CONSTRUCTION OF THE HALF-LINE POTENTIAL FROM THE JOST FUNCTION

Tuncay Aktosun
Department of Mathematics and Statistics
Mississippi State University
Mississippi State, MS 39762

Abstract: For the one-dimensional Schrödinger equation, the analysis is provided to recover the portion of the potential lying to the right (left) of any chosen point. The scattering data used consists of the left (right) Jost solution or its spatial derivative evaluated at that point, or the amplitudes of such functions. Various uniqueness and nonuniqueness results are established, and the recovery is illustrated with some explicit examples.

PACS (2003): 02.30.Zz, 03.65.Nk
Mathematics Subject Classification (2000): 34A55, 34L25, 34L40, 47A40, 81U05
Keywords: Schrödinger equation, Inverse scattering, Jost solution, Recovery of potential
Short title: Construction of the half-line potential

1. INTRODUCTION

In this paper we consider the one-dimensional Schrödinger equation and analyze the recovery of the portion of the potential lying to the right (left) of any chosen point. As our scattering data we use the amplitude of the Jost solution from the left (right) evaluated at that chosen point, or we use the amplitude of the spatial derivative of that Jost solution. As the alternate scattering data, we also consider using the Jost solution evaluated at the chosen point rather than its amplitude, or the spatial derivative of that Jost solution rather than the amplitude of that spatial derivative. For brevity, we refer to the quantities "from the left" ("from the right") as the "left" ("right") quantities.

Without loss of any generality, our problem can be posed as the recovery of the potential with support in $\mathbf{R}^+$ in terms of $|f_l(k,0)|$, or $|f_l'(k,0)|$, or $f_l(k,0)$, or $f_l'(k,0)$ known for $k\in\mathbf{R}$, where $f_l(k,x)$ denotes the left Jost solution to the Schrödinger equation

$$\Psi''(k,x)+k^2\Psi(k,x)=V(x)\,\Psi(k,x),\qquad x\in\mathbf{R}, \tag{1.1}$$

and the prime denotes the derivative with respect to the spatial coordinate x. Throughout our paper we assume that the potential V belongs to class $\mathcal{W}$; that is,

$$V(x)=c\,\delta(x)+U(x), \tag{1.2}$$

where c is a real constant, $\delta(x)$ is the Dirac delta distribution, and U is a real-valued potential having no bound states, vanishing for $x<0$, and belonging to $L^1_1(\mathbf{R}^+)$. By $L^1_n(\mathbf{R}^+)$ we denote the class of Lebesgue-measurable potentials U such that the integral $\int_0^\infty dx\,(1+|x|^n)\,|U(x)|$ is finite. Recall that a bound state for (1.1) corresponds to a solution that belongs to $L^2(\mathbf{R})$ in the x-variable.

The motivation for this paper came from the following question raised by Roy Pike while the author was attending the RCP264 Workshop on Inverse Problems and Nonlinearity in Montpellier, France in June 2000. When $c=0$ and there are no bound states, does $f_l'(\cdot, 0)$ uniquely determine U? This inverse problem arises in the acoustical analysis of the human vocal tract [1]. When the vocal tract is stimulated by a sinusoidal input volume velocity at the glottis, the impulse response at the lips is essentially given by $f_l'(\cdot, 0)$ [cf. (70) in [1]]. Such an inverse problem is equivalent to determining a scaled curvature of the duct of the vocal tract when a constant-frequency sound is uttered, and it has important applications in speech recognition [1]. The inclusion of $c\delta(x)$ in (1.2) allows a jump discontinuity in the rate of change of the radius of the vocal tract at the glottis. The mathematical complication caused by a nonzero c in (1.2) can be handled (see, e.g., Chapter I.3, especially Theorem 3.1.1 of [2]) with a minor modification to the existing theory for the selfadjoint Schrödinger operator on the line [3-8] by simply allowing a jump discontinuity in $f_l'(k, \cdot)$ at $x = 0$, as indicated in (2.17). While the author was attending the IMA Summer Program on Geometric Methods in Inverse Problems in Minneapolis in July 2001, a solution to Pike's question was suggested by John Sylvester in the case $c = 0$, based on the observation that the real part of $[1 + L(k)]/[1 - L(k)]$ can be expressed as $k^2/|f_l'(k, 0)|^2$, where L is the left reflection coefficient corresponding to (1.1).

Let us use $\mathbf{C}^+$ for the upper half complex plane and put $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$. A key element in our analysis is the use of (4.3) and (5.3), namely, the Schwarz integral formula (also known as the Poisson integral formula for the half plane) [9]. Recall that a function g that is analytic in $\mathbf{C}^+$ and continuous and bounded in $\overline{\mathbf{C}^+}$ can be constructed uniquely if the real part of that function is known on the real axis $\mathbf{R}$, and this can be done by using the Schwarz integral formula

$$g(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t - k - i0^+} \operatorname{Re}[g(t)], \qquad k \in \overline{\mathbf{C}^+}, \tag{1.3}$$

where the quantity $i0^+$ indicates that the value of $g(k)$ for real k should be evaluated as a limit with k approaching $\mathbf{R}$ from $\mathbf{C}^+$. Using the Schwarz integral formula as in (4.3), (5.3), and (5.11), we are able to construct L as in (4.6), (5.4), and (5.13), respectively. Having L in hand, we are able to tell whether there exists a corresponding potential V in class $\mathcal{W}$

and whether there is more than one such potential. In Theorem 3.2 we characterize L so that the existence and uniqueness of a corresponding potential in class $\mathcal{W}$ is assured.

In order to state our results precisely, let us introduce the following four data sets:

$$\mathcal{D} := \{f_l(k,0) : \, k \in \mathbf{R}\}, \quad \mathcal{D}' := \{f_l'(k,0^-) : \, k \in \mathbf{R}\}, \tag{1.4}$$

$$\mathcal{E} := \{|f_l(k,0)| : \, k \in \mathbf{R}\}, \quad \mathcal{E}' := \{|f_l'(k,0^-)| : \, k \in \mathbf{R}\}. \tag{1.5}$$

Our paper is organized as follows. In Section 2, under the assumption that the potential belongs to class $\mathcal{W}$, we present the relevant properties of the corresponding Jost solutions and scattering coefficients. In Section 3 we show how to construct $f_l(\cdot,0)$ and $f_l'(\cdot,0^-)$ from the data sets $\mathcal{E}$ and $\mathcal{E}'$, respectively, and we prove that $\mathcal{D}$ and $\mathcal{E}$ are equivalent, but $\mathcal{D}'$ and $\mathcal{E}'$ are equivalent only when V does not have a bound state. In that section we also show that a potential V in class $\mathcal{W}$ is uniquely determined from the corresponding left reflection coefficient L, and in Theorem 3.2 we characterize L so that a corresponding potential in $\mathcal{W}$ exists and is unique. In Section 4 we show how $f_l'(\cdot,0^-)$ and L are constructed from the data $\mathcal{D}$ or $\mathcal{E}$. In Section 5 we analyze the construction of $f_l(\cdot,0)$ and L from $\mathcal{D}'$ or $\mathcal{E}'$. Finally, in Section 6 we describe some methods to reconstruct V and present some explicit examples to illustrate the reconstruction of the potential from the data sets given in (1.4) and (1.5).

Our findings are summarized in the following theorem.

Theorem 1.1 *Assume V belongs to class $\mathcal{W}$, and let f_l, c, $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{E}'$ denote the corresponding left Jost solution, the strength of the Dirac delta distribution in V, and the data sets in (1.4) and (1.5), respectively. Then:*

(i) *The data sets $\mathcal{D}$ and $\mathcal{E}$ are equivalent.*

(ii) *The potential V has no bound states if and only if $c \geq \tilde{c}$, where $\tilde{c}$ is the constant defined in (2.30). The potential V has exactly one bound state if and only if $c < \tilde{c}$.*

(iii) *If* $c \geq \tilde{c}$ *then the data sets* $\mathcal{D}'$ *and* $\mathcal{E}'$ *are equivalent. If* $c < \tilde{c}$ *then* $\mathcal{D}'$ *and* $\mathcal{E}' \cup \{\beta\}$ *are equivalent, where the* $i\beta$ *is the (only) zero of* $f_l'(\cdot, 0^-)$ *in* $\mathbf{C}^+$*.*

(iv) *There is a one-parameter family of potentials in* $\mathcal{W}$ *corresponding to* $\mathcal{E}$ *if the value of* c *is not specified. The augmented data* $\mathcal{E} \cup \{c\}$ *uniquely identifies* V*.*

(v) *If* $c \geq \tilde{c}$ *then* V *is the only potential corresponding to the data* $\mathcal{D}'$*. If* $c < \tilde{c}$*, then either of the equivalent sets* $\mathcal{D}' \cup \{c\}$ *and* $\mathcal{D}' \cup \{f_l(0,0)\}$ *uniquely identifies* V*.*

(vi) *If* $c < \tilde{c}$ *then there exists a two-parameter family of potentials in* $\mathcal{W}$ *corresponding to the data* $\mathcal{E}'$*. The correspondence can be made one-to-one by further specifying any two of the three quantities* $f_l(0,0)$*,* c*, and the bound state energy. Alternatively, for a one-to-one correspondence, one can specify any two of the three quantities* $f_l(0,0)$*,* c*, and the (only) zero of* $f_l'(\cdot, 0^-)$ *in* $\mathbf{C}^+$*.*

Returning to Pike's question, we obtain the following conclusion from the above theorem. A potential with support in $\mathbf{R}^+$ having no bound states and no Dirac delta distributions is uniquely determined from either $f_l'(k,0)$ or $|f_l'(k,0)|$ known for $k \in \mathbf{R}$.

2. PRELIMINARIES

Recall that the left Jost solution f_l to (1.1) satisfies the asymptotics

$$f_l(k,x) = e^{ikx}[1 + o(1)], \quad f_l'(k,x) = ik\, e^{ikx}[1 + o(1)], \qquad x \to +\infty, \tag{2.1}$$

and the right Jost solution f_r to (1.1) satisfies

$$f_r(k,x) = e^{-ikx}[1 + o(1)], \quad f_r'(k,x) = -ik\, e^{-ikx}[1 + o(1)], \qquad x \to -\infty. \tag{2.2}$$

Since we assume $V \equiv 0$ for $x < 0$, from (1.1) and (2.2) we get

$$f_r(k,x) = e^{-ikx}, \qquad x \leq 0; \quad f_r'(k,x) = -ik\, e^{-ikx}, \qquad x < 0.$$

Note that $f_l(\cdot, 0)$ and $f_l'(\cdot, 0^+)$ are determined by U alone. It is known [3-8] that they are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$ and satisfy

$$f_l(k,0) = f_l(0,0) + o(1), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{2.3}$$

$$f_1(0,x) = 1 + o(1), \qquad x \to +\infty, \tag{2.4}$$

$$f_1(-k^*,0) = f_1(k,0)^*, \quad f_1'(-k^*,0^+) = f_1'(k,0^+)^*, \qquad k \in \overline{\mathbf{C}^+}, \tag{2.5}$$

where the asterisk denotes complex conjugation. Moreover, from (2.24) of [10] we have

$$\frac{f_1'(k,0^+)}{f_1(k,0)} = \frac{f_1'(0,0^+)}{f_1(0,0)} + \frac{ik}{f_1(0,0)^2} + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{2.6}$$

where we can use $O(k^2)$ instead of $o(k)$ if U further satisfies $U \in L^1_2(\mathbf{R}^+)$. As $k \to \infty$ in $\overline{\mathbf{C}^+}$ we have [4]

$$f_1(k,0) = 1 - \frac{1}{2ik}\int_0^\infty dx\, U(x) + \frac{1}{2ik}\int_0^\infty dx\, e^{2ikx} U(x) + O(1/k^2), \tag{2.7}$$

$$f_1'(k,0^+) = ik - \frac{1}{2}\int_0^\infty dx\, U(x) - \frac{1}{2}\int_0^\infty dx\, e^{2ikx} U(x) + O(1/k), \tag{2.8}$$

$$\frac{f_1'(k,0^+)}{f_1(k,0)} = ik - \int_0^\infty dx\, e^{2ikx} U(x) + o(1/k). \tag{2.9}$$

Since $U \in L^1(\mathbf{R}^+)$, the Riemann-Lebesgue lemma implies that $\int_0^\infty dx\, e^{2ikx} U(x) = o(1)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, which we will implicitly use throughout our paper.

Let T, L, and R denote the transmission coefficient, the left reflection coefficient, and the right reflection coefficient associated with V. Note that T and L can be defined with the help of the asymptotics

$$f_1(k,x) = \frac{e^{ikx}}{T(k)} + \frac{L(k)\, e^{-ikx}}{T(k)} + o(1), \qquad x \to -\infty, \tag{2.10}$$

and R can be defined via

$$R(k) = -\frac{L(-k)\, T(k)}{T(-k)}, \qquad k \in \mathbf{R}. \tag{2.11}$$

When $c = 0$, the coefficients T, L, and R reduce to the corresponding scattering coefficients for U, which we denote as τ, ℓ, and ρ, respectively. Since we assume $U \equiv 0$ on $\mathbf{R}^-$ and it has no bound states, it is known [4,7] that τ and ℓ are analytic in $\mathbf{C}^+$ and (cf. (3.8) of [10]) continuous in $\overline{\mathbf{C}^+}$, and

$$|\tau(k)|^2 + |\ell(k)|^2 = 1, \qquad k \in \mathbf{R}, \tag{2.12}$$

$$\rho(k) = -\frac{\ell(-k)\,\tau(k)}{\tau(-k)}, \qquad k \in \mathbf{R}, \tag{2.13}$$

$$\tau(k) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \frac{\log(1-|\ell(t)|^2)}{t-k-i0^+}\right), \qquad k \in \overline{\mathbf{C^+}}. \tag{2.14}$$

The reader is referred to Section 3 of [10] regarding the continuity of the scattering coefficients τ, ℓ, and ρ at $k=0$ in the generic and exceptional cases. If there were any bound states of U, they would correspond [3-8] to the (simple) poles of τ on $\mathbf{I}^+$, where we use $\mathbf{I}^+ := i(0,+\infty)$ for the positive imaginary axis. As $k\to\infty$ in $\overline{\mathbf{C^+}}$ it is known [4] that

$$\tau(k) = 1 + \frac{1}{2ik}\int_0^\infty dx\, U(x) - \frac{1}{8k^2}\left(\int_0^\infty dx\, U(x)\right)^2 + o(1/k^2), \tag{2.15}$$

$$\ell(k) = \frac{1}{2ik}\int_0^\infty dx\, e^{2ikx} U(x) + o(1/k^2), \tag{2.16}$$

and hence $\tau(k) = 1 + O(1/k)$ and $\ell(k) = o(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C^+}}$.

As mentioned earlier a nonzero c causes a jump discontinuity [2] in $f_l'(k,\cdot)$ at $x=0$, while leaving $f_l(k,0)$ unaffected:

$$f_l'(k,0^+) - f_l'(k,0^-) = c\, f_l(k,0). \tag{2.17}$$

Since we assume that U vanishes on $\mathbf{R}^-$, with the help of (2.10) we obtain

$$f_l(k,0) = \frac{1+L(k)}{T(k)} = \frac{1+\ell(k)}{\tau(k)}, \tag{2.18}$$

$$f_l'(k,0^-) = ik\,\frac{1-L(k)}{T(k)}, \quad f_l'(k,0^+) = ik\,\frac{1-\ell(k)}{\tau(k)}. \tag{2.19}$$

Using (2.17)-(2.19) we can express T and L explicitly in terms of τ and ℓ and vice versa. We have

$$\ell(k) = \frac{2ik\,L(k) - c[1+L(k)]}{2ik + c[1+L(k)]}, \quad L(k) = \frac{2ik\,\ell(k) + c[1+\ell(k)]}{2ik - c[1+\ell(k)]}, \tag{2.20}$$

$$\tau(k) = \frac{2ik\,T(k)}{2ik + c[1+L(k)]}, \quad T(k) = \frac{2ik\,\tau(k)}{2ik - c[1+\ell(k)]}. \tag{2.21}$$

Using the fact $\ell(k) = o(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C^+}}$ in (2.20), we see that the value of c can be recovered from L via

$$c = \lim_{k\to\infty}\left[2ik\,L(k)\right], \tag{2.22}$$

where the limit can be taken in any way in $\overline{\mathbf{C}^+}$.

Proposition 2.1 *Assume that V given in (1.2) belongs to class $\mathcal{W}$, and let f_l, T, and L denote the corresponding left Jost solution, the transmission coefficient, and the left reflection coefficient, respectively. Then:*

(i) *$f_l(\cdot,0)$ and $f_l'(\cdot,0^-)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, and they satisfy*

$$f_l(-k^*,0)=f_l(k,0)^*,\quad f_l'(-k^*,0^-)=f_l'(k,0^-)^*,\qquad k\in\overline{\mathbf{C}^+}. \tag{2.23}$$

(ii) *The quantities $k/T(k)$ and $kL(k)/T(k)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, and we have*

$$L(-k^*)=L(k)^*,\quad T(-k^*)=T(k)^*,\qquad k\in\overline{\mathbf{C}^+}. \tag{2.24}$$

(iii) *Neither $f_l(\cdot,0)$ nor T vanishes in $\overline{\mathbf{C}^+}\setminus\{0\}$.*

PROOF: As already mentioned, $f_l(\cdot,0)$ and $f_l'(\cdot,0^+)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Hence, (i) is obtained from (2.5) and (2.17). Using (2.18) and (2.19) we obtain

$$\frac{2ik}{T(k)}=ik\,f_l(k,0)+f_l'(k,0^-),\quad \frac{2ik\,L(k)}{T(k)}=ik\,f_l(k,0)-f_l'(k,0^-), \tag{2.25}$$

and hence, with the help of (i) we get (ii). Since $U\equiv 0$ for $x\in\mathbf{R}^-$ and it has no bound states, it is known [11] that ℓ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, $|\ell(k)|<1$ for $k\in\mathbf{R}\setminus\{0\}$, and $\ell(k)=o(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Using the maximum modulus principle on ℓ in $\overline{\mathbf{C}^+}$ we conclude that $|\ell(k)|<1$ on $\mathbf{C}^+$, and hence, via (2.18), we obtain $f_l(k,0)\neq 0$ for $k\in\overline{\mathbf{C}^+}\setminus\{0\}$. From the continuity of $k/T(k)$ in $\overline{\mathbf{C}^+}$, we conclude that $T(k)\neq 0$ for $k\in\overline{\mathbf{C}^+}\setminus\{0\}$. ∎

With the help of (2.23), we see that, in the data to recover V, it is sufficient to supply $|f_l(k,0)|$ or $|f_l'(k,0^-)|$ only for $k\in\mathbf{R}^+$. Alternatively, in the scattering data it is enough to know $f_l(k,0)$ or $f_l'(k,0^-)$ only on a subinterval of $\mathbf{R}$, which is a consequence of the uniqueness of the analytic continuation from $\mathbf{R}$ to $\mathbf{C}^+$.

Proposition 2.2 *Assume that V given in (1.2) belongs to class $\mathcal{W}$, and let $f_{\rm l}$, T, and L be the corresponding left Jost solution, transmission coefficient, and left reflection coefficient. As $k \to \infty$ in $\overline{\mathbf{C}^+}$ we have:*

$$f_{\rm l}'(k,0^-) = ik - c - \frac{1}{2}\int_0^\infty dx\, U(x) - \frac{1}{2}\int_0^\infty dx\, e^{2ikx} U(x) + O(1/k), \tag{2.26}$$

$$L(k) = \frac{c}{2ik} + \frac{1}{2ik}\int_0^\infty dx\, e^{2ikx} U(x) - \frac{c^2}{4k^2} + o(1/k^2), \tag{2.27}$$

$$\begin{aligned} T(k) =& 1 + \frac{c}{2ik} + \frac{1}{2ik}\int_0^\infty dx\, U(x) - \frac{c^2}{4k^2} \\ & - \frac{c}{4k^2}\int_0^\infty dx\, U(x) - \frac{1}{8k^2}\left(\int_0^\infty dx\, U(x)\right)^2 + o(1/k^2). \end{aligned} \tag{2.28}$$

PROOF: We obtain (2.26) by using (2.7) and (2.8) in (2.17). By using (2.15) and (2.16) in the second identities in (2.20) and (2.21), we get (2.27) and (2.28), respectively. ▮

We need to make a distinction between the generic and exceptional cases. The generic case occurs if $f_{\rm l}(0,\cdot)$ and $f_{\rm r}(0,\cdot)$ are linearly independent on $\mathbf{R}$, and the exceptional case occurs in case of linear dependence. An exceptional case signals the borderline at which the number of bound states changes by one. It is shown below that $V \in \mathcal{W}$ has either zero or one bound state (cf. [12]).

Proposition 2.3 *Assume that V given in (1.2) belongs to class $\mathcal{W}$, and let T be the corresponding transmission coefficient. Then, a bound state of (1.1) occurs only at a zero of $1/T$ on $\mathbf{I}^+$, and there is either one bound state or none.*

PROOF: Let H_c denote the selfadjoint realization [13] of the Hamiltonian $-d^2/dx^2 + V(x)$ in $L^2(\mathbf{R})$. As a result of selfadjointness, which is a consequence [2] of the realness of c in (1.2), all eigenvalues of H_c are real. Note that $f_{\rm l}(k,\cdot)$ and $f_{\rm r}(k,\cdot)$ are linearly independent for each $k \in \mathbf{R}\setminus\{0\}$ because using (2.1) and (2.2) we get $[f_{\rm r}(k,\cdot); f_{\rm l}(k,\cdot)] = 2ik$, where $[F;G] := FG' - F'G$ denotes the Wronskian. Moreover, (2.1) and (2.2) indicate that no linear combinations of $f_{\rm l}(k,\cdot)$ and $f_{\rm r}(k,\cdot)$ can be in $L^2(\mathbf{R})$ for any $k \in \mathbf{R}\setminus\{0\}$. Thus, H_c cannot have any positive eigenvalues; i.e., there are no L^2-solutions to (1.1) for any positive value of k^2. For $k = 0$, any solution to (1.1) must be an affine function of x on $\mathbf{R}^-$ [cf. (2.29)] and hence cannot be in $L^2(\mathbf{R})$ unless it identically vanishes on $\mathbf{R}^-$. However, the only such solution would be the trivial solution on $\mathbf{R}$ and hence $k^2 = 0$ cannot correspond to an eigenvalue of H_c. Thus, any eigenvalue must be negative and this can happen only when $k \in \mathbf{I}^+$. From (2.1) and (2.2) it is seen that an L^2-solution is possible if and only if $f_{\rm l}(k, \cdot)$ and $f_{\rm r}(k, \cdot)$ are linearly dependent, and comparing (2.2) and (2.10) we see that the eigenvalues of H_c occur exactly at the zeros of $1/T$ on $\mathbf{I}^+$. The eigenvalues of H_c can be analyzed in terms of quadratic forms [14,15]. The bilinear form associated with the Hamiltonian H_c is given by

$$Q_c(\varphi, \psi) = \langle \varphi', \psi' \rangle + \langle U\varphi, \psi \rangle + c\, \varphi(0)\, \psi(0)^*,$$

with domain being the standard Sobolev space $W^{1,2}(\mathbf{R})$ [16] and $\langle \cdot, \cdot \rangle$ denoting the standard inner product in $L^2(\mathbf{R})$. The difference of resolvents of H_c for two distinct values of c is a rank-one operator and that the min-max theorem and the spectral mapping theorem [4] indicate that the eigenvalues for different c values must interlace. Since we assume that H_0 has no eigenvalues, it follows that H_c has either one eigenvalue or none. ∎

We remark that the continuity of $f_{\rm l}(k, \cdot)$ at $x = 0$ follows from an embedding theorem for Sobolev spaces in one dimension.

Proposition 2.4 *Assume that V given in (1.2) belongs to class $\mathcal{W}$, and let $f_{\rm l}$, c, T, and L denote the corresponding left Jost solution, the strength of the Dirac delta distribution in V, the transmission coefficient, and the left reflection coefficient, respectively, with $\tilde{c}$ denoting the constant in (2.30). Then:*

(i) $f_{\rm l}(iz, 0) > 0$ *for* $z \geq 0$.

(ii) *If* $c > \tilde{c}$, *then* V *is generic and* $f_{\rm l}'(0, 0^-) < 0$.

(iii) *If* $c = \tilde{c}$, *then* V *is exceptional and* $f_{\rm l}'(0, 0^-) = 0$.

(iv) *If* $c < \tilde{c}$, *then* V *is generic and* $f_{\rm l}'(0, 0^-) > 0$.

(v) *T and L are continuous at $k=0$. If $c\neq\tilde c$ then $T(0)=0$ and $L(0)=-1$. If $c=\tilde c$ then $T(0)>0$ and $L(0)\in(-1,1)$.*

PROOF: By Proposition 2.1, when $k\in\mathbf{I}^+$ we know that $f_{\rm l}(\cdot,0)$ is continuous, nonzero, and real valued [cf. (2.23)]. Thus, with the help of (2.7) we conclude that $f_{\rm l}(iz,0)>0$ for $z>0$. The argument below shows that $f_{\rm l}(0,0)>0$ and hence we obtain (i). Since U is assumed to have no bound states, we have $f_{\rm l}(0,x)\neq 0$ for $x\geq 0$, as indicated in Theorem 3 on p. 163 of [4]. Then, using (2.4) and the continuity of $f_{\rm l}(0,\cdot)$ on $\mathbf{R}$, we conclude that $f_{\rm l}(0,0)>0$. Moreover, since we assume that $V\equiv 0$ for $x<0$, from (1.1) it follows that

$$f_{\rm l}(0,x)=x\,f_{\rm l}'(0,0^-)+f_{\rm l}(0,0),\qquad x<0. \tag{2.29}$$

Define

$$\tilde c:=\frac{f_{\rm l}'(0,0^+)}{f_{\rm l}(0,0)}. \tag{2.30}$$

Note that $\tilde c$ is uniquely determined by U alone. Since $f_{\rm l}(0,0)>0$ and U is assumed to have no bound states, the aforementioned theorem of [4] implies that $f_{\rm l}'(0,0^+)\leq 0$, and hence we conclude that $\tilde c\leq 0$. Using (2.17) and (2.30) we get

$$f_{\rm l}'(0,0^-)=(\tilde c-c)\,f_{\rm l}(0,0). \tag{2.31}$$

Thus, from (2.31) and the positivity of $f_{\rm l}(0,0)$ we see that the statements on the sign of $f_{\rm l}'(0,0^-)$ stated in (ii), (iii), and (iv) hold. From (2.2) we see that $f_{\rm r}(0,x)=1$ for $x\leq 0$, and hence a comparison with (2.29) shows that $f_{\rm l}(0,\cdot)$ and $f_{\rm r}(0,\cdot)$ are linearly dependent if and only if $f_{\rm l}'(0,0^-)=0$. Thus, in the light of (2.31) we obtain the facts stated in (ii), (iii), (iv) related to the generic or exceptional case. Finally, we will prove (v) as follows. From (2.17) and (2.25), we get

$$T(k)=\frac{2ik}{f_{\rm l}(k,0)\,[ik-c+f_{\rm l}'(k,0^+)/f_{\rm l}(k,0)]}, \tag{2.32}$$

$$L(k)=\frac{ik+c-f_{\rm l}'(k,0^+)/f_{\rm l}(k,0)}{ik-c+f_{\rm l}'(k,0^+)/f_{\rm l}(k,0)}. \tag{2.33}$$

Using (2.3), (2.6), and (2.30) in (2.32) and (2.33), as $k \to 0$ in $\overline{\mathbf{C}^+}$ we have

$$T(k) = \begin{cases} \dfrac{2ik}{(\tilde{c} - c)\, f_l(0,0)}[1 + o(1)], & c \neq \tilde{c}, \\[2ex] \dfrac{2\, f_l(0,0)}{1 + f_l(0,0)^2}[1 + o(1)], & c = \tilde{c}, \end{cases} \tag{2.34}$$

$$L(k) = \frac{c - \tilde{c} + ik[1 - 1/f_l(0,0)^2] + o(k)}{-c + \tilde{c} + ik[1 + 1/f_l(0,0)^2] + o(k)}. \tag{2.35}$$

Thus, using $f_l(0,0) > 0$, with the help of (2.34) and (2.35) we obtain the continuity of T and L at $k = 0$ and also establish the remaining assertions in (v). ∎

Next we present an exact criterion on the number of bound states of V.

Proposition 2.5 *Assume that V given in (1.2) belongs to class $\mathcal{W}$, and let N and c denote the corresponding number of bound states and the strength of the Dirac delta distribution, respectively, with $\tilde{c}$ denoting the constant in (2.30). We have $N = 0$ if and only if $c \geq \tilde{c}$, and we have $N = 1$ if and only if $c < \tilde{c}$.*

PROOF: By Proposition 2.3 we know that N is equal to the number of zeros of $1/T$ on $\mathbf{I}^+$ and that $N = 0$ or $N = 1$. From (2.34), as $k \to 0^+$ in $\mathbf{R}$ we obtain

$$\arg T(0^+) = \begin{cases} \pi/2 \mod 2\pi, & c < \tilde{c}, \\ 0 \mod 2\pi, & c = \tilde{c}, \\ -\pi/2 \mod 2\pi, & c > \tilde{c}. \end{cases} \tag{2.36}$$

where mod indicates the congruence modulo and arg denotes the continuous branch of the argument function with $\arg T(+\infty) = 0$. With the help of Propositions 2.1 and 2.3, (2.28), and the second identity in (2.24), by applying the argument principle to T, we conclude that N, the number of bound states of V, which is also the number of poles of T in $\mathbf{C}^+$, is related to $\arg T$ as

$$\arg T(0^+) = \left(N - \frac{d}{2}\right)\pi, \tag{2.37}$$

where $d = 1$ if $T(0) = 0$ and $d = 0$ if $T(0) \neq 0$. Comparing (2.36) and (2.37) we see that $N = 1$ if $c < \tilde{c}$ and that $N = 0$ if $c \geq \tilde{c}$. ∎

Proposition 2.6 *Assume that V given in (1.2) belongs to class $\mathcal{W}$, and let f_l, T, L, and N denote the corresponding left Jost solution, the transmission coefficient, the left reflection coefficient, and the number of bound states, respectively. Then, the following hold:*

(i) $|T(k)|^2 + |L(k)|^2 = 1$ *for* $k \in \mathbf{R}$.

(ii) *The quantities* $1/T$ *and* $f_l'(\cdot, 0^-)$ *have the same number of zeros in* $\mathbf{C}^+$, *and that number is either zero or one. A zero of* $1/T$ *and a zero of* $f_l'(\cdot, 0^-)$ *in* $\mathbf{C}^+$ *occur only on* $\mathbf{I}^+$.

(iii) *If* $N = 0$ *then* $T(iz) > 0$ *for* $z > 0$. *If* $N = 1$ *then* $T(iz)$ *is positive for* $z > \kappa$ *and negative for* $z \in (0, \kappa)$, *where the* $i\kappa$ *denotes the zero of* $1/T$ *on* $\mathbf{I}^+$.

(iv) *If* $N = 0$ *then* $f_l'(iz, 0^-) < 0$ *for* $z > 0$. *If* $N = 1$ *then* $f_l'(iz, 0^-)$ *negative for* $z > \beta$ *and positive for* $z \in (0, \beta)$, *where the* $i\beta$ *denotes the zero of* $f_l'(\cdot, 0^-)$ *on* $\mathbf{I}^+$.

(v) *If* $N = 1$ *then* $\kappa < \beta$.

PROOF: The proof of (i) is obtained directly by using the first formulas of (2.20) and (2.21) in (2.12) as well as the continuity of T and L at $k = 0$, as asserted in Proposition 2.4(v). To prove (ii) we will use Rouché's theorem as follows. Let $\Omega_\epsilon := \mathbf{C}^+ \setminus \{|k| \le \epsilon\}$ for any $\epsilon > 0$, and set

$$F_1(k) := \frac{2}{T(k)}, \quad F_2(k) := \frac{f_l'(k, 0^-)}{ik}.$$

As indicated in Proposition 2.1, F_1 and F_2 are analytic in Ω_ϵ and continuous on $\overline{\Omega_\epsilon}$, the closure of Ω_ϵ. As seen by the first identity in (2.25), we have

$$F_1(k) - F_2(k) = f_l(k, 0), \tag{2.38}$$

which is nonzero on the boundary $\partial\Omega_\epsilon$ as indicated in Proposition 2.1. From (2.8) and (2.28), as $k \to \infty$ in $\overline{\mathbf{C}^+}$ we obtain

$$F_1(k) = 2 + O(1/k), \quad F_1(k) - F_2(k) = 1 + O(1/k). \tag{2.39}$$

By Proposition 2.1(iii) we have $T(k) \neq 0$ for $k \in \mathbf{R} \setminus \{0\}$, and hence from (i) it follows that $|L(k)| < 1$ for $k \in \mathbf{R} \setminus \{0\}$. Thus, we have

$$0 < \left| \frac{1 + L(k)}{T(k)} \right| \leq \frac{1 + |L(k)|}{|T(k)|} < \frac{2}{|T(k)|}, \qquad k \in \mathbf{R} \setminus \{0\}. \tag{2.40}$$

With the help of (2.18) and (2.38), as seen from (2.39) and (2.40), we have $|F_1(k) - F_2(k)| < |F_1(k)|$ on the boundary $\partial\Omega_\epsilon$, and by Rouché's theorem, we conclude that F_1 and F_2 have the same number of zeros in Ω_ϵ. Then, from Proposition 2.3 we conclude that $1/T$ and $f_1'(\cdot, 0^-)$ each have N zeros in $\mathbf{C}^+$, where $N = 0$ or $N = 1$. By Proposition 2.3 we already know that a possible zero of $1/T$ in $\mathbf{C}^+$ occurs on $\mathbf{I}^+$, and from the second identity in (2.23) we conclude that a possible zero of $f_1'(\cdot, 0^-)$ in $\mathbf{C}^+$ must occur on $\mathbf{I}^+$ because otherwise $f_1'(\cdot, 0^-)$ would have two or more zeros in $\mathbf{C}^+$. Thus, we have proved (ii). From (2.23) and (2.24) we see that $f_1'(\cdot, 0^-)$ and T are real valued on $\mathbf{I}^+$, and their sign as $k \to \infty$ on $\mathbf{I}^+$ are determined by (2.26) and (2.28), respectively, as negative and positive. Thus we get the assertions in (iii) and (iv). With the help of (iv), we see that (v) is established if we can show that $f_1'(i\kappa, 0^-) > 0$, and this follows from the first identity in (2.25) by using $1/T(i\kappa) = 0$ and $f_1(i\kappa, 0) > 0$. ∎

3. CONSTRUCTION OF $f_1(\cdot, 0)$ AND $f_1'(\cdot, 0^-)$ FROM THEIR AMPLITUDES

In this section we show how to construct $f_1(\cdot, 0)$ and $f_1'(\cdot, 0^-)$ for $k \in \overline{\mathbf{C}^+}$ from their amplitudes known for $k \in \mathbf{R}$. We then analyze the properties of M defined in (3.4), which plays an important role in solving our inverse problem of recovery of V from any of the four data sets given in (1.4) and (1.5). We prove that $\mathcal{D}$ and $\mathcal{E}$ are equivalent, but $\mathcal{D}'$ and $\mathcal{E}'$ are equivalent only when V does not have a bound state. We also characterize the left reflection coefficient L so that a corresponding potential in $\mathcal{W}$ exists and is unique.

In class $\mathcal{W}$, we can uniquely construct $f_1(\cdot, 0)$ for $k \in \overline{\mathbf{C}^+}$ from its amplitude known for $k \in \mathbf{R}$ by solving the Riemann-Hilbert problem

$$f_1(k, 0)\, f_1(-k, 0) = |f_1(k, 0)|^2, \qquad k \in \mathbf{R},$$

which follows from the first identity in (2.23). We know from (2.7) and Propositions 2.1 and 2.4(i) that $f_l(\cdot,0)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, $f_l(k,0)\neq 0$ in $\overline{\mathbf{C}^+}$ and $f_l(k,0)=1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Proposition 5.2 of [17] indicates that $f_l(\cdot,0)$ is constructed from the data $\mathcal{E}$ given in (1.5) as

$$f_l(k,0)=\exp\left(\frac{1}{\pi i}\int_{-\infty}^{\infty}dt\,\frac{\log|f_l(t,0)|}{t-k-i0^+}\right),\qquad k\in\overline{\mathbf{C}^+}. \tag{3.1}$$

In other words, as stated in Theorem 1.1(i), the data sets $\mathcal{D}$ and $\mathcal{E}$ are equivalent.

Now let us consider the construction of $f_l'(\cdot,0^-)$ for $k\in\overline{\mathbf{C}^+}$ from its amplitude known for $k\in\mathbf{R}$. Note that (2.23) implies

$$f_l'(k,0^-)\,f_l'(-k,0^-)=|f_l'(k,0^-)|^2,\qquad k\in\mathbf{R}. \tag{3.2}$$

From Propositions 2.1, 2.4, and 2.6 we know that $f_l'(\cdot,0^-)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, it is nonzero in $\overline{\mathbf{C}^+}$ with the exceptions that it has a simple zero at $k=0$ if $c=\tilde c$ and a zero at $k=i\beta$ if $c<\tilde c$. Moreover, from (2.26) it follows that $f_l'(k,0^-)=ik+O(1)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Thus, with the help of Proposition 5.2 of [17], we solve the Riemann-Hilbert problem associated with (3.2) and obtain for $k\in\overline{\mathbf{C}^+}$

$$f_l'(k,0^-)=\begin{cases} ik\,\exp\left(\dfrac{-1}{\pi i}\displaystyle\int_{-\infty}^{\infty}dt\,\frac{\log|t/f_l'(t,0^-)|}{t-k-i0^+}\right), & c\geq\tilde c,\\[2ex] ik\left(\dfrac{k-i\beta}{k+i\beta}\right)\exp\left(\dfrac{-1}{\pi i}\displaystyle\int_{-\infty}^{\infty}dt\,\frac{\log|t/f_l'(t,0^-)|}{t-k-i0^+}\right), & c<\tilde c.\end{cases} \tag{3.3}$$

From (3.3) we observe the following. If $c\geq\tilde c$ then the data sets $\mathcal{D}'$ and $\mathcal{E}'$ are equivalent, and if $c<\tilde c$ then $\mathcal{D}'$ and $\mathcal{E}'\cup\{\beta\}$ are equivalent.

Let us now define a key quantity, namely

$$M(k):=\frac{f_l'(k,0^-)}{ik\,f_l(k,0)}. \tag{3.4}$$

From (2.25) we see that

$$M(k)=\frac{1-L(k)}{1+L(k)},\quad L(k)=\frac{1-M(k)}{1+M(k)}. \tag{3.5}$$

We summarize the relevant properties of M below.

Theorem 3.1 *Assume V belongs to class $\mathcal{W}$. Let f_l, M, β, c, and $\tilde{c}$ be the corresponding left Jost solution, the quantity in (3.4), the constant in Proposition 2.6(iv), the strength of the Dirac delta distribution in V, and the constant in (2.30), respectively. Then the following hold:*

(i) *M is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$.*

(ii) *If $c=\tilde{c}$ then M is continuous at $k=0$, and if $c\neq\tilde{c}$ then M has a singularity of $O(1/k)$ at $k=0$. As $k\to 0$ in $\overline{\mathbf{C}^+}$ we get*

$$M(k)=\begin{cases}\dfrac{1}{f_l(0,0)^2}+o(1), & c=\tilde{c},\\[2ex] \dfrac{1}{ik}\,\dfrac{f_l'(0,0^-)}{f_l(0,0)}+\dfrac{1}{f_l(0,0)^2}+o(1), & c\neq\tilde{c},\end{cases}$$

where we can replace $o(1)$ by $O(k)$ if we further have $U\in L^1_2(\mathbf{R}^+)$.

(iii) *M is nonzero in $\overline{\mathbf{C}^+}\setminus\{0\}$ if $c\geq\tilde{c}$. If $c<\tilde{c}$ then M has exactly one zero in $\overline{\mathbf{C}^+}\setminus\{0\}$, and that zero occurs at $k=i\beta$ on $\mathbf{I}^+$.*

(iv) *If $c\neq\tilde{c}$, then as $k\to 0$ in $\overline{\mathbf{C}^+}$ we get*

$$\frac{1}{M(k)}=ik\,\frac{f_l(0,0)}{f_l'(0,0^-)}+\frac{k^2}{f_l'(0,0^-)^2}+o(k^2),\qquad c\neq\tilde{c},$$

where we can replace $o(k^2)$ by $O(k^3)$ if we further have $U\in L^1_2(\mathbf{R}^+)$.

(v) *The real part of M is given by* $\operatorname{Re}[M(k)]=\dfrac{1}{|f_l(k,0)|^2}$ *for $k\in\mathbf{R}$.*

(vi) *The real part of $1/M$ is given by* $\operatorname{Re}\left[\dfrac{1}{M(k)}\right]=\dfrac{k^2}{|f_l'(k,0^-)|^2}$ *for $k\in\mathbf{R}$.*

(vii) *As $k\to\infty$ in $\overline{\mathbf{C}^+}$ we have*

$$M(k)=1+\frac{ic}{k}+\frac{i}{k}\int_0^\infty dx\,e^{2ikx}U(x)+o(1/k^2),\tag{3.6}$$

$$\frac{1}{M(k)}=1-\frac{ic}{k}-\frac{i}{k}\int_0^\infty dx\,e^{2ikx}U(x)+o(1/k^2).\tag{3.7}$$

PROOF: We obtain (i) directly from the properties of $f_l(\cdot, 0)$ and $f_l'(\cdot, 0^-)$ given in Proposition 2.1. Using (2.6) and (2.17) in (3.4) we get (ii). Propositions 2.5 and 2.6 imply (iii). Note that (iv) directly follows from (ii). The properties in (v) and (vi) are directly obtained from (3.4) by using (2.23) for $k \in \mathbf{R}$ and the fact that the Wronskian of $f_l(-k, \cdot)$ and $f_l(k, \cdot)$ is independent of x and is equal to $2ik$, which follows from (2.1). Finally, (3.6) is obtained by using (2.9) and (2.17) in (3.4), and (3.7) readily follows from (3.6). ∎

Let us now show that the left reflection coefficient L uniquely determines the corresponding potential in class $\mathcal{W}$ and any quantity related to that potential can be obtained from L. For example, the quantities c, τ, ℓ, ρ, T, R, and U can all be obtained from L as follows. First, use (2.22) to obtain c. Next, use the first identity in (2.20) to get ℓ. Then, use (2.14) to construct τ. Next, use (2.13) to get ρ. Then, use the second identity in (2.21) to obtain T. Next, use (2.11) to construct R. Finally, use ℓ to construct U by using one of the available methods [7,8] indicated in Section 6.

Let us define

$$\hat{\ell}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \ell(k)\, e^{ik\alpha}, \quad \hat{\rho}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \rho(k)\, e^{ik\alpha}. \tag{3.8}$$

The following theorem gives the characterization of class $\mathcal{W}$ in terms of the scattering data consisting of L alone. Given L satisfying such characterization conditions, the existence and uniqueness of a potential $V \in \mathcal{W}$ are assured, and that potential can be obtained by using one of the methods described in Section 6.

Theorem 3.2 *The left reflection coefficient L corresponds to a unique potential V in $\mathcal{W}$ if and only if the following conditions hold:*

(i) *L is continuous for $k \in \mathbf{R}$, and $L(-k) = L(k)^*$ for $k \in \mathbf{R}$.*

(ii) *$|L(k)| \le 1 - Ck^2/(1+k^2)$ on $\mathbf{R}$ for some positive constant C.*

(iii) *$L(0) \in [-1, 1)$.*

(iv) *L either has an analytic extension from $\mathbf{R}$ to $\mathbf{C}^+$ or it has a meromorphic extension* *to* $\mathbf{C}^+$ *with a single simple pole occurring on the positive imaginary axis, say at* $k = i\kappa$ *for some* $\kappa > 0$*. In the latter case, the residue of* L *at* $k = i\kappa$ *satisfies* $\text{Res}[L(i\kappa)] = ic_{\text{r}}^2$ *for some positive* c_{r}.

(v) $L(k) = \dfrac{c}{2ik} + o(1/k)$ *as* $k \to \infty$ *in* $\overline{\mathbf{C}^+}$, *where* c *is the strength of the delta distribution in* V *at* $x = 0$.

(vi) *The function* $k/\tau(k)$, *where* τ *is the quantity defined in (2.14), is continuous in* $\overline{\mathbf{C}^+}$.

(vii) *The function* $\hat{\ell}(\alpha)$, *defined in (3.8) is absolutely continuous and* $\hat{\ell}'(\alpha) \in L_1^1(-\infty, 0)$; *the function* $\hat{\rho}(\alpha)$ *defined in (3.8) is absolutely continuous and* $\hat{\rho}'(\alpha) \in L_1^1(a, +\infty)$ *for every* $a < 0$.

PROOF: The relationships between the scattering coefficients for V and those for U are explicitly available from (2.20) and (2.21), and the bound state information for V is known explicitly as stated in Proposition 2.3. Moreover, V has the form given in (1.2), and hence it is sufficient to characterize the real-valued potentials that vanish for $x < 0$ and that belong to $L_1^1(\mathbf{R}^+)$. Hence, we will use the characterization conditions on ℓ given in Theorem 3.3 of [11] by taking into account the presence of the Dirac delta distribution at $x = 0$ as well as the absence of bound states for U, and we will thus obtain the corresponding conditions on L. Using (i)-(iii) of Theorem 3.3 of [11] and the second identity in (2.20) we get (i)-(iii). By comparing the second identities in (2.20) and (2.21) or by using Proposition 2.1(ii), we see that the poles of T and L in $\mathbf{C}^+$ coincide. Thus, (iv) of Theorem 3.3 of [11] yields (iv). We obtain (v) by using (2.27) with the help of (v) of Theorem 3.3 of [11]. Finally, (vi) and (vii) are directly obtained from the corresponding items in Theorem 3.3 of [11]. ∎

4. CONSTRUCTION OF $f_{\text{l}}'(\cdot, 0^-)$ AND L FROM $\mathcal{D}$ OR $\mathcal{E}$

In Section 3 we have seen the equivalence of the data sets $\mathcal{D}$ and $\mathcal{E}$ that are defined in (1.4) and (1.5), respectively. In this section we show that either of these data sets corresponds to a one-parameter family for each of $f_{\text{l}}'(\cdot, 0^-)$ and L. We will see that if the value of c, the strength of the Dirac delta distribution in the potential, is further specified then $f_1'(\cdot, 0^-)$ and L are uniquely determined. Then, the existence and uniqueness of V are assured if L satisfies the characterization conditions given in Theorem 3.2

In terms of M given in (3.4), let us define

$$\Gamma(k) := M(k) + \frac{i}{k} \frac{f_1'(0,0^-)}{f_1(0,0)} - 1. \tag{4.1}$$

From Proposition 2.4(i) and Theorem 3.1 it follows that Γ is analytic in $\mathbf{C}^+$, bounded and continuous in $\overline{\mathbf{C}^+}$, and

$$\Gamma(k) = \frac{i}{k}\left[c + \frac{f_1'(0,0^-)}{f_1(0,0)}\right] + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}, \tag{4.2}$$

$$\operatorname{Re}[\Gamma(k)] = \frac{1}{|f_1(k,0)|^2} - 1, \qquad k \in \mathbf{R}.$$

Thus, from either $\mathcal{D}$ or $\mathcal{E}$, using (1.3) we can construct Γ uniquely as

$$\Gamma(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t-k-i0^+} \left[\frac{1}{|f_1(t,0)|^2} - 1\right], \qquad k \in \overline{\mathbf{C}^+}. \tag{4.3}$$

Having constructed Γ, using (4.2), we obtain

$$\frac{f_1'(0,0^-)}{f_1(0,0)} = -c - i \lim_{k\to\infty} [k\,\Gamma(k)], \qquad k \in \overline{\mathbf{C}^+}, \tag{4.4}$$

and hence with the help of (4.1) and (4.4) we get M as

$$M(k) = \Gamma(k) + 1 + \frac{ic}{k} - \frac{1}{k} \lim_{k\to\infty} [k\,\Gamma(k)], \qquad k \in \overline{\mathbf{C}^+}. \tag{4.5}$$

Next, using (4.5) in the second identity of (3.5), we construct L as

$$L(k) = \frac{-k\,\Gamma(k) - ic + \lim_{k\to\infty} [k\,\Gamma(k)]}{2k + k\,\Gamma(k) + ic - \lim_{k\to\infty} [k\,\Gamma(k)]}, \qquad k \in \overline{\mathbf{C}^+}. \tag{4.6}$$

Using (4.6) in the first identity in (2.20) we obtain

$$\ell(k) = \frac{-k\,\Gamma(k) + \lim_{k\to\infty} [k\,\Gamma(k)]}{2k + k\,\Gamma(k) - \lim_{k\to\infty} [k\,\Gamma(k)]}, \qquad k \in \overline{\mathbf{C}^+}, \tag{4.7}$$

which also verifies the fact that ℓ can be obtained by putting $c=0$ in L.

If our data is $\mathcal{D}$, we already have $f_l(\cdot,0)$ in hand. On the other hand, if our data is $\mathcal{E}$ then we construct $f_l(\cdot,0)$ by using (3.1). Next, using (3.4) and (4.5) we construct $f_l'(\cdot,0^-)$ as

$$f_l'(k,0^-)=i\,f_l(k,0)\left[k+k\,\Gamma(k)+ic-\lim_{k\to\infty}[k\,\Gamma(k)]\right],\qquad k\in\overline{\mathbf{C}^+}. \tag{4.8}$$

Also, using (4.8) and the first identity of (2.25) we get

$$T(k)=\frac{2k}{f_l(k,0)\left[2k+k\,\Gamma(k)+ic-\lim_{k\to\infty}[k\,\Gamma(k)]\right]},\qquad k\in\overline{\mathbf{C}^+}. \tag{4.9}$$

Using (4.6) and (4.9) in the first identity of (2.21) we obtain

$$\tau(k)=\frac{2k}{f_l(k,0)\left[2k+k\,\Gamma(k)-\lim_{k\to\infty}[k\,\Gamma(k)]\right]},\qquad k\in\overline{\mathbf{C}^+},$$

which also verifies the fact that τ can be obtained by putting $c=0$ in T.

Note that each of L, T, and $f_l'(\cdot,0^-)$ depends on c explicitly, whereas none of ℓ, τ, and $f_l(\cdot,0)$ contain c, and neither of the data sets $\mathcal{D}$ and $\mathcal{E}$ has c in them. Thus, each of these data sets corresponds to a one-parameter family of potentials. More precisely, either of $\mathcal{D}$ and $\mathcal{E}$ uniquely determines U but not V, and each of the data sets $\mathcal{D}\cup\{c\}$ and $\mathcal{E}\cup\{c\}$ uniquely determines V. Note also that (2.31) and (4.4) imply that the constant $\tilde{c}$ defined in (2.30) can directly be obtained from Γ as

$$\tilde{c}=-i\lim_{k\to\infty}[k\,\Gamma(k)]. \tag{4.10}$$

5. CONSTRUCTION OF $f_l(\cdot,0)$ AND L FROM $\mathcal{D}'$ OR $\mathcal{E}'$

In Section 4 we have presented the construction of $f_l'(\cdot,0^-)$ and L from either of the equivalent data sets $\mathcal{D}$ and $\mathcal{E}$. In this section we analyze the construction of $f_l(\cdot,0)$ and L in terms of the data sets $\mathcal{D}'$ and $\mathcal{E}'$. Recall from Section 3 that V is uniquely determined once we have in hand L satisfying the characterization conditions of Theorem 3.2.

We will consider two distinct cases, namely, the case $c \geq \tilde{c}$ and the case $c < \tilde{c}$, where $\tilde{c}$ is the constant in (2.30). As we have seen in Section 3, $\mathcal{D}'$ and $\mathcal{E}'$ are equivalent in the former case, and $\mathcal{D}'$ and $\mathcal{E}' \cup \{\beta\}$ are equivalent in the latter case, where the $i\beta$ corresponds to the zero of $f_1'(\cdot, 0^-)$ in $\mathbf{C}^+$.

Let us first consider the case $c \geq \tilde{c}$. In that case we will show that either of the equivalent data sets $\mathcal{D}'$ and $\mathcal{E}'$ uniquely determines $f_1(\cdot, 0)$, L, and in turn V, as explained in Section 3. Define

$$\Lambda(k) := \frac{1}{M(k)} - 1, \tag{5.1}$$

where M is the quantity in (3.4). From Theorem 3.1 it follows that Λ is analytic in $\mathbf{C}^+$, bounded and continuous in $\overline{\mathbf{C}^+}$, and

$$\Lambda(k) = -\frac{i\,c}{k} + o(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}, \tag{5.2}$$

$$\text{Re}\,[\Lambda(k)] = \frac{k^2}{|f_1'(k, 0^-)|^2} - 1, \qquad k \in \mathbf{R}.$$

Thus, via the Schwarz integral formula (1.3), either of the data sets $\mathcal{D}'$ and $\mathcal{E}'$ allows the unique construction of Λ as

$$\Lambda(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t - k - i0^+} \left[\frac{t^2}{|f_1'(t, 0^-)|^2} - 1 \right], \qquad k \in \overline{\mathbf{C}^+}. \tag{5.3}$$

Having Λ in hand, using (5.1) and the second identity in (3.5) we get

$$M(k) = \frac{1}{\Lambda(k) + 1}, \quad L(k) = \frac{\Lambda(k)}{2 + \Lambda(k)}, \qquad k \in \overline{\mathbf{C}^+}, \tag{5.4}$$

and from (5.2) we obtain

$$c = i \lim_{k \to \infty} [k\,\Lambda(k)], \qquad k \in \overline{\mathbf{C}^+}. \tag{5.5}$$

Using (5.4) and (5.5) in the first identity of (2.20) we have

$$\ell(k) = \frac{k\,\Lambda(k) - [1 + \Lambda(k)] \lim_{k \to \infty} [k\,\Lambda(k)]}{k[2 + \Lambda(k)] + [1 + \Lambda(k)] \lim_{k \to \infty} [k\,\Lambda(k)]}, \qquad k \in \overline{\mathbf{C}^+}. \tag{5.6}$$

If our data is $\mathcal{E}'$ rather than $\mathcal{D}'$, by using the first line of (3.3) we construct $f_l'(0,0^-)$. Then, with the help of (3.4), (5.1), (5.4), (5.5), and the first identities in each of (2.21) and (2.25), we get

$$f_l(k,0)=\frac{f_l'(k,0^-)}{ik}\,[\Lambda(k)+1]\,,\qquad k\in\overline{\mathbf{C}^+},\tag{5.7}$$

$$T(k)=\frac{2ik}{f_l'(k,0^-)\,[2+\Lambda(k)]},\qquad k\in\overline{\mathbf{C}^+},\tag{5.8}$$

$$\tau(k)=\frac{2ik^2}{f_l'(k,0^-)\left[k[2+\Lambda(k)]+[1+\Lambda(k)]\displaystyle\lim_{k\to\infty}[k\,\Lambda(k)]\right]},\qquad k\in\overline{\mathbf{C}^+}.$$

From (5.5) and (5.6) we observe that $\mathcal{D}'$ or $\mathcal{E}'$ uniquely determines c and ℓ. Hence, V and U as well as all the scattering coefficients for V and U are uniquely determined by $\mathcal{D}'$ or $\mathcal{E}'$. Note also that, with the help of (2.31), (5.5), and (5.7), the critical value $\tilde{c}$ defined in (2.30) can be directly obtained from Λ as

$$\tilde{c}=i\lim_{k\to\infty}[k\,\Lambda(k)]+\lim_{k\to 0}\left[\frac{ik}{1+\Lambda(k)}\right],$$

where the limits can be evaluated in any way in $\overline{\mathbf{C}^+}$.

Next, let us study the recovery of $f_l(\cdot,0)$ and L from $\mathcal{D}'$ or $\mathcal{E}'$ in the case $c<\tilde{c}$. If we use $\mathcal{E}'$ as our data rather than $\mathcal{D}'$, via the second line of (3.3) we get $\mathcal{D}'$ provided the value of β is specified. In either case, whether β is known or interpreted as a yet unspecified positive parameter, let us define

$$\Omega(k):=\frac{k^2+\beta^2}{k^2}\,\frac{1}{M(k)}-\frac{i\beta^2}{k}\,\frac{f_l(0,0)}{f_l'(0,0^-)}-1.\tag{5.9}$$

From Proposition 2.4(iv) and Theorem 3.1 it follows that Ω is analytic in $\mathbf{C}^+$, bounded and continuous in $\overline{\mathbf{C}^+}$, and we have

$$\Omega(k)=-\frac{i}{k}\left[c+\beta^2\frac{f_l(0,0)}{f_l'(0,0^-)}\right]+o(1/k),\qquad k\to\infty \text{ in } \overline{\mathbf{C}^+},\tag{5.10}$$

$$\text{Re}\,[\Omega(k)]=\frac{k^2+\beta^2}{|f_l'(k,0^-)|^2}-1,\qquad k\in\mathbf{R}.$$

Thus, via (1.3), either of the equivalent sets $\mathcal{D}'$ and $\mathcal{E}' \cup \{\beta\}$ allows us to construct Ω as

$$\Omega(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t-k-i0^+} \left[\frac{t^2+\beta^2}{|f_1'(t,0^-)|^2} - 1 \right], \qquad k \in \overline{\mathbf{C}^+}. \tag{5.11}$$

Having Ω in hand, using (5.10) we obtain

$$\beta^2 \, \frac{f_1(0,0)}{f_1'(0,0^-)} = -c + i \lim_{k\to\infty} [k\,\Omega(k)]. \tag{5.12}$$

From Proposition 2.4(iv), we already know that $f_1'(0,0^-) > 0$ and hence we can rewrite (5.12) as

$$\beta^2 \, \frac{f_1(0,0)}{|f_1'(0,0^-)|} = -c + i \lim_{k\to\infty} [k\,\Omega(k)],$$

from which we conclude that, given $\mathcal{E}'$, the three sets $\{f_1(0,0), \beta\}$, $\{f_1(0,0), c\}$, and $\{c, \beta\}$ are equivalent.

Having constructed Ω from $\mathcal{D}'$ or $\mathcal{E}' \cup \{\beta\}$, with the help of (3.4),(5.9), (5.12), and the second identity in (3.5), for $k \in \overline{\mathbf{C}^+}$ we get

$$M(k) = \frac{k^2+\beta^2}{k^2 + k^2\,\Omega(k) - ick - k \lim_{k\to\infty} [k\,\Omega(k)]},$$

$$L(k) = \frac{k^2\,\Omega(k) - \beta^2 - ick - k \lim_{k\to\infty} [k\,\Omega(k)]}{2k^2 + k^2\,\Omega(k) + \beta^2 - ick - k \lim_{k\to\infty} [k\,\Omega(k)]}, \tag{5.13}$$

$$f_1(k,0) = \frac{f_1'(k,0^-) \left[k + k\,\Omega(k) - ic - \lim_{k\to\infty} [k\,\Omega(k)] \right]}{i(k^2+\beta^2)}. \tag{5.14}$$

Using (2.20), (2.21), (5.13), and the first identity in (2.25), for $k \in \overline{\mathbf{C}^+}$ we obtain

$$T(k) = \frac{2ik(k^2+\beta^2)}{f_1'(k,0^-) \left[2k^2 + k^2\,\Omega(k) + \beta^2 - ick - k \lim_{k\to\infty} [k\,\Omega(k)] \right]}, \tag{5.15}$$

$$\ell(k) = \frac{k(k+ic)\,\Omega(k) - \beta^2 + c^2 + (k-ic) \lim_{k\to\infty} [k\,\Omega(k)]}{2k^2 + k(k-ic)\,\Omega(k) + \beta^2 - c^2 - 2ick - (k-ic) \lim_{k\to\infty} [k\,\Omega(k)]}, \tag{5.16}$$

$$\tau(k)=\frac{2ik(k^2+\beta^2)}{f_{\rm l}'(k,0^-)\left[2k^2+k(k-ic)\,\Omega(k)+\beta^2-c^2-2ick-(k-ic)\displaystyle\lim_{k\to\infty}[k\,\Omega(k)]\right]}. \quad (5.17)$$

From (5.13)-(5.17) we have the following observations. When $c<\tilde c$, each of T, L, $f_{\rm l}(\cdot,0)$, τ, and ℓ contains the parameters c and β. Hence, the data $\mathcal{E}'$ yields a two-parameter family for each of V and U. On the other hand, if we use $\mathcal{D}'$ as our data, then having β in hand, we see that there is a one-parameter family for each of V and U, where c is the parameter. Since we assume that U has no bound states, by Theorem 3.3(iv) of [11] ℓ cannot have any poles on $\mathbf{I}^+$. Thus, the values of β and c used in the equivalent data sets $\mathcal{D}'\cup\{c\}$ and $\mathcal{E}'\cup\{\beta,c\}$ are constrained so that the denominator in (5.16) does not vanish for $k\in\mathbf{I}^+$.

6. RECONSTRUCTION OF V

There are various ways to reconstruct the potential V starting from any one of the data sets $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{E}$, or $\mathcal{E}'$ defined in (1.4) and (1.5). In this section we briefly discuss some of such methods.

If our starting data is one of the equivalent sets $\mathcal{D}$ and $\mathcal{E}$, as we have seen in Section 4, one needs $\mathcal{D}\cup\{c\}$ or $\mathcal{E}\cup\{c\}$ to uniquely reconstruct V because $\mathcal{D}$ or $\mathcal{E}$ uniquely reconstructs the corresponding U but not the value of c. One way to recover V is to use (1.2), where we can obtain U by one of several methods, such as the radial Gel'fand-Levitan method [6,7,18] using the data $|f_{\rm l}(k,0)|$ for $k\in\mathbf{R}$, the radial Marchenko method [6,7,18] using the data $S(k):=f_{\rm l}(-k,0)/f_{\rm l}(k,0)$ for $k\in\mathbf{R}$, and the full-line Faddeev-Marchenko method [3-8] using the data $\ell(k)$ for $k\in\mathbf{R}$ that is given in (4.7). Another way to recover V is to use L and T given in (4.6) and (4.9), respectively, to form the corresponding R via (2.11), and then to obtain $f_{\rm l}(k,x)$ and $f_{\rm r}(k,x)$ by solving the Riemann-Hilbert problems [5,7,8,19]

$$f_{\rm l}(-k,x)=T(k)\,f_{\rm r}(k,x)-R(k)\,f_{\rm l}(k,x),\qquad k\in\mathbf{R}, \quad (6.1)$$

$$f_{\rm r}(-k,x)=-L(k)\,f_{\rm r}(k,x)+T(k)\,f_{\rm l}(k,x),\qquad k\in\mathbf{R}, \quad (6.2)$$

and obtain $V(x)$ for all $x \in \mathbf{R}$ via (1.1) at any particular value of k. If L satisfies the conditions in Theorem 3.2, the existence and uniqueness of V in class $\mathcal{W}$ are assured, and hence (6.1) and (6.2) are uniquely solvable for the corresponding $f_{\rm l}(k,x)$ and $f_{\rm r}(k,x)$.

If our starting data is $\mathcal{D}'$ or $\mathcal{E}'$, then we proceed by using the method of Section 5. There are two separate cases to consider, i.e. the case where V has no bound states and the case where V has one bound state. In the former case the value of c can be obtained via (5.5) and U can be reconstructed by using the radial Gel'fand-Levitan method with input (5.7), the radial Marchenko method with input (5.7), or the full-line Faddeev-Marchenko method with input (5.6). Alternatively, V can be reconstructed directly by solving the Riemann-Hilbert problems (6.1) and (6.2) with input consisting of the second identity in (5.4), (5.8), and (2.11). In the case where V has a bound state, as indicated in Section 5, one needs $\mathcal{D}' \cup \{c\}$ or $\mathcal{E}' \cup \{c, \beta\}$ for the unique reconstruction. The reconstruction can be achieved, for example, by recovering U via the full-line Faddeev-Marchenko method with input (5.16). Alternatively, V can be directly reconstructed by solving the Riemann-Hilbert problems (6.1) and (6.2) with input consisting of (5.13), (5.15), and (2.11).

Let us illustrate the recovery with two examples. In these examples the scattering coefficients are rational functions of k, allowing us to explicitly solve the integral equations of radial Gel'fand-Levitan, radial Marchenko, and full-line Faddeev-Marchenko integral equations with degenerate kernels. Alternatively, the Riemann-Hilbert problems in (6.1) and (6.2) can be solved explicitly with the method of [20].

Example 6.1 Let our data $\mathcal{E}$ be such that $|f_{\rm l}(k,0)|^2 = \dfrac{k^2+49}{k^2+25}$ for $k \in \mathbf{R}$. From (3.1) we get $f_{\rm l}(k,0) = \dfrac{k+7i}{k+5i}$ for $k \in \overline{\mathbf{C^+}}$. The method of Section 4 gives us $\Gamma(k) = \dfrac{-24i}{7(k+7i)}$ and

$$T(k) = \frac{2k(k+5i)}{2k^2+(14+c)ik-(7c+24)},$$

$$L(k) = \frac{-ick+(7c+24)}{2k^2+(14+c)ik-(7c+24)},$$

$$\tau(k) = \frac{k(k+5i)}{(k+3i)(k+4i)}, \quad \ell(k) = \frac{12}{(k+3i)(k+4i)}.$$

The potential and the left Jost solution can be obtained as described in the second paragraph of this section, and we obtain

$$V(x)=c\delta(x)+\theta(x)\frac{1200\,e^{10x}}{(6\,e^{10x}-1)^2},\qquad x\in\mathbf{R},$$

$$f_l(k,x)=e^{ikx}\left[1+\frac{10i}{k+5i}\frac{1}{6\,e^{10x}-1}\right],\qquad x\geq 0,$$

where $\theta(x)$ denotes the Heaviside function. Thus, V and its scattering coefficients depend on the parameter c, whereas U and its scattering coefficients are uniquely determined by $\mathcal{E}$. From (4.10) we see that $\tilde{c}=-24/7$, and hence in our example V has one bound state if $c<-24/7$ and none if $c\geq -24/7$.

Example 6.2 Let our data $\mathcal{D}'$ be such that $f_l'(k,0^-)=i(k-i\beta)$ for some positive β so that V has one bound state. From (5.11) we obtain $\Omega(k)=0$ for $k\in\overline{\mathbf{C}^+}$. Using (5.13)-(5.17), for $k\in\overline{\mathbf{C}^+}$ we get

$$f_l(k,0)=\frac{k-ic}{k+i\beta},\quad T(k)=\frac{2k(k+i\beta)}{2k^2-2ick+\beta^2},\quad L(k)=\frac{-(ick+\beta^2)}{2k^2-2ick+\beta^2},$$

$$\tau(k)=\frac{2k(k+i\beta)}{2k^2-2ick+\beta^2-c^2},\quad \ell(k)=\frac{c^2-\beta^2}{2k^2-2ick+\beta^2-c^2},$$

where $c\leq-\beta$ guarantees that U has no bound states. The potential and the left Jost solution are obtained as

$$V(x)=c\delta(x)+\theta(x)\frac{8\beta^2\,\dfrac{c-\beta}{c+\beta}\,e^{2\beta x}}{\left(\dfrac{c-\beta}{c+\beta}\,e^{2\beta x}-1\right)^2},\qquad x\in\mathbf{R},$$

$$f_l(k,x)=e^{ikx}\left[1+\frac{2i\beta}{k+i\beta}\frac{1}{\dfrac{c-\beta}{c+\beta}\,e^{2\beta x}-1}\right],\qquad x\geq 0.$$

Thus, V depends on the two parameters c and β. Note that the pole of T on $\mathbf{I}^+$ occurs at $k=i\kappa$ with $\kappa=(c+\sqrt{c^2+2\beta^2})/2$.

Let us finally remark that, as also noted by Ricardo Weder, if our sole purpose is to reconstruct U from the data $\mathcal{D}$ or $\mathcal{E}$, then one does not need to construct $f_l'(\cdot, 0^-)$, and instead one can simply use the radial Gel'fand-Levitan method or the radial Marchenko method and directly obtain U. In that regard, the result of Section 4, in addition to being an alternative method to reconstruct V, can be viewed as the analysis of the existence, uniqueness, and direct construction of $f_l'(\cdot, 0^-)$ when one knows $|f_l(k, 0)|$ for $k \in \mathbf{R}$. Similarly, the method of Section 5 using the data $|f_l'(k, 0^-)|$ or $f_l'(k, 0^-)$ for $k \in \mathbf{R}$ can also be interpreted as the analysis of the existence, uniqueness, and direct reconstruction of $f_l(\cdot, 0)$ as well as a method to reconstruct the potential V.

Acknowledgments. The author has benefited from discussions with Roy Pike, Barbara Forbes, John Sylvester, and Ricardo Weder. The research leading to this article was supported in part by the National Science Foundation under grant DMS-0204437 and the Department of Energy under grant DE-FG02-01ER45951.

REFERENCES

[1] B. J. Forbes, E. R. Pike, and D. B. Sharp, *The acoustical Klein-Gordon equation: The wave-mechanical step and barrier potential functions,* J. Acoust. Soc. Am. **114**, 1291–1302 (2003).

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics,* Springer, New York, 1988.

[3] L. D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation,* Amer. Math. Soc. Transl. (Ser. 2) **65**, 139–166 (1967).

[4] P. Deift and E. Trubowitz, *Inverse scattering on the line,* Comm. Pure Appl. Math. **32**, 121–251 (1979).

[5] R. G. Newton, *The Marchenko and Gel'fand-Levitan methods in the inverse scattering*

problem in one and three dimensions, In: J. B. Bednar et al. (eds), *Conference on inverse scattering: theory and application,* SIAM, Philadelphia, 1983, pp. 1–74.

[6] V. A. Marchenko, *Sturm-Liouville operators and applications,* Birkhäuser, Basel, 1986.

[7] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory,* 2nd ed., Springer, New York, 1989.

[8] T. Aktosun and M. Klaus, *Chapter 2.2.4, Inverse theory: problem on the line,* In: E. R. Pike and P. C. Sabatier (eds.), *Scattering,* Academic Press, London, 2001, pp. 770–785.

[9] L. Ahlfors, *Complex analysis,* 2nd ed., McGraw-Hill, New York, 1966.

[10] T. Aktosun and M. Klaus, *Small-energy asymptotics for the Schrödinger equation on the line,* Inverse Problems **17**, 619–632 (2001).

[11] T. Aktosun and V. G. Papanicolaou, *Recovery of a potential from the ratio of reflection and transmission coefficients,* J. Math. Phys. **44**, 4875–4883 (2003).

[12] T. Aktosun, M. Klaus, and C. van der Mee, *On the number of bound states for the one-dimensional Schrödinger equation,* J. Math. Phys. **39**, 4249–4256 (1998).

[13] J. Weidmann, *Spectral theory of ordinary differential operators,* Lecture Notes in Math. **1258**, Springer, Berlin, 1987.

[14] M. Reed and B. Simon, *Methods of modern mathematical physics, I, Functional analysis,* Academic Press, New York, 1980.

[15] T. Kato, *Perturbation theory for linear operators,* 2nd ed., Springer, Berlin, 1976.

[16] R. A. Adams, *Sobolev spaces,* Academic Press, New York, 1975.

[17] T. Aktosun and P. E. Sacks, *Phase recovery with nondecaying potentials,* Inverse Problems **16**, 821–838 (2000).

[18] K. Chadan, *Chapter 2.2.1, Radial inverse scattering problems,* In: E. R. Pike and P.

C. Sabatier (eds.), *Scattering,* Academic Press, London, 2001, pp. 726–741.

[19] T. Aktosun, M. Klaus, and C. van der Mee, *On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation,* J. Math. Phys. **34**, 2651–2690 (1993).

[20] T. Aktosun, M. Klaus, and C. van der Mee, *Explicit Wiener-Hopf factorization for certain nonrational matrix functions,* Integral Equations Operator Theory **15**, 879–900 (1992).