

# Inverse Scattering on the Line with Incomplete Scattering Data

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ABSTRACT. The Schrödinger equation is considered on the line when the potential is real valued, compactly supported, and square integrable. The nonuniqueness is analyzed in the recovery of such a potential from the data consisting of the ratio of a corresponding reflection coefficient to the transmission coefficient. It is shown that there are a discrete number of potentials corresponding to the data and that their  $L^2$ -norms are related to each other in a simple manner. All those potentials are identified, and it is shown how an additional estimate on the  $L^2$ -norm in the data can uniquely identify the corresponding potential. The recovery is illustrated with some explicit examples.

## 1. Introduction

In this paper we analyze the recovery of the potential in the Schrödinger equation on the line from a set of scattering data containing no information on bound states. Our work is motivated by the following question of Paul Sacks: Consider two potentials in the Schrödinger equation where one potential is obtained from the other by adding a bound state. Can we compare the  $L^2$ -norms of these two potentials, and can we conclude that the potential with fewer bound states has a smaller  $L^2$ -norm? By using (2.13) and (2.15) these questions can be answered as follows: Take a square-integrable potential and add a bound state with bound-state energy  $-\kappa^2$  and any bound-state norming constant. The new potential will have a larger  $L^2$ -norm differing from the previous  $L^2$ -norm by the exact value of  $16\kappa^3/3$ . Note that such a difference is independent of the value of the norming constant used, and hence  $L^2$ -norms of square-integrable potentials are affected only by bound-state energies and not by norming constants.

Our work is also motivated by the work of Rundell and Sacks [1], where it was shown that a bounded, real-valued, compactly-supported potential with a sufficiently small  $L^2$ -norm is uniquely determined by the corresponding ratio of a

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reflection coefficient to the transmission coefficient. With the help of the results in [2], our work here quantifies the smallness of the  $L^2$ -norm in the result of [1]. In Section 3 we present the exact least upper bound for that  $L^2$ -norm, below which we are assured the unique determination of a real-valued, compactly-supported, square-integrable potential in terms of the ratio of a reflection coefficient to the transmission coefficient; we do not require the potential to be bounded.

Let us now establish our notation. We consider the Schrödinger equation

$$(1.1) \quad \psi''(k, x) + k^2 \psi(k, x) = V(x) \psi(k, x), \quad x \in \mathbf{R},$$

where the potential  $V$  belongs to the Faddeev class, i.e. it is real valued, measurable, and in  $L^1_1(\mathbf{R})$ , the class of measurable functions on the real axis  $\mathbf{R}$  such that  $\int_{-\infty}^{\infty} dx (1 + |x|) |V(x)|$  is finite. The prime is used for the derivative with respect to the spatial coordinate  $x$ . The Jost solutions  $f_l$  and  $f_r$ , from the left and right, respectively, satisfy the respective boundary conditions

$$e^{-ikx} f_l(k, x) = 1 + o(1), \quad e^{-ikx} f'_l(k, x) = ik + o(1), \quad x \rightarrow +\infty,$$

$$e^{ikx} f_r(k, x) = 1 + o(1), \quad e^{ikx} f'_r(k, x) = -ik + o(1), \quad x \rightarrow -\infty,$$

and the transmission coefficient  $T$ , and the reflection coefficients  $L$  and  $R$ , from the left and right, respectively, are obtained from the spatial asymptotics

$$f_l(k, x) = \frac{e^{ikx}}{T(k)} + \frac{L(k) e^{-ikx}}{T(k)} + o(1), \quad x \rightarrow -\infty,$$

$$f_r(k, x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k) e^{ikx}}{T(k)} + o(1), \quad x \rightarrow +\infty.$$

A bound state of (1.1) is a square-integrable solution, and such states occur only at the  $k$ -values on  $\mathbf{I}^+$  in the upper half complex  $k$ -plane  $\mathbf{C}^+$  where  $T(k)$  has (simple) poles. Note that  $\mathbf{I}^+ := i(0, +\infty)$  denotes the positive imaginary axis. Later we will let  $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$  and  $\mathbf{I}^- := i(-\infty, 0)$ . The behavior at  $k = 0$  tells us whether the potential in (1.1) is generic or exceptional: The generic case occurs if  $T(0) = 0$  and the exceptional case occurs if  $T(0) \neq 0$ . For a review of scattering and bound states of (1.1), the reader is referred to [3-9] and the references therein.

Our paper is organized as follows: In Section 2 we briefly review the effect of adding a bound state to a potential and show that certain integrals of the resulting potential remain unaffected by the bound-state norming constant but affected only by the bound-state energy and in a rather simple manner. In Section 3 we analyze a consequence of the result of Section 2 in the recovery of a real-valued, compactly-supported, square-integrable potential in terms of the data  $L(k)/T(k)$ . We show that, corresponding to that data, there are a discrete number of potentials, and an additional estimate on the  $L^2$ -norm in the data allows the unique identification of a potential among all. We also illustrate the recovery with some explicit examples.

## 2. Effect of Bound States on Norms of a Potential

Let  $V^{[0]}$  denote a potential in the Faddeev class with no bound states. We use  $V^{[N]}$  for the potential obtained from  $V^{[0]}$  by adding  $N$  bound states at  $k = i\kappa_j$  with the corresponding bound-state dependency constants  $\gamma_j$ , where we have the ordering  $0 < \kappa_1 < \dots < \kappa_N$ . The superscript  $[j]$  refers to quantities associated with the potential  $V^{[j]}$ ; for example,  $T^{[j]}$ ,  $R^{[j]}$ , and  $L^{[j]}$  denote the scattering coefficients,

and  $f_1^{[j]}$  and  $f_r^{[j]}$  denote the left and right Jost solutions. Recall [4,9] that the dependency constants  $\gamma_j$  are defined as

$$\gamma_j := \frac{f_1^{[N]}(i\kappa_j, x)}{f_r^{[N]}(i\kappa_j, x)}, \quad 1 \leq j \leq N,$$

and the sign of  $\gamma_j$  is such that  $(-1)^{N-j}\gamma_j > 0$ . It is already known that

$$(2.1) \quad T^{[N]}(k) = T^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j},$$

$$(2.2) \quad R^{[N]}(k) = (-1)^N R^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}, \quad L^{[N]}(k) = (-1)^N L^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}.$$

For the known facts listed in this section, we refer the reader to [4], where it is shown that bound states can be added to a potential via the Darboux transformation. We have

$$(2.3) \quad V^{[j]}(x) - V^{[j-1]}(x) = -2\mu_j'(x), \quad 1 \leq j \leq N,$$

where we have defined

$$(2.4) \quad \mu_j(x) := \frac{\chi_j'(x)}{\chi_j(x)}, \quad \chi_j(x) := f_1^{[j-1]}(i\kappa_j, x) + |\gamma_j| f_r^{[j-1]}(i\kappa_j, x).$$

It is known that  $\chi_j(x)$  is continuous, strictly positive, and differentiable. In fact, as seen from (2.4) we have

$$\mu_j'(x) = V^{[j-1]}(x) + \kappa_j^2 - \mu_j(x)^2,$$

and hence from (2.3) it follows that

$$(2.5) \quad V^{[j]}(x) + V^{[j-1]}(x) = 2[\mu_j(x)^2 - \kappa_j^2].$$

Define

$$(2.6) \quad I_{j,n}(x) := \left[ V^{[j]}(x) - V^{[j-1]}(x) \right] \left[ V^{[j]}(x) + V^{[j-1]}(x) \right]^n, \quad n \geq 0, \quad 1 \leq j \leq N.$$

**THEOREM 2.1.** *Let  $V^{[j]}$  be the potential obtained from  $V^{[0]}$  by adding bound states of energy  $-\kappa_1^2, \dots, -\kappa_j^2$ , and assume that  $V^{[0]}$  belongs to the Faddeev class without any bound states. We then have*

$$(2.7) \quad \int_{-\infty}^{\infty} dx I_{j,n}(x) = (-1)^{n+1} 2^{n+2} \kappa_j^{2n+1} \frac{n!}{(2n+1)!!}, \quad n \geq 0, \quad 1 \leq j \leq N,$$

where  $(2n+1)!! := (1)(3)(5) \cdots (2n+1)$ .

PROOF. Taking the  $n$ th power in (2.5) and expanding the result, from (2.3) and (2.5) we get

$$(2.8) \quad I_{j,n}(x) = -2^{n+1} \frac{d}{dx} \sum_{p=0}^n (-1)^{n-p} \kappa_j^{2(n-p)} \binom{n}{p} \frac{\mu_j(x)^{2p+1}}{2p+1},$$

where  $\binom{n}{p} := \frac{n!}{p!(n-p)!}$  is the binomial coefficient. It is already known that

$$(2.9) \quad \mu_j(x) = \begin{cases} \kappa_j + o(1), & x \rightarrow +\infty, \\ -\kappa_j + o(1), & x \rightarrow -\infty. \end{cases}$$

Integrating (2.6) on  $\mathbf{R}$  and using (2.9), we get

$$(2.10) \quad \int_{-\infty}^{\infty} dx I_{j,n}(x) = (-1)^{n+1} 2^{n+2} \kappa_j^{2n+1} \sum_{p=0}^n (-1)^p \binom{n}{p} \frac{1}{2p+1}.$$

Note that the summation in (2.10) can be evaluated explicitly with the help of

$$(2.11) \quad \sum_{p=0}^n (-1)^p \binom{n}{p} \frac{1}{2p+1} = \int_0^1 dx (1-x^2)^n = \frac{n!}{(2n+1)!!}, \quad n \geq 0.$$

Thus, using (2.11) in (2.10) we establish (2.7).  $\square$

The result in (2.7) is remarkable in the sense that even though the integrand  $I_{j,n}(x)$  depends on the bound-state data  $\{\kappa_p, \gamma_p\}_{p=1}^j$ , its integral given in (2.7) is independent of the bound-state data, except for a rather simple  $\kappa_j$ -dependence.

For  $n=0$  and  $n=1$ , respectively, from (2.7) we get

$$(2.12) \quad \int_{-\infty}^{\infty} dx [V^{[j]}(x) - V^{[j-1]}(x)] = -4\kappa_j, \quad 1 \leq j \leq N,$$

$$(2.13) \quad \int_{-\infty}^{\infty} dx [V^{[j]}(x)^2 - V^{[j-1]}(x)^2] = \frac{16}{3} \kappa_j^3, \quad 1 \leq j \leq N.$$

By summing both sides in each of (2.12) and (2.13) over  $j$ , we get the following:

COROLLARY 2.2. *Let  $V^{[0]}$  be a potential in the Faddeev class with no bound states; add  $N$  bound states with energy  $-\kappa_1^2, \dots, -\kappa_N^2$ , resulting in the potential  $V^{[N]}$ . We then have*

$$(2.14) \quad \int_{-\infty}^{\infty} dx [V^{[N]}(x) - V^{[0]}(x)] = -4 \sum_{j=1}^N \kappa_j,$$

$$(2.15) \quad \int_{-\infty}^{\infty} dx [V^{[N]}(x)^2 - V^{[0]}(x)^2] = \frac{16}{3} \sum_{j=1}^N \kappa_j^3.$$

Let us indicate some resemblance between the result in (2.7) and the conserved quantities for an evolution equation that is exactly solvable by the inverse scattering transform [12-14]. For example, consider the time-evolution of the scattering data of (1.1) as  $T(k) \mapsto T(k)$ ,  $L(k) \mapsto L(k) e^{-8ik^3t}$ ,  $\kappa_j \mapsto \kappa_j$ , and  $\gamma_j \mapsto \gamma_j e^{-8\kappa_j^3t}$ . The

potential of (1.1) then evolves as  $V(x) \mapsto u(x, t)$ , where  $u(x, t)$  satisfies the initial-value problem for the Korteweg-de Vries equation (KdV)

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbf{R}, \quad t > 0; \quad u(x, 0) = V(x).$$

It is known [12-14] that  $\int_{-\infty}^{\infty} dx u(x, t)$ ,  $\int_{-\infty}^{\infty} dx u(x, t)^2$ , and an infinite number of other integrals are independent of  $t$  even though their integrands contain  $t$  explicitly. Such quantities are known as the conserved quantities for the KdV. Consider now, for example, (2.14) and (2.15), and let us time evolve the potentials  $V^{[0]}(x)$  and  $V^{[N]}(x)$  to obtain the corresponding solutions  $u^{[0]}(x, t)$  and  $u^{[N]}(x, t)$  of the KdV. Due to the fact that the bound-state energies  $-\kappa_j^2$  remain unchanged during the time evolution and that the right hand sides in (2.14) and (2.15) do not contain the dependency constants  $\gamma_j$ , we have

$$\int_{-\infty}^{\infty} dx \left[ u^{[N]}(x, t) - u^{[0]}(x, t) \right] = -4 \sum_{j=1}^N \kappa_j,$$

$$\int_{-\infty}^{\infty} dx \left[ u^{[N]}(x, t)^2 - u^{[0]}(x, t)^2 \right] = \frac{16}{3} \sum_{j=1}^N \kappa_j^3.$$

Other similar conserved quantities for the KdV can be obtained with the help of (2.7).

### 3. Recovery of the Potential from $L(k)/T(k)$

In [1] the recovery of a bounded, real-valued, compactly-supported potential is considered in terms of the data  $\mathcal{D}(k) := L(k)/T(k)$  known for  $k \in \mathbf{R}$ . In the class of such potentials corresponding to the same  $\mathcal{D}(k)$ , it was shown (cf. Theorem 2.3 of [1]) that there exists a positive constant  $C$  such that if  $V_1$  and  $V_2$  are two potentials with  $L^2$ -norms not exceeding  $C$  then  $V_1 \equiv V_2$ . The uniqueness and the reconstruction were obtained by transforming the problem into an equivalent time-domain problem; however, the value of  $C$  was left unspecified. In this section, we show how the value of  $C$  can be specified.

Recently, we have analyzed [2] the recovery of the potential  $V$  of (1.1) from  $\mathcal{D}(k)$  when  $V$  belongs to the Faddeev class. In this inverse problem, the construction of  $V$  is equivalent to the construction of the data  $\{L(k), N, \{\kappa_j\}, \{\gamma_j\}\}$ , where  $L$  is the left reflection coefficient,  $N$  is the number of bound states, the set  $\{-\kappa_j^2\}_{j=1}^N$  corresponds to the bound-state energies, and the set  $\{\gamma_j\}_{j=1}^N$  corresponds to the bound-state dependency constants. We have four cases to consider:

- (a) No information is available on the support of  $V$ , and the only data available is  $\mathcal{D}(k)$ .
- (b) In addition to  $\mathcal{D}(k)$ , it is known that the support of  $V$  is confined to a half line. In this case, there is no loss of generality in assuming that  $V \equiv 0$  for  $x < 0$ .
- (c) In addition to  $\mathcal{D}(k)$ , it is known that the support of  $V$  is confined a finite interval. In this case, there is no loss of generality in assuming that  $V \equiv 0$  for  $x \notin [0, 1]$ .
- (d) In addition to  $\mathcal{D}(k)$  and knowledge that  $V \equiv 0$  for  $x \notin [0, 1]$ , it is known that  $V$  is square integrable and some information related to the  $L^2$ -norm

is available. Such additional information may be in the form of a positive constant  $C$  which acts as an upper bound on the  $L^2$ -norm.

Let us consider the construction of  $V$  or equivalently of  $\{L(k), N, \{\kappa_j\}, \{\gamma_j\}\}$  in each of these four cases. For the analysis in the first three cases we refer the reader to [2] and give a brief summary below. Our results show that in case (c), given  $\mathcal{D}(k)$  for  $k \in \mathbf{R}$ , we are able to determine all the corresponding potentials, there are a discrete number of such potentials, the  $L^2$ -norm of each such potential is readily evaluated with the help of (2.15), and appropriate additional information on the  $L^2$ -norm enables us to further restrict the set of potentials corresponding to  $\mathcal{D}(k)$ . We also explain how the constant  $C$  in Theorem 2.3 of [1] arises: That constant allows us to identify the potential with the smallest  $L^2$ -norm among all those corresponding to the same  $\mathcal{D}(k)$ . By analyzing the inverse problem stated in (d), we show how to determine the precise values of  $C$  that can be used in [1].

**Case (a): Recovery of  $V$  from  $\mathcal{D}$  with no Support Information.**

If no information other than  $\mathcal{D}(k)$  is available, we have the following:

- (a.i) If  $\mathcal{D}(k)$  is bounded at  $k = 0$ , then there is no restriction on  $N$  and hence  $N \in \{0, 1, 2, \dots\}$ . Note that this case corresponds to the exceptional case for (1.1).
- (a.ii) If  $\mathcal{D}(k)$  is unbounded at  $k = 0$ , then  $\lim_{k \rightarrow 0} [2ik \mathcal{D}(k)]$  is either a positive constant or a negative constant. Thus, either  $\mathcal{D}(k) \rightarrow -\infty$  or  $\mathcal{D}(k) \rightarrow +\infty$  as  $k \rightarrow 0$  on  $\mathbf{I}^+$ . In the former case  $N$  must be even, i.e.  $N \in \{0, 2, 4, \dots\}$ ; in the latter case  $N$  must be odd, i.e.  $N \in \{1, 3, 5, \dots\}$ . Note that both these correspond to the generic case for (1.1).
- (a.iii) For each  $N$ -value resulting from (i) or (ii), given  $\mathcal{D}(k)$  there corresponds a  $2N$ -parameter family of potentials where the parameter set is  $\{\kappa_j, \gamma_j\}_{j=1}^N$ . There are no restrictions on the  $\kappa_j$  other than  $0 < \kappa_1 < \dots < \kappa_N$ . There are no restrictions on the  $\gamma_j$  other than  $(-1)^{N-j} \gamma_j > 0$ .

From the data  $\mathcal{D}(k)$  known for  $k \in \mathbf{R}$ , one uniquely constructs

$$(3.1) \quad T^{[0]}(k) = \exp \left( \frac{-1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\log(1 + |\mathcal{D}(s)|^2)}{s - k - i0^+} \right), \quad k \in \overline{\mathbf{C}^+}.$$

Then, with the help of (2.1), it is seen that the set  $\{\mathcal{D}(k), N, \{\kappa_j\}\}$  leads to the left reflection coefficient given by

$$(3.2) \quad L(k) = \mathcal{D}(k) T^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}, \quad k \in \mathbf{R}.$$

Note that  $T^{[0]}(k)$  appearing in (3.1) and (3.2) corresponds to the transmission coefficient for the potential  $V^{[0]}$ , which is obtained by removing all the  $N$  bound states from  $V$ . The left and right reflection coefficients,  $L^{[0]}(k)$  and  $R^{[0]}(k)$ , respectively, corresponding to  $V^{[0]}$  are uniquely determined only in the generic case as [cf. (2.2)]

$$(3.3) \quad L^{[0]}(k) = (-1)^N \mathcal{D}(k) T^{[0]}(k), \quad R^{[0]}(k) = (-1)^{N-1} \mathcal{D}(-k) T^{[0]}(k), \quad k \in \mathbf{R},$$

because only in the generic case  $(-1)^N$  is uniquely determined from  $\mathcal{D}(k)$ . In the exceptional case, the value of  $(-1)^N$  cannot be determined from  $\mathcal{D}(k)$  and hence there are two choices for  $V^{[0]}$ , which we denote by  $V_1^{[0]}$  and  $V_2^{[0]}$ , respectively, with

the corresponding scattering coefficients determined in terms of  $\mathcal{D}(k)$  and  $T^{[0]}(k)$  in (3.1) as follows:

$$T_1^{[0]}(k) = T^{[0]}(k), \quad L_1^{[0]}(k) = \mathcal{D}(k) T^{[0]}(k), \quad R_1^{[0]}(k) = -\mathcal{D}(-k) T^{[0]}(k),$$

$$T_2^{[0]}(k) = T^{[0]}(k), \quad L_2^{[0]}(k) = -\mathcal{D}(k) T^{[0]}(k), \quad R_2^{[0]}(k) = \mathcal{D}(-k) T^{[0]}(k).$$

For the comparison of two potentials with the same transmission coefficient but with reflection coefficients differing in sign, the reader is referred to [9-11]. As the next proposition shows, even though  $V_1^{[0]} \neq V_2^{[0]}$ , some of their characteristic features are related.

**PROPOSITION 3.1.** *Let  $V_1^{[0]}$  and  $V_2^{[0]}$  be two exceptional potentials in the Faddeev class with no bound states, and assume that  $T_1^{[0]} \equiv T_2^{[0]}$ ,  $L_1^{[0]} \equiv -L_2^{[0]}$ , and  $R_1^{[0]} \equiv -R_2^{[0]}$ , i.e. their reflection coefficients differ in sign and their transmission coefficients are the same for all  $k \in \mathbf{R}$ . Then we have the following:*

- (i)  $\int_{-\infty}^{\infty} dx \left[ V_2^{[0]}(x) - V_1^{[0]}(x) \right] \left[ V_2^{[0]}(x) + V_1^{[0]}(x) \right]^n = 0, \quad n \geq 0.$
- (ii)  $V_1^{[0]}$  vanishes on a half line if and only if  $V_2^{[0]}$  vanishes on the same half line. Consequently,  $V_1^{[0]}$  vanishes outside some interval if and only if  $V_2^{[0]}$  vanishes outside that interval.
- (iii) If  $V_1^{[0]}$  vanishes on  $\mathbf{R}^-$  and is continuous on the interval  $(0, \delta)$  for some  $\delta > 0$ , then  $V_1^{[0]}(0^+) = -V_2^{[0]}(0^+)$ . Similarly, if  $V_1^{[0]}$  vanishes on  $\mathbf{R}^+$  and is continuous on the interval  $(-\delta, 0)$  for some  $\delta > 0$ , then  $V_1^{[0]}(0^-) = -V_2^{[0]}(0^-)$ .

**PROOF.** From (2.24) and (2.25) of [11] we have

$$(3.4) \quad V_2^{[0]}(x) = V_1^{[0]}(x) - 2\rho_1'(x) = 2\rho_1(x)^2 - V_1^{[0]}(x),$$

where

$$\rho_1(x) := \frac{f_{1l}^{[0]'}(0, x)}{f_{1l}^{[0]}(0, x)} = \frac{f_{1r}^{[0]'}(0, x)}{f_{1r}^{[0]}(0, x)},$$

with  $f_{1l}^{[0]}(k, x)$  and  $f_{1r}^{[0]}(k, x)$  being the left and right Jost solutions for the potential  $V_1^{[0]}$ . It is known that  $f_{1l}^{[0]}(0, x)$  is continuous and strictly positive and that  $\rho_1(x) = o(1/x)$  as  $x \rightarrow \pm\infty$ . Hence, with the help of (3.4) we get [cf. (2.8)]

$$\left[ V_2^{[0]}(x) - V_1^{[0]}(x) \right] \left[ V_2^{[0]}(x) + V_1^{[0]}(x) \right]^n = -\frac{2^{n+1}}{2n+1} [\rho_1(x)^{2n+1}]', \quad n \geq 0,$$

and integrating both sides over  $\mathbf{R}$  we obtain (i). To prove (ii), notice that there is no loss of generality in choosing the half line as  $\mathbf{R}^-$ . If  $V_1^{[0]} \equiv 0$  for  $x < 0$ , then  $f_{1r}^{[0]}(0, x) = 1$  for  $x \leq 0$ , and hence  $\rho_1(x) = 0$  for  $x \leq 0$ . Thus, from (3.4) it follows that  $V_2^{[0]} \equiv 0$  for  $x < 0$  as well. Conversely, it follows that  $V_1^{[0]} \equiv 0$  on  $\mathbf{R}^-$  whenever  $V_2^{[0]} \equiv 0$  there; thus, we have proved (ii). From (3.4) we see that the first statement in (iii) holds whenever  $\rho_1(0^+) = 0$ , which is the case due to the continuity of  $\rho_1(x)$  at  $x = 0$  and  $\rho_1(x) = 0$  for  $x \leq 0$ , which is satisfied when  $V_1^{[0]} \equiv 0$  for  $x < 0$ . The second statement in (iii) is obtained in a similar manner.  $\square$

Note that Proposition 3.1(i) holds even when  $n$  is a noninteger. By letting  $n = 0$  and  $n = 1$  there and using the fact that a potential in the Faddeev class is

integrable, we obtain

$$(3.5) \quad \int_{-\infty}^{\infty} dx V_1^{[0]}(x) = \int_{-\infty}^{\infty} dx V_2^{[0]}(x), \quad \int_{-\infty}^{\infty} dx \left[ V_1^{[0]}(x)^2 - V_2^{[0]}(x)^2 \right] = 0.$$

For smooth potentials, we refer the reader to (2.101) of [10] for results similar to (3.5) and their generalizations.

**Case (b): Recovery of  $V$  from  $\mathcal{D}$  with Half-line Support.**

If  $\mathcal{D}(k)$  is given for  $k \in \mathbf{R}$  and if it is also known that  $V \equiv 0$  for  $x < 0$ , then, in addition to all the results given in case (a), in particular, in addition to (a.i) and (a.ii), we have the following improvements:

- (b.iii)  $\mathcal{D}(k)$  has a unique analytic extension to  $k \in \mathbf{C}^+$  and such an extension is uniquely determined by our data  $\mathcal{D}(k)$  known for  $k \in \mathbf{R}$ . The value of  $N$  must satisfy  $N \leq Z + 1$ , where  $Z$  denotes the number of zeros of  $\mathcal{D}(k)$  on  $\mathbf{I}^+$ . In fact, from the proof of Proposition 3.1 in [2] it follows that if  $\mathcal{D}(k)$  has multiple zeros on  $\mathbf{I}^+$ , then  $Z$  is actually the number of distinct zeros of odd multiplicity, without counting the multiplicities.
- (b.iv) For each  $N$ -value resulting from restrictions (a.i), (a.ii), and (b.iii), given  $\mathcal{D}(k)$  for  $k \in \mathbf{R}$ , there corresponds an  $N$ -parameter family of potentials where the parameter set is  $\{\kappa_j\}_{j=1}^N$ . The  $\kappa_j$  satisfy the restrictions  $0 < \kappa_1 < \dots < \kappa_N$  and  $(-1)^{N-j} \mathcal{D}(i\kappa_j) > 0$  for  $j = 1, \dots, N$ . The latter restriction confines the  $\kappa_j$  to subintervals whose endpoints are uniquely determined by the zeros of  $\mathcal{D}(k)$  on  $\mathbf{I}^+$ . The dependency constants  $\gamma_j$  are uniquely determined as  $\gamma_j = \mathcal{D}(i\kappa_j)$  and hence they are not free parameters. The left reflection coefficient  $L(k)$  given in (3.2) becomes meromorphic in  $\mathbf{C}^+$  with simple poles at  $k = i\kappa_j$  for  $j = 1, \dots, N$ . Thus, (3.2) now holds for  $k \in \overline{\mathbf{C}^+}$  and the left reflection coefficient  $L^{[0]}(k)$  given in (3.3) becomes analytic in  $\mathbf{C}^+$ .

**Case (c): Recovery of  $V$  from  $\mathcal{D}$  with Compact Support.**

If  $\mathcal{D}(k)$  for  $k \in \mathbf{R}$  is given and if it is also known that  $V \equiv 0$  for  $x \notin [0, 1]$ , then, in addition to all the results in cases (a) and (b), in particular, in addition to (a.i) and (a.ii), we have the following improvements:

- (c.iii) The quantity  $k \mathcal{D}(k)$  has a unique analytic extension to the entire complex plane, and such an extension is uniquely determined by our data  $\mathcal{D}(k)$  known for  $k \in \mathbf{R}$ . Moreover, as in (b.iii) the value of  $N$  must satisfy  $N \leq Z + 1$ , where  $Z$  is the number of zeros of  $\mathcal{D}(k)$  on  $\mathbf{I}^+$  having odd multiplicities, without counting the multiplicities.
- (c.iv) For each  $N$ -value resulting from restrictions (a.i), (a.ii), and (c.iii), given  $\mathcal{D}(k)$  for  $k \in \mathbf{R}$ , there correspond a discrete number of potentials where the discrete parameter set is  $\{\kappa_j\}_{j=1}^N$ . The set  $\{\kappa_j\}_{j=1}^N$  must be a subset of  $\{\beta_m\}$  and satisfy the additional restrictions  $0 < \kappa_1 < \dots < \kappa_N$  and  $(-1)^{N-j} \mathcal{D}(i\kappa_j) > 0$  for  $j = 1, \dots, N$ . Here, each  $k = -i\beta_m$  corresponds to a zero of  $1/T^{[0]}(k)$  on  $\mathbf{I}^-$ , where  $T^{[0]}(k)$  is the quantity in (3.1), and  $k/T^{[0]}(k)$  is now entire on  $\mathbf{C}$  and uniquely constructed via (3.1) from our data  $\mathcal{D}(k)$  known for  $k \in \mathbf{R}$ . The values  $k = -i\beta_m$  correspond to the (real) resonances of  $V^{[0]}$ . For an answer to the question whether the set  $\{\beta_m\}$  is a finite set or an infinite set, we refer the reader to [15]. Informally speaking, if  $V^{[0]} \in C_0^\infty[0, 1]$  and the order of the zero of  $V^{[0]}$



at  $x = 0$  or at  $x = 1$  is infinite, then the set  $\{\beta_m\}$  may be infinite; otherwise, it is a finite set.

**Case (d): Recovery of  $V$  from  $\mathcal{D}$  with Compact Support and  $L^2$ -norm.**

Let us assume that  $\mathcal{D}(k)$  is given for  $k \in \mathbf{R}$  and it is known that  $V \equiv 0$  for  $x \notin [0, 1]$ ,  $V \in L^2[0, 1]$ , and  $\|V\| \leq C$ , where we denote the  $L^2$ -norm of  $V$  as  $\|V\| := \sqrt{\int_{-\infty}^{\infty} dx V(x)^2}$ . We will determine the precise values of  $C$  that assure a unique or nonunique determination of  $V$  from  $\mathcal{D}$ .

As outlined below (3.3) in case (a), given  $\mathcal{D}(k)$  for  $k \in \mathbf{R}$ , we are able to uniquely determine  $V^{[0]}$  when  $\mathcal{D}$  is singular at  $k = 0$ , and we determine two distinct potentials  $V_1^{[0]}$  and  $V_2^{[0]}$  if  $\mathcal{D}$  is finite at  $k = 0$ . In the latter case, we know from (3.5) that  $\|V_1^{[0]}\| = \|V_2^{[0]}\|$ . Thus,  $\mathcal{D}(k)$  uniquely determines the  $L^2$ -norm of  $V^{[0]}$ , even though there are two distinct choices for  $V^{[0]}$  in the exceptional case. Let us denote that unique value by  $\|V^{[0]}\|$ .

As seen from (c.iv), for each allowed integer  $N$ ,  $\mathcal{D}(k)$  uniquely [2] determines a discrete number of ordered sets  $\{\kappa_j\}_{j=1}^N$  with the ordering  $0 < \kappa_1 < \dots < \kappa_N$  related to the bound states of  $V^{[N]}$ . Let us define

$$C_0 := \|V^{[0]}\|; \quad C_N := \left[ \|V^{[0]}\|^2 + \frac{16}{3} \sum_{j=1}^N \kappa_j^3 \right]^{1/2}.$$

Thus, for each  $N$ ,  $C_N$  consists of a sequence of values. Clearly,  $C_0$  consists of a single number. By listing all the elements in  $C_N$  for all allowed  $N$ -values, we obtain a discrete set of ordered positive numbers consisting of various  $\kappa_j$  values, and we denote that discrete set by  $\{C_N\}$ . This set is a subset of  $\{\beta_m\}$ , as indicated in (c.iv). The smallest number in the ordered set  $\{C_N\}$  is strictly less than the next larger number due to the fact that each set  $\{\kappa_j\}_{j=1}^N$  with the largest allowable  $N$  consists of distinct positive elements. This allows us to determine the value of  $C$  in the inequality  $\|V\| \leq C$  in order to determine a unique potential  $V$  corresponding to our data  $\mathcal{D}$ . By choosing  $C$  as greater than or equal to the smallest number in the set  $\{C_N\}$  but strictly less than the next larger element, we will uniquely determine the potential  $V$ . Next we illustrate this determination with some explicit examples.

As our scattering data let us use  $\mathcal{D}(k) = \frac{-\epsilon e^{ik} \sin \sqrt{k^2 + \epsilon}}{2ik \sqrt{k^2 + \epsilon}}$ , where  $\epsilon$  is a positive parameter. In fact, one corresponding potential is the square well of depth  $\epsilon$  supported on the interval  $[0, 1]$ . For each value of  $\epsilon$ , let us obtain all the potentials corresponding to  $\mathcal{D}(k)$  with support confined to  $[0, 1]$  and specify their  $L^2$ -norms. We have  $\lim_{k \rightarrow 0} [2ik \mathcal{D}(k)] = -\sqrt{\epsilon} \sin \sqrt{\epsilon}$ , and hence the exceptional case occurs when  $\sqrt{\epsilon}/\pi$  is a positive integer and the generic case occurs otherwise. The zeros of  $\mathcal{D}(k)$  on  $\mathbf{I}^+$  occur when  $\sin \sqrt{k^2 + \epsilon} = 0$ , and hence these are all simple zeros occurring at  $k = i\sqrt{\epsilon - (j-1)^2\pi^2}$  for  $j = 1, \dots, Z$ , with  $Z$  being equal to  $\lfloor \sqrt{\epsilon}/\pi \rfloor$ , i.e. the greatest integer less than or equal to  $\sqrt{\epsilon}/\pi$ . As  $k \rightarrow \infty$  on  $\mathbf{I}^+$ , we have  $\mathcal{D}(k) \rightarrow 0^+$ . As  $k \rightarrow 0$  on  $\mathbf{I}^+$ , we get  $(-1)^Z \mathcal{D}(k) \rightarrow 0^+$  in the exceptional case, and  $(-1)^Z \mathcal{D}(k) \rightarrow +\infty$  in the generic case. Define

$$\frac{1}{\tau(k)} := e^{ik} \left[ \cos \sqrt{k^2 + \epsilon} + \frac{2k^2 + \epsilon}{2ik \sqrt{k^2 + \epsilon}} \sin \sqrt{k^2 + \epsilon} \right].$$

Note that  $\tau(k)$  corresponds to the transmission coefficient of the square-well potential of depth  $\epsilon$  supported on  $[0, 1]$ . It is known that  $1/\tau(k)$  has exactly  $Z + 1$  (simple) zeros on  $\mathbf{I}^+$ , which we denote by  $\xi_j$  with the ordering  $0 < \xi_1 < \dots < \xi_{Z+1}$ . The quantity in (3.1) is obtained as

$$(3.6) \quad \frac{1}{T^{[0]}(k)} = \frac{1}{\tau(k)} \prod_{j=1}^{Z+1} \frac{k + i\xi_j}{k - i\xi_j}.$$

EXAMPLE 3.1. When  $\epsilon = 5$ , we are in the generic case and  $Z = 0$ . Hence,  $N \leq 1$ , but  $\mathcal{D}(k) \rightarrow +\infty$  on  $\mathbf{I}^+$  indicates that  $N$  must be odd. Thus,  $N = 1$  is the only allowed value. In this case,  $1/T^{[0]}(k)$  given in (3.6) has two zeros on  $\mathbf{I}^-$  at  $k = -i\beta_j$  with  $\beta_1 = 1.5433\overline{4}$  and  $\beta_2 = 1.585\overline{7}$ . We use an overline to indicate roundoff. In (3.6) we have  $\xi_1 = \beta_2$ . Corresponding to  $\mathcal{D}(k)$  we have two potentials  $V_1^{[1]}$  and  $V_2^{[1]}$ , having bound states at  $k = i\beta_1$  and  $k = i\beta_2$ , respectively. Note that  $V_2^{[1]}$  is the square well of depth  $\epsilon$ . We have  $\|V_1^{[1]}\| = 4.8312\overline{6}$  and  $\|V_2^{[1]}\| = 5$ . Thus, knowledge of any  $C$  satisfying  $\|V_1^{[1]}\| \leq C < \|V_2^{[1]}\|$  helps us to identify  $V_1^{[1]}$  or  $V_2^{[1]}$  as the unique potential corresponding to  $\mathcal{D}(k)$ . The left reflection coefficients  $L_1^{[1]}$  and  $L_2^{[1]}$  corresponding to  $V_1^{[1]}$  and  $V_2^{[1]}$ , respectively, are obtained from (3.2) as

$$L_j^{[1]}(k) = \mathcal{D}(k) T^{[0]}(k) \frac{k + i\beta_j}{k - i\beta_j}, \quad j = 1, 2.$$

Note that  $V_1^{[1]}$  and  $V_2^{[1]}$  can uniquely be constructed [11] from  $L_1^{[1]}$  and  $L_2^{[1]}$ , respectively, because they vanish for  $x < 0$ .

EXAMPLE 3.2. When  $\epsilon = \pi^2$ , we are in the exceptional case and  $Z = 0$ . Hence, both  $N = 0$  and  $N = 1$  are allowed. In this case  $1/T^{[0]}(k)$  given in (3.6) has only one zero on  $\mathbf{I}^-$  at  $k = -i\beta_1$  with  $\beta_1 = 2.52258\overline{8}$ . Thus, in (3.6) we have  $\xi_1 = \beta_1$ . Corresponding to  $\mathcal{D}(k)$  we have two potentials  $V^{[0]}$  and  $V^{[1]}$ , the former with no bound states and the latter with one bound state at  $k = i\beta_1$ . Note that  $V^{[1]}$  is the square well of depth  $\epsilon$ . We have  $\|V^{[0]}\| = 3.3853\overline{7}$  and  $\|V^{[1]}\| = \pi^2$ . Thus, knowledge of any  $C$  satisfying  $\|V^{[0]}\| \leq C < \|V^{[1]}\|$  helps us to identify either  $V^{[0]}$  or  $V^{[1]}$  as the unique potential corresponding to  $\mathcal{D}(k)$ . The left reflection coefficients  $L^{[0]}$  and  $L^{[1]}$  corresponding to  $V^{[0]}$  and  $V^{[1]}$ , respectively, are obtained from (3.2) as

$$L^{[0]}(k) = \mathcal{D}(k) T^{[0]}(k), \quad L^{[1]}(k) = \mathcal{D}(k) T^{[0]}(k) \frac{k + i\beta_1}{k - i\beta_1}.$$

Having  $L^{[0]}$  and  $L^{[1]}$  at hand, the potentials  $V^{[0]}$  and  $V^{[1]}$  can be uniquely constructed.

EXAMPLE 3.3. When  $\epsilon = 20$ , we are in the generic case and  $Z = 1$ . Hence,  $N \leq 2$ , but  $\mathcal{D}(k) \rightarrow -\infty$  on  $\mathbf{I}^+$  indicates that  $N$  must be even. Thus,  $N = 0$  and  $N = 2$  are the only possibilities. In this case  $1/T^{[0]}(k)$  given in (3.6) has two zeros on  $\mathbf{I}^-$  at  $k = -i\beta_1$  with  $\beta_1 = \xi_1 = 1.9302\overline{1}$  and  $k = -i\beta_2$  with  $\beta_2 = \xi_2 = 3.9255\overline{6}$ . When  $N = 2$ , the only potential  $V^{[2]}$  corresponding to  $\mathcal{D}(k)$  is the square well of depth  $\epsilon$  with support  $[0, 1]$ . When  $N = 0$ , the corresponding potential  $V^{[0]}$  is uniquely determined from  $\mathcal{D}(k)$  and its left reflection coefficient  $L^{[0]}(k)$  is obtained from (3.2) as

$$L^{[0]}(k) = \mathcal{D}(k) T^{[0]}(k) \frac{(k - i\beta_1)(k - i\beta_2)}{(k + i\beta_1)(k + i\beta_2)}.$$

In this case we have  $\|V^{[0]}\| = 6.2463\bar{5}$  and  $\|V^{[2]}\| = 20$ . Thus, an appropriate specification of the upper limit on the  $L^2$ -norm of the potential allows the unique identification of  $V^{[0]}$  or  $V^{[2]}$  from  $\mathcal{D}(k)$ .

EXAMPLE 3.4. When  $\epsilon = 130$ , the allowed values for  $N$  are 0, 2, and 4. In this case  $1/T^{[0]}(k)$  given in (3.6) has six zeros on  $\mathbf{I}^-$  at  $k = -i\beta_j$  with  $\beta_1 = 4.8729\bar{5}$ ,  $\beta_2 = 8.2260\bar{7}$ ,  $\beta_3 = 8.3286\bar{5}$ ,  $\beta_4 = 10.087\bar{9}$ ,  $\beta_5 = 10.740\bar{7}$ ,  $\beta_6 = 11.08\bar{5}$ . For  $N = 0$ , the only potential corresponding to  $\mathcal{D}(k)$  has norm  $\|V^{[0]}\| = 23.96\bar{8}$ . For  $N = 2$ , there are five potentials corresponding to  $\mathcal{D}(k)$  with norms  $\|V_1^{[2]}\| = 64.50\bar{9}$ ,  $\|V_2^{[2]}\| = 65.366\bar{8}$ ,  $\|V_3^{[2]}\| = 91.956\bar{6}$ ,  $\|V_4^{[2]}\| = 115.38\bar{7}$ ,  $\|V_5^{[2]}\| = 120.19\bar{7}$ , where  $V_1^{[2]}$  has bound states  $\{-\beta_1^2, -\beta_2^2\}$ ,  $V_2^{[2]}$  has  $\{-\beta_1^2, -\beta_3^2\}$ ,  $V_3^{[2]}$  has  $\{-\beta_1^2, -\beta_6^2\}$ ,  $V_4^{[2]}$  has  $\{-\beta_4^2, -\beta_6^2\}$ , and  $V_5^{[2]}$  has  $\{-\beta_5^2, -\beta_6^2\}$ . For  $N = 4$ , there are four potentials corresponding to  $\mathcal{D}(k)$  with norms  $\|V_1^{[4]}\| = 130$ ,  $\|V_2^{[4]}\| = 130.43\bar{2}$ ,  $\|V_3^{[4]}\| = 134.28\bar{7}$ ,  $\|V_4^{[4]}\| = 134.70\bar{5}$ , where  $V_1^{[4]}$  has bound states  $\{-\beta_1^2, -\beta_2^2, -\beta_4^2, -\beta_6^2\}$ ,  $V_2^{[4]}$  has  $\{-\beta_1^2, -\beta_3^2, -\beta_4^2, -\beta_6^2\}$ ,  $V_3^{[4]}$  has  $\{-\beta_1^2, -\beta_2^2, -\beta_5^2, -\beta_6^2\}$ , and finally  $V_4^{[4]}$  has  $\{-\beta_1^2, -\beta_3^2, -\beta_5^2, -\beta_6^2\}$ . Thus, some appropriate knowledge on the  $L^2$ -norm of the potential allows us to pick a unique potential among all these 16 potentials corresponding to the same  $\mathcal{D}(k)$ . Note that  $V_1^{[4]}$  is the square well of depth  $\epsilon$ .

### References

- [1] W. Rundell and P. Sacks, *On the determination of potentials without bound state data*, J. Comput. Appl. Math. **55** (1994), 325–2347.
- [2] T. Aktosun and V. Papanicolaou, *Recovery of a potential from the ratio of reflection and transmission coefficients*, J. Math. Phys. **44** (2003), 4875–4883.
- [3] L. D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Amer. Math. Soc. Transl. (Ser. 2) **65** (1967), 139–166.
- [4] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251.
- [5] R. G. Newton, *The Marchenko and Gel'fand-Levitan methods in the inverse scattering problem in one and three dimensions*, Conference on inverse scattering: theory and application, ed. by J. B. Bednar et al., SIAM, Philadelphia, 1983, pp. 1–74.
- [6] A. Melin, *Operator methods for inverse scattering on the real line*, Comm. Partial Differential Equations **10** (1985), 677–766.
- [7] V. A. Marchenko, *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.
- [8] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory*, 2nd ed., Springer, New York, 1989.
- [9] T. Aktosun and M. Klaus, *Chapter 2.2.4, Inverse theory: problem on the line*, Scattering, ed. by E. R. Pike and P. C. Sabatier, Academic Press, London, 2001, pp. 770–785.
- [10] A. Degasperis and P. C. Sabatier, *Extension of the one-dimensional scattering theory, and ambiguities*, Inverse Problems **3** (1987), 73–109.
- [11] T. Aktosun, M. Klaus, and C. van der Mee, *On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation*, J. Math. Phys. **34** (1993), 2651–2690.
- [12] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968), 467–490.
- [13] G. L. Lamb, Jr., *Elements of soliton theory*, Wiley, New York, 1980.
- [14] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.
- [15] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73** (1987), 277–296.

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