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Integral Equations and Operator Theory

Inverse Scattering with Rational Scattering Coefficients and Wave Propagation in Nonhomogeneous Media

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Dedicated to Israel Gohberg on the occasion of his 75th birthday

Abstract. The inverse scattering problem for the one-dimensional Schrödinger equation is considered when the potential is real valued and integrable and has a finite first-moment and no bound states. Corresponding to such potentials, for rational reflection coefficients with only simple poles in the upper half complex plane, a method is presented to recover the potential and the scattering solutions explicitly. A numerical implementation of the method is developed. For such rational reflection coefficients, the scattering wave solutions to the plasma-wave equation are constructed explicitly. The discontinuities in these wave solutions and in their spatial derivatives are expressed explicitly in terms of the potential.

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1. Introduction

Consider the Schrödinger equation

$$\psi''(k,x) + k^2\psi(k,x) = V(x)\,\psi(k,x), \qquad x \in \mathbf{R}, \tag{1.1}$$

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where the prime denotes the x-derivative, and the potential V is assumed to have no bound states and to belong to the Faddeev class. The bound states of (1.1) correspond to its square-integrable solutions. By the Faddeev class we mean the set of real-valued and measurable potentials for which $\int_{-\infty}^{\infty} dx\,(1+|x|)|V(x)|$ is finite.

Via the Fourier transformation

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\psi(k,x)\,e^{-ikt},$$

we can transform (1.1) into the plasma-wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial^2 u(x,t)}{\partial t^2}=V(x)\,u(x,t),\qquad x,t\in\mathbf{R}. \tag{1.2}$$

In the absence of bound states, (1.1) does not have any bounded solutions for $k^2<0$. The solutions for $k^2>0$ are known as the scattering solutions. Each scattering solution can be expressed as a linear combination of the two (linearly-independent) Jost solutions from the left and the right, denoted by $f_{\rm l}$ and $f_{\rm r}$, respectively, satisfying the respective asymptotic conditions

$$f_{\rm l}(k,x)=e^{ikx}\left[1+o(1)\right],\quad f_{\rm l}'(k,x)=ik\,e^{ikx}\left[1+o(1)\right],\qquad x\to+\infty,$$

$$f_{\rm r}(k,x)=e^{-ikx}\left[1+o(1)\right],\quad f_{\rm r}'(k,x)=-ik\,e^{-ikx}\left[1+o(1)\right],\qquad x\to-\infty.$$

We have

$$f_{\rm l}(k,x)=\frac{1}{T(k)}\,e^{ikx}+\frac{L(k)}{T(k)}\,e^{-ikx}+o(1),\qquad x\to-\infty,$$

$$f_{\rm r}(k,x)=\frac{1}{T(k)}\,e^{-ikx}+\frac{R(k)}{T(k)}\,e^{ikx}+o(1),\qquad x\to+\infty,$$

where L and R are the left and right reflection coefficients, respectively, and T is the transmission coefficient.

The solutions to (1.1) for $k=0$ require special attention. Generically, $f_{\rm l}(0,x)$ and $f_{\rm r}(0,x)$ are linearly independent on $\mathbf{R}$, and we have

$$T(0)=0,\quad R(0)=L(0)=-1.$$

In the exceptional case, $f_{\rm l}(0,x)$ and $f_{\rm r}(0,x)$ are linearly dependent on $\mathbf{R}$ and we have

$$T(0)=\sqrt{1-R(0)^2}>0,\quad -1<R(0)=-L(0)<1.$$

When V belongs to the Faddeev class and has no bound states, it is known [1-5] that either one of the reflection coefficients R and L contains the appropriate information to construct the other reflection coefficient, the transmission coefficient T, the potential V, and the Jost solutions $f_{\rm l}$ and $f_{\rm r}$. Our aim in this paper is to present explicit formulas for such a construction when the reflection coefficients are rational functions of k with simple poles on the upper half complex plane $\mathbf{C}^+$. We will use $\mathbf{C}^-$ to denote the lower half complex plane and let $\overline{\mathbf{C}^+}:=\mathbf{C}^+\cup\mathbf{R}$ and $\overline{\mathbf{C}^-}:=\mathbf{C}^-\cup\mathbf{R}$.

The recovery of V from a reflection coefficient constitutes the inverse scattering problem for (1.1). There has been a substantial amount of previous work [2,6-14] done on the inverse scattering problem with rational reflection coefficients. The solution to this inverse problem can, for example, be obtained by solving the Marchenko integral equation [1-5]. Another way to solve this inverse problem is to use Sabatier's method [2,12-14] utilizing transformations resembling Darboux transformations [1-3]. Dolveck-Guilpart developed [7] a numerical implementation of Sabatier's method. Yet another method is based on the Wiener-Hopf factorization of a 2×2 matrix [6] related to the scattering matrix for (1.1). It is also possible to use [15,16] a minimal realization of a rational reflection coefficient and to recover the potential explicitly. The method discussed in our paper is closely related to that given in [6]. Here, we are able to write down the Jost solutions explicitly in terms of the poles in $\mathbf{C}^+$ and the corresponding residues of the reflection coefficients. This also enables us to construct explicitly certain solutions to (1.2), which we call the Jost waves.

Our paper is organized as follows. In Section 2 we present the preliminary material needed for later sections, including an outline of the construction of T and L from the right reflection coefficient R. In Section 3 we present the explicit construction of the potential and the Jost solutions for $x > 0$ in terms of the poles in $\mathbf{C}^+$ and the corresponding residues of R. Having constructed the left reflection coefficient L in terms of R, in Section 4 we present the explicit construction of the potential and the Jost solutions for $x < 0$ in terms of the poles in $\mathbf{C}^+$ and the corresponding residues of L. In Section 5 we turn our attention to (1.2) and explicitly construct its solutions by using the Fourier transforms of the Jost solutions to (1.1). In Section 6 we analyze the discontinuities in such wave solutions and in their x-derivatives at each fixed t. Finally, in Section 7 we remark on the numerical implementation of our method.

2. Preliminaries

For convenience, we introduce the Faddeev functions from the left and right, denoted by m_{l} and m_{r}, respectively, defined as

$$m_{\mathrm{l}}(k,x) := e^{-ikx} f_{\mathrm{l}}(k,x), \quad m_{\mathrm{r}}(k,x) := e^{ikx} f_{\mathrm{r}}(k,x). \tag{2.1}$$

From (1.1) and (2.1) it follows that

$$m_{\mathrm{l}}''(k,x) + 2ik\, m_{\mathrm{l}}'(k,x) = V(x)\, m_{\mathrm{l}}(k,x), \qquad x \in \mathbf{R}, \tag{2.2}$$

$$m_{\mathrm{r}}''(k,x) - 2ik\, m_{\mathrm{r}}'(k,x) = V(x)\, m_{\mathrm{r}}(k,x), \qquad x \in \mathbf{R}.$$

It is known [1-5] that

$$f_{\mathrm{l}}(-k,x) = -R(k)\, f_{\mathrm{l}}(k,x) + T(k)\, f_{\mathrm{r}}(k,x), \qquad k \in \mathbf{R}, \tag{2.3}$$

$$f_{\mathrm{r}}(-k,x) = T(k)\, f_{\mathrm{l}}(k,x) - L(k)\, f_{\mathrm{r}}(k,x), \qquad k \in \mathbf{R}, \tag{2.4}$$

or equivalently

$$m_{\mathrm{l}}(-k,x) = -R(k)\, e^{2ikx} m_{\mathrm{l}}(k,x) + T(k)\, m_{\mathrm{r}}(k,x), \qquad k \in \mathbf{R}, \tag{2.5}$$

$$m_{\rm r}(-k,x)=T(k)\,m_{\rm l}(k,x)-L(k)\,e^{-2ikx}m_{\rm r}(k,x),\qquad k\in\mathbf{R}. \tag{2.6}$$

When R and L are rational functions of k, their domains can be extended meromorphically from $\mathbf{R}$ to the entire complex plane $\mathbf{C}$. Similarly, the Jost solutions and the Faddeev functions have extensions that are analytic in $\mathbf{C}^+$ and meromorphic in $\mathbf{C}^-$. We have

$$f_{\rm l}(-k^*,x)=f_{\rm l}(k,x)^*,\quad f_{\rm r}(-k^*,x)=f_{\rm r}(k,x)^*,\qquad k\in\mathbf{C},$$

$$T(-k^*)=T(k)^*,\quad R(-k^*)=R(k)^*,\quad L(-k^*)=L(k)^*,\qquad k\in\mathbf{C},$$

where the asterisk denotes complex conjugation. Note, in particular, that

$$|R(k)|^2=R(k)\,R(-k),\qquad k\in\mathbf{R}. \tag{2.7}$$

The scattering coefficients satisfy

$$T(k)\,T(-k)+R(k)\,R(-k)=1,\qquad k\in\mathbf{R}, \tag{2.8}$$

$$T(k)\,T(-k)+L(k)\,L(-k)=1,\qquad k\in\mathbf{R},$$

$$L(k)\,T(-k)+R(-k)\,T(k)=0,\qquad k\in\mathbf{R}, \tag{2.9}$$

with appropriate meromorphic extensions to $\mathbf{C}$.

In the rest of this section we outline the construction of T and L from R. From (2.7) and (2.8) we get

$$T(k)=\left[1-|R(k)|^2\right]\frac{1}{T(-k)},\qquad k\in\mathbf{R}. \tag{2.10}$$

If $R(k)$ is a rational function of $k\in\mathbf{R}$, so is $1-|R(k)|^2$. When V belongs to the Faddeev class and has no bound states, it is known [1-5] that $T(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, nonzero in $\overline{\mathbf{C}^+}\setminus\{0\}$, and $1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Generically $T(k)$ has a simple zero at $k=0$, and $T(0)\neq 0$ in the exceptional case. We have $R(k)=o(1/k)$ as $k\to\pm\infty$ in $\mathbf{R}$, and hence the rationality of R implies that $R(k)=O(1/k^2)$ as $k\to\infty$ in $\mathbf{C}$. With the help of (2.10), by factoring both the numerator and the denominator of $1-|R(k)|^2$, we can obtain $T(k)$ by separating the zeros and poles of $1-|R(k)|^2$ in $\mathbf{C}^+$ and in $\mathbf{C}^-$.

In the exceptional case it is known [1-5] that $|R(k)|<1$ for $k\in\mathbf{R}$. Thus, in that case we get

$$1-|R(k)|^2=\frac{\prod(k-k_a^+)}{\prod(k-k_m^+)}\,\frac{\prod(k-k_b^-)}{\prod(k-k_n^-)},\qquad k\in\mathbf{R}, \tag{2.11}$$

where $k_a^+,k_m^+\in\mathbf{C}^+$ and $k_b^-,k_n^-\in\mathbf{C}^-$. Note that the left hand side in (2.11) is an even function of k and it converges to 1 as $k\to\pm\infty$. As a result we find that the degrees of the four polynomials $\prod(k-k_a^+)$, $\prod(k-k_b^-)$, $\prod(k-k_m^+)$, and $\prod(k-k_n^-)$ are all the same. Hence, from (2.10) and (2.11) we get

$$T(k)\,\frac{\prod(k-k_n^-)}{\prod(k-k_b^-)}=\frac{1}{T(-k)}\,\frac{\prod(k-k_a^+)}{\prod(k-k_m^+)}, \tag{2.12}$$

where the left hand side is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and $1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Similarly, the right hand side of (2.12) is analytic in $\mathbf{C}^-$, continuous in $\overline{\mathbf{C}^-}$, and $1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^-}$. With the help of Morera's theorem, we conclude that each side of (2.12) must be equal to an entire function on $\mathbf{C}$ that converges to 1 at ∞. By Liouville's theorem both sides must then be equal to 1. Therefore, we obtain

$$T(k)=\frac{\prod(k-k_b^-)}{\prod(k-k_n^-)},\qquad k\in\mathbf{C}. \tag{2.13}$$

The argument given above can easily be adapted to the generic case. In the generic case, it is known [1-5] that $T(k)=O(k)$ as $k\to 0$, and the construction of T from R is similarly obtained by replacing exactly one of k_b^- with zero and exactly one of k_a^+ with zero in the above argument.

Let us write the expression in (2.13) for $T(k)$ in a slightly different notation which will be useful in Section 5:

$$T(k)=\begin{cases}\dfrac{k\prod_{j=1}^{n_z}(k+z_j)}{\prod_{j=1}^{n_z+1}(k+p_j)}, & T(0)=0,\\[2ex] \dfrac{\prod_{j=1}^{n_z}(k+z_j)}{\prod_{j=1}^{n_z}(k+p_j)}, & T(0)\neq 0,\end{cases} \tag{2.14}$$

where the z_j for $1\le j\le n_z$ correspond to the zeros of $T(-k)$ in $\mathbf{C}^+$ and the p_j correspond to the poles there. Thus, the poles of $1/T(-k)$ in $\mathbf{C}^+$ occur at $k=z_j$, and let us use τ_j to denote the residues there:

$$\tau_j:=\operatorname{Res}\left(\frac{1}{T(-k)},z_j\right),\qquad j=1,\dots,n_z. \tag{2.15}$$

Once $T(k)$ is constructed, with the help of (2.9) we obtain

$$L(k)=-\frac{R(-k)\,T(k)}{T(-k)}.$$

3. Construction on the Positive Half Line

In this section, when $x>0$ we explicitly construct the Jost solutions and the potential in terms of the poles and residues of $R(k)$ in $\mathbf{C}^+$. We use n_{l} to denote the number of poles of $R(k)$ in $\mathbf{C}^+$, assume that such poles are simple and occur at $k=k_{\mathrm{l}j}$ in $\mathbf{C}^+$, and use $\rho_{\mathrm{l}j}$ to denote the corresponding residues.

We define

$$B_{\mathrm{l}}(x,\alpha):=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,[m_{\mathrm{l}}(k,x)-1]\,e^{-ik\alpha}. \tag{3.1}$$

When $\alpha<0$, we have $B_{\mathrm{l}}(x,\alpha)=0$ due to, for each fixed x, the analyticity of $m_{\mathrm{l}}(k,x)$ in $\mathbf{C}^+$, the continuity of $m_{\mathrm{l}}(k,x)$ in $\overline{\mathbf{C}^+}$, and the fact that $m_{\mathrm{l}}(k,x)=$

$1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. From (2.5) we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,[m_{\rm l}(-k,x)-1]\,e^{ik\alpha} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,R(k)\,m_{\rm l}(k,x)e^{ik(2x+\alpha)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,[T(k)\,m_{\rm r}(k,x)-1]\,e^{ik\alpha}. \tag{3.2}$$

The second integral on the right hand side of (3.2) vanishes when $\alpha>0$ due to the fact that $T(k)$ and $m_{\rm r}(k,x)$ are analytic for $k\in\mathbf{C}^+$ and continuous for $k\in\overline{\mathbf{C}^+}$, and $T(k)\,m_{\rm r}(k,x)=1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Thus, from (3.1) and (3.2) we obtain

$$B_{\rm l}(x,\alpha)=-\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,R(k)\,e^{2ikx+ik\alpha}m_{\rm l}(k,x), \qquad \alpha>0. \tag{3.3}$$

From (3.1) and the fact that $B_{\rm l}(x,\alpha)=0$ for $\alpha<0$, we have

$$m_{\rm l}(k,x)=1+\int_0^{\infty} d\alpha\,B_{\rm l}(x,\alpha)\,e^{ik\alpha}. \tag{3.4}$$

When $2x+\alpha>0$, the integral in (3.3) can be evaluated as a contour integral along the boundary of $\mathbf{C}^+$, to which the only contribution comes from the poles of $R(k)$ in $\mathbf{C}^+$. Since such poles are assumed to be simple, we get

$$B_{\rm l}(x,\alpha)=-i\sum_{j=1}^{n_{\rm l}}\rho_{{\rm l}j}\,e^{2ik_{{\rm l}j}x+ik_{{\rm l}j}\alpha}m_{\rm l}(k_{{\rm l}j},x), \qquad 2x+\alpha>0, \quad \alpha>0. \tag{3.5}$$

Using (3.5) in (3.4), with the help of

$$\int_0^{\infty} d\alpha\,e^{i(k+k_{{\rm l}j})\alpha}=\frac{i}{k+k_{{\rm l}j}}, \qquad k\in\overline{\mathbf{C}^+},$$

we get

$$m_{\rm l}(k,x)=1+\sum_{j=1}^{n_{\rm l}}\frac{\rho_{{\rm l}j}\,e^{2ik_{{\rm l}j}x}}{k+k_{{\rm l}j}}\,m_{\rm l}(k_{{\rm l}j},x), \qquad x\ge 0. \tag{3.6}$$

We are interested in determining $m_{\rm l}(k_{{\rm l}j},x)$ appearing in (3.5) and (3.6). To do so, we put $k=k_{{\rm l}p}$ in (3.6) for $1\le p\le n_{\rm l}$. Then, for $x\ge 0$ we obtain

$$m_{\rm l}(k_{{\rm l}p},x)=1+\sum_{j=1}^{n_{\rm l}}\frac{\rho_{{\rm l}j}\,e^{2ik_{{\rm l}j}x}}{k_{{\rm l}p}+k_{{\rm l}j}}m_{\rm l}(k_{{\rm l}j},x), \qquad p=1,\dots,n_{\rm l}. \tag{3.7}$$

Notice that (3.7) is a linear algebraic system and can be written as

$$\mathbf{M}_{\rm l}(x)\begin{bmatrix} m_{\rm l}(k_{{\rm l}1},x)\\ \vdots \\ m_{\rm l}(k_{{\rm l}n_{\rm l}},x)\end{bmatrix}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}, \tag{3.8}$$

where $\mathbf{M}_{\rm l}(x)$ is the $n_{\rm l}\times n_{\rm l}$ matrix-valued function whose (p,q)-entry is given by

$$[\mathbf{M}_{\rm l}(x)]_{pq}:=\delta_{pq}-\frac{\rho_{{\rm l}q}\,e^{2ik_{{\rm l}q}x}}{k_{{\rm l}p}+k_{{\rm l}q}}, \tag{3.9}$$

with δ_{pq} denoting the Kronecker delta. The unique solvability of the linear system in (3.8) and hence the invertibility of $\mathbf{M}_{\rm l}(x)$ follow from Corollary 4.2 of [6]. Using (3.8) in (3.6) we get for $x \geq 0$

$$m_{\rm l}(k,x) = 1 + \begin{bmatrix} \dfrac{\rho_{{\rm l}1}\, e^{2ik_{{\rm l}1}x}}{k+k_{{\rm l}1}} & \dots & \dfrac{\rho_{{\rm l}n_{\rm l}}\, e^{2ik_{{\rm l}n_{\rm l}}x}}{k+k_{{\rm l}n_{\rm l}}} \end{bmatrix} \mathbf{M}_{\rm l}(x)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{3.10}$$

We can simplify (cf. p.12 of [17]) the bilinear form in (3.10) and obtain for $x \geq 0$

$$m_{\rm l}(k,x) = 1 - \frac{1}{\det \mathbf{M}_{\rm l}(x)} \begin{vmatrix} 0 & \dfrac{\rho_{{\rm l}1}\, e^{2ik_{{\rm l}1}x}}{k+k_{{\rm l}1}} & \dots & \dfrac{\rho_{{\rm l}n_{\rm l}}\, e^{2ik_{{\rm l}n_{\rm l}}x}}{k+k_{{\rm l}n_{\rm l}}} \\ 1 & & & \\ \vdots & & \mathbf{M}_{\rm l}(x) & \\ 1 & & & \end{vmatrix}, \tag{3.11}$$

and hence we have written $m_{\rm l}(k,x)-1$ as the ratio of two determinants that are constructed solely in terms of the $k_{{\rm l}j}$ and $\rho_{{\rm l}j}$ with $1 \leq j \leq n_{\rm l}$.

Similarly, from (3.5) and (3.8) we get

$$B_{\rm l}(x,\alpha) = -i \begin{bmatrix} \rho_{{\rm l}1}\, e^{ik_{{\rm l}1}(2x+\alpha)} & \dots & \rho_{{\rm l}n_{\rm l}}\, e^{ik_{{\rm l}n_{\rm l}}(2x+\alpha)} \end{bmatrix} \mathbf{M}_{\rm l}(x)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

or equivalently

$$B_{\rm l}(x,\alpha) = \frac{\det \Gamma_{\rm l}(x,\alpha)}{\det \mathbf{M}_{\rm l}(x)}, \qquad x \geq 0, \quad \alpha > 0, \tag{3.12}$$

where $\Gamma_{\rm l}(x,\alpha)$ is the $(n_{\rm l}+1)\times(n_{\rm l}+1)$ matrix defined as

$$\Gamma_{\rm l}(x,\alpha) := \begin{bmatrix} 0 & i\rho_{{\rm l}1}\, e^{2ik_{{\rm l}1}x+ik_{{\rm l}1}\alpha} & \dots & i\rho_{{\rm l}n_{\rm l}}\, e^{2ik_{{\rm l}n_{\rm l}}x+ik_{{\rm l}n_{\rm l}}\alpha} \\ 1 & & & \\ \vdots & & \mathbf{M}_{\rm l}(x) & \\ 1 & & & \end{bmatrix}. \tag{3.13}$$

It is pleasantly surprising that we have

$$\det \Gamma_{\rm l}(x,0^+) = \frac{d}{dx} \det \mathbf{M}_{\rm l}(x). \tag{3.14}$$

The proof of (3.14) is somehow involved, and we briefly describe the basic steps in the proof. First, in the matrix $\Gamma_{\rm l}(x,0^+)$, multiply the $(j+1)$st column by $e^{-ik_{{\rm l}j}x}$ and the $(j+1)$st row by $e^{ik_{{\rm l}j}x}$ for all $1 \leq j \leq n_{\rm l}$. The determinant remains unchanged. Then, use the cofactor expansion of the resulting determinant with respect to the first column and get

$$\det \Gamma_{\rm l}(x,0^+) = 0\,|\cdot| - e^{ik_{{\rm l}1}x}\,|\cdot| + e^{ik_{{\rm l}2}x}\,|\cdot| - ... + (-1)^{n_{\rm l}} e^{ik_{{\rm l}n_{\rm l}}x}\,|\cdot|, \tag{3.15}$$

where $|\cdot|$ denotes the appropriate subdeterminant. Next, put each coefficient $e^{ik_{1j}x}$ on the right hand side of (3.15) into the first row of the corresponding subdeterminant. We need to show that the resulting quantity is equal to the x-derivative of $\det \mathbf{M}_{\rm l}(x)$. In order to do so, in the matrix $\mathbf{M}_{\rm l}(x)$ multiply the jth row by $e^{ik_{1j}x}$ and the jth column by $e^{-ik_{1j}x}$ for $1 \le j \le n_{\rm l}$, which results in no change in $\det \mathbf{M}_{\rm l}(x)$. Now take the x-derivative of the resulting determinant and write it as a sum where the jth term is the determinant of a matrix obtained by taking the x-derivative of the jth row of $\mathbf{M}_{\rm l}(x)$. Then rewrite each term in the summation so that the row whose derivative has been evaluated is moved to the first row while the remaining rows are left in the same order. By comparison, we then conclude (3.14).

Using (3.14) in (3.12) we get

$$B_{\rm l}(x,0^+) = \frac{\frac{d}{dx}\det \mathbf{M}_{\rm l}(x)}{\det \mathbf{M}_{\rm l}(x)}, \qquad x \ge 0. \tag{3.16}$$

It is known [1-5] that

$$V(x) = -2\frac{d}{dx} B_{\rm l}(x,0^+), \qquad x \in \mathbf{R}. \tag{3.17}$$

Therefore, using (3.16) in (3.17) we get

$$V(x) = -2\frac{d}{dx}\left[\frac{\frac{d}{dx}\det \mathbf{M}_{\rm l}(x)}{\det \mathbf{M}_{\rm l}(x)}\right], \qquad x > 0. \tag{3.18}$$

From (3.9) we see that $\mathbf{M}_{\rm l}(x)$ is uniquely constructed in terms of the poles and residues of $R(k)$ in $\mathbf{C}^+$. Thus, in (3.18) we have expressed $V(x)$ for $x > 0$ in terms of the $(2n_{\rm l})$ constants $k_{\rm l j}$ and $\rho_{\rm l j}$ alone.

Alternatively, having constructed $m_{\rm l}(k,x)$ for $x \ge 0$ as in (3.11), we can use (2.2) and obtain the potential for $x > 0$ as

$$V(x) = \frac{m_{\rm l}''(k,x) + 2ik\, m_{\rm l}'(k,x)}{m_{\rm l}(k,x)}. \tag{3.19}$$

Note that even though the parameter k appears in the individual terms on the right hand side in (3.19), it is absent from the right hand side as a whole. In particular, using $k = 0$ in (3.19) we can evaluate $V(x)$ for $x > 0$ as

$$V(x) = \frac{m_{\rm l}''(0,x)}{m_{\rm l}(0,x)}. \tag{3.20}$$

We have shown that, starting with a rational right reflection coefficient R, one can explicitly construct $m_{\rm l}(k,x)$ for $x \ge 0$ and $V(x)$ for $x > 0$. We can then obtain $f_{\rm l}(k,x)$ via (2.1). This means that we also have $f_{\rm l}(-k,x)$ in hand. Then, using (2.3) we also get $f_{\rm r}(k,x)$ for $x \ge 0$ as

$$f_{\rm r}(k,x) = \frac{f_{\rm l}(-k,x) + R(k)\, f_{\rm l}(k,x)}{T(k)},$$

with $T(k)$ as in (2.14).

4. Construction on the Negative Half Line

In this section we present the explicit construction of the Jost solutions and the potential when $x<0$. Finding $m_{\rm r}(k,x)$ and $V(x)$ for $x<0$ in terms of $L(k)$ is similar to the construction of $m_{\rm l}(k,x)$ and $V(x)$ for $x>0$ from R as outlined in Section 3. As we shall see in (4.6) and (4.10), the explicit formulas for $m_{\rm r}(k,x)$ and $V(x)$ for $x<0$ are written in terms of the poles and residues of $L(k)$ in $\mathbf{C}^+$. We let $n_{\rm r}$ denote the number of poles of $L(k)$ in $\mathbf{C}^+$, use $k_{{\rm r}j}$ and $\rho_{{\rm r}j}$ to denote those (simple) poles and the corresponding residues, respectively.

Let

$$B_{\rm r}(x,\alpha):=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,[m_{\rm r}(k,x)-1]\,e^{-ik\alpha}\,dk. \tag{4.1}$$

When $\alpha<0$, we get $B_{\rm r}(x,\alpha)=0$ because, for each fixed x, $m_{\rm r}(k,x)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and $1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Starting with (2.6) we show [cf. (3.3)] that

$$B_{\rm r}(x,\alpha)=-\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,L(k)\,e^{-2ikx+ik\alpha}m_{\rm r}(k,x),\qquad \alpha>0, \tag{4.2}$$

and thus, [cf. (3.4)] we obtain

$$m_{\rm r}(k,x)=1+\int_0^{\infty} d\alpha\,B_{\rm r}(x,\alpha)\,e^{ik\alpha}. \tag{4.3}$$

In order to evaluate the integral in (4.2), we use a contour integration along the infinite semicircle which is the boundary of $\mathbf{C}^+$. Since the poles of $L(k)$ in $\mathbf{C}^+$ are assumed to be simple, we obtain [cf. (3.5)]

$$B_{\rm r}(x,\alpha)=-i\sum_{j=1}^{n_{\rm r}}\rho_{{\rm r}j}\,e^{-2ik_{{\rm r}j}x+ik_{{\rm r}j}\alpha}m_{\rm r}(k_{{\rm r}j},x),\qquad -2x+\alpha>0,\quad \alpha>0. \tag{4.4}$$

Using (4.4) in (4.3) we get [cf. (3.6)]

$$m_{\rm r}(k,x)=1+\sum_{j=1}^{n_{\rm r}}\frac{\rho_{{\rm r}j}\,e^{-2ik_{{\rm r}j}x}}{k+k_{{\rm r}j}}\,m_{\rm r}(k_{{\rm r}j},x),\qquad x\le 0. \tag{4.5}$$

Proceeding as in Section 3 leading to (3.10), we get for $x\le 0$

$$m_{\rm r}(k,x)=1+\begin{bmatrix}\dfrac{\rho_{{\rm r}1}\,e^{-2ik_{{\rm r}1}x}}{k+k_{{\rm r}1}} & \dots & \dfrac{\rho_{{\rm r}n_{\rm r}}\,e^{-2ik_{{\rm r}n_{\rm r}}x}}{k+k_{{\rm r}n_{\rm r}}}\end{bmatrix}\mathbf{M}_{\rm r}(x)^{-1}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix},$$

or equivalently

$$m_{\rm r}(k,x)=1-\frac{1}{\det\mathbf{M}_{\rm r}(x)}\begin{vmatrix}0 & \dfrac{\rho_{{\rm r}1}\,e^{-2ik_{{\rm r}1}x}}{k+k_{{\rm r}1}} & \dots & \dfrac{\rho_{{\rm r}n_{\rm r}}\,e^{-2ik_{{\rm r}n_{\rm r}}x}}{k+k_{{\rm r}n_{\rm r}}}\\ 1 & & & \\ \vdots & & \mathbf{M}_{\rm r}(x) & \\ 1 & & & \end{vmatrix}, \tag{4.6}$$

where $\mathbf{M}_{\rm r}(x)$ is the $n_{\rm r} \times n_{\rm r}$ matrix-valued function whose (p, q)-entry is given by

$$[\mathbf{M}_{\rm r}(x)]_{pq} := \delta_{pq} - \frac{\rho_{{\rm r}q}\, e^{-2ik_{{\rm r}q}x}}{k_{{\rm r}p} + k_{{\rm r}q}}. \tag{4.7}$$

Let us remark that the invertibility of $\mathbf{M}_{\rm r}(x)$ follows from Corollary 4.2 of [6]. Then [cf. (3.12)]

$$B_{\rm r}(x, \alpha) = \frac{\det \Gamma_{\rm r}(x, \alpha)}{\det \mathbf{M}_{\rm r}(x)}, \qquad x \le 0, \quad \alpha > 0, \tag{4.8}$$

where $\Gamma_{\rm r}(x, \alpha)$ is the $(n_{\rm r} + 1) \times (n_{\rm r} + 1)$ matrix defined as

$$\Gamma_{\rm r}(x, \alpha) := \begin{bmatrix} 0 & i\rho_{{\rm r}1}\, e^{-2ik_{{\rm r}1}x + ik_{{\rm r}1}\alpha} & \dots & i\rho_{{\rm r}n_{\rm r}}\, e^{-2ik_{{\rm r}n_{\rm r}}x + ik_{{\rm r}n_{\rm r}}\alpha} \\ 1 & & & \\ \vdots & & \mathbf{M}_{\rm r}(x) & \\ 1 & & & \end{bmatrix}. \tag{4.9}$$

Similarly as in the proof of (3.14), it can be shown that

$$\det \Gamma_{\rm r}(x, 0^+) = -\frac{d}{dx} \det \mathbf{M}_{\rm r}(x),$$

and hence (4.8) implies

$$B_{\rm r}(x, 0^+) = \frac{-\frac{d}{dx} \det \mathbf{M}_{\rm r}(x)}{\det \mathbf{M}_{\rm r}(x)}, \qquad x \le 0.$$

It is known [1-5] that

$$V(x) = 2\frac{d}{dx} B_{\rm r}(x, 0^+), \qquad x \in \mathbf{R},$$

and hence we obtain

$$V(x) = -2\frac{d}{dx} \left[\frac{\frac{d}{dx} \det \mathbf{M}_{\rm r}(x)}{\det \mathbf{M}_{\rm r}(x)} \right], \qquad x < 0. \tag{4.10}$$

Alternatively, having constructed $m_{\rm r}(k, x)$ as in (4.6) for $x \le 0$, we can evaluate $V(x)$ for $x < 0$ via [cf. (3.19) and (3.20)]

$$V(x) = \frac{m_{\rm r}''(k, x) - 2ik\, m_{\rm r}'(k, x)}{m_{\rm r}(k, x)}, \tag{4.11}$$

$$V(x) = \frac{m_{\rm r}''(0, x)}{m_{\rm r}(0, x)}. \tag{4.12}$$

Note that the right hand side in (4.11) is independent of k as a whole.

If we start with R, we can construct T and L as in Section 2. Then, using the poles and residues of $L(k)$ in $\mathbf{C}^+$, we can construct $m_{\rm r}(k, x)$ for $x \le 0$ and $V(x)$ for $x < 0$. We then obtain $f_{\rm r}(k, x)$ with the help of (2.1) and (4.6). Having $f_{\rm r}(k, x)$ and $f_{\rm r}(-k, x)$ in hand, via (2.4) we construct $f_{\rm l}(k, x)$ for $x \le 0$ by using

$$f_{\rm l}(k, x) = \frac{f_{\rm r}(-k, x) + L(k)\, f_{\rm r}(k, x)}{T(k)}.$$

5. Wave Propagation and Jost Waves

We now wish to analyze certain solutions to the plasma-wave equation (1.2) when V belongs to the Faddeev class, there are no bound states, and the corresponding reflection coefficients are rational functions of k with simple poles in $\mathbf{C}^+$.

We define the Jost wave from the left as

$$J_{\rm l}(x,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, f_{\rm l}(k,x)\, e^{-ikt}. \tag{5.1}$$

Using (2.1) in (5.1), we get

$$J_{\rm l}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-t)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, [m_{\rm l}(k,x)-1]\, e^{ik(x-t)}. \tag{5.2}$$

From (3.1), (5.2), and the fact that $\int_{-\infty}^{\infty} dk\, e^{ik\alpha} = 2\pi\,\delta(\alpha)$, we obtain

$$J_{\rm l}(x,t) = \delta(x-t) + B_{\rm l}(x,t-x), \qquad x,t\in\mathbf{R}, \tag{5.3}$$

where $\delta(x)$ denotes the Dirac delta distribution. Note that $B_{\rm l}(x,t-x)=0$ if $t-x<0$ because $B_{\rm l}(x,\alpha)=0$ for $\alpha<0$, as we have seen in Section 3. Thus,

$$J_{\rm l}(x,t) = 0, \qquad x-t>0. \tag{5.4}$$

Comparing (5.3) with (3.12), we see that when $x\ge 0$ and $t-x>0$, we can express $B_{\rm l}(x,t-x)$ as the ratio of two determinants that are constructed explicitly with the help of (3.9) and (3.13). Hence, we have

$$J_{\rm l}(x,t) = \frac{\det \Gamma_{\rm l}(x,t-x)}{\det \mathbf{M}_{\rm l}(x)}, \qquad x\ge 0, \quad t-x>0. \tag{5.5}$$

We also need $J_{\rm l}(x,t)$ in the region with $x-t<0$ and $x+t<0$. Towards that goal, we can use (2.6) in (5.2) and get

$$J_{\rm l}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-t)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left[\frac{m_{\rm r}(-k,x)}{T(k)}-1\right] e^{ik(x-t)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{L(k)}{T(k)}\, m_{\rm r}(k,x)\, e^{-ik(x+t)},$$

or equivalently

$$J_{\rm l}(x,t) = \delta(x-t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left[\frac{m_{\rm r}(k,x)}{T(-k)}-1\right] e^{-ik(x-t)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{L(k)}{T(k)}\, m_{\rm r}(k,x)\, e^{-ik(x+t)}. \tag{5.6}$$

This separation causes each of the two integrands in (5.6) to have a simple pole at $k=0$ in the generic case. The zeros of $T(-k)$ in $\mathbf{C}^+$ contribute to the first integral in (5.6). The poles of $L(k)$ in $\mathbf{C}^+$ contribute to the second integral in (5.6). We find that the contribution from $k = 0$ to the two integrals on the right hand side of (5.6) is given by

$$i\,\theta(-x)\,[\theta(x+t)-\theta(t-x)]\;m_{\rm r}(0,x)\,\text{Res}\,(1/T(k),0)\,, \tag{5.7}$$

where $\theta(x)$ is the Heaviside function defined as

$$\theta(x) := \begin{cases} 1, & x>0,\\ 0, & x<0. \end{cases}$$

Note that (5.3) can be evaluated by using the fact that $L(0) = -1$ in the generic case and that $T(k)$ is analytic and nonzero in $\mathbf{C}^+$.

As in Section 4, let us use $k_{{\rm r}j}$ to denote the poles of $L(k)$ in $\mathbf{C}^+$ and $\rho_{{\rm r}j}$ the residues there. Similarly, as in (2.14) and (2.15) let us use z_j for the poles of $1/T(-k)$ in $\mathbf{C}^+$ and τ_j for the corresponding residues for $1 \le j \le n_z$. The contributions to the right hand side of (5.6) from the zeros of $T(-k)$ and the poles of $L(k)$ in $\mathbf{C}^+$ can be evaluated by using a contour integration along the infinite semicircle enclosing $\mathbf{C}^+$. Hence, in the region with $t-x>0$ and $x+t<0$, that contribution is given by

$$i\sum_{j=1}^{n_z}\tau_j\,m_{\rm r}(z_j,x)\,e^{-iz_j(x-t)}+i\sum_{j=1}^{n_{\rm r}}\frac{\rho_{{\rm r}j}\,e^{-ik_{{\rm r}j}(x+t)}}{T(k_{{\rm r}j})}\,m_{\rm r}(k_{{\rm r}j},x). \tag{5.8}$$

From (5.7) and (5.8) we see that, in the region with $t-x>0$ and $x+t<0$, the Jost wave from the left is given by

$$\begin{aligned} J_{\rm l}(x,t) = -i\,m_{\rm r}(0,x)\,\text{Res}\,(1/T(k),0)&+i\sum_{j=1}^{n_z}\tau_j\,m_{\rm r}(z_j,x)\,e^{-iz_j(x-t)}\\ &+i\sum_{j=1}^{n_{\rm r}}\frac{\rho_{{\rm r}j}\,e^{-ik_{{\rm r}j}(x+t)}}{T(k_{{\rm r}j})}\,m_{\rm r}(k_{{\rm r}j},x). \end{aligned} \tag{5.9}$$

Note that $m_{\rm r}(z_j,x)$ and $m_{\rm r}(k_{{\rm r}j},x)$ can be evaluated explicitly by using (4.6) and $T(k_{{\rm r}j})$ by using (2.14).

Finally, we will evaluate $J_{\rm l}(x,t)$ in the region with $x<0$ and $x+t>0$. In that region, the contribution to the first integral in (5.6) from the zeros of $T(-k)$ in $\mathbf{C}^+$ is evaluated with the help of a contour integration along the boundary of $\mathbf{C}^+$ and we get the first summation term in (5.8). However, the second integral in (5.6) needs to be evaluated as a contour integral along the boundary of $\mathbf{C}^-$ due to the presence of the exponential term $e^{-ik(x+t)}$ in the integrand. With the help of (2.9) we write that integral as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\frac{L(k)}{T(k)}\,m_{\rm r}(k,x)\,e^{-ik(x+t)}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\frac{R(k)}{T(k)}\,m_{\rm r}(-k,x)\,e^{ik(x+t)}, \tag{5.10}$$

where the right hand side can now be evaluated as a contour integral along the boundary of $\mathbf{C}^+$. Let us now evaluate the contribution to that integral coming from the poles of $R(k)$ in $\mathbf{C}^+$ and also the poles of $m_{\rm r}(-k,x)$ in $\mathbf{C}^+$. From Section 3 and

(4.5) we see that the former poles occur at $k = k_{\mathrm{l}j}$ and the latter occur at $k = k_{\mathrm{r}j}$. As a result, such contributions to the right hand side of (5.10) can be explicitly evaluated. For example, when the sets $\{k_{\mathrm{l}j}\}_{j=1}^{n_{\mathrm{l}}}$ and $\{k_{\mathrm{r}j}\}_{j=1}^{n_{\mathrm{r}}}$ do not intersect, we get

$$-i\sum_{j=1}^{n_{\mathrm{l}}} \frac{\rho_{\mathrm{l}j}\, e^{ik_{\mathrm{l}j}(x+t)}}{T(k_{\mathrm{l}j})}\, m_{\mathrm{r}}(-k_{\mathrm{l}j},x) - i\sum_{j=1}^{n_{\mathrm{r}}} \frac{R(k_{\mathrm{r}j})\, e^{ik_{\mathrm{r}j}(x+t)}}{T(k_{\mathrm{r}j})} \mathrm{Res}\,(m_{\mathrm{r}}(-k,x),k_{\mathrm{r}j})\,. \tag{5.11}$$

From (4.5) we see that

$$\mathrm{Res}\,(m_{\mathrm{r}}(-k,x),k_{\mathrm{r}j}) = -\rho_{\mathrm{r}j}\, e^{-2ik_{\mathrm{r}j}x}\, m_{\mathrm{r}}(k_{\mathrm{r}j},x),$$

and thus, in the region with $x<0$ and $x+t>0$, with the help of (5.6)-(5.11) we obtain

$$\begin{aligned} J_{\mathrm{l}}(x,t) = i\sum_{j=1}^{n_{\mathrm{z}}} \tau_j\, m_{\mathrm{r}}(z_j,x)\, e^{-iz_j(x-t)} - i\sum_{j=1}^{n_{\mathrm{l}}} \frac{\rho_{\mathrm{l}j}\, e^{ik_{\mathrm{l}j}(x+t)}}{T(k_{\mathrm{l}j})}\, m_{\mathrm{r}}(-k_{\mathrm{l}j},x) \\ + i\sum_{j=1}^{n_{\mathrm{r}}} \frac{\rho_{\mathrm{r}j}\, R(k_{\mathrm{r}j})\, e^{-ik_{\mathrm{r}j}(x-t)}}{T(k_{\mathrm{r}j})}\, m_{\mathrm{r}}(k_{\mathrm{r}j},x), \end{aligned} \tag{5.12}$$

where we note that there is no contribution to $J_{\mathrm{l}}(x,t)$ from the poles at $k=0$ [cf. (5.7)] in the region with $x<0$ and $x+t>0$. In case the sets $\{k_{\mathrm{l}j}\}_{j=1}^{n_{\mathrm{l}}}$ and $\{k_{\mathrm{r}j}\}_{j=1}^{n_{\mathrm{r}}}$ partially or wholly overlap, the integral on the right hand side of (5.10) can be evaluated explicitly in a similar way as a contour integral along the boundary of $\mathbf{C}^+$ and the result in (5.12) can be modified appropriately.

We can write the Jost wave $J_{\mathrm{l}}(x,t)$ by combining (5.3)-(5.5), (5.9), and (5.12) as

$$\begin{aligned} J_{\mathrm{l}}(x,t) =& \delta(x-t) + \theta(x)\,\theta(t-x)\frac{\det\Gamma_{\mathrm{l}}(x,t-x)}{\det\mathbf{M}_{\mathrm{l}}(x)} \\ &+ i\,\theta(-x)\,\theta(t-x)\sum_{j=1}^{n_{\mathrm{z}}} \tau_j\, m_{\mathrm{r}}(z_j,x)\, e^{-iz_j(x-t)} \\ &- i\,\theta(t-x)\,\theta(-x-t)\, m_{\mathrm{r}}(0,x)\,\mathrm{Res}\,(1/T(k),0) \\ &+ i\,\theta(t-x)\,\theta(-x-t)\sum_{j=1}^{n_{\mathrm{r}}} \frac{\rho_{\mathrm{r}j}\, e^{-ik_{\mathrm{r}j}(x+t)}}{T(k_{\mathrm{r}j})}\, m_{\mathrm{r}}(k_{\mathrm{r}j},x) \\ &- i\,\theta(-x)\,\theta(x+t)\sum_{j=1}^{n_{\mathrm{l}}} \frac{\rho_{\mathrm{l}j}\, e^{ik_{\mathrm{l}j}(x+t)}}{T(k_{\mathrm{l}j})}\, m_{\mathrm{r}}(-k_{\mathrm{l}j},x) \\ &+ i\,\theta(-x)\,\theta(x+t)\sum_{j=1}^{n_{\mathrm{r}}} \frac{\rho_{\mathrm{r}j}\, R(k_{\mathrm{r}j})\, e^{-ik_{\mathrm{r}j}(x-t)}}{T(k_{\mathrm{r}j})}\, m_{\mathrm{r}}(k_{\mathrm{r}j},x), \end{aligned}$$

and hence, in terms of the quantities that have been constructed explicitly once the rational reflection coefficient R is known, we have constructed $J_{\mathrm{l}}(x,t)$ for all $x,t\in\mathbf{R}$ except when $x=t$ and also when $x=-t$ with $x<0$. In Section 6, we will see that $J_{\rm l}(x,t)$ may have jump discontinuities when $x=t$ and also when $x=-t$ with $x<0$, and we will evaluate those discontinuities.

We define the Jost wave from the right in a similar manner, by letting

$$J_{\rm r}(x,t):=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,f_{\rm r}(k,x)\,e^{-ikt}. \tag{5.13}$$

Using (2.1) in (5.13), we get [cf. (5.2)]

$$J_{\rm r}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{-ik(x+t)}+\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,[m_{\rm r}(k,x)-1]\,e^{-ik(x+t)}, \tag{5.14}$$

which can be written as [cf. (4.1)]

$$J_{\rm r}(x,t)=\delta(x+t)+B_{\rm r}(x,x+t). \tag{5.15}$$

Note that $B_{\rm r}(x,x+t)=0$ if $x+t<0$ due to, for each fixed x, the analyticity in $\mathbf{C}^+$, the continuity in $\overline{\mathbf{C}^+}$, and the $O(1/k)$-behavior as $k\to\infty$ in $\overline{\mathbf{C}^+}$ of $m_{\rm r}(k,x)-1$. Thus,

$$J_{\rm r}(x,t)=0, \qquad x+t<0. \tag{5.16}$$

Comparing (4.8) and (5.15), when $x<0$ and $x+t>0$, we see that we can write $B_{\rm r}(x,x+t)$ as the ratio of two determinants and obtain [cf. (5.5)]

$$J_{\rm r}(x,t)=\frac{\det\Gamma_{\rm r}(x,x+t)}{\det\mathbf{M}_{\rm r}(x)}, \qquad x\le 0, \quad x+t>0, \tag{5.17}$$

where $\mathbf{M}_{\rm r}(x)$ and $\Gamma_{\rm r}(x,\alpha)$ are as in (4.7) and (4.9), respectively.

Next, we will obtain $J_{\rm r}(x,t)$ in the region with $x+t>0$ and $x-t>0$. Using (2.5) in (5.14), we get [cf. (5.6)]

$$\begin{aligned}J_{\rm r}(x,t)=\delta(x+t)+&\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\left[\frac{m_{\rm l}(k,x)}{T(-k)}-1\right]e^{ik(x+t)}\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\frac{R(k)}{T(k)}\,m_{\rm l}(k,x)\,e^{ik(x-t)}.\end{aligned} \tag{5.18}$$

The zeros of $T(-k)$ in $\mathbf{C}^+$ contribute to the first integral on the right hand side of (5.18). As in (5.6), we evaluate that integral as a contour integral along the boundary of $\mathbf{C}^+$. The poles of $R(k)$ in $\mathbf{C}^+$ contribute to the second integral in (5.18). In the generic case, each of the two integrands in (5.18) has a simple pole at $k=0$ because of the simple zero of $T(k)$ there; the contribution from $k=0$ in the two integrals in (5.18) can be evaluated as in (5.7) and we get

$$i\,\theta(x)\,[\theta(t-x)-\theta(x+t)]\,m_{\rm l}(0,x)\,\mathrm{Res}\,(1/T(k),0)\,. \tag{5.19}$$

Recalling that $\{k_{\mathrm{l}j}\}$ is the set poles of $R(k)$ in $\mathbf{C}^+$ and $\{z_j\}$ is the set of zeros of $T(-k)$ in $\mathbf{C}^+$, in the region with $x+t>0$ and $x-t>0$ we get [cf. (5.9)]

$$\begin{aligned} J_{\mathrm{r}}(x,t) = -i\, m_{\mathrm{l}}(0,x)\,\mathrm{Res}\,(1/T(k),0) + i\sum_{j=1}^{n_{\mathrm{z}}} \tau_j\, m_{\mathrm{l}}(z_j,x)\, e^{iz_j(x+t)} \\ + i\sum_{j=1}^{n_{\mathrm{l}}} \frac{\rho_{\mathrm{l}j}\, e^{ik_{\mathrm{l}j}(x-t)}}{T(k_{\mathrm{l}j})}\, m_{\mathrm{l}}(k_{\mathrm{l}j},x), \end{aligned} \tag{5.20}$$

where the first term on the right hand side is the contribution from (5.19). Note that $m_{\mathrm{l}}(z_j,x)$ and $m_{\mathrm{l}}(k_{\mathrm{l}j},x)$ can be evaluated explicitly from (3.11) and $T(k_{\mathrm{l}j})$ from (2.14).

Finally, let us evaluate $J_{\mathrm{r}}(x,t)$ in the region with $x>0$ and $t-x>0$. From (5.19) we see that the contribution from the pole at $k=0$ to $J_{\mathrm{r}}(x,t)$ is nil when $x>0$ and $t-x>0$. To obtain $J_{\mathrm{r}}(x,t)$ when $x>0$ and $t-x>0$, we can use (5.18) and evaluate the first integral there via a contour integration along the boundary of $\mathbf{C}^+$ to get the first summation term in (5.20). To evaluate the second integral in (5.18), since $t-x>0$, with the help of (2.9) we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{R(k)}{T(k)} m_{\mathrm{l}}(k,x)\, e^{ik(x-t)} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{L(k)}{T(k)} m_{\mathrm{l}}(-k,x)\, e^{-ik(x-t)},$$

where the right hand side is to be evaluated as a contour integral along the boundary of $\mathbf{C}^+$ with the contributions coming from the poles $k_{\mathrm{r}j}$ of $L(k)$ in $\mathbf{C}^+$ and the poles of $m_{\mathrm{l}}(-k,x)$ in $\mathbf{C}^+$. From (3.6) or (3.10) we see that the latter poles occur at exactly $k=k_{\mathrm{l}j}$, which are the poles of $R(k)$ in $\mathbf{C}^+$. If the sets $\{k_{\mathrm{l}j}\}_{j=1}^{n_{\mathrm{l}}}$ and $\{k_{\mathrm{r}j}\}_{j=1}^{n_{\mathrm{r}}}$ do not intersect, we get [cf. (5.11)]

$$\begin{aligned} -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{L(k)}{T(k)}\, m_{\mathrm{l}}(-k,x)\, e^{-ik(x-t)} = -i\sum_{j=1}^{n_{\mathrm{r}}} \frac{\rho_{\mathrm{r}j}\, e^{-ik_{\mathrm{r}j}(x-t)}}{T(k_{\mathrm{r}j})}\, m_{\mathrm{l}}(-k_{\mathrm{r}j},x) \\ - i\sum_{j=1}^{n_{\mathrm{l}}} \frac{L(k_{\mathrm{l}j})\, e^{-ik_{\mathrm{l}j}(x-t)}}{T(k_{\mathrm{l}j})}\mathrm{Res}\,(m_{\mathrm{l}}(-k,x),k_{\mathrm{l}j})\,. \end{aligned} \tag{5.21}$$

From (3.6) we see that

$$\mathrm{Res}\,(m_{\mathrm{l}}(-k,x),k_{\mathrm{l}j}) = -\rho_{\mathrm{l}j}\, e^{2ik_{\mathrm{l}j}x}\, m_{\mathrm{l}}(k_{\mathrm{l}j},x). \tag{5.22}$$

In case the sets $\{k_{\mathrm{l}j}\}_{j=1}^{n_{\mathrm{l}}}$ and $\{k_{\mathrm{r}j}\}_{j=1}^{n_{\mathrm{r}}}$ overlap, the result in (5.21) can appropriately be modified.

By combining (5.15)-(5.22), we can write the Jost wave $J_{\rm r}(x,t)$ for any $x,t\in\mathbf{R}$ as

$$\begin{aligned}J_{\rm r}(x,t)=&\delta(x+t)+\theta(-x)\,\theta(x+t)\,\frac{\det\Gamma_{\rm r}(x,x+t)}{\det\mathbf{M}_{\rm r}(x)}\\&+i\,\theta(x)\,\theta(x+t)\sum_{j=1}^{n_{\rm z}}\tau_j\,m_{\rm l}(z_j,x)\,e^{iz_j(x+t)}\\&-i\,\theta(x+t)\,\theta(x-t)\,m_{\rm l}(0,x)\,\text{Res}\,(1/T(k),0)\\&+i\,\theta(x+t)\,\theta(x-t)\sum_{j=1}^{n_{\rm l}}\frac{\rho_{{\rm l}j}\,e^{ik_{{\rm l}j}(x-t)}}{T(k_{{\rm l}j})}\,m_{\rm l}(k_{{\rm l}j},x)\\&-i\,\theta(x)\,\theta(t-x)\sum_{j=1}^{n_{\rm r}}\frac{\rho_{{\rm r}j}\,e^{-ik_{{\rm r}j}(x-t)}}{T(k_{{\rm r}j})}\,m_{\rm l}(-k_{{\rm r}j},x)\\&+i\,\theta(x)\,\theta(t-x)\sum_{j=1}^{n_{\rm l}}\frac{\rho_{{\rm l}j}\,L(k_{{\rm l}j})\,e^{ik_{{\rm l}j}(x+t)}}{T(k_{{\rm l}j})}\,m_{\rm l}(k_{{\rm l}j},x).\end{aligned}$$

Thus, in terms of the quantities that have been constructed explicitly once the rational reflection coefficient $R(k)$ is known, we have obtained $J_{\rm r}(x,t)$ for all $x,t\in\mathbf{R}$ except when $x=-t$ and also when $x=t$ with $x>0$. In the next section, we will see that $J_{\rm r}(x,t)$ may have jump discontinuities when $x=-t$ and also when $x=t$ with $x>0$, and we will evaluate those discontinuities.

6. Discontinuities in the Jost Waves

For each fixed $t\in\mathbf{R}$, let us now analyze the discontinuities in the Jost waves $J_{\rm l}(x,t)$ and $J_{\rm r}(x,t)$ and in their x-derivatives. From (3.9) and (3.18), we see that for $x>0$ the potential V is the ratio of linear combinations of various exponential functions of x and that ratio decays exponentially as $x\to+\infty$. Similarly, for $x<0$, from (4.7) and (4.10) we see that V is the ratio of linear combinations of various exponential functions and that ratio decays exponentially as $x\to-\infty$. In the absence of bound states, it is known [3] that $m_{\rm l}(0,x)>0$ and $m_{\rm r}(0,x)>0$ for $x\in\mathbf{R}$. Hence, from (3.20) and (4.12) we see that the only discontinuities in V and its derivatives can occur at $x=0$.

From (7.5)-(7.7) of [18], as $k\to\infty$ in $\overline{\mathbf{C}^+}$ we have

$$m_{\rm l}(k,x)=1-\frac{\gamma_{\rm l}(x)}{2ik}-\frac{1}{8k^2}\left[\gamma_{\rm l}(x)^2-2\,q_{\rm l}(k,x)\right]+O(1/k^3),\tag{6.1}$$

$$m_{\rm r}(k,x)=1-\frac{\gamma_{\rm r}(x)}{2ik}-\frac{1}{8k^2}\left[\gamma_{\rm r}(x)^2+2\,q_{\rm r}(k,x)\right]+O(1/k^3),\tag{6.2}$$

$$m_{\rm l}'(k,x)=\frac{\gamma_{\rm l}(x)}{2ik}+O(1/k^2),\quad m_{\rm r}'(k,x)=\frac{\gamma_{\rm r}(x)}{2ik}+O(1/k^2),\tag{6.3}$$

with

$$\gamma_{\rm l}(x) := \int_x^\infty dy\, V(y), \quad \gamma_{\rm r}(x) := \int_{-\infty}^x dy\, V(y),$$

$$q_{\rm l}(k,x) := V(x) + \theta(-x)\left[V(0^+) - V(0^-)\right] e^{-2ikx},$$

$$q_{\rm r}(k,x) := -V(x) + \theta(x)\left[V(0^+) - V(0^-)\right] e^{2ikx}.$$

Note that $q_{\rm l}(k,x)$ and $q_{\rm r}(k,x)$ are continuous at $x = 0$, and we have

$$q_{\rm l}(k,0) = V(0^+), \quad q_{\rm r}(k,0) = -V(0^-).$$

With the help of (5.2) and (5.3), let us define

$$U_{\rm l}(x,t) := J_{\rm l}(x,t) - \delta(x-t). \tag{6.4}$$

We will refer to $U_{\rm l}(x,t)$ as the tail of the Jost wave $J_{\rm l}(x,t)$. From (5.2) we see that the discontinuity in the tail $U_{\rm l}(x,t)$ is caused by the $(1/k)$-term in the expansion of $m_{\rm l}(k,x) - 1$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. By using

$$\frac{1}{2\pi i}\int_{-\infty}^\infty dk\, \frac{e^{ik\xi}}{k + i0^+} = -\theta(-\xi), \tag{6.5}$$

we evaluate that contribution as $(1/2)\,\gamma_{\rm l}(x)\,\theta(t-x)$, and hence the only discontinuity in the tail $U_{\rm l}(x,t)$ occurs at the wavefront $x = t$ and is given by

$$U_{\rm l}(t+0^+,t) - U_{\rm l}(t-0^+,t) = -\frac{1}{2}\int_t^\infty dy\, V(y).$$

In other words,

$$U_{\rm l}(x,t) = \begin{cases} 0, & x > t, \\ \dfrac{1}{2}\displaystyle\int_t^\infty dy\, V(y), & x = t - 0^+. \end{cases}$$

Next, let us analyze the discontinuities in $\partial U_{\rm l}(x,t)/\partial x$. From (5.2) and (6.4), we see that

$$\frac{\partial U_{\rm l}(x,t)}{\partial x} = \frac{1}{2\pi}\int_{-\infty}^\infty dk\, \left\{m_{\rm l}'(k,x) + ik\left[m_{\rm l}(k,x) - 1\right]\right\} e^{ik(x-t)}, \tag{6.6}$$

and for each fixed $t \in \mathbf{R}$, the discontinuity in $\partial U_{\rm l}(x,t)/\partial x$ is caused by the $(1/k)$-term in the expansion of the integrand of (6.6) as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Using (6.1), (6.3), and (6.5) in (6.6) we see that there are exactly two such discontinuities. The first discontinuity occurs at the wavefront $x = t$, and the second occurs at $x = -t$. At the wavefront, we get

$$\frac{\partial U_{\rm l}(x,t)}{\partial x} = \begin{cases} 0, & x > t, \\ \dfrac{V(x)}{4} - \dfrac{1}{2}\displaystyle\int_t^\infty dy\, V(y) - \dfrac{1}{8}\left[\int_t^\infty dy\, V(y)\right]^2, & x = t - 0^+. \end{cases}$$

The contribution to the discontinuity at $x = -t$ is obtained as

$$\frac{\partial U_{\rm l}(t+0^+,t)}{\partial x} - \frac{\partial U_{\rm l}(t-0^+,t)}{\partial x} = -\frac{1}{4}\,\theta(-x)\left[V(0^+) - V(0^-)\right].$$

In analogy to (6.4), with the help of (5.14) let us define

$$U_{\mathrm{r}}(x,t) := J_{\mathrm{r}}(x,t) - \delta(x+t). \tag{6.7}$$

We will refer to $U_{\mathrm{r}}(x,t)$ as the tail of the Jost wave $J_{\mathrm{r}}(x,t)$. Using (6.2) and (6.5) in (6.7), we see that for each fixed $t \in \mathbf{R}$, the only discontinuity in $U_{\mathrm{r}}(x,t)$ occurs at the wavefront $x = -t$ and is described by

$$U_{\mathrm{r}}(x,t) = \begin{cases} 0, & x < -t, \\ \dfrac{1}{2}\displaystyle\int_{-\infty}^{t} dy\, V(y), & x = -t + 0^+. \end{cases}$$

To determine the discontinuities in $\partial U_{\mathrm{r}}(x,t)/\partial x$, first from (5.14) we obtain

$$\frac{\partial U_{\mathrm{r}}(x,t)}{\partial x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \left\{m_{\mathrm{r}}'(k,x) - ik\left[m_{\mathrm{r}}(k,x) - 1\right]\right\} e^{-ik(x+t)}. \tag{6.8}$$

Using (6.2), (6.3), and (6.5) in (6.8), for each fixed $t \in \mathbf{R}$, we see that the discontinuities in $\partial U_{\mathrm{r}}(x,t)/\partial x$ may occur only at $x = -t$ and at $x = t$. The former occurs at the wavefront and is described by

$$\frac{\partial U_{\mathrm{r}}(x,t)}{\partial x} = \begin{cases} 0, & x < -t, \\ -\dfrac{V(x)}{4} - \dfrac{1}{2}\displaystyle\int_{-\infty}^{t} dy\, V(y) + \dfrac{1}{8}\left[\displaystyle\int_{-\infty}^{t} dy\, V(y)\right]^2, & x = -t + 0^+. \end{cases}$$

Finally, the discontinuity at $x = t$ is given by

$$\frac{\partial U_{\mathrm{r}}(t+0^+,t)}{\partial x} - \frac{\partial U_{\mathrm{r}}(t-0^+,t)}{\partial x} = \frac{1}{4}\,\theta(x)\left[V(0^+) - V(0^-)\right].$$

7. Numerical Implementation

One of the authors (Borkowski) has implemented the theoretical method described in this paper as a *Mathematica 4.2* notebook. The user inputs a rational function for $R(k)$ and instructs *Mathematica* to evaluate the notebook. *Mathematica* then calculates all of the quantities relevant to (1.1); namely, the Faddeev functions $m_{\mathrm{l}}(k,x)$ and $m_{\mathrm{r}}(k,x)$, Jost solutions $f_{\mathrm{l}}(k,x)$ and $f_{\mathrm{r}}(k,x)$, the potential $V(x)$, the scattering coefficients $T(k)$ and $L(k)$, and the quantities $B_{\mathrm{l}}(x,\alpha)$ and $B_{\mathrm{r}}(x,\alpha)$ given in (3.1) and (4.1), respectively.

The implemented program first reduces $R(k)$, then calculates $T(k)$ and $L(k)$ as described in Section 2, and then reduces those to cancel common factors appearing both in the numerator and in the denominator. The reduction in each scattering coefficient is achieved by computing all the zeros and poles in $\mathbf{C}^+$, comparing them within a chosen numerical precision, and cancelling the common factors appearing in the numerator and in the denominator. This reduction is necessary because *Mathematica* cannot usually cancel the terms by itself or cannot simplify enough in certain circumstances. Finally, the Faddeev functions and the

Jost solutions, along with the potential are determined for $x > 0$, and then for $x < 0$.

Our program has been able to duplicate the numerical results in [9]. However, we have not been able to duplicate the two numerical examples in [7] given by Dolveck-Guilpart. We have also verified that the result of our program agrees with the analytical example of Sabatier [14]. Prof. Paul Sacks of Iowa State University has used his *Matlab* program for the solution of the inverse scattering problem based on transforming the relevant inverse problem into an equivalent time-domain problem and solving it by a time-domain method [19]; he also was able to duplicate the results in [9], but not in [7], and he has confirmed to us that our results are in complete agreement with his as far as the two examples in [7] are concerned. Prof. Sabatier later has informed us that the two numerical examples given in [7] by Dolveck-Guilpart were indeed incorrect, and the rational reflection coefficients used in those two examples were outside the domain of the applicability of the method of [14].

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