EXACT SOLUTIONS TO THE SINE-GORDON EQUATION

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Abstract: A systematic method is presented to provide various equivalent solution formulas for exact solutions to the sine-Gordon equation. Such solutions are analytic in the spatial variable x and the temporal variable t, and they are exponentially asymptotic to integer multiples of 2π as $x \to \pm\infty$. The solution formulas are expressed explicitly in terms of a real triplet of constant matrices. The method presented is generalizable to other integrable evolution equations where the inverse scattering transform is applied via the use of a Marchenko integral equation. By expressing the kernel of that Marchenko equation as a matrix exponential in terms of the matrix triplet and by exploiting the separability of that kernel, an exact solution formula to the Marchenko equation is derived, yielding various equivalent exact solution formulas for the sine-Gordon equation.

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1. INTRODUCTION

Our goal in this paper is to derive, in terms of a triplet of constant matrices, explicit formulas for exact solutions to the sine-Gordon equation

$$u_{xt} = \sin u, \tag{1.1}$$

where u is real valued and the subscripts denote the partial derivatives with respect to the spatial coordinate x and the temporal coordinate t. Under the transformation

$$x \mapsto ax + \frac{t}{a}, \qquad t \mapsto ax - \frac{t}{a},$$

where a is a positive constant, (1.1) is transformed into the alternate form

$$u_{xx} - u_{tt} = \sin u, \tag{1.2}$$

and hence our explicit formulas can easily be modified to obtain explicit solutions to (1.2) as well. Let us note that one can omit a multiple of 2π from any solution to (1.1). We are interested in solutions to (1.1) where $u_x(x,t) \to 0$ as $x \to \pm\infty$ for each fixed t, and hence without any loss of generality we will normalize our solutions so that $u(x,t) \to 0$ as $x \to +\infty$.

The sine-Gordon equation arises in applications as diverse as the description of surfaces of constant mean curvature [10,16], one-dimensional crystal dislocation theory [17,23,40,41], magnetic flux propagation in Josephson junctions (gaps between two superconductors) [29,31], condensation of charge density waves [11,22,36], wave propagation in ferromagnetic materials [19,27,30], excitation of phonon modes [35], and propagation of deformations along the DNA double helix [18,26,38,43].

The literature on exact solutions to (1.1) is large, and we will mention only a few and refer the reader to those references and further ones therein. For a positive constant a, by substituting

$$u(x,t) = 4\,\tan^{-1}\left(\frac{U(ax + a^{-1}t)}{V(ax - a^{-1}t)}\right), \tag{1.3}$$

into (1.1) and solving the resulting partial differential equations for U and V, Steuerwald [42] has catalogued many exact solutions to the sine-Gordon equation in terms of elliptic functions. Some of these solutions, including the one-soliton solution, two-soliton solutions modeling a soliton-soliton and soliton-antisoliton collision, and the breather solution, can be written in terms of elementary functions [25,37], while the n-soliton solutions can be expressed as in (1.3) where U and V are certain determinants [34,39,45]. The same separation technique can also be used to find exact solutions to the sine-Gordon equation on finite $(x+t)$-intervals [13]. Solutions to the sine-Gordon equation with initial data specified on invariant algebraic manifolds of conserved quantities can be written explicitly in terms of Jacobi theta functions [24]. The ordered exponential integrals appearing in such solutions can be evaluated explicitly [9,28]. Let us also mention that some exact solutions to the sine-Gordon equations can be obtained via the Darboux or Bäcklund transformations [21,37] from already known exact solutions.

The sine-Gordon equation was the fourth nonlinear partial differential equation whose initial-value problem was discovered [2,3] to be solvable by the inverse scattering transform method. This method associates (1.1) with the first-order system of ordinary differential equations

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi - \dfrac{1}{2}\, u_x(x,t)\,\eta, \\ \dfrac{d\eta}{dx} = \dfrac{1}{2}\, u_x(x,t)\,\xi + i\lambda\eta, \end{cases} \tag{1.4}$$

where u_x appears in the coefficients as a potential. By exploiting the one-to-one correspondence between u_x and the corresponding scattering data for (1.4), the inverse scattering transform method determines the time evolution $u(x,0) \mapsto u(x,t)$ for (1.1) with the help of the solutions to the direct and inverse scattering problems for (1.4). The direct scattering problem for (1.4) amounts to finding the scattering coefficients (related to the asymptotics of scattering solutions to (1.4) as $x \to \pm\infty$) when $u(x,t)$ is known for all x. On the other hand, the inverse scattering problem consists of finding $u(x,t)$ from an appropriate set of scattering data for (1.4).

In this paper we provide several, but equivalent, explicit formulas for exact solutions to (1.1). The key idea to obtain such explicit formulas is to express the kernel of a related Marchenko integral equation arising in the inverse scattering problem for (1.4) in terms of a real triplet (A, B, C) of constant matrices and by using matrix exponentials. Such explicit formulas provide a compact and concise way to express our exact solutions, which can equivalently be expressed in terms of exponential, trigonometric (sine and cosine), and polynomial functions of x and t. This can be done by "unpacking" matrix exponentials in our explicit formulas. As the matrix size increases, the unpacked expressions become very long. However, such expressions can be evaluated explicitly for any matrix size either by hand or by using a symbolic software package such as Mathematica. One of the powerful features of our method comes from the fact that our concise and compact explicit solution formulas are valid for any matrix size in the matrix exponentials involved. In some other available methods, exact solutions are attempted in terms of elementary functions without the use of matrix exponentials, and hence exact solutions produced by such other methods will be relatively simple and we cannot expect those methods to produce our solutions when the matrix size is large.

Our method is generalizable and applicable to obtain similar explicit formulas for exact solutions to other integrable nonlinear partial differential equations, where a Marchenko integral equation is used to solve a related inverse scattering problem. We refer the reader to [5-7,14,15], where similar ideas are used to obtain explicit formulas for exact solutions to the Korteweg-de Vries equation on the half line and to the focusing nonlinear Schrödinger equation and its matrix generalizations.

In our method, with the help of the matrix triplet and matrix exponentials, we easily establish the separability of the kernel of the relevant Marchenko integral equation and thus solve it exactly by using linear algebra. We then obtain our exact solutions to the sine-Gordon equation by a simple integration of the solution to the Marchenko equation.

Our method easily handles complications arising from the presence of non-simple poles of the transmission coefficient in the related linear system (1.4). Dealing with non-simple poles without the use of matrix exponentials is very complicated, and this issue has also been a problem [33,44] in solving other integrable nonlinear partial differential equations such as the nonlinear Schrödinger equation.

Our paper is organized as follows. In Section 2 we establish our notation, introduce the relevant Marchenko integral equation, and mention how a solution to the sine-Gordon equation is obtained from the solution to the Marchenko equation by using the inverse scattering transform method. In Section 3 we outline the solution to the Marchenko integral equation when its kernel is represented in terms of a triplet of matrices (A, B, C) and thus we derive two solution formulas for exact solutions to the sine-Gordon equation. In Sections 4 and 5 we show that our explicit solution formulas hold when the input matrix triplets come from a larger family; we show that our solution formulas in the more general case can be obtained by constructing two auxiliary constant matrices Q and N satisfying the respective Lyapunov equations given in Section 4, or equivalently by constructing an auxiliary constant matrix P satisfying the Sylvester equation given in Section 5. In Section 4 we also show that the matrix triplet (A, B, C) used as input to construct our exact solutions to the sine-Gordon equation can be chosen in various equivalent ways and we prove that our exact solutions are analytic on the xt-plane. In Section 5 we also explore the relationship between the Lyapunov equations and the Sylvester equation and show how their solutions are related to each other in a simple but interesting way. In that section we also show that the two solution formulas derived in Section 3 are equivalent. In Section 6 we show that those two equivalent solution formulas can be represented in other equivalent forms. In Section 7 we evaluate the square of the spatial derivative of our solutions to (1.1) by providing some explicit formulas in terms of the matrix triplet (A, B, C), and we evaluate the asymptotics of our exact solutions as $x \to -\infty$ for each fixed t. In Section 8 we show that the reflection coefficients associated with such solutions are zero, and we also evaluate explicitly the corresponding transmission coefficient. Finally, in Section 9 we provide some specific examples of our exact solutions and their snapshots.

Let us remark on the logarithm and inverse tangent functions we use throughout our paper. The log function we use is the principal branch of the complex-valued logarithm function and it has its branch cut along the negative real axis while $\log(1) = 0$. The $\tan^{-1}$ function we use is the single-valued branch related to the principal branch of the logarithm as

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right), \qquad \log z = 2i \tan^{-1}\left(\frac{i(1-z)}{1+z}\right), \tag{1.5}$$

and its branch cut is $(-i\infty, -i] \cup [i, +i\infty)$. For any square matrix M not having eigenvalues on that branch cut, we define

$$\tan^{-1} M := \frac{1}{2\pi i} \int_\gamma dz \, [\tan^{-1} z](zI - M)^{-1}, \tag{1.6}$$

where the contour γ encircles each eigenvalue of M exactly once in the positive direction and avoids the branch cut of $\tan^{-1} z$. If all eigenvalues of M have modulus less than 1, we then have the familiar series expansion

$$\tan^{-1} M := M - \frac{1}{3}M^3 + \frac{1}{5}M^5 - \frac{1}{7}M^7 + \dots.$$

For real-valued $h(x)$ that vanishes as $x \to +\infty$, the function $\tan^{-1}(h(x))$ always has range $(-\pi/2, \pi/2)$ when x values are restricted to $(x_0, +\infty)$ for some large x_0 value; our $\tan^{-1}(h(x))$ is the continuous extension of that piece from $x \in (x_0, +\infty)$ to $x \in (-\infty, +\infty)$.

2. PRELIMINARIES

In this section we briefly review the scattering and inverse scattering theory for (1.4) by introducing the scattering coefficients and a Marchenko integral equation associated with (1.4). We assume that u is real valued and that u_x is integrable in x for each fixed t. We also mention how a solution to the sine-Gordon equation is obtained from the solution to the Marchenko equation. We refer the reader to the generic references such as [1,4,25,32] for the details.

Two linearly independent solutions to (1.4) known as the Jost solutions from the left and from the right, denoted by $\psi(\lambda, x, t)$ and $\phi(\lambda, x, t)$, respectively, are those solutions satisfying the respective spatial asymptotics

$$\psi(\lambda, x, t) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.1}$$

$$\phi(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty.$$

The scattering coefficients for (1.4), i.e. the transmission coefficient T, the right reflection coefficient R, and the left reflection coefficient L, can be defined through the spatial asymptotics

$$\psi(\lambda, x, t) = \begin{bmatrix} \dfrac{L(\lambda, t)\, e^{-i\lambda x}}{T(\lambda)} \\ \dfrac{e^{i\lambda x}}{T(\lambda)} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{2.2}$$

$$\phi(\lambda, x, t) = \begin{bmatrix} \dfrac{e^{-i\lambda x}}{T(\lambda)} \\ \dfrac{R(\lambda, t)\, e^{i\lambda x}}{T(\lambda)} \end{bmatrix} + o(1), \qquad x \to +\infty,$$

where T does not depend on t, and R and L depend on t as

$$R(\lambda, t) = R(\lambda, 0)\, e^{-it/(2\lambda)}, \qquad L(\lambda, t) = L(\lambda, 0)\, e^{it/(2\lambda)}.$$

We recall that a bound state corresponds to a square-integrable solution to (1.4) and such solutions can only occur at the poles of the meromorphic extension of T to the upper half complex λ-plane denoted by $\mathbf{C}^+$. Because $u(x, t)$ is real valued, such poles can occur either on the positive imaginary axis, or for each pole not on the positive imaginary axis there corresponds a pole symmetrically located with respect to the imaginary axis. Furthermore, such poles are not necessarily simple. If u_x is integrable in x for each fixed t and if the transmission coefficient T is continuous for real values of λ, it can be proved by elementary means that the number of such poles and their multiplicities are finite.

With the convention $u(x,t) \to 0$ as $x \to +\infty$, it is known that $u(x,t)$ in (1.4) can be determined as

$$u(x,t) = -4 \int_x^\infty dr\, K(r,r,t), \tag{2.3}$$

or equivalently we have

$$u_x(x,t) = 4K(x,x,t),$$

where $K(x,y,t)$ is the solution to the Marchenko integral equation

$$K(x,y,t) - \Omega(x+y,t)^* + \int_x^\infty dv \int_x^\infty dr\, K(x,v,t)\, \Omega(v+r,t)\, \Omega(r+y,t)^* = 0, \qquad y > x, \tag{2.4}$$

where the asterisk is used to denote complex conjugation (without taking the matrix transpose) and

$$\Omega(y,t) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, R(\lambda,t)\, e^{i\lambda y} + \sum_{j=1}^n c_j\, e^{i\lambda_j y - it/(2\lambda_j)}, \tag{2.5}$$

provided the poles λ_j of the transmission coefficient are all simple.

The inverse scattering transform procedure can be summarized via the following diagram:

$$\begin{tikzcd}
u(x,0) \arrow[r] \arrow[d, "\text{sine-Gordon solution}"'] & u_x(x,0) \arrow[rr, "\text{direct scattering at } t=0"] & & \{R(\lambda,0),\{\lambda_j,c_j\}\} \arrow[d, "\text{time evolution}"] \\
u(x,t) & u_x(x,t) \arrow[l] & & \{R(\lambda,t),\{\lambda_j,c_j\, e^{-it/(2\lambda_j)}\}\} \arrow[ll, "\text{inverse scattering at } t"]
\end{tikzcd}$$

We note that in general the summation term in (2.5) is much more complicated, and the expression we have provided for it in (2.5) is valid only when the transmission coefficient T has simple poles at λ_j with $j = 1, \ldots, n$ on $\mathbf{C}^+$. In case of bound states with nonsimple poles, it is unknown to us if the norming constants with the appropriate time dependence have ever been presented in the literature. Extending our previous results for the nonlinear Schrödinger equation [6,7,12,14] to the sine-Gordon equation, it is possible to obtain the norming constants with appropriate dependence on the parameter t in the most general case, whether the bound-state poles occur on the positive imaginary axis or occur pairwise located symmetrically with respect to the positive imaginary axis, and whether any such poles are simple or have multiplicities. In fact, in Section 8 we present the norming constants and their proper time dependence on t as well as the most general form of the summation term that should appear in (2.5).

When u is real valued, it is known that for real λ we have

$$R(-\lambda,t)=R(\lambda,t)^*, \qquad L(-\lambda,t)=L(\lambda,t)^*, \qquad T(-\lambda)=T(\lambda)^*.$$

Because u is real valued, as we verify in Section 3, both the kernel $\Omega(y,t)$ and the solution $K(x,y,t)$ in (2.4) are also real valued, i.e.

$$\Omega(y,t)^*=\Omega(y,t), \tag{2.6}$$

$$K(x,y,t)^*=K(x,y,t). \tag{2.7}$$

3. EXPLICIT SOLUTIONS TO THE SINE-GORDON EQUATION

Our goal in this section is to obtain some exact solutions to the sine-Gordon equation in terms of a triplet of constant matrices. Following the main idea of [6,7] we will replace the summation term in (2.5) by a compact expression in terms of a matrix triplet (A,B,C), i.e. we will replace $\Omega(y,t)$ when $R=0$ by

$$\Omega(y,t)=Ce^{-Ay-A^{-1}t/2}B, \tag{3.1}$$

where A, B, C are real and constant matrices of sizes $p\times p$, $p\times 1$, and $1\times p$, respectively, for some positive integer p.

Recall that any rational function $f(\lambda)$ that vanishes as $\lambda \to \infty$ in the complex λ-plane has a matrix realization in terms of three constant matrices A, B, C as

$$f(\lambda) = -iC(\lambda I - iA)^{-1}B, \tag{3.2}$$

where I is the $p \times p$ identity matrix, A has size $p \times p$, B has size $p \times 1$, and C has size $1 \times p$ for some p. We will refer to (A, B, C) as a matrix triplet of size p. It is possible to pad A, B, C with zeros or it may be possible to change them and increase or decrease the value of p without changing $f(\lambda)$. The smallest positive integer p yielding $f(\lambda)$ gives us a "minimal" realization for $f(\lambda)$, and it is known [8] that a minimal realization is unique up to a similarity transformation. Thus, without any loss of generality we can always assume that our triplet (A, B, C) corresponds to a minimal realization, and we will refer to such a triplet as a minimal triplet. Note that the poles of $f(\lambda)$ correspond to the eigenvalues of the matrix (iA). By taking the Fourier transform of both sides of (3.2), where the Fourier transform is defined as

$$\hat{f}(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, f(\lambda)\, e^{i\lambda y},$$

we obtain

$$\hat{f}(y) = Ce^{-Ay}B. \tag{3.3}$$

We note that under the similarity transformation $(A, B, C) \mapsto (S^{-1}AS, S^{-1}B, CS)$ for some invertible matrix S, the quantities $f(\lambda)$ and $\hat{f}(y)$ remain unchanged.

Comparing (3.1) and (3.3) we see that they are closely related to each other. As mentioned earlier, without loss of any generality we assume that the real triplet (A, B, C) in (3.1) corresponds to a minimal realization in (3.2). For the time being, we will also assume that all eigenvalues of A in (3.1) have positive real parts. However, in later sections we will relax the latter assumption and choose our triplet in a less restrictive way, i.e. in the admissible class $\mathcal{A}$ defined in Section 4.

Let us use a dagger to denote the matrix adjoint (complex conjugation and matrix transpose). Although the adjoint and the transpose are equal to each other for real matrices, we will continue to use the dagger notation even for the matrix transpose of real matrices so that we can utilize the previous related results in [5,6] obtained for the Zakharov-Shabat system and the nonlinear Schrödinger equation. Since Ω appearing in (3.1) is a scalar we have $\Omega^\dagger = \Omega^*$; thus, we get

$$\Omega(y,t)^* = B^\dagger e^{-A^\dagger y-(A^\dagger)^{-1}t/2}C^\dagger.$$

We note that when Ω is given by (3.1), the Marchenko equation is exactly solvable by using linear algebra. This follows from the separability property of the kernel, i.e.

$$\Omega(x+y,t) = Ce^{-Ax}e^{-Ay-A^{-1}t/2}B, \tag{3.4}$$

indicating the separability in x and y; thus, (3.4) allows us to try a solution to (2.4) in the form

$$K(x,y,t) = H(x,t)\, e^{-A^\dagger y-(A^\dagger)^{-1}t/2}C^\dagger. \tag{3.5}$$

Using (3.5) in (2.4) we get

$$H(x,t)\,\Gamma(x,t) = B^\dagger e^{-A^\dagger x}, \tag{3.6}$$

or equivalently

$$H(x,t) = B^\dagger e^{-A^\dagger x}\Gamma(x,t)^{-1}, \tag{3.7}$$

where we have defined

$$\Gamma(x,t) := I + e^{-A^\dagger x-(A^\dagger)^{-1}t/2}Qe^{-2Ax-A^{-1}t/2}Ne^{-A^\dagger x}, \tag{3.8}$$

with the constant $p\times p$ matrices Q and N defined as

$$Q := \int_0^\infty ds\, e^{-A^\dagger s}C^\dagger Ce^{-As}, \qquad N := \int_0^\infty dr\, e^{-Ar}BB^\dagger e^{-A^\dagger r}. \tag{3.9}$$

It is seen from (3.9) that Q and N are selfadjoint, i.e.

$$Q = Q^\dagger, \qquad N = N^\dagger. \tag{3.10}$$

In fact, since the triplet (A, B, C) is real, the matrices Q and N are also real and hence they are symmetric matrices. Using (3.7) in (3.5) we obtain

$$K(x, y, t) = B^\dagger e^{-A^\dagger x} \Gamma(x, t)^{-1} \, e^{-A^\dagger y - (A^\dagger)^{-1} t/2} C^\dagger, \tag{3.11}$$

or equivalently

$$K(x, y, t) = B^\dagger F(x, t)^{-1} e^{-A^\dagger (y-x)} C^\dagger, \tag{3.12}$$

where we have defined

$$F(x, t) := e^{\beta^\dagger} + Q \, e^{-\beta} N, \tag{3.13}$$

with the quantity β defined as

$$\beta := 2Ax + \frac{1}{2} \, A^{-1} t. \tag{3.14}$$

From (2.3) and (3.12) we see that

$$u(x, t) = -4 \int_x^\infty dr \, B^\dagger F(r, t)^{-1} C^\dagger. \tag{3.15}$$

The procedure described in (3.4)-(3.15) is exactly the same procedure used in [5,6] with the only difference of using $A^{-1}t/2$ in the matrix exponential in (3.15) instead of $4iA^2 t$ used in [5,6]. However, such a difference does not affect the solution to the Marchenko integral equation at all thanks to the fact that A and A^{-1} commute with each other. In fact, the solution to the Marchenko equation is obtained the same way if one replaces $A^{-1}/2$ by any function of the matrix A because such a matrix function commutes with A.

We will later prove that $F(x, t)$ given in (3.13) is invertible on the entire xt-plane and that $F(x, t)^{-1} \to 0$ exponentially as $x \to \pm\infty$ and hence $u(x, t)$ given in (3.15) is well defined on the entire xt-plane. We note that, as a result of (2.6), the solution $K(x, y, t)$ to the Marchenko equation (2.4) is real and hence (2.7) is satisfied. Hence, from (2.3) we see that $u(x, t)$ is real valued, and by taking the adjoint of both sides of (3.15) we get

$$u(x, t) = -4 \int_x^\infty dr \, C [F(r, t)^\dagger]^{-1} B. \tag{3.16}$$

Instead of using (2.6) at the last stage, let us instead use it from the very beginning when we solve the Marchenko equation (2.4). Replacing Ω^* by Ω in the two occurrences in (2.4), we can solve (2.4) in a similar way as in (3.4)-(3.15) and obtain

$$K(x,y,t) = CE(x,t)^{-1}e^{-A(y-x)}B, \tag{3.17}$$

where we have defined

$$E(x,t) := e^{\beta} + P\,e^{-\beta}P, \tag{3.18}$$

with β as in (3.14) and the constant matrix P given by

$$P := \int_0^\infty ds\, e^{-As}BCe^{-As}. \tag{3.19}$$

Thus, from (2.3) and (3.17) we obtain

$$u(x,t) = -4\int_x^\infty dr\, CE(r,t)^{-1}B. \tag{3.20}$$

We will show in Section 5 that the two explicit solutions to the sine-Gordon equation given by (3.15) and (3.20) are identical by proving that

$$E(x,t) = F(x,t)^\dagger. \tag{3.21}$$

4. EXACT SOLUTIONS USING THE LYAPUNOV EQUATIONS

In Section 3 we have derived (3.15) and (3.20) by assuming that we start with a real minimal triplet (A,B,C) where the eigenvalues of A have positive real parts. In this section we show that the explicit formula (3.15) for exact solutions to the sine-Gordon equation remains valid if the matrix triplet (A,B,C) used to construct such solutions is chosen in a larger class. Starting with a more arbitrary triplet we will construct the matrix F given in (3.13), where the auxiliary matrices Q and N are no longer given by (3.9) but obtained by uniquely solving the respective Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \tag{4.1}$$

$$AN + NA^\dagger = BB^\dagger. \tag{4.2}$$

Many of the proofs in this section are similar to those obtained earlier for the nonlinear Schrödinger equation [5,6] and hence we will refer the reader to those references for the details of some of the proofs.

Definition 4.1 *We say that the triplet (A, B, C) of size p belongs to the admissible class $\mathcal{A}$ if the following conditions are met:*

(i) *The matrices A, B, and C are all real valued.*

(ii) *The triplet (A, B, C) corresponds to a minimal realization for $f(\lambda)$ when that triplet is used on the right hand side of (3.2).*

(iii) *None of the eigenvalues of A are purely imaginary and no two eigenvalues of A can occur symmetrically with respect to the imaginary axis in the complex λ-plane.*

We note that, since A is real valued, the condition stated in (iii) is equivalent to the condition that zero is not an eigenvalue of A and that no two eigenvalues of A are located symmetrically with respect to the origin in the complex plane. Equivalently, (iii) can be stated as A and $(-A)$ not having any common eigenvalues. We will say that a triplet is admissible if it belongs to the admissible class $\mathcal{A}$.

Starting with a triplet (A, B, C) in the admissible class $\mathcal{A}$, we will obtain exact solutions to the sine-Gordon equation as follows:

(a) Using A, B, C as input, construct the auxiliary matrices Q and N by solving the respective Lyapunov equations (4.1) and (4.2). As the next theorem shows, the solutions to (4.1) and (4.2) are unique and can be obtained as

$$Q = \frac{1}{2\pi} \int_\gamma d\lambda \, (\lambda I + iA^\dagger)^{-1} C^\dagger C (\lambda I - iA)^{-1}, \tag{4.3}$$

$$N = \frac{1}{2\pi} \int_\gamma d\lambda \, (\lambda I - iA)^{-1} BB^\dagger (\lambda I + iA^\dagger)^{-1}, \tag{4.4}$$

where γ is any positively oriented simple closed contour enclosing all eigenvalues of (iA) and leaving out all eigenvalues of $(-iA^\dagger)$. If all eigenvalues of A have positive real parts, then Q and N can also be evaluated as in (3.9).

(b) Using the auxiliary matrices Q and N and the triplet (A, B, C), form the matrix $F(x, t)$ as in (3.13) and obtain the scalar $u(x, t)$ as in (3.15), which becomes a solution to (1.1).

Theorem 4.2 *Consider any triplet (A, B, C) belonging to the admissible class $\mathcal{A}$ described in Definition 4.1. Then:*

(i) *The Lyapunov equations (4.1) and (4.2) are uniquely solvable, and their solutions are given by (4.3) and (4.4), respectively.*

(ii) *The constant matrices Q and N given in (4.3) and (4.4), respectively, are selfadjoint; i.e. $Q^\dagger = Q$ and $N^\dagger = N$. In fact, since the triplet (A, B, C) is real, the matrices Q and N are also real. Furthermore, both Q and N are invertible.*

(iii) *The resulting matrix $F(x, t)$ formed as in (3.13) is real valued and invertible on the entire xt-plane, and the function $u(x, t)$ defined in (3.15) is a solution to the sine-Gordon equation everywhere on the xt-plane. Moreover, $u(x, t)$ is analytic on the entire xt-plane and $u_x(x, t)$ decays to zero exponentially as $x \to \pm\infty$ at each fixed $t \in \mathbf{R}$.*

PROOF: The proof of (i) follows from Theorem 4.1 of Section 4.1 of [20]. It is directly seen from (4.1) that $Q^\dagger$ is also a solution whenever Q is a solution, and hence the uniqueness of the solution assures $Q = Q^\dagger$. Similarly, as a result of the realness of the triplet (A, B, C), one can show that Q^* is also a solution to (4.1) and hence $Q = Q^*$. The selfadjointness and realness of N are established the same way. The invertibility of Q and N is a result of the minimality of the triplet (A, B, C) and a proof can be found in the proofs of Theorems 3.2 and 3.3 of [5] by replacing (2.2) of [5] with (3.13) in the current paper, completing the proof of (ii). From (3.13) and (3.14) it is seen that the realness of the triplet (A, B, C) and of Q and N implies the realness of F. The proof of the invertibility of F is similar to the proof of Proposition 4.1 (a) of [5] and the rest of the proof of (iii) is obtained as in Theorem 3.2 (d) and (e) of [5]. ∎

We will say that two triplets (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are equivalent if they lead to the same $u(x, t)$ given in (3.15). The next result shows that two admissible triplets are closely related to each other and can always be transformed into each other.

Theorem 4.3 *For any admissible triplet $(\tilde{A}, \tilde{B}, \tilde{C})$, there corresponds an equivalent admissible triplet (A, B, C) in such a way that all eigenvalues of A have positive real parts.*

PROOF: The proof is similar to the proof of Theorem 3.2 of [5], where the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ is expressed explicitly when one starts with the triplet (A, B, C). Below we provide the explicit formulas of constructing (A, B, C) by starting with $(\tilde{A}, \tilde{B}, \tilde{C})$; i.e., by providing the inverse transformation formulas for those given in [5]. Without loss of any generality, we can assume that $(\tilde{A}, \tilde{B}, \tilde{C})$ has the form

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \qquad \tilde{C} = [\, \tilde{C}_1 \quad \tilde{C}_2 \,],$$

where all eigenvalues of $\tilde{A}_1$ have positive real parts and all eigenvalues of $\tilde{A}_2$ have negative real parts, and for some $0 \le q \le p$, the sizes of the matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{C}_1$, $\tilde{C}_2$ are $q \times q$, $(p-q) \times (p-q)$, $q \times 1$, $(p-q) \times 1$, $1 \times q$, and $1 \times (p-q)$, respectively. We first construct the matrices $\tilde{Q}$ and $\tilde{N}$ by solving the respective Lyapunov equations

$$\begin{cases} \tilde{Q}\tilde{A} + \tilde{A}^\dagger \tilde{Q} = \tilde{C}^\dagger \tilde{C}, \\ \tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger. \end{cases}$$

Writing $\tilde{Q}$ and $\tilde{N}$ in block matrix forms of appropriate sizes as

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ \tilde{Q}_3 & \tilde{Q}_4 \end{bmatrix}, \qquad \tilde{N} = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_3 & \tilde{N}_4 \end{bmatrix}, \tag{4.5}$$

and, for appropriate block matrix sizes, by letting

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = [\, C_1 \quad C_2 \,], \tag{4.6}$$

we obtain

$$A_1 = \tilde{A}_1, \qquad A_2 = -\tilde{A}_2^\dagger, \qquad B_1 = \tilde{B}_1 - \tilde{N}_2 \tilde{N}_4^{-1} \tilde{B}_2, \qquad B_2 = \tilde{N}_4^{-1} \tilde{B}_2, \tag{4.7}$$

$$C_1 = \tilde{C}_1 - \tilde{C}_2 \tilde{Q}_4^{-1} \tilde{Q}_3, \qquad C_2 = \tilde{C}_2 \tilde{Q}_4^{-1}, \tag{4.8}$$

yielding (A, B, C) by starting with $(\tilde{A}, \tilde{B}, \tilde{C})$. ∎

When the triplet (A, B, C) is decomposed as in (4.6), let us decompose the corresponding solutions Q and N to the respective Lyapunov equations (4.1) and (4.2), in an analogous manner to (4.5), as

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}, \qquad N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}. \tag{4.9}$$

The relationship between (4.5) and (4.9) is summarized in the following theorem.

Theorem 4.4 *Under the transformation $(A, B, C) \mapsto (\tilde{A}, \tilde{B}, \tilde{C})$ specified in Theorem 4.3, the quantities Q, N, F, E appearing in (4.1), (4.2), (3.13), (3.18), respectively, are transformed as*

$$(Q, N, F, E) \mapsto (\tilde{Q}, \tilde{N}, \tilde{F}, \tilde{E}),$$

where

$$\tilde{Q} = \begin{bmatrix} Q_1 - Q_2 Q_4^{-1} Q_3 & -Q_2 Q_4^{-1} \\ -Q_4^{-1} Q_3 & -Q_4^{-1} \end{bmatrix}, \qquad N = \begin{bmatrix} N_1 - N_2 N_4^{-1} N_3 & -N_2 N_4^{-1} \\ -N_4^{-1} N_3 & -N_4^{-1} \end{bmatrix},$$

$$\tilde{F} = \begin{bmatrix} I & -Q_2 Q_4^{-1} \\ 0 & -Q_4^{-1} \end{bmatrix} F \begin{bmatrix} I & 0 \\ -N_4^{-1} N_3 & -N_4^{-1} \end{bmatrix}, \tag{4.10}$$

$$\tilde{E} = \begin{bmatrix} I & -N_2 N_4^{-1} \\ 0 & -N_4^{-1} \end{bmatrix} E \begin{bmatrix} I & 0 \\ -Q_4^{-1} Q_3 & -Q_4^{-1} \end{bmatrix}. \tag{4.11}$$

PROOF: The proof can be obtained in a similar manner to the proof of Theorem 3.2 of [5] by using

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2^\dagger \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} I & -N_2 N_4^{-1} \\ 0 & -N_4^{-1} \end{bmatrix} B, \qquad \tilde{C} = C \begin{bmatrix} I & 0 \\ -Q_4^{-1} Q_3 & -Q_4^{-1} \end{bmatrix},$$

corresponding to the transformation specified in (4.7) and (4.8). ∎

As the following theorem shows, for an admissible triplet (A, B, C), there is no loss of generality in assuming that all eigenvalues of A have positive real parts and B has a special form consisting of zeros and ones.

Theorem 4.5 *For any admissible triplet $(\tilde{A}, \tilde{B}, \tilde{C})$, there correspond a special admissible triplet (A, B, C), where A is in a Jordan canonical form with each Jordan block containing a distinct eigenvalue having a positive real part, the entries of B consist of zeros and ones, and C has constant real entries. More specifically, for some appropriate positive integer m we have*

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \qquad C = [\, C_1 \quad C_2 \quad \cdots \quad C_m \,], \tag{4.12}$$

where in the case of a real (positive) eigenvalue ω_j of A_j the corresponding blocks are given by

$$C_j := [\, c_{jn_j} \quad \cdots \quad c_{j2} \quad c_{j1} \,], \tag{4.13}$$

$$A_j := \begin{bmatrix} \omega_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & \omega_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_j \end{bmatrix}, \qquad B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{4.14}$$

with A_j having size $n_j \times n_j$, B_j size $n_j \times 1$, C_j size $1 \times n_j$, and the constant c_{jn_j} is nonzero. In the case of complex eigenvalues, which must appear in pairs as $\alpha_j \pm i\beta_j$ with $\alpha_j > 0$, the corresponding blocks are given by

$$C_j := [\, \gamma_{jn_j} \quad \epsilon_{jn_j} \quad \ldots \quad \gamma_{j1} \quad \epsilon_{j1} \,], \tag{4.15}$$

$$A_j := \begin{bmatrix} \Lambda_j & -I_2 & 0 & \ldots & 0 & 0 \\ 0 & \Lambda_j & -I_2 & \ldots & 0 & 0 \\ 0 & 0 & \Lambda_j & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \Lambda_j & -I_2 \\ 0 & 0 & 0 & \ldots & 0 & \Lambda_j \end{bmatrix}, \quad B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{4.16}$$

where γ_{js} and ϵ_{js} for $s = 1, \dots, n_j$ are real constants with $(\gamma_{jn_j}^2 + \epsilon_{jn_j}^2) > 0$, I_2 denotes the 2×2 unit matrix, each column vector B_j has $2n_j$ components, each A_j has size $2n_j \times 2n_j$, and each 2×2 matrix Λ_j is defined as

$$\Lambda_j := \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}. \tag{4.17}$$

PROOF: The real triplet (A, B, C) can be chosen as described in Section 3 of [7]. ∎

5. EXACT SOLUTIONS USING THE SYLVESTER EQUATION

In Section 3, starting from a minimal triplet (A, B, C) with all eigenvalues of A having positive real parts, we have obtained the exact solution formula (3.20) to the sine-Gordon equation by constructing the matrix $E(x, t)$ in (3.18) with the help of the auxiliary matrix P in (3.19). In this section we show that the explicit formula (3.20) for exact solutions to the sine-Gordon equation remains valid if the matrix triplet (A, B, C) used to construct such solutions comes from a larger class, namely from the admissible class $\mathcal{A}$ specified in Definition 4.1.

Starting with any triplet (A, B, C) in the admissible class $\mathcal{A}$, we obtain exact solutions to the sine-Gordon equation as follows:

(a) Using A, B, C as input, construct the auxiliary matrix P by solving the Sylvester equation

$$AP + PA = BC. \tag{5.1}$$

The unique solution to (5.1) can be obtained as

$$P = \frac{1}{2\pi} \int_\gamma d\lambda\, (\lambda I - iA)^{-1} BC (\lambda I + iA)^{-1}, \tag{5.2}$$

where γ is any positively oriented simple closed contour enclosing all eigenvalues of (iA) and leaving out all eigenvalues of $(-iA)$. If all eigenvalues of A have positive real parts, then P can be evaluated as in (3.19).

(b) Using the auxiliary matrix P and the triplet (A,B,C), form the matrix $E(x,t)$ as in (3.18) and then form the scalar $u(x,t)$ as in (3.20).

Theorem 5.1 *Consider any triplet (A,B,C) belonging to the admissible class $\mathcal{A}$ described in Definition 4.1. Then, the Sylvester equation (5.1) is uniquely solvable, and its solution is given by (5.2). Furthermore, that solution is real valued.*

PROOF: The unique solvability of (5.1) is already known [20]. For the benefit of the reader we outline the steps below. From (5.1) we get

$$-(\lambda I - iA)P + P(\lambda I + iA) = iBC,$$

or equivalently

$$-P(\lambda I + iA)^{-1} + (\lambda I - iA)^{-1}P = i(\lambda I - iA)^{-1}BC(\lambda I + iA)^{-1}. \tag{5.3}$$

Dividing both sides of (5.3) by (2π) and then integrating along γ, and using

$$\frac{1}{2\pi i}\int_\gamma d\lambda\,(\lambda I - iA)^{-1} = I, \qquad \frac{1}{2\pi i}\int_\gamma d\lambda\,(\lambda I + iA)^{-1} = 0,$$

we obtain (5.2) as the unique solution to (5.1). Since the admissible triplet (A,B,C) is real, by taking complex conjugate of both sides of (5.1) we see that P^* also solves (5.1). From the uniqueness of the solution to (5.1), it then follows the $P^* = P$. ∎

Next we show that, for any triplet (A,B,C) in our admissible class $\mathcal{A}$, there is a close relationship between the matrix P given in (5.2) and the matrices Q and N appearing in (4.3) and (4.4), respectively.

Theorem 5.2 *Let the triplet (A,B,C) of size p belong to the admissible class specified in Definition 4.1. Then the solution P to the Sylvester equation (5.1) and the solutions Q and N to the respective Lyapunov equations (4.1) and (4.2) satisfy*

$$NQ = P^2. \tag{5.4}$$

PROOF: Note that (5.4) is valid when the matrix A in the triplet is diagonal. To see this, note that the use of the triplet (A, B, C) with

$$A = \text{diag}\{a_1, \cdots, a_p\}, \qquad B = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}, \qquad C = [\, c_1 \quad \cdots \quad c_p \,],$$

in (4.1), (4.2), and (5.1) yields

$$P_{jk} = \frac{b_j c_k}{a_j + a_k}, \qquad Q_{jk} = \frac{c_j c_k}{a_j + a_k}, \qquad N_{jk} = \frac{b_j b_k}{a_j + a_k},$$

where the subscript jk denotes the (j, k) entry of the relevant matrix. Hence,

$$(NQ)_{jk} = \sum_{s=1}^{p} \frac{b_j b_s c_s c_k}{(a_j + a_s)(a_s + a_k)}, \quad (P^2)_{jk} = \sum_{s=1}^{p} \frac{b_j c_s b_s c_k}{(a_j + a_s)(a_s + a_k)},$$

establishing (5.4). Next, let us assume that A is not diagonal but diagonalizable through a real-valued invertible matrix S so that $\tilde{A} = S^{-1}AS$ and $\tilde{A}$ is diagonal. Then, under the transformation

$$(A, B, C) \mapsto (\tilde{A}, \tilde{B}, \tilde{C}) = (S^{-1}AS, S^{-1}B, CS),$$

we get

$$(Q, N, P) \mapsto (\tilde{Q}, \tilde{N}, \tilde{P}) = (S^{\dagger}QS, S^{-1}N(S^{\dagger})^{-1}, S^{-1}PS),$$

where $\tilde{Q}$, $\tilde{N}$, and $\tilde{P}$ satisfy (4.1), (4.2), and (5.1), respectively, when (A, B, C) is replaced with $(\tilde{A}, \tilde{B}, \tilde{C})$ in those three equations. We note that $(\tilde{A}, \tilde{B}, \tilde{C})$ is an admissible triplet when (A, B, C) is admissible because the eigenvalues of A and $\tilde{A}$ coincide. Since $\tilde{A}$ is diagonal, we already have $\tilde{N}\tilde{Q} = \tilde{P}^2$, which easily reduces to $NQ = P^2$ given in (5.4). In case A is not diagonalizable, we proceed as follows. There exists a sequence of admissible triplets (A_k, B, C) converging to (A, B, C) as $k \to +\infty$ such that each A_k is diagonalizable. Let the triplet (Q_k, N_k, P_k) correspond to the solutions to (4.1), (4.2), and (5.1), respectively, when (A, B, C) is replaced with (A_k, B, C) in those three equations. We then have $N_k Q_k = P_k^2$, and hence $(Q_k, N_k, P_k) \to (Q, N, P)$ yields (5.4). Note that we have used the stability of solutions to (4.1), (4.2), and (5.1). In fact, that stability directly follows from the unique solvability of the matrix equations (4.1), (4.2), (5.1) and the fact that their unique solvability is preserved under a small perturbation of A. ∎

Theorem 5.3 *Let the triplet* (A, B, C) *belong to the admissible class specified in Definition 4.1. Then, the solution* P *to the Sylvester equation (5.1) and the solutions* Q *and* N *to the respective Lyapunov equations (4.1) and (4.2) satisfy*

$$N(A^\dagger)^j Q = PA^j P, \qquad j = 0, \pm 1, \pm 2, \dots . \tag{5.5}$$

PROOF: Under the transformation

$$(A, B, C) \mapsto (\tilde{A}, \tilde{B}, \tilde{C}) = (A, A^j B, C),$$

we get

$$(Q, N, P) \mapsto (\tilde{Q}, \tilde{N}, \tilde{P}) = (Q, A^j N (A^\dagger)^j, A^j P),$$

where $\tilde{Q}$, $\tilde{N}$, and $\tilde{P}$ satisfy (4.1), (4.2), and (5.1), respectively, when (A, B, C) is replaced with $(\tilde{A}, \tilde{B}, \tilde{C})$ in those three equations. Since $(\tilde{A}, \tilde{B}, \tilde{C})$ is also admissible, (5.4) implies that $\tilde{N}\tilde{Q} = \tilde{P}^2$, which yields (5.5) after a minor simplification. ∎

Next, given any admissible triplet (A, B, C), we prove that the corresponding solution P to (5.1) is invertible and that the matrix $E(x, t)$ given in (3.18) is invertible and that (3.21) holds everywhere on the xt-plane.

Theorem 5.4 *Let the triplet* (A, B, C) *belong to the admissible class specified in Definition 4.1, and let the matrices* Q, N, P *be the corresponding solutions to (4.1), (4.2), and (5.2), respectively. Then:*

(i) *The matrix* P *is invertible.*

(ii) *The matrices* F *and* E *given in (3.13) and (3.18), respectively, are real valued and satisfy (3.21).*

(iii) *The matrix* $E(x, t)$ *is invertible on the entire* xt*-plane.*

PROOF: The invertibility of P follows from (5.4) and the fact that both Q and N are invertible, as stated in Theorem 4.2 (ii); thus, (i) is established. To prove (ii) we proceed as follows. The real-valuedness of F has already been established in Theorem 4.2 (iii). From (3.18) it is seen that the real-valuedness of the triplet (A, B, C) and of P implies that E is real valued. From (3.13), (3.14), and (3.18) we see that (3.21) holds if and only if we have

$$Ne^{-\beta^\dagger}Q = Pe^{-\beta}P, \tag{5.6}$$

where we have already used $N^\dagger = N$ and $Q^\dagger = Q$, as established in Theorem 4.2 (ii). Since (5.5) implies

$$N(-\beta^\dagger)^j Q = P(-\beta)^j P, \qquad j = 0, 1, 2, \ldots,$$

we see that (5.6) holds. Having established (3.21), the invertibility of $E(x,t)$ on the entire xt-plane follows from the invertibility of $F(x,t)$, which has been established in Theorem 4.2 (iii). ∎

Next, we show that the explicit formulas (3.15), (3.16), and (3.20) are all equivalent to each other.

Theorem 5.5 *Consider any triplet (A, B, C) belonging to the admissible class $\mathcal{A}$ described in Definition 4.1. Then:*

(i) *The explicit formulas (3.15), (3.16), and (3.20) yield equivalent exact solutions to the sine-Gordon equation (1.1) everywhere on the entire xt-plane.*

(ii) *The equivalent solution $u(x,t)$ given in (3.15), (3.16), and (3.20) is analytic on the entire xt-plane, and $u_x(x,t)$ decays to zero exponentially as $x \to \pm\infty$ at each fixed $t \in \mathbf{R}$.*

PROOF: Because $u(x,t)$ is real and scalar valued, we already have the equivalence of (3.15) and (3.16). The equivalence of (3.16) and (3.20) follows from (3.21). We then have (ii) as a consequence of Theorem 4.2 (iii). ∎

6. FURTHER EQUIVALENT FORMS FOR EXACT SOLUTIONS

In Theorem 5.5 we have shown that the exact solutions given by the explicit formulas (3.15), (3.16), and (3.20) are equivalent. In this section we show that our exact solutions can be written in various other equivalent forms. We first present two propositions that will be useful in later sections.

Proposition 6.1 *If (A, B, C) is admissible, then the quantities F^{-1} and E^{-1}, appearing in (3.13) and (3.18), respectively, vanish exponentially as $x \to \pm\infty$.*

PROOF: It is sufficient to give the proof when the eigenvalues of A have all positive real parts because, as seen from (4.10) and (4.11), the same result also holds when some or all eigenvalues of A have negative real parts. When the eigenvalues of A have positive real parts, from (3.13) we get

$$F^{-1} = e^{-\beta^\dagger/2}[I + e^{-\beta^\dagger/2}Qe^{-\beta}Ne^{-\beta^\dagger/2}]^{-1}e^{-\beta^\dagger/2}, \tag{6.1}$$

where the invertibility of Q and N is guaranteed by Theorem 4.2 (ii). Hence, (6.1) implies that $F^{-1} \to 0$ exponentially as $x \to +\infty$. From (3.21) and the realness of E and F we also get $E^{-1} \to 0$ exponentially as $x \to +\infty$. To obtain the asymptotics as $x \to -\infty$, we proceed as follows. From (3.13) we obtain

$$Q^{-1}FN^{-1} = e^{-\beta/2}[I + e^{\beta/2}Q^{-1}e^{\beta^\dagger}N^{-1}e^{\beta/2}]e^{-\beta/2},$$

and hence

$$F^{-1} = N^{-1}e^{\beta/2}[I + e^{\beta/2}Q^{-1}e^{\beta^\dagger}N^{-1}e^{\beta/2}]^{-1}e^{\beta/2}Q^{-1},$$

and thus $F^{-1} \to 0$ exponentially as $x \to -\infty$. From (3.21) and the realness of E and F we also get $E^{-1} \to 0$ exponentially as $x \to -\infty$. ∎

Proposition 6.2 *The quantity $E(x, t)$ defined in (3.18) satisfies*

$$E_x = 2AE - 2BCe^{-\beta}P, \qquad Ee^{-\beta}P = Pe^{-\beta}E, \qquad e^{\beta}P^{-1}E = P + e^{\beta}P^{-1}e^{\beta}. \tag{6.2}$$

If (A,B,C) is admissible and all eigenvalues of A have positive real parts, then $E^{-1}Pe^{-\beta} \to P^{-1}$ exponentially as $x \to -\infty$.

PROOF: We obtain the first equality (6.2) by taking the x-derivative of (3.18) and by using (5.1). The second equality can be verified directly by using (3.18) in it. The third equality is obtained by a direct premultiplication from (3.18). The limit as $x \to -\infty$ is seen from the last equality in (6.2) with the help of (3.14). ∎

Let us start with a triplet (A,B,C) of size p belonging to the admissible class specified in Definition 4.1. Letting

$$M(x,t) := e^{-\beta/2}Pe^{-\beta/2}, \tag{6.3}$$

where β as in (3.14) and P is the unique solution to the Sylvester equation (5.1), we can write (3.18) also as

$$E(x,t) = e^{\beta/2}\Lambda e^{\beta/2}, \tag{6.4}$$

where we have defined

$$\Lambda(x,t) := I + [M(x,t)]^2. \tag{6.5}$$

Using (5.1) in (6.3), we see that the x-derivative of $M(x,t)$ is given by

$$M_x(x,t) = -e^{-\beta/2}BCe^{-\beta/2}. \tag{6.6}$$

Proposition 6.3 *The eigenvalues of the matrix M defined in (6.3) cannot occur on the imaginary axis in the complex plane. Furthermore, the matrices $(I - iM)$ and $(I + iM)$ are invertible on the entire xt-plane.*

PROOF: From (6.4) and (6.5) we see that

$$(I - iM)(I + iM) = e^{-\beta/2}Ee^{-\beta/2},$$

and by Theorem 5.4 (iii) the matrix E is invertible on the entire xt-plane. Thus, both $(I - iM)$ and $(I + iM)$ are invertible, and consequently M cannot have eigenvalues $\pm i$. For any real, nonzero c, consider the transformation $(A,B,C)\mapsto(A,cB,cC)$ of an admissible triple (A,B,C). The resulting triple is also admissible, and as seen from (5.1) and (6.3) we have $(P,M,I+M^2)\mapsto(c^2P,c^2M,I+c^4M^2)$. Thus, M cannot have any purely imaginary eigenvalues. Since P is known to be invertible by Theorem 5.4 (i), as seen from (6.3) the matrix M is invertible on the entire xt-plane and hence cannot have zero as its eigenvalue. ∎

Theorem 6.4 *The solution to the sine-Gordon equation given in the equivalent forms (3.15), (3.16), and (3.20) can also be written as*

$$u(x,t)=-4\mathrm{Tr}[\tan^{-1}M(x,t)], \tag{6.7}$$

$$u(x,t)=2i\log\left(\frac{\det(I+iM(x,t))}{\det(I-iM(x,t))}\right), \tag{6.8}$$

$$u(x,t)=4\tan^{-1}\left(i\,\frac{\det(I+iM(x,t))-\det(I-iM(x,t))}{\det(I+iM(x,t))+\det(I-iM(x,t))}\right), \tag{6.9}$$

where M is the matrix defined in (6.3) and Tr denotes the matrix trace (the sum of diagonal entries).

PROOF: Let us note that the equivalence of (6.8) and (6.9) follows from the second equality in (1.5) by using $z=\det(I+iM)/\det(I-iM)$ there. To show the equivalence of (6.7) and (6.8), we use the matrix identity

$$\tan^{-1}M=\frac{1}{2i}\log\left((I+iM)(I-iM)^{-1}\right),$$

which is closely related to the first identity in (1.5), and the matrix identity

$$\mathrm{Tr}[\log z]=\log\det z,$$

with the invertible matrix $z=(I+iM)(I-iM)^{-1}$. Thus, we have established the equivalence of (6.7), (6.8), and (6.9). We will complete the proof by showing that (3.20) is equivalent to (6.7). Using the fact that for any $m\times n$ matrix α and any $n\times m$ matrix γ we have

$$\mathrm{Tr}[\alpha\gamma]=\mathrm{Tr}[\gamma\alpha], \tag{6.10}$$

from (6.4)-(6.6) we get

$$-4CE^{-1}B = 4\mathrm{Tr}[M_x(I+M^2)^{-1}]. \tag{6.11}$$

By Proposition 6.1 we know that E^{-1} vanishes exponentially as $x \to +\infty$. Hence, with the help of (6.11) we see that we can write (3.20) as

$$u(x,t) = 4\mathrm{Tr}\left[\int_x^\infty dr\, M_r(r,t)[I+M(r,t)\,M(r,t)]^{-1}\right],$$

which yields (6.7). ∎

Theorem 6.5 *The solution to the sine-Gordon equation given in the equivalent forms (3.15), (3.16), (3.20), (6.7)-(6.9) can also be written as*

$$u(x,t) = -4\sum_{j=1}^{p} \tan^{-1} \kappa_j(x,t), \tag{6.12}$$

where the scalar functions $\kappa_j(x,t)$ correspond to the eigenvalues of the matrix $M(x,t)$ defined in (6.3) and the repeated eigenvalues are allowed in the summation.

PROOF: At a fixed (x,t)-value, using the matrix identity

$$\mathrm{Tr}[M(x,t)^s] = \sum_{j=1}^{p} [\kappa_j(x,t)]^s, \qquad s = 1,2,3,\dots,$$

for large $|z|$ values in the complex z-plane we obtain

$$\mathrm{Tr}[(zI-M)^{-1}] = \sum_{s=0}^{\infty} z^{-s-1}\mathrm{Tr}[M^s] = \sum_{s=0}^{\infty}\sum_{j=1}^{p} z^{-s-1}\kappa_j^s = \sum_{j=1}^{p}(z-\kappa_j)^{-1}, \tag{6.13}$$

where we dropped the arguments of M and κ_j for simplicity. Choosing the contour γ as in (1.6) so that each eigenvalue $\kappa_j(x,t)$ is encircled exactly once in the positive direction, we can extend (6.13) to $z \in \gamma$ by an analytic continuation with respect to z. Using (6.12) in (1.6), we then obtain

$$\frac{1}{2\pi i}\int_\gamma dz\,[\tan^{-1} z]\,\mathrm{Tr}[(zI-M)^{-1}] = \sum_{j=1}^{p}\frac{1}{2\pi i}\int_\gamma dz\,[\tan^{-1} z]\,(z-\kappa_j)^{-1},$$

or equivalently

$$\operatorname{Tr}[\tan^{-1} M(x,t)] = \sum_{j=1}^{p} \tan^{-1} \kappa_j(x,t),$$

which yields (6.12) in view of (6.7). ∎

Let us note that the equivalence of (6.7)-(6.9), and (6.12) implies that one can replace M by its Jordan canonical form in any of those four expressions without changing the value of $u(x,t)$. This follows from the fact that $u(x,t)$ in (6.8) remains unchanged if M is replaced by its Jordan canonical form and is confirmed in (6.12) by the fact that the eigenvalues remain unchanged under a similarity transformation on a matrix.

The next result shows that we can write our explicit solution given in (6.12) yet another equivalent form, which is expressed in terms of the coefficients in the characteristic polynomial of the matrix $M(x,t)$ given in (6.8). Let that characteristic polynomial be given by

$$\det\left(zI - M(x,t)\right) = \prod_{j=1}^{p} [z - \kappa_j(x,t)] = \sum_{j=0}^{p} (-1)^j \sigma_j(x,t)\, z^{p-j},$$

where the coefficients $\sigma_j(x,t)$ can be written in terms of the eigenvalues $\kappa_j(x,t)$ as

$$\sigma_0 = 1, \qquad \sigma_1 = \sum_{j=1}^{p} \kappa_j, \qquad \sigma_2 = \sum_{1\le j<k\le p} \kappa_j \kappa_k, \qquad \dots \quad, \qquad \sigma_p = \kappa_1 \cdots \kappa_p, \tag{6.14}$$

where we have dropped the arguments and have written κ_j and σ_j for $\kappa_j(x,t)$ and $\sigma_j(x,t)$, respectively, for simplicity.

Theorem 6.6 *The solution to the sine-Gordon equation given in the equivalent forms (3.15), (3.16), (3.20), (6.7)-(6.9), and (6.12) can also be written as*

$$u(x,t) = -4\tan^{-1}\left(\frac{\displaystyle\sum_{s=0}^{\lfloor (p-1)/2 \rfloor} (-1)^s \sigma_{2s+1}(x,t)}{\displaystyle\sum_{s=0}^{\lfloor p/2 \rfloor} (-1)^s \sigma_{2s+1}(x,t)} \right), \tag{6.15}$$

where $\lfloor j \rfloor$ denotes the greatest integer function of j and the quantities $\sigma_j(x,t)$ are those given in (6.14).

PROOF: When $p=2$, by letting $\eta_j := \tan^{-1}\kappa_j(x,t)$ and using the addition formula for the tangent function, we obtain

$$\tan(\eta_1+\eta_2)=\frac{\tan\eta_1+\tan\eta_2}{1-(\tan\eta_1)(\tan\eta_2)}=\frac{\kappa_1+\kappa_2}{1-\kappa_1\kappa_2}=\frac{\sigma_1}{\sigma_0-\sigma_2}, \tag{6.16}$$

and hence the application of the inverse tangent function on both sides of (6.16) yields (6.15). For larger values of p, we proceed by induction with respect to p and by the further use of the addition formula for the tangent function. ∎

7. FURTHER PROPERTIES OF OUR EXACT SOLUTIONS

In this section we derive an explicit expression, in terms of a matrix triplet, for the square of the spatial derivative of our exact solutions to (1.1) and analyze further properties of such solutions.

Theorem 7.1 *If (A,B,C) is admissible, then the solution to the sine-Gordon equation given in the equivalent forms (3.15), (3.16), (3.20), (6.7)-(6.9), (6.12), and (6.15) satisfy*

$$[u_x(x,t)]^2=\mathrm{Tr}[(\Lambda^{-1}\Lambda_x)_x]=\mathrm{Tr}[(E^{-1}E_x)_x]=\mathrm{Tr}[(F^{-1}F_x)_x], \tag{7.1}$$

where Λ, E, and F are the quantities appearing in (6.5), (3.18), and (3.13), respectively. Consequently, we have

$$[u_x(x,t)]^2=\frac{\partial^2\log(\det\Lambda(x,t))}{\partial x^2}=\frac{\partial^2\log(\det E(x,t))}{\partial x^2}=\frac{\partial^2\log(\det F(x,t))}{\partial x^2}. \tag{7.2}$$

PROOF: Let us use the notation of Theorems 4.3 and 4.4 and use a tilde to denote the quantities associated with the triplet $(\tilde A,\tilde B,\tilde C)$, where some or all eigenvalues of $\tilde A$ have negative real parts. Because of the equivalence stated in Theorems 4.3 and 4.4, we can convert the starting triplet $(\tilde A,\tilde B,\tilde C)$ into an admissible triplet (A,B,C) where the matrix A has eigenvalues with positive real parts. We will first establish (7.1) and (7.2) for the quantities associated with the triplet (A,B,C) and then show that those formulas remain valid when we use $(\tilde{A}, \tilde{B}, \tilde{C})$ as the input triplet. We exploit the connection between (1.4) and the Zakharov-Shabat system given in (2.1) of [6], where $q = -iu_x/2$ and u is real valued. From (2.4) and (2.10) of [6] we see that

$$[u_x(x,t)]^2 = 8\frac{\partial G(x,x,t)}{\partial x}, \qquad \int_x^\infty dr\,[u_r(r,t)]^2 = -8G(x,x,t), \tag{7.3}$$

where we have

$$G(x,y,t) = -\int_x^\infty drK(x,r,t)^*\,\Omega(r+y,t)^*, \tag{7.4}$$

with $K(x,y,t)$ given in the equivalent forms (3.12) or (3.17), and $\Omega(r+y,t)$ given in (3.4). Since our triplet (A,B,C) is real, both K and Ω are real valued and we can ignore the complex conjugations in the integrand in (7.4). Thus, we get

$$G(x,y,t) = -CE(x,t)^{-1}\int_x^\infty dr\, e^{-A(r-x)}BCe^{-A(r+y)-A^{-1}t/2}B, \tag{7.5}$$

which is evaluated with the help of (3.19) as

$$G(x,y,t) = -CE(x,t)^{-1}Pe^{-\beta}e^{-A(y-x)}B, \tag{7.6}$$

where β is the quantity in (3.14). Omitting the arguments (x,t) and using (7.6) in (7.3) we get

$$u_x^2 = -8[CE^{-1}Pe^{-\beta}B]_x. \tag{7.7}$$

Using (6.3) and (6.4) in (7.7) we obtain

$$u_x^2 = -8[Ce^{-\beta/2}\Lambda^{-1}Me^{-\beta/2}B]_x, \tag{7.8}$$

where M is the quantity defined in (6.3). With the help of (6.10) we write (7.8) as

$$u_x^2 = -8\mathrm{Tr}[e^{-\beta/2}BCe^{-\beta/2}\Lambda^{-1}M]_x,$$

or equivalently, after using (6.6), we get

$$u_x^2 = -8\mathrm{Tr}[M_x\Lambda^{-1}M]_x,$$

Using (6.5) and the fact that M and Λ^{-1} commute, we obtain the first equality in (7.1). With the help of (6.4) we obtain

$$\Lambda_x = -A\Lambda - \Lambda A + e^{-\beta/2} E_x e^{-\beta/2},$$

$$\Lambda^{-1}\Lambda_x = -\Lambda^{-1}A\Lambda - A + \Lambda^{-1}e^{-\beta/2} E_x e^{-\beta/2}, \tag{7.9}$$

and hence using (6.4) and (6.10), from (7.9) we obtain

$$\mathrm{Tr}[\Lambda^{-1}\Lambda_x] = -2\mathrm{Tr}[A] + \mathrm{Tr}[E^{-1}E_x], \tag{7.10}$$

establishing the second equality in (7.1). With the help of (3.21) and the fact that E and F are real valued, we establish the third equality in (7.1). Using the matrix identity

$$\mathrm{Tr}[\alpha^{-1}\alpha_x] = \frac{1}{\det\alpha}\frac{\partial\det\alpha}{\partial x} = \frac{\partial\log(\det\alpha)}{\partial x},$$

we write (7.1) in the equivalent form of (7.2). Now, if we use $(\tilde{A}, \tilde{B}, \tilde{C})$ instead of (A, B, C), we see from (4.11) that, for some constant invertible matrices Y and Z, we have

$$\tilde{E} = YEZ, \qquad \tilde{E}^{-1} = Z^{-1}E^{-1}Y^{-1}, \qquad \tilde{E}_x = YE_xZ, \tag{7.11}$$

and hence, with the help of (6.10) and (7.11) we get

$$\mathrm{Tr}[\tilde{E}^{-1}\tilde{E}_x] = \mathrm{Tr}[E^{-1}E_x]. \tag{7.12}$$

Similarly, (4.10) yields

$$\tilde{F} = Z^\dagger F Y^\dagger, \qquad \tilde{F}^{-1} = (Y^\dagger)^{-1}F^{-1}(Z^\dagger)^{-1}, \qquad \tilde{F}_x = Z^\dagger F_x Y^\dagger,$$

which yields

$$\mathrm{Tr}[\tilde{F}^{-1}\tilde{F}_x] = \mathrm{Tr}[F^{-1}F_x]. \tag{7.13}$$

Note that from (7.10) and (7.12) we get

$$\mathrm{Tr}[\tilde{\Lambda}^{-1}\tilde{\Lambda}_x] + 2\,\mathrm{Tr}[\tilde{A}] = \mathrm{Tr}[\Lambda^{-1}\Lambda_x] + 2\,\mathrm{Tr}[A]. \tag{7.14}$$

Thus, by taking the x-derivatives of both sides in (7.12), (7.13), and (7.14), we establish (7.1) and (7.2) without any restriction on the sign of the real parts of the eigenvalues of A. ∎

Next, we show that the proof of Theorem 7.1 can be obtained directly without using (7.3)-(7.6). For this purpose, it is sufficient for us to show that (7.7) can directly be derived from (3.20).

Proposition 7.2 *The equality in (3.20) implies (7.7), i.e. we have*

$$-8(CE^{-1}PE^{-\beta}B)_x = 16CE^{-1}BCE^{-1}B. \tag{7.15}$$

PROOF: We directly evaluate the left hand side of (7.15) by taking the x-derivative of $E^{-1}Pe^{-\beta}$. We simplify the resulting expression by using the first two equalities given in (6.2), and we obtain the right hand side in (7.15). ∎

The next result shows that $u(-\infty,t)$ must be an integer multiple of 2π. In fact, we have $u(-\infty,t)=2\pi j$, where $j\in\{-p,-p+1,\dots,0,\dots,p-1,p\}$, with p denoting the size of the triplet (A,B,C) used to construct our exact solutions.

Theorem 7.3 *If (A,B,C) is admissible and the eigenvalues of A have positive real parts, then the solution to the sine-Gordon equation given in the equivalent forms (3.15), (3.16), (3.20), (6.7)-(6.9),(6.12), and (6.15) satisfies*

$$\int_{-\infty}^{\infty} dr\,[u_r(r,t)]^2 = 16\,\mathrm{Tr}[A], \tag{7.16}$$

and $u(x,t)$ converges to an integer multiple of (2π) as $x\to-\infty$.

PROOF: From (7.6) and the second equation in (7.3) we see that

$$\int_x^{\infty} dr\,[u_r(r,t)]^2 = 8CE(x,t)^{-1}Pe^{-\beta}B,$$

and hence with the help of Proposition 6.1, (5.1), and (6.10) we get

$$\int_{-\infty}^{\infty} dr\,[u_r(r,t)]^2 = 8CP^{-1}B = 8\mathrm{Tr}[BCP^{-1}] = 8\mathrm{Tr}[(AP+PA)P^{-1}] = 16\mathrm{Tr}[A],$$

yielding (7.16). By taking the time derivative of both sides of (7.16), we get

$$0=\int_{-\infty}^{\infty} dr\, u_r(r,t)\, u_{rt}(r,t) = \int_{-\infty}^{\infty} dr\, u_r(r,t)\, \sin(u(r,t)) = \cos(u(-\infty,t)) - \cos(u(+\infty,t)),$$

which proves that $u(-\infty,t)$ is an integer multiple of (2π) because we use the convention that $u(+\infty,t)=0$. ∎

8. TRANSMISSION COEFFICIENT AND NORMING CONSTANTS

In this section we show that our exact solutions given in equivalent forms (3.15), (3.16), (3.20), (6.7)-(6.9), (6.12), and (6.15) correspond to zero reflection coefficients in (1.4), we evaluate that corresponding Jost solution explicitly in terms of our triplet (A,B,C), determine the transmission coefficient explicitly in terms of the matrix A, and we also relate our triplet to the norming constants for (1.4) and to their time evolutions. As we have seen in Section 4 there is no loss of generality in choosing our triplet in the special form specified in Theorem 4.5, and hence in this section we will assume that (A,B,C) has the particular form given in (4.12)-(4.17).

The Jost solution $\psi(\lambda,x,t)$ satisfying the asymptotics (2.1) is given, as in (2.9) of [6], by

$$\psi(\lambda,x,t)=\begin{bmatrix} 0 \\ e^{i\lambda x}\end{bmatrix} + \int_x^{\infty} dy \begin{bmatrix} K(x,y,t) \\ G(x,y,t)\end{bmatrix} e^{i\lambda y}, \tag{8.1}$$

where $K(x,y,t)$ and $G(x,y,t)$ are the quantities in (3.17) and (7.6), respectively. Using (3.17) and (7.6) in (8.1) we obtain

$$\psi(\lambda,x,t)=e^{i\lambda x}\begin{bmatrix} iCE(x,t)^{-1}(\lambda I+iA)^{-1}B \\ 1-iCE(x,t)^{-1}Pe^{-\beta}(\lambda I+iA)^{-1}B\end{bmatrix}, \tag{8.2}$$

where E and β are the quantities appearing in (3.18) and (3.14), respectively. With the help of Propositions 5.1 and 5.2, by taking the limit of (8.2) as $x\to-\infty$ and by comparing the result with (2.2), we see that $L(\lambda,t)=0$, and hence [1,4,32] also $R(\lambda,t)=0$, and

$$\frac{1}{T(\lambda)} = 1 - iCP^{-1}(\lambda I+iA)^{-1}B. \tag{8.3}$$

Using (5.1), with the help of Proposition 4.2 of [6], we can invert (8.3) to get

$$T(\lambda) = 1 + iC(\lambda I - iA)^{-1} P^{-1} B. \tag{8.4}$$

By using Proposition 4.3 of [6] and (5.1), we can write (8.4) as the ratio of two determinants as

$$T(\lambda) = \frac{\det(\lambda I + iA)}{\det(\lambda I - iA)}. \tag{8.5}$$

Having determined the transmission coefficient T in terms of the matrix A appearing in (4.12), let us clarify the relationship between A and the poles and zeros of T in $\mathbf{C}^+$. From (8.5) we see that the zeros and poles of T occur exactly at the eigenvalues of $(-iA)$ and of (iA), respectively, and that the poles of T occur either on the positive imaginary axis or they are pairwise symmetrically located with respect to the imaginary axis in $\mathbf{C}^+$. A comparison of T given in (8.5) with A_j given in (4.14) shows that a bound-state pole λ_j of T located on the positive imaginary axis is related to the eigenvalue ω_j of A_j in the form $\lambda_j = i\omega_j$. A comparison of the poles of T given in (8.5) with A_j of (4.16) reveals the relationship between the poles off the imaginary axis and the real constants α_j and β_j appearing in A_j; namely, the pair of bound-state poles of T symmetrically located with respect to the imaginary axis in $\mathbf{C}^+$ occur at $\lambda = \lambda_j$ and $\lambda = -\lambda_j^*$, where

$$\lambda_j = -\beta_j + i\alpha_j, \qquad -\lambda_j^* = \beta_j + i\alpha_j. \tag{8.6}$$

Having clarified the relationship between the matrix A appearing in (4.12) and the bound-state poles in $\mathbf{C}^+$ of the transmission coefficient T, let us now discuss the relationship between the bound-state norming constants and the row vector C appearing in (4.12). In case of nonsimple bound-state poles of T, the bound state norming constants can be introduced [6,12] in such a way that the generalization from the simple to the nonsimple bound states is the most natural. The summation term in (2.5) assumes that there are n simple bound-state poles of T at $\lambda = \lambda_j$ with the norming constants $c_j e^{-it/(2\lambda_j)}$. Let us now generalize it to the case where each bound-state pole λ_j has multiplicity n_j, i.e. when there are n_j linearly independent solutions to (1.4) for $\lambda = \lambda_j$. The most natural generalization is obtained by the association

$$c_j \mapsto C_j, \qquad \lambda_j \mapsto iA_j, \qquad 1 \mapsto B_j,$$

where A_j, B_j, C_j are the matrices appearing in (4.12). The summation term (2.5) then generalizes to one of the equivalent terms given in the set of equalities

$$Ce^{-A^{-1}t/2}e^{-Ay}B = \sum_{j=1}^{n} C_j e^{-A_j^{-1}t/2}e^{-A_j y}B_j = \sum_{j=1}^{n}\sum_{s=1}^{n_j} \frac{1}{(s-1)!}\, y^{s-1}\theta_{js}(t)\, e^{i\lambda_j y}, \tag{8.7}$$

where (A, B, C) is the special triplet appearing in (4.12) and $\theta_{js}(t)$ are the norming constants associated with the eigenvalue λ_j with multiplicity n_j.

From (8.7) we observe the relationship between the bound-state norming constants $\theta_{js}(t)$ and the vectors C_j appearing in (4.12). If λ_j occurs on the positive imaginary axis, then we see that $\theta_{js}(0)$ is the same as c_{js} appearing in (4.13) and hence the time evolution $\theta_{js}(0) \mapsto \theta_{js}(t)$ is governed by

$$[\,\theta_{jn_j}(t) \quad \cdots \quad \theta_{j1}(t)\,] = [\,\theta_{jn_j}(0) \quad \cdots \quad \theta_{j1}(0)\,]\, e^{-A_j^{-1}t/2}, \tag{8.8}$$

where A_j is the matrix obtained as in (4.14) by using $\omega_j = -i\lambda_j$ there. We note that the norming constants $\theta_{js}(t)$ are all real (positive, negative, or zero) with the understanding that $c_{jn_j}(t) \neq 0$.

Because of the real valuedness stated in (2.6), if the bound-state pole λ_j of T occurring off the positive imaginary axis has $\theta_{js}(t)$ as the norming constants, then the bound-state pole occurring at $(-\lambda_j^*)$ has $\theta_{js}(t)^*$ as the norming constants. In this case (8.6) holds, and a comparison of (8.8) with (4.15) and (4.16) reveals that the contribution from the pair λ_j and $(-\lambda_j^*)$ is given by one of the equivalent forms

$$\sum_{s=1}^{n_j} \frac{1}{(s-1)!}\left[\theta_{js}(t)\, y^{s-1}e^{i\lambda_j y} + \theta_{js}(t)^*\, y^{s-1}e^{-i\lambda_j^* y}\right] = C_j e^{-A_j y - A_j^{-1}t/2}B_j,$$

where (A_j, B_j, C_j) is the real triplet of size $2n_j$ appearing in (4.15) and (4.16). Thus, we see that the real constants ϵ_{js} and γ_{js} appearing in (4.15) are related to the real and imaginary parts of the norming constants $\theta_{js}(t)$ as

$$\epsilon_{js} = \mathrm{Re}[\theta_{js}(0)], \qquad \gamma_{js} = -\mathrm{Im}[\theta_{js}(0)].$$

Defining the real $1 \times (2n_j)$ vector

$$\theta_j(t) := [\, -\mathrm{Im}[\theta_{jn_j}(t)] \quad \mathrm{Re}[\theta_{jn_j}(t)] \quad \cdots \quad \cdots \quad -\mathrm{Im}[\theta_{j1}(t)] \quad \mathrm{Re}[\theta_{j1}(t)]\,],$$

we obtain the time evolution $\theta_{js}(0) \mapsto \theta_{js}(t)$ as

$$\theta_j(t) = \theta_j(0)\, e^{-A_j^{-1}t/2},$$

where A_j is the $(2n_j) \times (2n_j)$ matrix appearing in (4.16).

Let us note that, by using (8.8), we can describe the time evolution of the (complex) norming constants $\theta_{js}(t)$ for $s = 1, \dots, n_j$ corresponding to the complex λ_j given in (8.6) by simply replacing the real matrix A_j of size $n_j \times n_j$ given in (4.14) with a complex-valued A_j of the same size. That complex A_j is simply obtained by replacing ω_j in (4.14) by the complex quantity $(-i\lambda_j)$. In that case, the time evolution of the norming constants $\theta_{js}(t)^*$ for $s = 1, \dots, n_j$ corresponding to the complex $-\lambda_j^*$ given in (8.6) is simply obtained by taking the complex conjugate of both sides of (8.8).

In short, in the most general case the summation term in (2.5) is given by the expression $Ce^{-Ay-A^{-1}t/2}$, where the triplet (A, B, C) has the form (4.12).

9. EXAMPLES

Example 9.1 The triplet (A, B, C) with

$$A = [\, a\,], \qquad B = [\,1\,], \qquad C = [\,c\,],$$

where $a > 0$ and $c \neq 0$, through the use of (3.19) and (6.3), yields

$$P = \left[\frac{c}{2a} \right], \qquad M = \left[\frac{c}{2a} \, e^{-2ax - t/(2a)} \right],$$

and hence from (6.7) we get

$$u(x,t) = -4 \tan^{-1} \left(\frac{c}{2a} \, e^{-2ax - t/(2a)} \right). \tag{9.1}$$

If $c > 0$, the solution in (9.1) is known as a "kink" [25]; it moves to the left with speed $1/(4a^2)$ and $u(x,t) \to -2\pi$ as $x \to -\infty$. If $c < 0$, the solution in (9.1) is known as an "antikink" [25]; it moves to the left with speed $1/(4a^2)$ and $u(x,t) \to 2\pi$ as $x \to -\infty$.

Example 9.2 The triplet (A, B, C) with

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} c_2 & c_1 \end{bmatrix},$$

where $a > 0$, $b \neq 0$, and $c_2 \neq 0$, through the use of (3.19), (6.3), and (6.9) yields

$$u(x,t) = -4 \tan^{-1} \left(\frac{\text{num}}{\text{den}} \right), \tag{9.2}$$

where

$$\text{num} := 8a^2 e^{a\zeta_+} \left[(ac_1 - bc_2) \cos(b\zeta_-) - (bc_1 + ac_2) \sin(b\zeta_-) \right],$$

$$\text{den} := b^2(c_1^2 + c_2^2) + 16a^2(a^2 + b^2) e^{2a\zeta_+}, \qquad \zeta_\pm := 2x \pm \frac{t}{2(a^2 + b^2)}.$$

The solution in (9.2) corresponds to a "breather" [25] and $u(x,t) \to 0$ as $x \to -\infty$. For example, the choice $a = 1$, $b = 2$, $c_1 = 2$, $c_2 = 1$ simplifies (9.2) to

$$u(x,t) = 4 \tan^{-1} \left(\frac{2e^{2x + t/10} \sin(4x - t/5)}{1 + 4e^{4x + t/5}} \right).$$

Example 9.3 The triplet (A, B, C) with

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 & c_2 \end{bmatrix},$$

where a_1 and a_2 are distinct positive constants, and c_1 and c_2 are real nonzero constants, by proceeding the same way as in the previous example, yields (9.2) with

$$\text{num} := 2(a_1+a_2)^2\left(a_1c_2e^{2a_1x+t/(2a_1)}+a_2c_1e^{2a_2x+t/(2a_2)}\right),$$

$$\text{den} := -(a_1-a_2)^2c_1c_2+4a_1a_2(a_1+a_2)^2e^{(a_1+a_2)(2x+t/(2a_1a_2))}. \tag{9.3}$$

If $(c_1c_2)<0$ then the quantity in (9.3) never becomes zero; the corresponding solution is known as a "soliton-antisoliton" [25] interaction. On the other hand, if $(c_1c_2)>0$ then the quantity in (9.3) becomes zero on a curve on the xt-plane and the corresponding solution is known as a "soliton-soliton" [25] interaction. For example, the choice $a=1$, $b=2$, $c_1=\pm1$, $c_2=\mp1$ yields

$$u(x,t)=\pm4\tan^{-1}\left(\frac{18e^{2x+t/2}-36e^{4x+t/4})}{1+72e^{6x+3t/4}}\right),$$

with $u(x,t)\to0$ as $x\to-\infty$. On the other hand, the choice $a=1$, $b=2$, $c_1=\pm1$, $c_2=\pm1$ yields the solution

$$u(x,t)=\mp4\tan^{-1}\left(\frac{18e^{2x+t/2}+36e^{4x+t/4})}{-1+72e^{6x+3t/4}}\right),$$

with $u(x,t)\to\mp4\pi$ as $x\to-\infty$.

Example 9.4 The triplet (A,B,C) with

$$A=\begin{bmatrix}a&-1&0\\0&a&-1\\0&0&a\end{bmatrix},\qquad B=\begin{bmatrix}0\\0\\1\end{bmatrix},\qquad C=\begin{bmatrix}c_3&c_2&c_1\end{bmatrix},$$

where $a>0$, and c_1, c_2, c_3 are real constants with $c_3\neq0$, by proceeding the same way as in the previous example, yields $u(x,t)$ in the form of (9.2), where

$$\text{num} := c_3^3e^{-4ax-t/a}+32g,\qquad \text{den} := 4ae^{-2ax-t/(2a)}[128a^8e^{4ax+t/a}+h_1+h_2],$$

$$g := (8a^4c_1+8a^3c_2+8a^2c_3)-(4a^2c_2+8ac_3)t+c_3t^2+(16a^4c_2+16a^3c_3)x-8a^2c_3xt+16a^4c_3x^2,$$

$$h_1 := (8a^4c_2^2 - 8a^4c_1c_3 + 16a^3c_2c_3 + 14a^2c_3^2) - (4a^2c_2c_3 + 4ac_3^2)t,$$

$$h_2 := c_3^2t^2 + (16a^4c_2c_3 + 32a^3c_3^2)x - 8a^2c_3^2tx + 16a^4c_3^2x^2.$$

The choice $a = 1$, $c_1 = -1$, $c_2 = -1$, $c_3 = -2$ yields

$$u(x,t) = -4\tan^{-1}\left(\frac{e^{-4x-t} + 8(16 - 10t + t^2 + 24x - 8tx + 16x^2)}{2e^{-2x-t/2}[32e^{4x+t} + 20 - 6t + t^2 - 8tx + 40x + 16x^2]}\right),$$

with $u(x,t) \to 2\pi$ as $x \to -\infty$. On the other hand, the choice $a = 1$, $c_1 = 0$, $c_2 = 0$, $c_3 = 1$ yields

$$u(x,t) = -4\tan^{-1}\left(\frac{e^{-4x-t} + 32(8 - 8t + t^2 + 16x - 8tx + 16x^2)}{4e^{-2x-t/2}[128e^{4x+t} + 14 - 4t + t^2 - 8tx + 32x + 16x^2]}\right),$$

with $u(x,t) \to -2\pi$ as $x \to -\infty$.

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