

Inverse Scattering Transform, KdV, and Solitons

Tuncay Aktosun

Abstract. In this review paper, the Korteweg-de Vries equation (KdV) is considered, and it is derived by using the Lax method and the AKNS method. An outline of the inverse scattering problem and of its solution is presented for the associated Schrödinger equation on the line. The inverse scattering transform is described in order to solve the initial-value problem for the KdV, and the time evolution of the corresponding scattering data is obtained. Soliton solutions to the KdV are derived in several ways.

1. Introduction

The *Korteweg-de Vries equation* (KdV, for short) is used to model propagation of water waves in long, narrow, and shallow canals. It was first formulated [1] in 1895 by the Dutch mathematicians Diederik Johannes Korteweg and Gustav de Vries. Korteweg was a well known mathematician of his time, and de Vries wrote a doctoral thesis on the subject under Korteweg.

After some scaling, it is customary to write the KdV in the form

$$(1.1) \quad \frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \mathbf{R}, \quad t > 0,$$

where $-u(x, t)$ corresponds to the vertical displacement of the water from the equilibrium at the location x at time t . Replacing u by $-u$ amounts to replacing -6 by $+6$ in (1.1). Also, by scaling x , t , and u , i.e. by multiplying them with some positive constants, it is possible to change the constants in front of each of the three terms on the left-hand side of (1.1) at will.

Note that the KdV is a *nonlinear* partial differential equation (PDE, for short) due to the presence of the uu_x term, where we use a subscript to denote the partial derivative. The u_{xxx} term makes it *dispersive*, i.e. in general an initial

Received by the editors December 1, 2001.

2000 *Mathematics Subject Classification.* Primary 35Q53; Secondary 35Q51 81U40.

Key words and phrases. Korteweg-de Vries equation, Inverse scattering transform, Soliton.

The research leading to this article was supported in part by the National Science Foundation under grants DMS-0296022 and DMS-0204437 and the Department of Energy under grant DE-FG02-01ER45951.

wave $u(x, 0)$ will broaden in space as time progresses. In addition to its solutions showing behavior of nonlinearity and dispersiveness, the KdV possesses certain special solutions, known as *solitary wave solutions*, which would not be expected from a nonlinear and dispersive PDE. A single solitary wave solution to a PDE retains its shape in time and moves either to the left or right. It turns out that such a solution to (1.1) only moves to the right, and any general solitary wave solution asymptotically resembles a train of single solitary wave solutions.

It was a Scottish engineer named John Scott Russell who first observed a solitary water wave. This happened in 1834 on the Edinburgh-to-Glasgow canal, some 60 years before the formulation of the KdV as a model for water waves. Russell reported [2] his observation to the British Association for the Advancement of Science in September 1844. The details of Russell's observation, including the rivalry between him and George Airy who did not believe in the existence of solitary water waves, can be found in [3-6].

As outlined in the famous preprint [7], which was never published as a journal article, Enrico Fermi in his summer visits to Los Alamos, jointly with J. Pasta and S. Ulam, studied a one-dimensional (1-D, for short) dynamical system of 64 particles with forces between neighbors containing nonlinear terms. These computational studies were carried out on the Los Alamos computer named Maniac I. The primary aim of the study was to determine the rate of approach to the equipartition of energy among various degrees of freedom. Contrary to expectations, Fermi, Pasta, and Ulam observed very little, if any, tendency towards the equipartition of energy, but instead the almost ongoing recurrence to the initial state, which was puzzling. After Fermi's death in November 1954, Pasta and Ulam completed their last few computational examples. Their preprint appears in Fermi's Collected Papers [8] and is also available on the internet [9].

The importance of the KdV arose in 1965, when Zabusky and Kruskal [10] were able to explain the Fermi-Pasta-Ulam puzzle in terms of solitary-wave solutions to the KdV. In their analysis of numerical solutions to the KdV, Zabusky and Kruskal observed *solitary-wave pulses*, named such pulses *solitons* because of their particle-like behavior, and noted that such pulses interact with each other nonlinearly but come out of their interaction virtually unaffected in size or shape. Such unusual nonlinear interactions among soliton solutions to the KdV created a lot of excitement, but at that time no one knew how to solve such a nonlinear PDE, except numerically.

In their celebrated paper [11] of 1967, Gardner, Greene, Kruskal, and Miura presented a method, now known as the *inverse scattering transform*, to solve the initial-value problem for the KdV, assuming that the initial value $u(x, 0)$ approaches a constant sufficiently rapidly as $x \rightarrow \pm\infty$. There is no loss of generality in choosing that constant as zero. They showed that $u(x, t)$ can be obtained from $u(x, 0)$ with the help of the solution to the *inverse scattering problem* for the 1-D Schrödinger equation with the *time-evolved scattering data*. They also explained that soliton solutions to the KdV correspond to the case of zero reflection coefficient in the scattering data. They observed from various numerical studies of the

KdV that, for large t , $u(x, t)$ in general consists of a finite train of solitons traveling in the positive x direction and an oscillatory train spreading in the opposite direction.

In our paper, we present an elementary review of the inverse scattering transform for the KdV. We consider the time-evolved Schrödinger equation, where $V(x)$ in (2.1) is replaced by $u(x, t)$; namely, we deal with

$$(1.2) \quad \frac{d^2 \psi(k, x; t)}{dx^2} + k^2 \psi(k, x; t) = u(x, t) \psi(k, x; t), \quad x \in \mathbf{R},$$

where $t > 0$ is a parameter that is usually interpreted as time. Thus, we view $V(x)$ as the initial value $u(x, 0)$ of the potential $u(x, t)$ and look at $\psi(k, x; t)$ as the time evolution of $\psi(k, x)$ of (2.1) from the initial time $t = 0$. The *scattering coefficients* $T(k; t)$, $R(k; t)$, and $L(k; t)$ associated with (1.2) are viewed as evolving from the corresponding coefficients $T(k)$, $R(k)$, and $L(k)$ of (2.1), respectively, from $t = 0$. Thus, our notation is such that

$$\begin{aligned} V(x) &= u(x, 0), & \psi(k, x) &= \psi(k, x; 0), \\ T(k) &= T(k; 0), & R(k) &= R(k; 0), & L(k) &= L(k; 0). \end{aligned}$$

Our review paper is organized as follows. In Section 2 we consider *scattering solutions* and *bound state solutions* to the Schrödinger equation (2.1) and introduce the scattering coefficients, *bound-state norming constants*, and *dependency constants* corresponding to a potential in the so-called *Faddeev class*. In Section 3 we present an outline of the inverse scattering problem for (2.1) and review some solution methods based on solving an associated *Riemann-Hilbert problem*. In Section 4 we consider the time evolution of the scattering coefficients, bound-state norming constants, and dependency constants when the potential evolves from $u(x, 0)$ to $u(x, t)$. We also introduce the *Lax pair* associated with the KdV and derive the KdV by using the *Lax method*. In Section 5 we study the *AKNS method* and derive the KdV via that method. In Section 6 we present some of the methods to solve the initial value problem for the KdV. In Section 7 we concentrate on soliton solutions to the KdV and obtain various representations of the N -soliton solution. Finally, in Section 8 we provide certain remarks on the *Bäcklund transformation*, the *conserved quantities*, and some other aspects related to the KdV.

2. Schrödinger Equation and the Scattering Data

Consider the Schrödinger equation

$$(2.1) \quad \frac{d^2 \psi(k, x)}{dx^2} + k^2 \psi(k, x) = V(x) \psi(k, x), \quad x \in \mathbf{R},$$

where V is real valued and belongs to $L^1_1(\mathbf{R})$. Here, $L^1_n(\mathbf{R})$ denotes the class of measurable potentials such that $\int_{-\infty}^{\infty} dx (1 + |x|^n) |V(x)|$ is finite. The class of real-valued potentials in $L^1_1(\mathbf{R})$ is sometimes called the Faddeev class, after Ludwig Faddeev's analysis [12] of the inverse scattering problem for (2.1) within that

class of potentials. In appropriate units, (2.1) describes the quantum mechanical behavior of a particle of total energy k^2 under the influence of the potential V . The inverse scattering problem for (2.1) consists of the determination of V from an appropriate set of scattering data.

There are two types of solutions to (2.1). The scattering solutions consist of linear combinations of e^{ikx} and e^{-ikx} as $x \rightarrow \pm\infty$, and they occur for $k \in \mathbf{R} \setminus \{0\}$. Real k values correspond to positive energies of the particle, and a particle of positive energy can be visualized as being capable of escaping to $x = \pm\infty$ as a result of scattering from V . Heuristically, since $V(x)$ vanishes at $x = \pm\infty$, the particle will still have some kinetic energy at infinity and hence is allowed to be at infinity. On the other hand, a bound state of (2.1) is a solution that belongs to $L^2(\mathbf{R})$ in the x variable. It turns out that, when V belongs to the Faddeev class, the bound-state solutions to (2.1) decay exponentially as $x \rightarrow \pm\infty$, and they can occur only at certain k -values on the imaginary axis in \mathbf{C}^+ . We use \mathbf{C}^+ to denote the upper-half complex plane and put $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$. Each bound state corresponds to a negative total energy of the particle, and as a result the particle is bound by the potential and does not have sufficient kinetic energy to escape to infinity. We will use N to denote the number of bound states, which is known to be finite when V is in the Faddeev class, and suppose that the bound states occur at $k = i\kappa_j$ with the ordering $0 < \kappa_1 < \dots < \kappa_N$.

Among the scattering solutions to (2.1) are the *Jost solution from the left*, f_l , and the *Jost solution from the right*, f_r , satisfying the respective boundary conditions

$$(2.2) \quad e^{-ikx} f_l(k, x) = 1 + o(1), \quad e^{-ikx} f_l'(k, x) = ik + o(1), \quad x \rightarrow +\infty,$$

$$(2.3) \quad e^{ikx} f_r(k, x) = 1 + o(1), \quad e^{ikx} f_r'(k, x) = -ik + o(1), \quad x \rightarrow -\infty,$$

where the prime is used for the derivative with respect to the spatial coordinate x . From the spatial asymptotics

$$(2.4) \quad f_l(k, x) = \frac{e^{ikx}}{T(k)} + \frac{L(k)e^{-ikx}}{T(k)} + o(1), \quad x \rightarrow -\infty,$$

$$(2.5) \quad f_r(k, x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k)e^{ikx}}{T(k)} + o(1), \quad x \rightarrow +\infty,$$

we obtain the scattering coefficients, namely, the *transmission coefficient* T , and the *reflection coefficients* L and R , *from the left and right*, respectively. It is also possible to express the scattering coefficients in terms of certain Wronskians [12-16] involving f_l and f_r . We have

$$(2.6) \quad T(k) = \frac{2ik}{[f_r(k, x); f_l(k, x)]},$$

$$L(k) = \frac{[f_l(k, x); f_r(-k, x)]}{[f_r(k, x); f_l(k, x)]}, \quad R(k) = \frac{[f_l(-k, x); f_r(k, x)]}{[f_r(k, x); f_l(k, x)]},$$

where the Wronskian is defined as $[F; G] := FG' - F'G$.

It is known [12-16] that, for each fixed $x \in \mathbf{R}$, the Jost solutions $f_1(\cdot, x)$ and $f_r(\cdot, x)$ have analytic extensions in k to \mathbf{C}^+ . Moreover,

$$f_1(-k^*, x) = f_1(k, x)^*, \quad f_r(-k^*, x) = f_r(k, x)^*, \quad k \in \overline{\mathbf{C}^+},$$

$$T(-k) = T(k)^*, \quad R(-k) = R(k)^*, \quad L(-k) = L(k)^*, \quad k \in \mathbf{R},$$

where the asterisk denotes complex conjugation. We also have

$$(2.7) \quad R(k)T(k)^* = -L(k)^*T(k), \quad k \in \mathbf{R},$$

$$(2.8) \quad |T(k)|^2 + |L(k)|^2 = 1 = |T(k)|^2 + |R(k)|^2, \quad k \in \mathbf{R}.$$

Thus, the scattering coefficients cannot exceed one in absolute value for real k . Furthermore, $T(k) \neq 0$ if $k \in \mathbf{R} \setminus \{0\}$, and hence the reflection coefficients are strictly less than one in absolute value when $k \in \mathbf{R} \setminus \{0\}$. In general, R and L are defined only for real k values, but T has a meromorphic extension to \mathbf{C}^+ . For large k one has

$$T(k) = 1 + O(1/k), \quad k \rightarrow \infty \text{ in } \overline{\mathbf{C}^+},$$

$$R(k) = o(1/k), \quad L(k) = o(1/k), \quad k \rightarrow \pm\infty.$$

Each bound state corresponds to a pole of T in \mathbf{C}^+ and vice versa. It is known that the bound states are simple, i.e. at each $k = i\kappa_j$ there exists only one linearly independent solution to (2.1) belonging to $L^2(\mathbf{R})$. The bound-state norming constants c_{lj} and c_{rj} are defined as

$$c_{lj} := \left[\int_{-\infty}^{\infty} dx f_1(i\kappa_j, x)^2 \right]^{-1/2}, \quad c_{rj} := \left[\int_{-\infty}^{\infty} dx f_r(i\kappa_j, x)^2 \right]^{-1/2},$$

and they are related to each other via the residues of T as

$$(2.9) \quad \text{Res}(T, i\kappa_j) = i c_{lj}^2 \gamma_j = i \frac{c_{rj}^2}{\gamma_j},$$

where γ_j is the dependency constant given by

$$(2.10) \quad \gamma_j := \frac{f_1(i\kappa_j, x)}{f_r(i\kappa_j, x)}.$$

The sign of γ_j is the same as that of $(-1)^{N-j}$ and hence

$$c_{rj} = (-1)^{N-j} \gamma_j c_{lj}.$$

The normalized bound-state solution $\varphi_j(x)$ at $k = i\kappa_j$ is defined as

$$\varphi_j(x) := c_{lj} f_1(i\kappa_j, x) = (-1)^{N-j} c_{rj} f_r(i\kappa_j, x).$$

The *scattering matrix* associated with (2.1) is given by

$$\mathbf{S}(k) := \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \quad k \in \mathbf{R},$$

and it can be constructed in terms of the bound-state energies and either one of the reflection coefficients R and L . Given $R(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, one can construct T as

$$(2.11) \quad T(k) = \left(\prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\log(1 - |R(s)|^2)}{s - k - i0^+} \right), \quad k \in \overline{\mathbf{C}^+},$$

and use (2.7) to construct $L(k)$ for $k \in \mathbf{R}$. Similarly, given $L(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, one can construct T [cf. (2.8) and (2.11)] as

$$T(k) = \left(\prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\log(1 - |L(s)|^2)}{s - k - i0^+} \right), \quad k \in \overline{\mathbf{C}^+},$$

and obtain $R(k)$ for $k \in \mathbf{R}$ via (2.7).

3. Inverse Scattering Problem

When there are no bound states, either one of the reflection coefficients R and L uniquely determines the corresponding potential in the Faddeev class. However, when there are bound states, for the unique determination of V , in addition to one reflection coefficient and the bound-state energies, one must also specify a bound-state norming constant or, equivalently, the dependency constant for each bound state. To recover V uniquely, as our scattering data we may use either the *left scattering data* $\{R, \{\kappa_j\}, \{c_{lj}\}\}$ or the *right scattering data* $\{L, \{\kappa_j\}, \{c_{rj}\}\}$; these two are equivalent to each other, and each is also equivalent to $\{\mathbf{S}, \{\gamma_j\}\}$.

A *characterization* for a specific class of potentials consists of specifying some necessary and sufficient conditions on the scattering data which guarantee that there exists a corresponding unique potential in that class. Such conditions are usually obtained by using the *Faddeev-Marchenko method* [12-19]; this method is also known as the *Marchenko method*, and sometimes in the literature it is referred to as the *Gel'fand-Levitán-Marchenko method* even though this is a misnomer [20]. The characterization conditions can be stated for the left scattering data, for the right scattering data, or for the combination of both. For a characterization in the class of real-valued potentials belonging to $L_2^1(\mathbf{R})$, the reader is referred to [14]. Various characterizations in the Faddeev class can be found in [13,17-19].

Since k appears as k^2 in (2.1), the functions $f_1(-k, x)$ and $f_r(-k, x)$ are also solutions to (2.1) and they can be expressed as linear combinations of the Jost solutions $f_1(k, x)$ and $f_r(k, x)$ as

$$\begin{aligned} f_1(-k, x) &= T(k) f_r(k, x) - R(k) f_1(k, x), & k \in \mathbf{R}, \\ f_r(-k, x) &= T(k) f_1(k, x) - L(k) f_r(k, x), & k \in \mathbf{R}, \end{aligned}$$

or equivalently as

$$(3.1) \quad m_1(-k, x) = T(k) m_r(k, x) - R(k) e^{2ikx} m_1(k, x), \quad k \in \mathbf{R},$$

$$(3.2) \quad m_r(-k, x) = T(k) m_l(k, x) - L(k) e^{-2ikx} m_r(k, x), \quad k \in \mathbf{R},$$

where m_l and m_r are the *Faddeev functions* defined as

$$(3.3) \quad m_l(k, x) := e^{-ikx} f_l(k, x), \quad m_r(k, x) := e^{ikx} f_r(k, x).$$

Each of (3.1) and (3.2) can be viewed as a Riemann-Hilbert problem [6,15], where, knowing the scattering coefficients for $k \in \mathbf{R}$, the aim is to construct m_l and m_r such that, for each $x \in \mathbf{R}$, $m_l(\cdot, x)$ and $m_r(\cdot, x)$ are analytic in \mathbf{C}^+ , continuous in $\overline{\mathbf{C}^+}$, and behave like $1 + O(1/k)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Once $m_l(k, x)$ or $m_r(k, x)$ is constructed, the potential can be obtained with the help of (2.1) and (3.3), namely, by using

$$(3.4) \quad V(x) = \frac{m_l''(k, x)}{m_l(k, x)} + 2ik \frac{m_l'(k, x)}{m_l(k, x)}, \quad V(x) = \frac{m_r''(k, x)}{m_r(k, x)} - 2ik \frac{m_r'(k, x)}{m_r(k, x)},$$

where the right-hand sides can be evaluated at any particular value of $k \in \overline{\mathbf{C}^+}$.

Alternatively, the potential can be constructed by the Faddeev-Marchenko method; namely, V can be obtained from the left scattering data $\{R, \{\kappa_j\}, \{c_{lj}\}\}$ by solving the *left Marchenko integral equation* or from the right scattering data $\{L, \{\kappa_j\}, \{c_{rj}\}\}$ by solving the *right Marchenko integral equation*.

The left Marchenko equation using the left scattering data as the input is given by

$$(3.5) \quad B_l(x, y) + \Omega_l(2x + y) + \int_0^\infty dz \Omega_l(2x + y + z) B_l(x, z) = 0, \quad y > 0,$$

where

$$\Omega_l(y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk R(k) e^{iky} + \sum_{j=1}^N c_{lj}^2 e^{-\kappa_j y}.$$

One can obtain (3.5) from (3.1) via a Fourier transformation. Once (3.5) is solved and $B_l(x, y)$ is obtained, the potential is recovered as

$$(3.6) \quad V(x) = -2 \frac{dB_l(x, 0^+)}{dx},$$

and the Faddeev function from the left is constructed as

$$(3.7) \quad m_l(k, x) = 1 + \int_0^\infty dy B_l(x, y) e^{iky}.$$

Similarly, via a Fourier transformation on (3.2), using the right scattering data as the input one obtains the right Marchenko equation

$$(3.8) \quad B_r(x, y) + \Omega_r(-2x + y) + \int_0^\infty dz \Omega_r(-2x + y + z) B_r(x, z) = 0, \quad y > 0,$$

where

$$\Omega_r(y) := \frac{1}{2\pi} \int_{-\infty}^\infty dk L(k) e^{iky} + \sum_{j=1}^N c_{rj}^2 e^{-\kappa_j y}.$$

Once (3.8) is solved, the potential is recovered by using

$$(3.9) \quad V(x) = 2 \frac{dB_r(x, 0^+)}{dx},$$

and the Faddeev function from the right is constructed as

$$(3.10) \quad m_r(k, x) = 1 + \int_0^\infty dy B_r(x, y) e^{iky}.$$

When the characterization conditions on the scattering data corresponding to potentials in the Faddeev class are satisfied, both the left and right Marchenko equations are uniquely solvable, and the right-hand sides of (3.6) and (3.9) are equal to each other and belong to $L^1_1(\mathbf{R})$. Thus, V can be obtained from either (3.6) or (3.9). There are various other methods to recover V from an appropriate set of scattering data. We refer the reader to [16,19] for a review of some of those methods.

4. Lax Method and Evolution of the Scattering Data

Soon after Gardner, Green, Kruskal, and Miura demonstrated [11] that the initial value problem for the KdV can be solved by the inverse scattering transform, Peter Lax gave [21] a criterion to show that the KdV can be viewed as a *compatibility condition* related to the time evolution of solutions to (1.2). Since Lax's criterion is applicable to other nonlinear PDEs solvable by an inverse scattering transform (e.g. the initial-value problem for the nonlinear Schrödinger equation can be solved [4,6,22] via the inverse scattering transform for the Zakharov-Shabat system), we first outline the general idea behind the Lax method and next demonstrate its application on the KdV.

Given a linear operator \mathcal{L} with $\mathcal{L}\psi = \lambda\psi$, we are interested in finding another operator \mathcal{A} (the operators \mathcal{A} and \mathcal{L} are said to form a Lax pair) such that:

- (i) The spectral parameter λ does not change in time.
- (ii) The quantity $\psi_t - \mathcal{A}\psi$ must remain a solution to $\mathcal{L}\psi = \lambda\psi$.
- (iii) The quantity $\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$ must be a multiplication operator.

As the following argument shows, for compatibility, we are forced to have

$$(4.1) \quad \mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0,$$

which is interpreted as an integrable PDE and in general is nonlinear. From condition (ii) above we see that

$$\mathcal{L}(\psi_t - \mathcal{A}\psi) = \lambda(\psi_t - \mathcal{A}\psi),$$

or equivalently,

$$\begin{aligned}
(4.2) \quad \mathcal{L}\psi_t - \mathcal{L}\mathcal{A}\psi &= \lambda\psi_t - \mathcal{A}(\lambda\psi) \\
&= \partial_t(\lambda\psi) - \mathcal{A}\mathcal{L}\psi \\
&= \partial_t(\mathcal{L}\psi) - \mathcal{A}\mathcal{L}\psi \\
&= \mathcal{L}_t\psi + \mathcal{L}\psi_t - \mathcal{A}\mathcal{L}\psi,
\end{aligned}$$

where we have used $\mathcal{L}\psi = \lambda\psi$ and $\lambda_t = 0$. After canceling the $\mathcal{L}\psi_t$ terms in (4.2), we get

$$(4.3) \quad (\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L})\psi = 0.$$

Because of condition (iii) listed above, from (4.3) we obtain the compatibility condition (4.1).

Let us write the Schrödinger equation (1.2) as

$$(4.4) \quad \mathcal{L}\psi = \lambda\psi, \quad \mathcal{L} := -\partial_x^2 + u(x, t),$$

and try

$$(4.5) \quad \mathcal{A} = \alpha\partial_x^3 + \beta\partial_x^2 + \xi\partial_x + \eta,$$

where the coefficients α , β , ξ , and η may depend on x and t , but not on λ . Note that $\mathcal{L}_t = u_t$ and λ is the same as k^2 . Using (4.4) and (4.5) we can write (4.1) explicitly as

$$(4.6) \quad ()\partial_x^5 + ()\partial_x^4 + ()\partial_x^3 + ()\partial_x^2 + ()\partial_x + () = 0,$$

where each $()$ in (4.6) denotes the appropriate coefficient. The coefficient of ∂_x^5 automatically vanishes. Setting the coefficients of ∂_x^j equal to zero for $j = 4, 3, 2, 1$, we obtain

$$\alpha = c_1, \quad \beta = c_2, \quad \xi = c_3 - \frac{3}{2}c_1u, \quad \eta = c_4 - \frac{3}{4}c_1u_x - c_2u,$$

where c_1 , c_2 , c_3 , and c_4 are arbitrary constants. Using $c_1 = -4$ and $c_3 = 0$ in the last term on the left-hand side of (4.6) and setting that last term equal to zero, we obtain the KdV equation given in (1.1). Further, by using $c_2 = c_4 = 0$, we obtain the operator \mathcal{A} associated with \mathcal{L} as

$$(4.7) \quad \mathcal{A} = -4\partial_x^3 + 6u\partial_x + 3u_x.$$

Let us remark that condition (ii) stated above allows us to determine the time evolution of any solution to the Schrödinger equation (1.2) as the initial potential $u(x, 0)$ evolves to $u(x, t)$. For example, let us find the time evolution of $f_1(k, x; t)$, the Jost solution from the left. By using (2.2) and (2.5) with $T(k)$ and $R(k)$ replaced by $T(k; t)$ and $R(k; t)$, respectively, we see that

$$(4.8) \quad e^{-ikx} f_1(k, x; t) = 1 + o(1), \quad e^{-ikx} f_1'(k, x; t) = ik + o(1), \quad x \rightarrow +\infty,$$

$$(4.9) \quad f_r(k, x; t) = \frac{e^{-ikx}}{T(k; t)} + \frac{R(k; t) e^{ikx}}{T(k; t)} + o(1), \quad x \rightarrow +\infty.$$

From condition (ii) of the Lax method and (4.7) we obtain

$$(4.10) \quad \begin{aligned} \partial_t f_1(k, x; t) - (-4\partial_x^3 + 6u\partial_x + 3u_x) f_1(k, x; t) \\ = p(k, t) f_1(k, x; t) + q(k, t) f_r(k, x; t), \end{aligned}$$

where we have used the fact that the quantity $\partial_t f_1 - \mathcal{A}f_1$ remains a solution to (1.2) and hence can be expressed as a linear combination of the two linearly independent Jost solutions $f_1(k, x; t)$ and $f_r(k, x; t)$ with coefficients $p(k, t)$ and $q(k, t)$, respectively. For each fixed t , assuming that $u(x, t) = o(1)$ and $u_x(x, t) = o(1)$ as $x \rightarrow +\infty$, the coefficients $p(k, t)$ and $q(k, t)$ can be evaluated by letting $x \rightarrow +\infty$ in (4.10). Using (4.8) and (4.9) in (4.10), we get

$$(4.11) \quad \partial_t e^{ikx} + 4\partial_x^3 e^{ikx} = p(k, t) e^{ikx} + q(k, t) \left[\frac{1}{T(k; t)} e^{-ikx} + \frac{R(k; t)}{T(k; t)} e^{ikx} \right].$$

From (4.11), by comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$(4.12) \quad q(k, t) = 0, \quad p(k, t) = -4ik^3.$$

Thus, the time evolution of $f_1(k, x; t)$ is determined by the linear third-order PDE

$$(4.13) \quad \partial_t f_1(k, x; t) - \mathcal{A}f_1(k, x; t) = -4ik^3 f_1(k, x; t).$$

Similarly, letting $x \rightarrow -\infty$ in (4.10), with the help of (2.3), (2.4), and (4.12) we obtain

$$(4.14) \quad \begin{aligned} \partial_t \left[\frac{1}{T(k; t)} e^{ikx} + \frac{L(k; t)}{T(k; t)} e^{-ikx} \right] \\ = [-4\partial_x^3 - 4ik^3] \left[\frac{1}{T(k; t)} e^{ikx} + \frac{L(k; t)}{T(k; t)} e^{-ikx} \right], \end{aligned}$$

where we have also used that $u(x, t) = o(1)$ and $u_x(x, t) = o(1)$ as $x \rightarrow -\infty$. From (4.14), comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$\partial_t T(k; t) = 0, \quad \partial_t L(k; t) = -8ik^3 L(k; t),$$

and hence

$$(4.15) \quad T(k; t) = T(k; 0) = T(k), \quad L(k; t) = L(k; 0) e^{-8ik^3 t} = L(k) e^{-8ik^3 t}.$$

Thus, the transmission coefficient remains unchanged and the reflection coefficient from the left undergoes a simple phase change as t progresses.

Proceeding in a similar manner, we can obtain the time evolution of the Jost solution $f_r(k, x; t)$ and the right reflection coefficient $R(k; t)$. With \mathcal{A} as in (4.7), we get

$$(4.16) \quad \partial_t f_r(k, x; t) - \mathcal{A}f_r(k, x; t) = 4ik^3 f_r(k, x; t),$$

$$(4.17) \quad R(k; t) = R(k; 0) e^{8ik^3 t} = R(k) e^{8ik^3 t}.$$

In order to evaluate the time evolution of the dependency constants $\gamma_j(t)$, we can substitute $\gamma_j(t)f_r(i\kappa_j, x; t)$ for $f_1(i\kappa_j, x; t)$ [cf. (2.10)] and evaluate (4.13) at $k = i\kappa_j$. We get

$$(4.18) \quad \begin{aligned} f_r(i\kappa_j, x; t) \partial_t \gamma_j(t) + \gamma_j(t) \partial_t f_r(i\kappa_j, x; t) - \gamma_j(t) \mathcal{A} f_r(i\kappa_j, x; t) \\ = -4\kappa_j^3 \gamma_j(t) f_r(i\kappa_j, x; t). \end{aligned}$$

On the other hand, from (4.16) at $k = i\kappa_j$, we obtain

$$(4.19) \quad \gamma_j(t) \partial_t f_r(i\kappa_j, x; t) - \gamma_j(t) \mathcal{A} f_r(i\kappa_j, x; t) = 4\kappa_j^3 \gamma_j(t) f_r(i\kappa_j, x; t).$$

Subtracting (4.19) from (4.18) we conclude that $\partial_t \gamma_j(t) = -8\kappa_j^3 \gamma_j(t)$, which leads to

$$(4.20) \quad \gamma_j(t) = \gamma_j(0) e^{-8\kappa_j^3 t} = \gamma_j e^{-8\kappa_j^3 t}.$$

Then, from (2.9) and (4.15) we obtain the time evolution of the norming constants $c_{lj}(t)$ and $c_{rj}(t)$ as

$$c_{lj}(t) = c_{lj}(0) e^{4\kappa_j^3 t} = c_{lj} e^{4\kappa_j^3 t}, \quad c_{rj}(t) = c_{rj}(0) e^{-4\kappa_j^3 t} = c_{rj} e^{-4\kappa_j^3 t}.$$

5. AKNS Method to Derive the KdV

In the previous section we have seen that the KdV arises as a compatibility condition in the Lax method. There are other methods to derive nonlinear PDEs that can be solved by the inverse scattering transform, i.e. by solving the inverse problem with the time-evolved scattering data for a corresponding linear differential equation. One of these methods was developed by Ablowitz, Kaup, Newel, and Segur, and it was first applied to the Sine-Gordon equation [23]. Here we outline the basic idea behind the method of Ablowitz, Kaup, Newel, and Segur (AKNS method, for short) and use it to derive the KdV equation.

Given a linear operator \mathcal{X} associated with the system $v_x = \mathcal{X}v$, we are interested in finding another operator \mathcal{T} (the operators \mathcal{X} and \mathcal{T} are said to form an *AKNS pair*) such that:

- (i) The spectral parameter λ does not change in time.
- (ii) The quantity $v_t - \mathcal{T}v$ must remain a solution to $v_x = \mathcal{X}v$.
- (iii) The quantity $\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X}$ must be a (matrix) multiplication operator.

Note that in general \mathcal{X} contains the spectral parameter λ , and hence \mathcal{T} also depends on λ as well. Usually, \mathcal{X} and \mathcal{T} are matrix-valued with entries depending on x , t , and λ . As the operator \mathcal{A} in the Lax method determines the time evolution of solutions to $\mathcal{L}\psi = \lambda\psi$, in the AKNS method the operator \mathcal{T} determines the time evolution of solutions to $v_x = \mathcal{X}v$ according to condition (ii) listed above.

As the following argument shows, for compatibility, we are forced to have

$$(5.1) \quad \mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} = 0,$$

which leads to an integrable PDE and is in general nonlinear. From condition (ii) above we see that

$$(v_t - \mathcal{T}v)_x = \mathcal{X}(v_t - \mathcal{T}v),$$

or equivalently,

$$\begin{aligned} v_{tx} - \mathcal{T}_x v - \mathcal{T}v_x &= \mathcal{X}v_t - \mathcal{X}\mathcal{T}v \\ (5.2) \qquad &= (\mathcal{X}v)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= (v_x)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= v_{xt} - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v. \end{aligned}$$

We expect v to be smooth enough so that $v_{tx} = v_{xt}$. Let us replace $\mathcal{T}v_x$ by $\mathcal{T}\mathcal{X}v$ on the left-hand side in (5.2), from which we get $(\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X})v = 0$, which in turn, as a result of condition (iii) listed above, gives us the compatibility condition (5.1).

Let us write the Schrödinger equation (1.2), by replacing the spectral parameter k^2 by λ , in the form of the first-order linear system $v_x = \mathcal{X}v$ by choosing

$$v = \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} 0 & u(x, t) - \lambda \\ 1 & 0 \end{bmatrix}.$$

We will construct \mathcal{T} so that \mathcal{T} and \mathcal{X} will form an AKNS pair. Let us try

$$\mathcal{T} = \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix},$$

where the entries α , β , ξ , and η may depend on x , t , and λ . The compatibility condition (5.1) leads to

$$\begin{bmatrix} 0 & u_t \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \alpha_x & \beta_x \\ \xi_x & \eta_x \end{bmatrix} + \begin{bmatrix} 0 & u - \lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix} - \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix} \begin{bmatrix} 0 & u - \lambda \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently

$$(5.3) \quad \begin{bmatrix} -\alpha_x - \beta + \xi(u - \lambda) & u_t - \beta_x + \eta(u - \lambda) - \alpha(u - \lambda) \\ -\xi_x + \alpha - \eta & -\eta_x + \beta - \xi(u - \lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From the (1, 1), (2, 1), and (2, 2) entries of the above matrix equation, we obtain

$$(5.4) \quad \beta = -\alpha_x + (u - \lambda)\xi, \quad \eta = \alpha - \xi_x, \quad \eta_x = -\alpha_x.$$

Then the (1, 2) entry in (5.3) is given by

$$(5.5) \quad u_t + \frac{1}{2}\xi_{xxx} - u_x\xi - 2\xi_x(u - \lambda) = 0.$$

Letting $\xi = \lambda\zeta + \mu$ in (5.5), where λ is the spectral parameter, we obtain

$$2\zeta_x\lambda^2 + \left(\frac{1}{2}\zeta_{xxx} - 2\zeta_x u + 2\mu_x - u_x\zeta\right)\lambda + \left(u_t + \frac{1}{2}\mu_{xxx} - 2\mu_x u - u_x\mu\right) = 0.$$

Equating the coefficients of each power of λ to zero, we get

$$(5.6) \quad \zeta = c_1, \quad \mu = \frac{1}{2}c_1 u + c_2, \quad u_t - \frac{3}{2}c_1 u u_x - c_2 u_x + \frac{1}{4}c_1 u_{xxx} = 0,$$

where c_1 and c_2 are arbitrary constants. Using $c_1 = 4$ and $c_2 = 0$, from (5.6) we obtain the KdV given in (1.1). Moreover, with the help of (5.4) we get

$$\alpha = u_x + c_3, \quad \beta = -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}, \quad \xi = 4\lambda + 2u, \quad \eta = c_3 - u_x,$$

where c_3 is an arbitrary constant. Letting $c_3 = 0$, we obtain

$$(5.7) \quad \mathcal{T} = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}.$$

It is possible to obtain the time evolution of the Jost solutions and of the scattering data via the AKNS method. Note that condition (ii) of the AKNS method is satisfied if we have

$$(5.8) \quad v_t - \mathcal{T}v = p(t, \lambda)v,$$

for some (scalar) coefficient $p(t, \lambda)$. For example, if we choose

$$v = \begin{bmatrix} f_1'(k, x; t) \\ f_1(k, x; t) \end{bmatrix},$$

by letting $x \rightarrow +\infty$ in (5.8) and by using (2.2), we obtain the time evolution of $f_1(k, x; t)$ given in (4.13). By using (2.4) with $T(x)$ replaced by $T(k; t)$ and with $L(k)$ by $L(k; t)$, and by letting $x \rightarrow -\infty$ in (5.8) and using (2.2), we get the time evolutions given in (4.15). In a similar way, with the help of (2.3), (4.9), and (5.7), by choosing

$$v = \begin{bmatrix} f_r'(k, x; t) \\ f_r(k, x; t) \end{bmatrix},$$

and by letting $x \rightarrow \pm\infty$ in (5.8), we obtain (4.16) and (4.17).

6. Solution to the Cauchy Problem for the KdV

The Cauchy problem (initial-value problem) for the KdV consists of finding $u(x, t)$ when $u(x, 0)$ is known. As shown by Gardner, Greene, Kruskal, and Miura [11], this problem can be solved by using the inverse scattering transform. Let $\mathcal{D}(t)$ denote any of the equivalents of the scattering data for the the Schrödinger equation with the time-evolved potential $u(x, t)$. In other words, we have

$$(6.1) \quad \mathcal{D}(t) := \{R(k; t), L(k; t), T(k; t), \{\kappa_j(t)\}, \{\gamma_j(t)\}, \{c_{lj}(t)\}, \{c_{rj}(t)\}\},$$

or equivalently,

$$\mathcal{D}(t) = \{R(k) e^{8ik^3 t}, L(k) e^{-8ik^3 t}, T(k), \{\kappa_j\}, \{\gamma_j e^{-8\kappa_j^3 t}\}, \{c_{lj} e^{4\kappa_j^3 t}\}, \{c_{rj} e^{-4\kappa_j^3 t}\}\}.$$

Note that $\mathcal{D}(0)$ corresponds to the initial scattering data associated with the potential $V(x)$, where $u(x, 0) = V(x)$.

Below we outline the solution to the Cauchy problem for the KdV.

- (i) Given $u(x, 0)$, determine the corresponding scattering data $\mathcal{D}(0)$. This is done by solving the *direct scattering problem* $V(x) \mapsto \mathcal{D}(0)$. The solution to this problem is essentially equivalent to solving (2.1) and obtaining the Jost solutions $f_l(k, x)$ and $f_r(k, x)$, from which $\mathcal{D}(0)$ can be constructed.
- (ii) Evolve in time the scattering data as $\mathcal{D}(0) \mapsto \mathcal{D}(t)$ in accordance with (6.1). Note that the time evolution of the scattering data is really simple. On the other hand, the time evolution of the potential $u(x, 0) \mapsto u(x, t)$ will be much more complicated. Similarly, we expect that the time evolution of the Jost solutions $f_l(k, x) \mapsto f_l(k, x; t)$ and $f_r(k, x) \mapsto f_r(k, x; t)$, governed by the PDEs (4.13) and (4.16), respectively, will be complicated.
- (iii) Having obtained the time-evolved scattering data $\mathcal{D}(t)$, solve the corresponding inverse scattering problem $\mathcal{D}(t) \mapsto u(x, t)$ for (1.2). This problem is known to be uniquely solvable [13] when the initial potential V belongs to the Faddeev class.

As indicated in Section 3, in step (iii) above one can obtain the solution to the inverse scattering problem by solving, for example, the time-evolved Riemann-Hilbert problem [cf. (3.1)]

$$(6.2) \quad m_l(-k, x; t) = T(k) m_r(k, x; t) - R(k) e^{2ikx + 8ik^3 t} m_l(k, x; t), \quad k \in \mathbf{R},$$

and recover $u(x, t)$ by using [cf. (3.4)]

$$(6.3) \quad u(x, t) = \frac{m_l''(k, x; t)}{m_l(k, x; t)} + 2ik \frac{m_l'(k, x; t)}{m_l(k, x; t)}, \quad k \in \overline{\mathbf{C}^+},$$

where the right-hand side can be evaluated at any k value, including $k = 0$ and $k = \pm\infty$. Equivalently, one can solve the time-evolved Riemann-Hilbert problem [cf. (3.2)]

$$m_r(-k, x; t) = T(k) m_l(k, x; t) - L(k) e^{-2ikx - 8ik^3 t} m_r(k, x; t), \quad k \in \mathbf{R},$$

and use [cf. (3.4)]

$$u(x, t) = \frac{m_r''(k, x; t)}{m_r(k, x; t)} - 2ik \frac{m_r'(k, x; t)}{m_r(k, x; t)}, \quad k \in \overline{\mathbf{C}^+},$$

where the right-hand side can be evaluated at any k value in $\overline{\mathbf{C}^+}$.

Alternatively, one can solve the time-evolved left Marchenko equation [cf. (3.5)]

$$(6.4) \quad B_l(x, y; t) + \Omega_l(2x + y; t) + \int_0^\infty dz \Omega_l(2x + y + z; t) B_l(x, z; t) = 0, \quad y > 0,$$

with

$$\Omega_l(y; t) := \frac{1}{2\pi} \int_{-\infty}^\infty dk R(k) e^{8ik^3 t + iky} + \sum_{j=1}^N c_{lj}^2 e^{8\kappa_j^3 t - \kappa_j y},$$

and recover $u(x, t)$ by using [cf. (3.6)]

$$(6.5) \quad u(x, t) = -2 \frac{\partial B_l(x, 0^+; t)}{\partial x}.$$

Equivalently, one can solve the time-evolved right Marchenko equation [cf. (3.8)]

$$(6.6) \quad B_r(x, y; t) + \Omega_r(-2x + y; t) + \int_0^\infty dz \Omega_r(-2x + y + z; t) B_r(x, z; t) = 0, \quad y > 0,$$

with

$$\Omega_r(\alpha; t) := \frac{1}{2\pi} \int_{-\infty}^\infty dk L(k) e^{-8ik^3t + ik\alpha} + \sum_{j=1}^N c_{rj}^2 e^{-8\kappa_j^3 t - \kappa_j \alpha},$$

and obtain $u(x, t)$ by using [cf. (3.9)]

$$(6.7) \quad u(x, t) = 2 \frac{\partial B_r(x, 0^+; t)}{\partial x}.$$

7. Soliton Solutions to the KdV

Consider the Cauchy problem for the KdV corresponding to the initial scattering data with zero reflection coefficients, N bound states at $k = i\kappa_j$, and dependency constant γ_j , where we have the ordering $0 < \kappa_1 < \dots < \kappa_N$. When $R \equiv 0$, from (2.11) we see that the transmission coefficient is given by

$$(7.1) \quad T(k) = \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}.$$

In this case, the inverse scattering problem $\mathcal{D}(t) \mapsto u(x, t)$ can be solved algebraically in a closed form, and the resulting solution $u(x, t)$ to the KdV is known as the N -soliton solution. Using (7.1) we can write (6.2) as

$$(7.2) \quad m_l(-k, x; t) \prod_{j=1}^N (k - i\kappa_j) = m_r(k, x; t) \prod_{j=1}^N (k + i\kappa_j), \quad k \in \mathbf{R}.$$

From the analyticity properties of m_l and m_r it follows that each side in (7.2) is entire in k with a polynomial growth of leading term k^N as $k \rightarrow \infty$ in the complex plane \mathbf{C} . Further, since k appears as ik in the Faddeev functions, both sides in (7.2) must be a polynomial of the form

$$(7.3) \quad k^N + ik^{N-1}a_{N-1}(x, t) + \dots + i^N a_0(x, t),$$

where $a_j(x, t)$ are real valued and to be determined by using [cf. (2.10), (3.3), and (4.20)]

$$(7.4) \quad \gamma_j e^{2\kappa_j x - 8\kappa_j^3 t} = \frac{m_l(i\kappa_j, x; t)}{m_r(i\kappa_j, x; t)}, \quad j = 1, \dots, N.$$

From (7.2) and (7.3) we get

$$(7.5) \quad m_l(k, x; t) = \frac{k^N - ik^{N-1}a_{N-1}(x, t) + \cdots + (-i)^N a_0(x, t)}{\prod_{j=1}^N (k + i\kappa_j)},$$

$$(7.6) \quad m_r(k, x; t) = \frac{k^N + ik^{N-1}a_{N-1}(x, t) + \cdots + i^N a_0(x, t)}{\prod_{j=1}^N (k + i\kappa_j)}.$$

Let us define

$$(7.7) \quad \omega_j := \gamma_j e^{2\kappa_j x - 8\kappa_j^3 t}, \quad j = 1, \dots, N,$$

where we recall that the sign of ω_j is the same [cf. (2.10)] as that of $(-1)^{N-j}$. Using (7.5)-(7.7) in (7.4) we get a system of linear algebraic equations for the N unknowns $a_j(x, t)$, namely

$$\omega_j = \frac{\kappa_j^N - \kappa_j^{N-1}a_{N-1}(x, t) + \cdots + (-1)^N a_0(x, t)}{\kappa_j^N + \kappa_j^{N-1}a_{N-1}(x, t) + \cdots + a_0(x, t)}, \quad j = 1, \dots, N,$$

which can be written as

$$(7.8) \quad \begin{bmatrix} \kappa_1^{N-1}(\omega_1 + 1) & \kappa_1^{N-2}(\omega_1 - 1) & \cdots & \omega_1 - (-1)^N \\ \kappa_2^{N-1}(\omega_2 + 1) & \kappa_2^{N-2}(\omega_2 - 1) & \cdots & \omega_2 - (-1)^N \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_N^{N-1}(\omega_N + 1) & \kappa_N^{N-2}(\omega_N - 1) & \cdots & \omega_N - (-1)^N \end{bmatrix} \begin{bmatrix} a_{N-1}(x, t) \\ a_{N-2}(x, t) \\ \vdots \\ a_0(x, t) \end{bmatrix} = \begin{bmatrix} \kappa_1^N(1 - \omega_1) \\ \kappa_2^N(1 - \omega_2) \\ \vdots \\ \kappa_N^N(1 - \omega_N) \end{bmatrix}.$$

Let $M^{(N)}$ denote the coefficient matrix in (7.8) whose (j, n) entry is given as

$$(7.9) \quad M_{jn}^{(N)} := \kappa_j^{N-n} [\omega_j - (-1)^n], \quad j, n = 1, \dots, N.$$

From (7.8), via Cramer's rule, it is possible to extract $a_{N-j}(x, t)$ explicitly as

$$a_{N-j}(x, t) = -\frac{\det P^{(N-j)}}{\det M^{(N)}}, \quad j = 1, \dots, N,$$

where the $(N+1) \times (N+1)$ matrix $P^{(N-j)}$ is given by

$$P^{(N-j)} := \begin{bmatrix} 0 & Y^{(N,j)} \\ Q^{(N)} & M^{(N)} \end{bmatrix},$$

with $Q^{(N)}$ being the $N \times 1$ matrix whose j th row contains the entry $\kappa_j^N(1 - \omega_j)$ for $j = 1, \dots, N$; and $M^{(N)}$ being the $N \times N$ matrix given in (7.9); and $Y^{(N,j)}$ being

the $1 \times N$ matrix whose j th column contains the entry 1 and all the remaining entries being zero. For example, we have

$$P^{(N-1)} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \kappa_1^N(1 - \omega_1) & \kappa_1^{N-1}(\omega_1 + 1) & \kappa_1^{N-2}(\omega_1 - 1) & \cdots & \omega_1 - (-1)^N \\ \kappa_2^N(1 - \omega_2) & \kappa_2^{N-1}(\omega_2 + 1) & \kappa_2^{N-2}(\omega_2 - 1) & \cdots & \omega_2 - (-1)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_N^N(1 - \omega_N) & \kappa_N^{N-1}(\omega_N + 1) & \kappa_N^{N-2}(\omega_N - 1) & \cdots & \omega_N - (-1)^N \end{bmatrix}.$$

We note that $u(x, t)$ can be constructed from $a_{N-1}(x, t)$ alone or from $a_0(x, t)$ alone. For example, using (6.3) and (7.5) in the limit $k \rightarrow \pm\infty$, we see that

$$(7.10) \quad u(x, t) = 2 \frac{\partial a_{N-1}(x, t)}{\partial x},$$

or using (6.3) and (7.5) at $k = 0$ we see that

$$(7.11) \quad u(x, t) = \frac{1}{a_0(x, t)} \frac{\partial^2 a_0(x, t)}{\partial x^2}.$$

In fact, using (7.5) in (6.3) we see that one can recover $u(x, t)$ by using any one of the following $(N + 1)$ equations:

$$u(x, t) = \frac{1}{a_{N-j}(x, t)} \left[\frac{\partial^2 a_{N-j}(x, t)}{\partial x^2} + 2 \frac{\partial a_{N-j-1}(x, t)}{\partial x} \right], \quad j = 0, 1, \dots, N,$$

where we have defined $a_{-1}(x, t) := 0$ and $a_N(x, t) := 1$.

Alternatively, we can obtain the N -soliton solution $u(x, t)$ via (6.5) by solving, in a closed form, the left Marchenko equation (6.4), which has a degenerate kernel thanks to the fact that $R \equiv 0$. Similarly, $u(x, t)$ can be obtained via (6.7) by solving, in a closed form, the right Marchenko equation (6.6), which has a degenerate kernel.

It is also possible to obtain the solutions to the Marchenko equations (6.4) and (6.6) in an algebraic manner, without really solving the integral equations themselves. Below we illustrate this for the recovery of $B_1(x, y; t)$ and obtain $u(x, t)$ via (6.5). Using $R \equiv 0$ we can write (6.2) as

$$m_l(-k, x; t) - 1 = [T(k) - 1] m_r(k, x; t) + [m_r(k, x; t) - 1], \quad k \in \mathbf{R}.$$

Taking the Fourier transform of both sides with $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{iky}$ and exploiting the analyticity properties of m_l and m_r , we obtain [cf. (3.7), (3.10)]

$$(7.12) \quad B_1(x, y; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk [T(k) - 1] m_r(k, x; t) e^{iky}, \quad y > 0.$$

The right-hand side of (7.12) can be evaluated as a residue integral along the semicircular arc that is the boundary of \mathbf{C}^+ . Using (7.1) in (7.12), we get

$$(7.13) \quad B_1(x, y; t) = i \sum_{j=1}^N [\text{Res}(T, i\kappa_j)] m_r(i\kappa_j, x; t) e^{-\kappa_j y}, \quad y > 0.$$

Using (2.9) and (7.4) in (7.13), we obtain

$$(7.14) \quad B_1(x, y; t) = - \sum_{j=1}^N \varepsilon_j m_1(i\kappa_j, x; t) e^{-\kappa_j y},$$

where we have defined

$$\varepsilon_j := c_{1j}^2 e^{-2\kappa_j x + 8\kappa_j^3 t}.$$

Using the time-evolved version of (3.7), from (7.14) we get

$$(7.15) \quad m_1(k, x; t) = 1 - \sum_{j=1}^N \frac{i\varepsilon_j}{k + i\kappa_j} m_1(i\kappa_j, x; t) e^{-\kappa_j y}.$$

Putting $k = i\kappa_n$ in (7.15) for $n = 1, \dots, N$, it is possible to recover the $m_1(i\kappa_j, x; t)$ by solving the linear algebraic system

$$(7.16) \quad \begin{bmatrix} 1 + \frac{\varepsilon_1}{\kappa_1 + \kappa_1} & \frac{\varepsilon_2}{\kappa_1 + \kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_1 + \kappa_N} \\ \frac{\varepsilon_1}{\kappa_2 + \kappa_1} & 1 + \frac{\varepsilon_2}{\kappa_2 + \kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_2 + \kappa_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\varepsilon_1}{\kappa_N + \kappa_1} & \frac{\varepsilon_2}{\kappa_N + \kappa_2} & \cdots & 1 + \frac{\varepsilon_N}{\kappa_N + \kappa_N} \end{bmatrix} \begin{bmatrix} m_1(i\kappa_1, x; t) \\ m_1(i\kappa_2, x; t) \\ \vdots \\ m_1(i\kappa_N, x; t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Let Γ denote the coefficient matrix in (7.16), namely, let its (n, j) entry be given by

$$\Gamma_{nj} = \delta_{nj} + \frac{\varepsilon_j}{\kappa_n + \kappa_j}, \quad n, j = 1, \dots, N,$$

where δ_{nj} denotes the Kronecker delta. Then, using (7.14) and (7.16), we obtain $B_1(x, 0^+; t)$ as the ratio of two determinants as

$$B_1(x, 0^+; t) = \frac{\det Z}{\det \Gamma},$$

where Z is the matrix defined as

$$Z := \begin{bmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_N \\ 1 & 1 + \frac{\varepsilon_1}{\kappa_1 + \kappa_1} & \frac{\varepsilon_2}{\kappa_1 + \kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_1 + \kappa_N} \\ 1 & \frac{\varepsilon_1}{\kappa_2 + \kappa_1} & 1 + \frac{\varepsilon_2}{\kappa_2 + \kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_2 + \kappa_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\varepsilon_1}{\kappa_N + \kappa_1} & \frac{\varepsilon_2}{\kappa_N + \kappa_2} & \cdots & 1 + \frac{\varepsilon_N}{\kappa_N + \kappa_N} \end{bmatrix}.$$

It can be shown that $\det Z = \partial_x(\det \Gamma)$ and hence with the help of (6.5), we get

$$(7.17) \quad u(x, t) = -2 \frac{\partial}{\partial x} \left[\frac{1}{\det \Gamma} \frac{\partial(\det \Gamma)}{\partial x} \right].$$

Some Mathematica notebooks based on (7.10), (7.11), and (7.17) for the evaluation of N -soliton solutions to the KdV and their animations are available at the author's web page [24].

We can use (7.10), (7.11), or (7.17) to analyze properties of solitons of the KdV. For example, when $N = 1$, from (7.10) or (7.11) we get the single soliton solution to (1.1) as

$$(7.18) \quad u(x, t) = -2\kappa_1^2 \operatorname{sech}^2 \left(\kappa_1 x - 4\kappa_1^3 t + \sqrt{\ln \gamma_1} \right).$$

It is seen that the amplitude of this wave is $2\kappa_1^2$, it moves in the positive x direction with speed $4\kappa_1^2$, and the dependency constant γ_1 plays a role in the initial location of the soliton. The width of the soliton is inversely proportional to κ_1 , which can be seen, e.g., by using the fact that $\int_{-\infty}^{\infty} dx \sqrt{-u(x, t)}$ is equal to $\sqrt{2}\pi$.

By exploiting the properties of one-soliton solutions to the KdV, one can show that as $t \rightarrow +\infty$, the N -soliton solution to the KdV resembles a train of N separate solitons each traveling with speed $4\kappa_j^2$. In this case, the KdV can be considered for all $t \in \mathbf{R}$ and it can be shown that each soliton emerges from the nonlinear interaction by experiencing only a change in the phase. For details, the reader is referred to [3,4,6,25].

8. Conclusion

In this section we will comment on three aspects of the KdV; namely, the time evolution stated in condition (ii) of the Lax method outlined in Section 4, the conserved quantities, and the Bäcklund transformation.

Some references carelessly state the time evolution associated with the Lax method. For example, in (1.2.10) of [3, page 6] it is stated that the evolution of the solutions to (1.2) is given by $\partial_t \psi = \mathcal{A}\psi$ with \mathcal{A} as in (4.7), instead of the correct statement (ii) of Section 4. The incorrectness of $\partial_t \psi = \mathcal{A}\psi$ can be demonstrated explicitly by an elementary example. Consider the 1-soliton solution to the KdV given in (7.18). Let us choose $\gamma_1 = \kappa_1 = 1$. A solution to (1.2) is obtained as $f_1(i, x; t)$. Let us call that solution ψ . We have

$$\psi(x, t) = \frac{1}{2} e^{-4t} \operatorname{sech}(x - 4t), \quad u(x, t) = -2 \operatorname{sech}^2(x - 4t).$$

It can directly be verified that ψ satisfies (1.2) with $k = i$, and $(\partial_t - \mathcal{A})\psi = -4\psi$, as indicated by the correct time evolution (4.13) with $k = i$; hence $\partial_t \psi \neq \mathcal{A}\psi$.

In (1.7.6) of [6, page 25] and (4) of [5, page 81] it is carelessly stated that the evolution of the solutions to (1.2) is given by $\partial_t \psi - \mathcal{A}\psi = c\psi$ with \mathcal{A} as in (4.7) and c is an arbitrary constant or a function of t . The incorrectness of this can be demonstrated, for example, by choosing ψ as $f_1(k, x; t) + f_r(k, x; t)$, i.e.

the sum of the Jost solutions to (1.2). From (4.13) and (4.16) it follows that, with this choice of ψ , the equation $\partial_t \psi - \mathcal{A}\psi = c\psi$ would hold if and only if $(4ik^3 + c)f_r(k, x; t) - (4ik^3 - c)f_l(k, x; t) = 0$ for $x \in \mathbf{R}$ and $t > 0$, which is impossible due to the linear independence of $f_l(k, x; t)$ and $f_r(k, x; t)$ on $x \in \mathbf{R}$. Note that (4.4) has two linearly independent (generalized) eigenfunctions for each $\lambda > 0$ and hence $\partial_t \psi - \mathcal{A}\psi$ in general is not expected to be a constant multiple of ψ .

Let us note that a potential in the Faddeev class need not even be continuous. On the other hand, from (1.1) we see that classical solutions to the KdV are thrice differentiable with respect to x . Informally speaking, the discontinuities that may be present in the initial value $u(x, 0)$ disappear and $u(x, t)$ becomes smoother for $t > 0$. On the other hand, even though $u(x, t)$ changes as t increases, certain integrals involving $u(x, t)$ with respect to x remain unchanged in time. Such quantities are known as conserved quantities for the KdV. They can either be obtained directly from (1.1) or from the expansion of $T(k)$ in powers of $1/k$ as $k \rightarrow \pm\infty$ by using the fact that $T(k; t)$ does not change in time [cf. (4.15)]. When $u(x, t)$ is smooth, with the help of (2.6) we obtain

$$T(k; t) = 1 + \frac{C_1}{2ik} - \frac{C_1^2}{8k^2} - \frac{C_1^3}{48ik^3} + \frac{C_2}{8ik^3} + O(1/k^4), \quad k \rightarrow \infty \text{ in } \overline{\mathbf{C}^+},$$

where we have defined

$$C_j := \int_{-\infty}^{\infty} dx u(x, t)^j, \quad j = 1, 2.$$

Thus, we have identified two of the infinite number of conserved quantities; i.e., C_1 and C_2 are independent of time and are equal to their values at $t = 0$. The time independence of C_1 and C_2 can also be obtained directly from (1.1). We can write (1.1) as $u_t = (3u^2 - u_{xx})_x$, and hence

$$\frac{dC_1}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} dx u(x, t) = 3u(x, t)^2 - u_{xx}(x, t) \Big|_{x=-\infty}^{\infty} = 0,$$

where we have used $u(x, t) = o(1)$ and $u_{xx}(x, t) = o(1)$ as $x \rightarrow \pm\infty$. Similarly, after multiplying (1.1) with $u(x, t)$, we can write the resulting equation as

$$(8.1) \quad (u^2)_t = (4u^3 - 2uu_{xx} + u_x^2)_x.$$

Integrating both sides of (8.1) on $x \in \mathbf{R}$, we get

$$\frac{dC_2}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} dx u(x, t)^2 = 4u(x, t)^3 - 2u(x, t)u_{xx}(x, t) + u_x(x, t)^2 \Big|_{x=-\infty}^{\infty} = 0,$$

verifying the time independence of C_2 .

Can we characterize the set of nonlinear PDEs solvable by an inverse scattering transform? In other words, can we find a set of necessary and sufficient conditions that guarantee an initial-value problem for a nonlinear PDE to be solvable via an inverse scattering transform related to a linear problem? There does not yet seem to be a satisfactory solution to this characterization problem. On the

other hand, nonlinear PDEs solvable by an inverse scattering transform seem to have some common characteristic features such as the Lax pair, the AKNS pair, soliton solutions, an infinite number of conserved quantities, a Hamiltonian formalism, the Painlevé property, and the Bäcklund transformation. Here, we only briefly explain the last feature and refer the reader to [6] for details and other features.

A Bäcklund transformation is a means to produce another integrable nonlinear PDE from a given one. The basic idea is as follows. Assume v satisfies the integrable nonlinear PDE $\mathcal{M}(v) = 0$, and u satisfies another integrable nonlinear PDE, say $\mathcal{Q}(u) = 0$. A relationship $\mathcal{P}(u, v) = 0$, which is called a Bäcklund transformation, involving v , u , and their derivatives allows us to obtain $\mathcal{Q}(u) = 0$ from $\mathcal{M}(v) = 0$. A Bäcklund transformation can also be used on the same nonlinear PDE to produce another solution from a given solution.

As an example, assume that v satisfies the *modified KdV* given by

$$(8.2) \quad v_t - 6v^2v_x + v_{xxx} = 0, \quad x \in \mathbf{R}, \quad t > 0.$$

Then, choosing

$$(8.3) \quad u = v_x + v^2,$$

one can show that

$$u_t - 6uu_x + u_{xxx} = (\partial_x + 2v)(v_t - 6v^2v_x + v_{xxx}), \quad x \in \mathbf{R}, \quad t > 0.$$

Thus, (8.2) and (8.3) imply (1.1). The Bäcklund transformation given in (8.3) is known as Miura's transformation [26]. For a Bäcklund transformation applied on the KdV to produce other solutions from a given solution, we refer the reader to [6,27].

Another interesting question is the determination of the linear problem associated with the inverse scattering transform. In other words, given a nonlinear PDE that is known to be solvable by an inverse scattering transform, can we determine the corresponding linear problem? There does not yet seem to be a completely satisfactory answer to this question. We mention that Wahlquist and Estabrook [28] developed the so-called *prolongation method* to derive the linear scattering problem associated with the KdV and refer the reader to [6] for details.

References

- [1] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary waves*, Phil. Mag. **39** (1895) 422–443.
- [2] J. S. Russell, *Report on waves*, Report of the 14th meeting of the British Association for the Advancement of Science, John Murray, London, 1845, pp. 311–390.
- [3] G. L. Lamb, Jr., *Elements of soliton theory*, Wiley, New York, 1980.
- [4] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.

- [5] P. G. Drazin, *Solitons*, Cambridge University Press, Cambridge, 1986.
- [6] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press, Cambridge, 1991.
- [7] E. Fermi, J. Pasta, and S. Ulam, *Studies of non linear problems*, Document LA-1940, Los Alamos National Laboratory, May 1955.
- [8] E. Fermi, *Collected papers, Vol. II: United States, 1939–1954*, University of Chicago Press, Chicago, 1965.
- [9] <http://www.osti.gov/accomplishments/pdf/A80037041/A80037041.pdf>
- [10] N. J. Zabusky and M. D. Kruskal, *Interaction of “solitons” in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett. **15** (1965) 240–243.
- [11] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. **19** (1967) 1095–1097.
- [12] L. D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Am. Math. Soc. Transl. (Ser. 2) **65** (1967) 139–166.
- [13] V. A. Marchenko, *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.
- [14] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979) 121–251.
- [15] R. G. Newton, *The Marchenko and Gel’fand-Levitan methods in the inverse scattering problem in one and three dimensions*, In: J. B. Bednar et al. (eds), *Conference on inverse scattering: theory and application*, SIAM, Philadelphia, 1983, pp. 1–74.
- [16] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory*, 2nd ed., Springer, New York, 1989.
- [17] A. Melin, *Operator methods for inverse scattering on the real line*, Comm. Partial Differential Equations **10** (1985) 677–766.
- [18] T. Aktosun and M. Klaus, *Small-energy asymptotics for the Schrödinger equation on the line*, Inverse Problems **17** (2001) 619–632.
- [19] T. Aktosun and M. Klaus, *Chapter 2.2.4, Inverse theory: problem on the line*, In: E. R. Pike and P. C. Sabatier (eds), *Scattering*, Academic Press, London, 2001, pp. 770–785.
- [20] R. G. Newton, *Inverse scattering. I. One dimension*, J. Math. Phys. **21** (1980) 493–505.
- [21] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968) 467–490.
- [22] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Soviet Phys. JETP **34** (1972) 62–69.
- [23] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Method for solving the sine-Gordon equation*, Phys. Rev. Lett. **30** (1973) 1262–1264.
- [24] <http://www.msstate.edu/~aktosun/kdv>
- [25] G. B. Whitham, *Linear and nonlinear waves*, Wiley, New York, 1974.
- [26] R. M. Miura, *Korteweg-de Vries equation and generalization. I. A remarkable explicit nonlinear transformation*, J. Math. Phys. **9** (1968) 1202–1204.

- [27] H. D. Wahlquist and F. B. Estabrook, *Bäcklund transformation for solutions of the Korteweg-de Vries equation*, Phys. Rev. Lett. **31** (1973) 1386–1390.
- [28] H. D. Wahlquist and F. B. Estabrook, *Prolongation structures and nonlinear evolution equations*, J. Math. Phys. **16** (1975) 1–7.

Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA

E-mail address: aktosun@math.msstate.edu