y,t)}=0,\qquad x<y<+\infty,$$

and obtain its solution $K(x,y,t)$.

(iii) Recover $u(x,t)$ from the solution $K(x,y,t)$ to the Marchenko equation via

$$u(x,t)=-2K(x,x,t).$$

(iv) Having determined $K(x,y,t)$, one can alternatively get $|u(x,t)|^2$ from

$$|u(x,t)|^2=2\,\frac{\partial G(x,x,t)}{\partial x},$$

where we have defined

$$G(x,y,t):=-\int_x^\infty dz\,\overline{K(x,z,t)}\,\overline{\Omega(z+y,t)}.$$

IX. Solitons

A soliton solution to an integrable NPDE is a solution $u(x,t)$ for which the reflection coefficient in the corresponding scattering data is zero. In other words, a soliton solution $u(x,t)$ to an integrable NPDE is nothing but a reflectionless potential in the associated LODE. When the reflection coefficient is zero, the kernel of the relevant Marchenko integral equation becomes separable. An integral equation with a separable kernel can be solved explicitly by transforming that linear equation into a system of linear algebraic equations. In that case, we get exact solutions to the integrable NPDE, which are known as soliton solutions.

For the KdV equation the N-soliton solution is obtained by using $R(k,t)=0$ in (8.1). In that case, letting

$$X(x):=\begin{bmatrix} e^{-\kappa_1 x} & e^{-\kappa_2 x} & \dots & e^{-\kappa_N x}\end{bmatrix},\quad Y(y,t):=\begin{bmatrix} c_1(t)\,e^{-\kappa_1 y}\\ c_2(t)\,e^{-\kappa_2 y}\\ \vdots \\ c_N(t)\,e^{-\kappa_N y}\end{bmatrix},$$

we get $\Omega(x+y,t) = X(x)\,Y(y,t)$. As a result of this separability the Marchenko integral equation can be solved algebraically and the solution has the form $K(x,y,t) = H(x,t)\,Y(y,t)$, where $H(x,t)$ is a row vector with N entries that are functions of x and t. A substitution in (8.2) yields

$$K(x,y,t) = -X(x)\,\Gamma(x,t)^{-1}Y(y,t), \tag{9.1}$$

where the $N \times N$ matrix $\Gamma(x,t)$ is given by

$$\Gamma(x,t) := I + \int_x^\infty dz\, Y(z,t)\,X(z), \tag{9.2}$$

with I denoting the $N \times N$ identity matrix. Equivalently, the (j,l)-entry of Γ is given by

$$\Gamma_{jl} = \delta_{jl} + \frac{c_j(0)\,e^{-2\kappa_j x + 8\kappa_j^3 t}}{\kappa_j + \kappa_l},$$

with δ_{jl} denoting the Kronecker delta. Using (9.1) in (8.3) we obtain

$$u(x,t) = 2\,\frac{\partial}{\partial x}\left[X(x)\,\Gamma(x,t)^{-1}Y(x,t)\right] = 2\,\mathrm{tr}\,\frac{\partial}{\partial x}\left[Y(x,t)\,X(x)\,\Gamma(x,t)^{-1}\right],$$

where tr denotes the matrix trace (the sum of diagonal entries in a square matrix). From (9.2) we see that $-Y(x,t)\,X(x)$ is equal to the x-derivative of $\Gamma(x,t)$ and hence the N-soliton solution can also be written as

$$u(x,t) = -2\,\mathrm{tr}\,\frac{\partial}{\partial x}\left[\frac{\partial \Gamma(x,t)}{\partial x}\,\Gamma(x,t)^{-1}\right] = -2\,\frac{\partial}{\partial x}\left[\frac{\frac{\partial}{\partial x}\det\Gamma(x,t)}{\det\Gamma(x,t)}\right], \tag{9.3}$$

where det denotes the matrix determinant. When $N = 1$, we can express the one-soliton solution $u(x,t)$ to the KdV equation in the equivalent form

$$u(x,t) = -2\,\kappa_1^2\,\mathrm{sech}^2\left(\kappa_1 x - 4\kappa_1^3 t + \theta\right),$$

with $\theta := \log\sqrt{2\kappa_1/c_1(0)}$.

Let us mention that, using matrix exponentials, we can express [6] the N-soliton solution appearing in (9.3) in various other equivalent forms such as

$$u(x,t) = -4Ce^{-Ax+8A^3t}\Gamma(x,t)^{-1}A\Gamma(x,t)^{-1}e^{-Ax}B,$$

where

$$A := \mathrm{diag}\{\kappa_1, \kappa_2, \dots, \kappa_N\},$$
$$B^\dagger := \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}, \quad C := \begin{bmatrix} c_1(0) & c_2(0) & \dots & c_N(0) \end{bmatrix}. \tag{9.4}$$

Note that a dagger is used for the matrix adjoint (transpose and complex conjugate), and B has N entries. In this notation we can express (9.2) as

$$\Gamma(x,t) = I + \int_x^\infty dz\, e^{-zA}BCe^{-zA}e^{8tA^3}.$$

As for the NLS equation, the well-known N-soliton solution (with simple bound-state poles) is obtained by choosing $R(\lambda,t) = 0$ and $n_j = 1$ in (8.4). Proceeding as in the KdV case, we obtain the N-soliton solution in terms of the triplet A, B, C with

$$A := \mathrm{diag}\{-i\lambda_1, -i\lambda_2, \dots, -i\lambda_N\}, \tag{9.5}$$

where the complex constants λ_j are the distinct poles of the transmission coefficient in $\mathbf{C}^+$, B and C are as in (9.4) except for the fact that the constants $c_j(0)$ are now allowed to be nonzero complex numbers. In terms of the matrices $P(x,t)$, M, and Q defined as

$$P(x,t) := \text{diag}\{e^{2i\lambda_1 x+4i\lambda_1^2 t}, e^{2i\lambda_2 x+4i\lambda_2^2 t}, \dots, e^{2i\lambda_N x+4i\lambda_N^2 t}\}, \quad M_{jl} := \frac{i}{\lambda_j - \overline{\lambda}_l}, \quad Q_{jl} := \frac{-i\overline{c}_j c_l}{\overline{\lambda}_j - \lambda_l}.$$

we construct the N-soliton solution $u(x,t)$ to the NLS equation as

$$u(x,t) = -2B^\dagger \left[I + P(x,t)^\dagger Q P(x,t)\, M\right]^{-1} P(x,t)^\dagger C^\dagger, \tag{9.6}$$

or equivalently as

$$u(x,t) = -2B^\dagger e^{-A^\dagger x}\Gamma(x,t)^{-1} e^{-A^\dagger x+4i(A^\dagger)^2 t} C^\dagger, \tag{9.7}$$

where we have defined

$$\Gamma(x,t) := I + \left[\int_x^\infty ds\, \left(Ce^{-As-4iA^2 t}\right)^\dagger \left(Ce^{-As-4iA^2 t}\right)\right]\left[\int_x^\infty dz\, \left(e^{-Az}B\right)\left(e^{-Az}B\right)^\dagger\right]. \tag{9.8}$$

Using (9.4) and (9.5) in (9.8), we get the (j,l)-entry of $\Gamma(x,t)$ as

$$\Gamma_{jl} = \delta_{jl} - \sum_{m=1}^N \frac{\overline{c}_j c_l\, e^{i(2\lambda_m - \overline{\lambda}_j - \overline{\lambda}_l)x + 4i(\lambda_m^2 - \overline{\lambda}_j^2)t}}{(\lambda_m - \overline{\lambda}_j)(\lambda_m - \overline{\lambda}_l)}.$$

Note that the absolute square of $u(x,t)$ is given by

$$|u(x,t)|^2 = \text{tr}\left[\frac{\partial}{\partial x}\left(\Gamma(x,t)^{-1}\frac{\partial \Gamma(x,t)}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\frac{\partial}{\partial x}\det\Gamma(x,t)}{\det\Gamma(x,t)}\right].$$

For the NLS equation, when $N=1$, from (9.6) or (9.7) we obtain the single-soliton solution

$$u(x,t) = \frac{-8\overline{c}_1(\text{Im}[\lambda_1])^2\, e^{-2i\overline{\lambda}_1 x - 4i(\overline{\lambda}_1)^2 t}}{4(\text{Im}[\lambda_1])^2 + |c_1|^2\, e^{-4x(\text{Im}[\lambda_1]) - 8t(\text{Im}[\lambda_1^2])}},$$

where Im denotes the imaginary part.

X. Future Directions

There are many issues related to the IST and solitons that cannot be discussed in such a short review. We will briefly mention only a few.

Can we characterize integrable NPDEs? In other words, can we find a set of necessary and sufficient conditions that guarantee that an IVP for a NPDE is solvable via an IST? Integrable NPDEs seem to have some common characteristic features [1] such as possessing Lax pairs, AKNS pairs, soliton solutions, infinite number of conserved quantities, a Hamiltonian formalism, the Painlevé property, and the Bäcklund transformation. Yet, there does not seem to be a satisfactory solution to their characterization problem.

Another interesting question is the determination of the LODE associated with an IST. In other words, given an integrable NPDE, can we determine the corresponding LODE? There does not yet seem to be a completely satisfactory answer to this question.

When the initial scattering coefficients are rational functions of the spectral parameter, representing the time-evolved scattering data in terms of matrix exponentials results in the separability of the kernel of the

Marchenko integral equation. In that case, one obtains explicit formulas [4,6] for exact solutions to some integrable NPDEs and such solutions are constructed in terms of a triplet of constant matrices A, B, C whose sizes are $p \times p$, $p \times 1$, and $1 \times p$, respectively, for any positive integer p. Some special cases of such solutions have been mentioned in Section IX, and it would be interesting to determine if such exact solutions can be constructed also when p becomes infinite.

XI. Bibliography

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