

**TIME EVOLUTION OF THE SCATTERING DATA  
FOR A FOURTH-ORDER LINEAR DIFFERENTIAL OPERATOR**

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**Abstract:** The time evolution of the scattering and spectral data is obtained for the differential operator  $\frac{d^4}{dx^4} + \frac{d}{dx}u(x, t)\frac{d}{dx} + v(x, t)$ , where  $u(x, t)$  and  $v(x, t)$  are real-valued potentials decaying exponentially as  $x \rightarrow \pm\infty$  at each fixed  $t$ . The result is relevant in a crucial step of the inverse scattering transform method that is used in solving the initial-value problem for a pair of coupled nonlinear partial differential equations satisfied by  $u(x, t)$  and  $v(x, t)$ .

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**Short title:** Time evolution for a fourth-order ODE

## 1. INTRODUCTION

Consider the fourth-order ordinary differential equation

$$\varphi'''' + (u\varphi')' + v\varphi = k^4\varphi, \quad x \in \mathbf{R}, \quad (1.1)$$

where the prime denotes the derivative with respect to the independent variable  $x$ , and the potentials  $u$  and  $v$  are real valued in such a way that  $u, u', v$  are integrable and they decay exponentially or faster [8,9] as  $x \rightarrow \pm\infty$ . Let us write (1.1) as  $\mathcal{L}\varphi = k^4\varphi$ , where  $\mathcal{L}$  is the linear operator defined as

$$\mathcal{L} := D^4 + DuD + v, \quad D := \frac{d}{dx}. \quad (1.2)$$

We will consider (1.1) in the sector  $\Omega$  in the complex  $k$ -plane where  $\arg k \in [0, \pi/4]$  with the understanding that we exclude the point  $k = 0$ . A complete analysis at  $k = 0$  for (1.1) does not exist and this deserves a careful future study. Some partial analysis at  $k = 0$  is available in [8]. We will omit the analysis at  $k = 0$  in this paper.

The differential equation appearing in (1.1) is the canonical equivalent [4,6] of the Euler-Bernoulli equation, and the former is obtained from the latter via a transformation of both independent and dependent variables [4]. Such fourth-order equations arise in analyzing vibrations of beams, whereas second-order equations are used in describing vibrations of strings.

We refer the reader to [8,9] and [7,10] for two complementary studies of (1.1). Under appropriate restrictions on the potentials  $u$  and  $v$ , Iwasaki [8,9] studied (1.1) by analyzing it in the sector  $\Omega$  under the assumption that there are no bound states and no spectral or nonspectral singularities. He obtained [8] various properties of solutions to (1.1) and formulated [9] the inverse problem of recovery of  $u$  and  $v$  from some appropriate scattering data. Iwasaki posed [9] the inverse scattering problem for (1.1) as a boundary-value problem where the scattering data consisted of a reflection coefficient and a connection coefficient specified on the boundary of  $\Omega$ , and he provided the proof of uniqueness for

the solution to that inverse problem. A special case of (1.1) was studied in [7,10] under the very restrictive assumption that the reflection and connection coefficients are all zero. The corresponding inverse problem was formulated [10] as a Riemann-Hilbert problem on the whole complex  $k$ -plane, where the set of scattering data is specified on the rays  $\arg k = (l - 1)\pi/4$  for  $l = 1, 2, \dots, 8$ . It seems to be the case that the authors of [7,10] have not been aware of [8,9]. Some examples of explicit solutions to (1.1) were provided in [7,10] under the restriction that the reflection and connection coefficients are all zero. We should indicate that the terminology used in [7,10] differs from that used in [8,9]. For the benefit of the reader the relationships between the quantities in [7,10] and those in [8,9] are indicated in Section 5 of our paper.

It is already known [5,10,12] that (1.1) is related to the coupled system of nonlinear partial differential equations given in (4.2), which is solvable by the inverse scattering transform method and is related to the Gelfand-Dickey hierarchy [5]. The time evolution of the scattering and bound-state data forms a crucial step in that inverse scattering transform.

One of our goals in this paper is to analyze the bound states for (1.1) by analyzing it in the sector  $\Omega$  of the complex  $k$ -plane, which was also the domain used in [8,9], and to obtain the time evolution of the corresponding bound-state data; namely, the time evolution of the dependency and norming constants. The time evolution of the bound-state norming constants is given in [10] by analyzing (1.1) in the entire complex  $k$ -plane. Another goal of ours is to obtain the time evolutions of the scattering and connection coefficients as well as other coefficients related to scattering, which is absent in [7,10]. In our paper we also clarify how the relevant quantities in [7,10] are related to those of [8,9], and we provide some illustrative examples of explicit solutions to (1.1) helping to understand the corresponding scattering and bound states better.

Our paper is organized as follows. In Section 2 we provide the preliminaries and introduce the Jost solutions  $\psi_+$  and  $\psi_-$  as well as the exponential solutions  $\phi_+$  and  $\phi_-$

to (1.1), and we present the scattering data in terms of the spatial asymptotics of the Jost and exponential solutions. In Section 3 we analyze the bound states associated with (1.1) and introduce the dependency constants and norming constants for each bound state. In Section 4 we consider the system of integrable nonlinear partial differential equations satisfied by the time-evolved potentials  $u(x, t)$  and  $v(x, t)$ , and we obtain the time evolution of the corresponding scattering and bound-state data. In Section 5 we explain how the quantities associated with (1.1) and used in [7,10] are related to those used in [8,9]. Finally, in Section 6 we provide some illustrative explicit solutions to (1.1).

## 2. PRELIMINARIES

Consider the case that the potentials  $u$  and  $v$  in (1.1) are functions of  $x$  alone and do not depend on the parameter  $t$ . In Section 4 we will consider the case where  $u$  and  $v$  depend on both  $x$  and  $t$ .

As in [8] let us introduce the Jost solutions  $\psi_+$  and  $\psi_-$  to (1.1) in the sector  $\arg k \in (0, \pi/4)$  with the asymptotics

$$\psi_{\pm}(k, x) = e^{\pm ikx} [1 + o(1)], \quad x \rightarrow \pm\infty, \quad (2.1)$$

in such a way that  $e^{\mp ikx} \psi_{\pm}(k, x)$  remains bounded for all  $x \in \mathbf{R}$ . Similarly, let us introduce the exponential solutions  $\phi_+$  and  $\phi_-$  in the sector  $\arg k \in (0, \pi/4)$  satisfying the asymptotics

$$\phi_{\pm}(k, x) = e^{\mp kx} [1 + o(1)], \quad x \rightarrow \pm\infty. \quad (2.2)$$

It follows that these solutions satisfy the respective integral relations given by

$$\begin{aligned} \phi_+(k, x) &= e^{-kx} + \frac{1}{4k^4} \sum_{j=1}^4 \int_x^{\infty} dy [k_j^3 u(y) - k_j^2 u'(y) + k_j v(y)] e^{k_j(x-y)} \phi_+(k, y), \\ \psi_+(k, x) &= e^{ikx} + \frac{1}{4k^4} \sum_{j=1,2,4} \int_x^{\infty} dy [k_j^3 u(y) - k_j^2 u'(y) + k_j v(y)] e^{k_j(x-y)} \psi_+(k, y) \\ &\quad - \frac{1}{4k^4} \int_{-\infty}^x dy [k_3^3 u(y) - k_3^2 u'(y) + k_3 v(y)] e^{k_3(x-y)} \psi_+(k, y), \end{aligned}$$

$$\begin{aligned}
\phi_-(k, x) &= e^{kx} - \frac{1}{4k^4} \sum_{j=1}^4 \int_{-\infty}^x dy [k_j^3 u(y) - k_j^2 u'(y) + k_j v(y)] e^{k_j(x-y)} \phi_-(k, y), \\
\psi_-(k, x) &= e^{-ikx} + \frac{1}{4k^4} \int_x^{\infty} dy [k_1^3 u(y) - k_1^2 u'(y) + k_1 v(y)] e^{k_1(x-y)} \psi_-(k, y) \\
&\quad - \frac{1}{4k^4} \sum_{j=2,3,4} \int_{-\infty}^x dy [k_j^3 u(y) - k_j^2 u'(y) + k_j v(y)] e^{k_j(x-y)} \psi_-(k, y),
\end{aligned}$$

where we have defined

$$k_1 := k, \quad k_2 := ik, \quad k_3 := -k, \quad k_4 := -ik. \quad (2.3)$$

The spatial asymptotics of the Jost solutions and the exponential solutions can be obtained with the help of the four integral relations given above. Below, we list those asymptotics, through which we introduce the scattering and connection coefficients for (1.1). For  $\arg k \in (0, \pi/4)$  we have

$$\begin{cases}
\psi_{\pm}(k, x) = e^{\pm ikx} [1 + o(1)], & x \rightarrow \pm\infty, \\
\psi_{\pm}(k, x) = e^{\pm ikx} \left[ \frac{1}{T(k)} + o(1) \right], & x \rightarrow \mp\infty, \\
\phi_{\pm}(k, x) = e^{\mp kx} [1 + o(1)], & x \rightarrow \pm\infty, \\
\phi_{\pm}(k, x) = e^{\mp kx} [A(k) + o(1)], & x \rightarrow \mp\infty.
\end{cases} \quad (2.4)$$

For  $\arg k = 0$  we have

$$\begin{cases}
\psi_{\pm}(k, x) = e^{\pm ikx} [1 + o(1)], & x \rightarrow \pm\infty, \\
\psi_{\pm}(k, x) = e^{\pm ikx} \left[ \frac{1}{T(k)} + \frac{R_{\pm}(k)}{T(k)} e^{\mp 2ikx} + o(1) \right], & x \rightarrow \mp\infty, \\
\phi_{\pm}(k, x) = e^{\mp kx} [1 + o(1)], & x \rightarrow \pm\infty, \\
\phi_{\pm}(k, x) = e^{\mp kx} [A(k) + o(1)], & x \rightarrow \mp\infty.
\end{cases} \quad (2.5)$$

For  $\arg k = \pi/4$  we have

$$\begin{cases}
\psi_{\pm}(k, x) = e^{\pm ikx} [1 + C_{\pm}(k) e^{\mp kx \mp ikx} + o(1)], & x \rightarrow \pm\infty, \\
\psi_{\pm}(k, x) = e^{\pm ikx} \left[ \frac{1}{T(k)} + o(1) \right], & x \rightarrow \mp\infty, \\
\phi_{\pm}(k, x) = e^{\mp kx} [1 + o(1)], & x \rightarrow \pm\infty, \\
\phi_{\pm}(k, x) = e^{\mp kx} [A(k) + B_{\pm}(k) e^{\pm kx \pm ikx} + o(1)], & x \rightarrow \mp\infty.
\end{cases} \quad (2.6)$$

We emphasize that  $A(k)$  and  $T(k)$  are defined for  $\arg k \in [0, \pi/4]$ ,  $R_+(k)$  and  $R_-(k)$  are defined for  $\arg k = 0$ , and the four coefficients  $B_+(k)$ ,  $B_-(k)$ ,  $C_+(k)$ , and  $C_-(k)$  are defined only for  $\arg k = \pi/4$ . The coefficient  $T$  is known as the transmission coefficient,  $R_+$  and  $R_-$  are the left and right reflection coefficients, respectively,  $C_+$  and  $C_-$  are known as the connection coefficients, and  $B_+$  and  $B_-$  are some coefficients that can be expressed in terms of  $A$ ,  $C_+$ , and  $C_-$ , as we will see. It is either known [8] or can easily be shown that for  $\arg k = 0$  we have

$$1 + \frac{|R_{\pm}(k)|^2}{|T(k)|^2} = \frac{1}{|T(k)|^2}, \quad \frac{R_{\pm}(k)}{T(k)} = -\frac{R_{\mp}(k)^*}{T(k)^*}, \quad A(k) = A(k)^*, \quad (2.7)$$

and for  $\arg k = \pi/4$  we have

$$\left\{ \begin{array}{l} B_{\mp}(k) = -iB_{\pm}(k)^*, \quad C_-(k) = \frac{iT(k)^*C_+(k)^*}{T(k)}, \\ |B_-(k)| = |B_+(k)|, \quad |C_-(k)| = |C_+(k)|, \\ \frac{1}{T(k)} = A(k)^* + B_{\pm}(k)C_{\pm}(k), \quad B_{\pm}(k)^* + A(k)C_{\pm}(k) = 0, \\ \frac{A(k)}{T(k)} = |A(k)|^2 - |B_{\pm}(k)|^2 = |A(k)|^2 (1 - |C_{\pm}(k)|^2), \end{array} \right. \quad (2.8)$$

where the asterisk denotes complex conjugation.

### 3. BOUND STATES

Since the coefficient of the third derivative in (1.1) is zero, it follows from the general theory of ordinary differential equations that the Wronskian of any four solutions to (1.1) is independent of  $x$ , and that Wronskian is zero if and only if those four solutions are linearly dependent. Recall that a Wronskian is defined with the help of a determinant. For example, the Wronskian involving the Jost and exponential solutions is given by

$$W_4[\psi_+, \psi_-, \phi_+, \phi_-] := \begin{vmatrix} \psi_+ & \psi_- & \phi_+ & \phi_- \\ \psi'_+ & \psi'_- & \phi'_+ & \phi'_- \\ \psi''_+ & \psi''_- & \phi''_+ & \phi''_- \\ \psi'''_+ & \psi'''_- & \phi'''_+ & \phi'''_- \end{vmatrix}.$$

Using (2.4)-(2.6) we obtain

$$W_4[\psi_+(k, x), \psi_-(k, x), \phi_+(k, x), \phi_-(k, x)] = -16ik^6 \frac{A(k)}{T(k)}, \quad \arg k \in [0, \pi/4]. \quad (3.1)$$

The linear independence and boundedness properties of various solutions to (1.1) help to identify bound-state solutions. Recall that eigenfunctions of  $\mathcal{L}$  correspond to square-integrable solutions to (1.1), which are also known as bound-state solutions. It is easy to verify that  $\mathcal{L}$  is selfadjoint and hence its eigenvalues can occur only for real values of  $k^4$ , i.e. when  $k$  lies on the boundary of the region  $\Omega$  introduced in Section 1. Thus, any positive eigenvalue of  $\mathcal{L}$  can occur only on the ray  $\arg k = 0$  and any negative eigenvalue can occur only on the ray  $\arg k = \pi/4$ . If  $A(k) = 0$  at some  $k$ -value on the boundary of the region  $\Omega$  and a square-integrable solution to (1.1) at that  $k$ -value does not exist, then we call that  $k$ -value a spectral singularity of (1.1). If  $A(k) = 0$  at some  $k$ -value in the interior of  $\Omega$ , then we call that  $k$ -value a nonspectral singularity of (1.1). By a singularity we refer to either a spectral or nonspectral singularity. It is already known that at a singular point the two integral relations given in Section 2 for  $\psi_+(k, x)$  and  $\psi_-(k, x)$ , respectively, are not solvable [8]. Spectral and nonspectral singularities for (1.1) may exist, and some explicit examples are illustrated in Section 6.

Note that a bound state in the region  $\Omega$  can occur only when  $A(k)/T(k) = 0$  somewhere on the ray  $\arg k = 0$  or  $\arg k = \pi/4$ . Otherwise, as seen from (3.1), the four solutions  $\psi_+$ ,  $\psi_-$ ,  $\phi_+$ , and  $\phi_-$  are linearly independent, and the asymptotics of those four solutions given in (2.5) and (2.6) indicate that no linear combination of them can decay simultaneously both as  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ .

If there is a bound state at  $k = \kappa$  on the ray  $\arg k = 0$ , then we must have  $A(\kappa) = 0$ . This follows from the first identity in (2.7) implying that  $1/T(\kappa) \neq 0$  and the fact that  $A(\kappa)/T(\kappa) = 0$  at the bound state.

Since four linearly independent solutions to (1.1) must have respective asymptotics proportional to  $e^{\kappa x}[1 + o(1)]$ ,  $e^{-\kappa x}[1 + o(1)]$ ,  $e^{i\kappa x}[1 + o(1)]$ ,  $e^{-i\kappa x}[1 + o(1)]$  as  $x \rightarrow +\infty$ , and appropriately similar asymptotics as  $x \rightarrow -\infty$ , it follows that a bound state at  $k = \kappa$  must

decay exponentially as both as  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ . In this case, we see from (2.5) that  $\psi_+(\kappa, x)$  and  $\psi_-(\kappa, x)$  are two linearly independent solutions to (1.1) and they do not vanish simultaneously both as  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ . Thus, from (2.5) we conclude that a bound-state eigenfunction  $\varphi(\kappa, x)$  must be in the form

$$\varphi(\kappa, x) = d_1\phi_+(\kappa, x) = d_2\phi_-(\kappa, x), \quad (3.2)$$

for some nonzero constants  $d_1$  and  $d_2$ . Since any constant multiple of an eigenfunction is still an eigenfunction of  $\mathcal{L}$ , only the ratio  $d_2/d_1$  is relevant and we can call it a dependency constant at  $k = \kappa$ , i.e.

$$\eta(\kappa) := \frac{\phi_+(\kappa, x)}{\phi_-(\kappa, x)}, \quad (3.3)$$

where  $\eta(\kappa)$  is the dependency constant at  $k = \kappa$  on the ray  $\arg k = 0$ . Defining the bound-state norming constants  $d_+$  and  $d_-$  as

$$d_{\pm}(\kappa) := \left[ \int_{-\infty}^{\infty} dx \phi_{\pm}(\kappa, x) \phi_{\pm}(\kappa, x)^* \right]^{-1/2}, \quad (3.4)$$

we see that  $d_- = \eta d_+$  and that  $d_{\pm}(\kappa) \phi_{\pm}(\kappa, x)$  is a normalized bound-state eigenfunction of the operator  $\mathcal{L}$ .

Having clarified the status of bound states on the ray  $\arg k = 0$ , let us now consider the bound states on the ray  $\arg k = \pi/4$ . If there is a bound state at  $k = \kappa$  on the ray  $\arg k = \pi/4$ , then we have three possibilities:

- (i)  $A(\kappa) = 0$  and  $1/T(\kappa) \neq 0$ . In this case, an argument similar to the case given on the ray  $\arg k = 0$  shows that  $\phi_+(\kappa, x)$  and  $\phi_-(\kappa, x)$  are linearly dependent and the bound state is simple and has the form given in (3.2). Recall that a bound state occurring at  $k = \kappa$  is simple if there is only one linearly independent square-integrable solution to (1.1) when  $k = \kappa$ .
- (ii)  $A(\kappa) \neq 0$  and  $1/T(\kappa) = 0$ . In this case, a similar argument indicates that a bound-state eigenfunction must have the form

$$\varphi(\kappa, x) = c_1\psi_+(\kappa, x) = c_2\psi_-(\kappa, x),$$

for some nonzero constants  $c_1$  and  $c_2$ . Since a bound-state eigenfunction is defined up to a constant multiple, only the ratio  $c_2/c_1$  is relevant and we can call that ratio a dependency constant at  $k = \kappa$ , i.e.

$$\gamma(\kappa) := \frac{\psi_+(\kappa, x)}{\psi_-(\kappa, x)}, \quad (3.5)$$

where  $\gamma(\kappa)$  is the dependency constant at  $k = \kappa$  on the ray  $\arg k = \pi/4$ . Defining the bound-state norming constants  $c_+$  and  $c_-$  as

$$c_{\pm}(\kappa) := \left[ \int_{-\infty}^{\infty} dx \psi_{\pm}(\kappa, x) \psi_{\pm}(\kappa, x)^* \right]^{-1/2}, \quad (3.6)$$

we see that  $c_- = \gamma c_+$  and that  $c_{\pm}(\kappa) \psi_{\pm}(\kappa, x)$  is a normalized bound-state eigenfunction of the operator  $\mathcal{L}$ . In this case there is only one linearly independent bound-state eigenfunction at  $k = \kappa$ , and hence the bound state is simple.

- (iii)  $A(\kappa) = 0$  and  $1/T(\kappa) = 0$ . In this case,  $\psi_+(\kappa, x)$  and  $\psi_-(\kappa, x)$  are linearly dependent solutions decaying exponentially as  $x \rightarrow +\infty$  as well as  $x \rightarrow -\infty$ , and  $\phi_+(\kappa, x)$  and  $\phi_-(\kappa, x)$  are also two linearly dependent solutions decaying exponentially as  $x \rightarrow +\infty$  as well as  $x \rightarrow -\infty$ . On the other hand,  $\psi_+(\kappa, x)$  and  $\phi_+(\kappa, x)$  are linearly independent, and hence they form a basis for the two-dimensional eigenspace consisting of bound-state solutions at  $k = \kappa$ . In other words, the multiplicity of the bound state in this case is two. The corresponding dependency constants  $\eta(\kappa)$  and  $\gamma(\kappa)$  and the norming constants  $c_+(\kappa)$ ,  $c_-(\kappa)$ ,  $d_+(\kappa)$ ,  $d_-(\kappa)$  are defined as in (3.3)-(3.6).

#### 4. TIME EVOLUTION OF SCATTERING AND OTHER COEFFICIENTS

In the previous sections we have considered the Jost and exponential solutions, the scattering and connection coefficients, and the bound states associated with (1.1) in the case where the potentials  $u$  and  $v$  are functions of  $x$  alone. Let us now assume that the potentials  $u$  and  $v$  appearing in (1.1) also depend on the extra parameter  $t$ . In that case all the relevant quantities associated with (1.1) may also depend on  $t$  as well. In this section we analyze such a dependence on  $t$  by interpreting  $t$  as the time variable.

Now consider the time evolution of the potentials  $u(x, t)$  and  $v(x, t)$  from their initial values  $u(x, 0)$  and  $v(x, 0)$ , respectively, so that the time-evolved linear operator  $\mathcal{L}$  corresponding to (1.2) is given by

$$\mathcal{L} := \partial_x^4 + \partial_x u(x, t) \partial_x + v(x, t), \quad (4.1)$$

where  $\partial_x = \partial/\partial x$ , and let [5,10,12]

$$\mathcal{A} := -8\partial_x^3 - 6u(x, t)\partial_x - 3u_x(x, t),$$

so that  $\mathcal{L}$  and  $\mathcal{A}$  form a Lax pair. As easily verified, the differential operator  $\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$  reduces to a scalar multiplication operator, and in fact we get

$$\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0,$$

which is equivalent to the system of nonlinear evolution equations

$$\begin{cases} u_t = 10u_{xxx} + 6uu_x - 24v_x, \\ v_t = 3u_{xxxx} + 3uu_{xxx} + 3u_x u_{xx} - 6uv_x - 8v_{xxx}. \end{cases} \quad (4.2)$$

Note that we use subscripts to denote the appropriate partial derivatives.

Since  $u$  and  $v$  vanish as  $x \rightarrow \pm\infty$  at each fixed  $t$ , we have  $\mathcal{A} \rightarrow -8\partial_x^3$  as  $x \rightarrow \pm\infty$ . From (4.1) we also see that

$$\mathcal{L}_t = u_t \partial_x^2 + u_{xt} \partial_x + v_t.$$

In order to solve the initial-value problem related to (4.2), i.e. to determine  $u(x, t)$  and  $v(x, t)$  that solve (4.2) when  $u(x, 0)$  and  $v(x, 0)$  are specified, we are interested in analyzing the time evolutions of the scattering and other coefficients associated with (1.1).

Towards our goal, we first analyze the time evolutions of the Jost solutions  $\psi_+(k, x, t)$  and  $\psi_-(k, x, t)$  and the exponential solutions  $\phi_+(k, x, t)$  and  $\phi_-(k, x, t)$ , from which the time evolutions of other relevant coefficients are easily extracted.

**Theorem 4.1** *In the region  $\arg k \in [0, \pi/4]$  the time evolutions of  $\psi_+(k, x, t)$ ,  $\psi_-(k, x, t)$ ,  $\phi_+(k, x, t)$ , and  $\phi_-(k, x, t)$  are given by*

$$[\partial_t - \mathcal{A}]\psi_{\pm} = \mp 8ik^3 \psi_{\pm}, \quad [\partial_t - \mathcal{A}]\phi_{\pm} = \mp 8k^3 \phi_{\pm}. \quad (4.3)$$

Moreover, the time evolutions of various coefficients appearing in (2.4)-(2.6) are given by

$$T(k, t) = T(k, 0), \quad A(k, t) = A(k, 0), \quad \arg k \in [0, \pi/4],$$

$$R_{\pm}(k, t) = R_{\pm}(k, 0)e^{\mp 16ik^3 t}, \quad \arg k = 0,$$

$$B_{\pm}(k, t) = B_{\pm}(k, 0)e^{\pm(8ik^3 - 8k^3)t}, \quad C_{\pm}(k, t) = C_{\pm}(k, 0)e^{\mp(8ik^3 - 8k^3)t}, \quad \arg k = \pi/4.$$

PROOF: The proofs in (4.3) can all be given as in the case of the time evolution of  $\psi_+$  for  $\arg k = \pi/4$ , which is outlined below. It is known [1-3,11] that  $[\partial_t - \mathcal{A}]\psi_+$  must satisfy  $\mathcal{L}\varphi = k^4\varphi$ , where  $\mathcal{L}$  is the operator in (4.1). Thus, we have

$$[\partial_t - \mathcal{A}]\psi_+ = c_1(k, t)\psi_+ + c_2(k, t)\psi_- + c_3(k, t)\phi_+ + c_4(k, t)\phi_-, \quad (4.4)$$

for some coefficients  $c_j(k, t)$  to be determined. By evaluating (4.4) as  $x \rightarrow -\infty$  and  $x \rightarrow +\infty$ , with the help of (2.6) we obtain

$$\left(\frac{e^{ikx}}{T}\right)_t - 8ik^3 \frac{e^{ikx}}{T} = c_1 \frac{e^{ikx}}{T} + c_2[e^{-ikx} + C_- e^{kx}] + c_3[Ae^{-kx} + B_+ e^{ikx}] + c_4 e^{kx}, \quad (4.5)$$

$$\begin{aligned} (C_+)_t e^{-kx} - 8ik^3 e^{ikx} - 8k^3 C_+ e^{-kx} \\ = c_1[e^{ikx} + C_+ e^{-kx}] + c_2 \frac{e^{-ikx}}{T} + c_3 e^{-kx} + c_4[Ae^{kx} + B_- e^{-ikx}]. \end{aligned} \quad (4.6)$$

By matching the corresponding coefficients of the exponential terms in (4.5) and (4.6) we get

$$c_1 = -8ik^3, \quad c_2 = c_3 = c_4 = 0, \quad T_t = 0, \quad (C_+)_t = (8k^3 - 8ik^3)C_+,$$

and hence the first equation in (4.3) for  $\psi_+$  is confirmed when  $\arg k = \pi/4$ , and we also get the time evolutions of  $T$  and  $C_+$  when  $\arg k = \pi/4$ , as stated. The remaining parts of the proof are obtained in a similar way. ■

The implication of Theorem 4.1 that  $T(k, t)$  and  $A(k, t)$  do not change in  $t$  is significant. As we have seen in Section 3, at a bound state  $k = \kappa$  we must have  $A(\kappa, t)/T(\kappa, t) = 0$ , and at a singularity  $k = \kappa$  we must have  $A(\kappa, t) = 0$ . Hence, the  $k$ -values corresponding to bound states or singularities of the operator  $\mathcal{L}$  of (4.1) also remain unchanged in time.

**Theorem 4.2** *Assume that  $k = \kappa$  corresponds to a bound state of (1.1). The time evolution of the bound-state dependency constants  $\gamma(\kappa, t)$  and  $\eta(\kappa, t)$  and the evolution of the norming constants  $c_{\pm}(\kappa, t)$  and  $d_{\pm}(\kappa, t)$  are given by*

$$c_{\pm}(\kappa, t) = c_{\pm}(\kappa, 0) e^{\pm 4(1+i)\kappa^3 t}, \quad d_{\pm}(\kappa, t) = d_{\pm}(\kappa, 0) e^{\pm 4(1+i)\kappa^3 t}, \quad (4.7)$$

$$\gamma(\kappa, t) = \gamma(\kappa, 0) e^{-8(1+i)\kappa^3 t}, \quad \eta(\kappa, t) = \eta(\kappa, 0) e^{-8(1+i)\kappa^3 t}. \quad (4.8)$$

PROOF: Let us assume that there is a bound state at  $k = \kappa$  somewhere on  $\arg k = \pi/4$  with  $1/T(\kappa, 0) = 0$ . Then,  $\psi_+(\kappa, x, t)$  is a bound-state solution and the norming constant  $c_+(\kappa, t)$  can be defined as in (3.6) via

$$c_+(\kappa, t) := \left[ \int_{-\infty}^{\infty} dx \psi_+(\kappa, x, t) \psi_+(\kappa, x, t)^* \right]^{-1/2}, \quad (4.9)$$

so that  $c_+(\kappa, t) \psi_+(\kappa, x, t)$  is normalized, i.e. its  $L^2$ -norm is equal to one. Let us now find the time evolution of  $c_+(\kappa, t)$ . From (4.3) and its complex conjugate we obtain

$$[\partial_t + 8\partial_x^3 + 6\partial_x u(x, t) + 3u_x(x, t)]\psi_+(\kappa, x, t) = -8i\kappa^3 \psi_+(\kappa, x, t), \quad (4.10)$$

$$[\partial_t + 8\partial_x^3 + 3\partial_x u(x, t) + 3u_x(x, t)]\psi_+(\kappa, x, t)^* = 8i(\kappa^*)^3 \psi_+(\kappa, x, t)^*, \quad (4.11)$$

where we recall that the potentials  $u$  and  $v$  are assumed to be real valued. Multiplying (4.10) by  $\psi_+(\kappa, x, t)^*$  and (4.11) by  $\psi_+(\kappa, x, t)$ , and adding the resulting equations we obtain

$$\partial_t |\psi_+|^2 + \partial_x [8\psi_+^* (\partial_x^2 \psi_+) + 8\psi_+ (\partial_x^2 \psi_+^*) - 8(\partial_x \psi_+) (\partial_x \psi_+^*) + 6u |\psi_+|^2] = -8(1+i)\kappa^3 |\psi_+|^2,$$

where we have used the fact that  $k^* = -ik$  on the ray  $\arg k = \pi/4$ . Integrating over the real axis and using the vanishing of  $\psi_+$ ,  $\partial_x \psi_+$ ,  $\partial_x^2 \psi_+$ , and  $u$  as  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ , we obtain

$$\frac{d}{dt} \int_{-\infty}^{\infty} dx |\psi_+(\kappa, x, t)|^2 = -8(1+i)\kappa^3 \int_{-\infty}^{\infty} dx |\psi_+(\kappa, x, t)|^2. \quad (4.12)$$

Using (4.9) we can write (4.12) as

$$\frac{d}{dt} \left[ \frac{1}{c_+(\kappa, t)^2} \right] = \frac{-8(1+i)\kappa^3}{c_+(\kappa, t)^2},$$

or equivalently we obtain

$$\frac{dc_+(\kappa, t)}{dt} = 4(1+i)\kappa^3 c_+(\kappa, t),$$

which yields

$$c_+(\kappa, t) = c_+(\kappa, 0) e^{4(1+i)\kappa^3 t}.$$

The time evolution of the norming constants  $c_-(\kappa, t)$ ,  $d_+(\kappa, t)$ , and  $d_-(\kappa, t)$  appearing in the analogs of (3.4) and (3.6) can be obtained in a similar way. With the help of (4.3) we obtain (4.7), and hence the dependency constants  $\gamma(\kappa, t)$  and  $\eta(\kappa, t)$  appearing in the analogs of (3.3) and (3.5), respectively, evolve according to (4.8). ■

## 5. A COMPARISON OF REFERENCES [8] AND [10]

As seen from (1.1), if  $f(k, x)$  is a solution to (1.1), so are  $f(-k, x)$ ,  $f(ik, x)$ , and  $f(-ik, x)$ . Thus, a solution to (1.1) known in the region  $\Omega$  in the complex  $k$ -plane can be extended to the three regions obtained by rotating  $\Omega$  by  $\pi/2$ ,  $\pi$ , and  $3\pi/2$ , respectively, around the origin of the complex  $k$ -plane. Moreover, since the potentials  $u$  and  $v$  are real valued,  $f(k^*, x)^*$  is also a solution and hence a solution known in a region in the complex  $k$ -plane can be extended to the symmetric region with respect to the real axis. Thus, solutions known in  $\Omega$  can be extended to the entire complex  $k$ -plane.

In [10], some four solutions  $\Psi_j(k, x)$  to (1.1) for  $j = 1, 2, 3, 4$  are presented on the whole complex  $k$ -plane with spatial asymptotics

$$\Psi_j(k, x) = \begin{cases} e^{k_j x} [1 + o(1)], & x \rightarrow +\infty, \\ a_j(k) e^{k_j x} [1 + o(1)], & x \rightarrow -\infty, \end{cases}$$

where the  $a_j(k)$  are certain coefficients and the  $k_j$  are as in (2.3). Comparing the asymptotics as  $x \rightarrow \pm\infty$ , we see that those four solutions are related to the Jost and exponential solutions appearing in (2.1) and (2.2) as follows:

$$\Psi_1(k, x) = \frac{\phi_-(k, x)}{A(k)}, \quad \Psi_2(k, x) = \psi_+(k, x),$$

$$\Psi_3(k, x) = \phi_+(k, x), \quad \Psi_4(k, x) = T(k)\psi_-(k, x),$$

where  $A$  and  $T$  are the coefficients appearing in some of (2.4)-(2.6). Then, as  $k$  moves to the boundary of  $\arg k \in (0, \pi/4)$  from the interior, we see that the reflection coefficients  $r_0(k)$ ,  $r_1(k)$ , and  $r_2(k)$  defined in [10] are related as follows to the coefficients used in [8] and in our paper:

$$r_0(k) = R_-(k), \quad r_1(k) = C_+(k)^*, \quad r_2(k) = \frac{B_-(k)^*}{A(k)^*}.$$

Moreover, the quantities  $a_j(k)$  appearing in [10] are related to the quantities used in [8] and in our paper as

$$a_1(k) = \frac{1}{A(k)}, \quad a_2(k) = \frac{1}{T(k)}, \quad a_3(k) = A(k), \quad a_4(k) = T(k).$$

Thus, the inverse problem has been analyzed in [10] in the special case  $R_\pm(k) = C_\pm(k) = B_\pm(k) = 0$ . In that case,  $A(k)$  and  $T(k)$  simply become rational functions of  $k$  with asymptotics  $1 + O(1/k)$  as  $k \rightarrow \infty$  and with appropriate jump conditions on the rays  $\arg k = (l-1)\pi/4$  for  $l = 1, 2, \dots, 8$ . One can then formulate the inverse problem on the entire complex  $k$ -plane as a Riemann-Hilbert problem and solve it explicitly.

## 6. EXAMPLES

In this section we present some explicit examples of solutions and relevant quantities associated with (1.1). Such examples should help to understand better the scattering and bound-state data for (1.1). It is already known [5,8,10] that if  $f(k, x)$  is a solution to the Schrödinger equation

$$-f''(k, x) + q(x)f(k, x) = k^2 f(k, x),$$

then  $f(k, x)$  is also a solution to (1.1) when

$$u(x) = -2q(x), \quad v(x) = q(x)^2 - q''(x),$$

because in that case we have

$$D^4 + DuD + v = (-D^2 + q)^2. \tag{6.1}$$

Note that (6.1) holds in our first and fourth examples below, but it does not hold for our second and third examples.

**Example 6.1** Let  $u(x) = v(x) = 0$  for  $x < 0$  and

$$u(x) = \frac{4e^x}{(1+e^x)^2}, \quad v(x) = \frac{2e^x(1-e^x)^2}{(1+e^x)^4}, \quad x > 0.$$

By using the continuity of  $\psi_+$ ,  $\psi'_+$ ,  $\psi''_+$  at  $x = 0$  and the jump condition

$$\psi_+'''(k, 0^+) - \psi_+'''(k, 0^-) = -u(0^+) \psi_+'(k, 0),$$

we can determine all the quantities relevant to (1.1). In terms of

$$f(k, x) := e^{ikx} \left[ 1 - \frac{2i}{(2k+i)(1+e^x)} \right],$$

we have the Jost and exponential solutions

$$\psi_+(k, x) = \begin{cases} f(k, x) + C_+(k) f(ik, x), & x \geq 0, \\ \frac{1}{T(k)} e^{ikx} + \frac{R_+(k)}{T(k)} e^{-ikx} + c_1(k) e^{kx}, & x \leq 0, \end{cases}$$

$$\phi_+(k, x) = \begin{cases} f(ik, x), & x \geq 0, \\ A(k) e^{-kx} + B_+(k) e^{ikx} + c_2(k) e^{kx} + c_3(k) e^{-ikx}, & x \leq 0, \end{cases}$$

where

$$A(k) = \frac{(8k^2+1)(2k-1)}{16k^3}, \quad B_+(k) = \frac{(4-4i)k^2+i}{16k^3(2k+1)}, \quad c_1(k) = \frac{(2i-2)k+(1+i)}{2k(8k^2+1)},$$

$$c_2(k) = \frac{-2k+1}{16k^3}, \quad c_3(k) = \frac{(4+4i)k^2-i}{16k^3(2k+1)}, \quad R_+(k) = \frac{16k^4+(2+4i)k^2-1}{(8k^2-i)(16k^4+2ik^2-1)},$$

$$C_+(k) = \frac{(4+4i)k^2+i}{(8k^2+1)(2k+i)(2k-1)}, \quad T(k) = \frac{2k(2k-1)(2k+1)(2k+i)(8k^2+1)}{(8k^2-i)(16k^4+2ik^2-1)}.$$

The remaining coefficients  $R_-$ ,  $B_-$ , and  $C_-$  can easily be evaluated by using (2.7) and (2.8).

In this example, there is exactly one simple bound state at  $k = \kappa$  with  $\kappa := (1+i)/4$ , where  $T(k)$  has a simple pole. A corresponding bound-state eigenfunction is a constant multiple of  $\psi_+(\kappa, x)$ , and it can be chosen as

$$\varphi(\kappa, x) = \begin{cases} \frac{e^{-x/4}}{1+e^x} \{ \cos(x/4) + \sin(x/4) + e^x [\cos(x/4) - 3\sin(x/4)] \}, & x \geq 0, \\ e^{x/4} [\cos(x/4) - 3\sin(x/4)], & x \leq 0. \end{cases}$$

Even though  $A$  and  $T$  each have a zero at  $k = 1/2$  on the ray  $\arg k = 0$ , their ratio  $A/T$  is nonzero at  $k = 1/2$ , which corresponds to a spectral singularity and not to a bound state.

**Example 6.2** Let  $u(x) = 0$  and  $v(x) = -\epsilon\delta(x)$ , with  $\delta(x)$  denoting the Dirac delta distribution and  $\epsilon$  being a real, nonzero parameter. We want  $\psi$ ,  $\psi'$ ,  $\psi''$  to be continuous at  $x = 0$ , and  $\psi'''(k, 0^-) = -\epsilon + \psi'''(k, 0^+)$ . We find

$$\psi_+(k, x) = \begin{cases} e^{ikx} - \frac{\epsilon}{4k^3 + \epsilon} e^{-kx}, & x \geq 0, \\ \frac{4k^3 + (1-i)\epsilon}{4k^3 + \epsilon} e^{ikx} + \frac{i\epsilon}{4k^3 + \epsilon} e^{-ikx} - \frac{\epsilon}{4k^3 + \epsilon} e^{kx}, & x \leq 0, \end{cases}$$

$$\phi_+(k, x) = \begin{cases} e^{-kx}, & x \geq 0, \\ -\frac{i\epsilon}{4k^3} e^{ikx} + \frac{i\epsilon}{4k^3} e^{-ikx} - \frac{\epsilon}{4k^3} e^{kx} + \frac{4k^3}{4k^3 + \epsilon} e^{-kx} & x \leq 0. \end{cases}$$

The coefficients related to the corresponding scattering problem are given by

$$\frac{1}{T(k)} = \frac{4k^3 + (1-i)\epsilon}{4k^3 + \epsilon}, \quad R_{\pm}(k) = \frac{i\epsilon}{4k^3 + (1-i)\epsilon},$$

$$A(k) = \frac{4k^3 + \epsilon}{4k^3}, \quad B_{\pm}(k) = -\frac{i\epsilon}{4k^3}, \quad C_{\pm}(k) = -\frac{\epsilon}{4k^3 + \epsilon}.$$

Note that  $A(k)/T(k)$  vanishes on the rays  $\arg k = 0$  and  $\arg k = \pi/4$  only when

$$4k^3 + (1-i)\epsilon = 0.$$

Since we assume  $\epsilon \neq 0$ , we find that there are no such  $k$  values if  $\epsilon < 0$ , and there exists exactly one  $k$  value lying on the ray  $\arg k = \pi/4$  when  $\epsilon > 0$ . Denoting that  $k$ -value by  $\kappa$ , we obtain a bound state of multiplicity one at  $k = \kappa$ , where

$$\kappa := \frac{1+i}{2} \sqrt[3]{\epsilon}.$$

Thus, a bound-state eigenfunction is obtained as

$$\psi_+(\kappa, x) = \begin{cases} e^{i\kappa x} + ie^{-\kappa x}, & x \geq 0, \\ e^{-i\kappa x} + ie^{\kappa x}, & x \leq 0. \end{cases}$$

Since  $\int_{-\infty}^{\infty} dx |\psi_+(\kappa, x)|^2 = 4/\sqrt[3]{\epsilon}$ , a normalized bound-state eigenfunction is given by

$$\varphi(\kappa, x) = \begin{cases} \sqrt[6]{\epsilon} e^{-\sqrt[3]{\epsilon} x/2} [\cos(\sqrt[3]{\epsilon} x/2) + \sin(\sqrt[3]{\epsilon} x/2)], & x \geq 0, \\ \sqrt[6]{\epsilon} e^{\sqrt[3]{\epsilon} x/2} [\cos(\sqrt[3]{\epsilon} x/2) - \sin(\sqrt[3]{\epsilon} x/2)], & x \leq 0. \end{cases}$$

**Example 6.3** For any positive constant  $c$ , consider

$$u(x) = \frac{16c^2[1 + \sqrt{2} \cosh(2cx)]}{[\sqrt{2} + \cosh(2cx)]^2},$$

$$v(x) = \frac{4c^4[\sqrt{2} \cosh(6cx) - 12 \cosh(4cx) - 5\sqrt{2} \cosh(2cx) + 4]}{[\sqrt{2} + \cosh(2cx)]^4}.$$

We then obtain

$$A(k) = \frac{[k - (1+i)c][k - (1-i)c]}{[k + (1+i)c][k + (1-i)c]}, \quad T(k) = \frac{[k + (1+i)c][k - (1-i)c]}{[k - (1+i)c][k + (1-i)c]},$$

$$R_{\pm}(k) = 0, \quad C_{\pm}(k) = 0, \quad B_{\pm}(k) = 0,$$

$$\psi_+(k, x) = e^{ikx} \left[ 1 + \frac{i\alpha(x)}{k + (1+i)c} + \frac{i\alpha(x)^*}{k - (1-i)c} \right], \quad \phi_+(k, x) = \psi_+(ik, x),$$

where we have defined

$$\alpha(x) := \frac{-\sqrt{2}c - (1+i)ce^{-2cx}}{\sqrt{2} + \cosh(2cx)}.$$

This example was presented in [10] in different terminology. Note that  $A(k)/T(k)$  has a double zero at  $k = \kappa$  with  $\kappa := (1+i)c$ , which corresponds to a bound state of multiplicity two. Two linearly independent eigenfunctions are given by  $\psi_+(\kappa, x)$  and  $\phi_+(\kappa, x)$ , or they can be chosen as real valued, e.g. as

$$\varphi_1(\kappa, x) = \frac{(\sqrt{2}e^{-cx} + 2e^{cx}) \cos(cx) + \sqrt{2}e^{-cx} \sin(cx)}{\sqrt{2} + \cosh(2cx)},$$

$$\varphi_2(\kappa, x) = \frac{(\sqrt{2}e^{-cx} + 2e^{cx}) \sin(cx) - \sqrt{2}e^{-cx} \cos(cx)}{\sqrt{2} + \cosh(2cx)}.$$

**Example 6.4** Consider the potentials

$$u(x) = \frac{-4}{(|x| + 1)^2}, \quad v(x) = \frac{-8}{(|x| + 1)^4}.$$

Using (2.4)-(2.6) and the continuity of the solutions to (1.1) and the continuity of their first, second, and third  $x$ -derivatives, we get

$$T(k) = \frac{k(k^4 + k^3 + k^2 + 2k + 2)}{(k+i)[k^4 + (1+i)k^3 + ik^2 + (1-i)k + 3]},$$

$$\begin{aligned}
A(k) &= \frac{(k+1)(k^4 + k^3 + k^2 + 2k + 2)}{k^5}, \\
C_+(k) &= \frac{-2(k+i)}{k^4 + k^3 + k^2 + 2k + 2}, \quad B_+(k) = \frac{-2i(k+1)(k+i)}{k^5}, \\
R_+(k) &= \frac{i(k^2 + k + 3)}{(k+i)[k^4 + (1+i)k^3 + ik^2 + (1-i)k + 3]}, \\
\psi_+(k, x) &= \begin{cases} f(k, x) + C_+(k) f(ik, x), & x \geq 0 \\ \frac{1}{T(k)} f(-k, -x) + \frac{R_+(k)}{T(k)} f(k, -x) + C_+(k) f(ik, -x), & x \leq 0, \end{cases}
\end{aligned}$$

and for  $x \geq 0$  we have  $\phi_+(k, x) = f(ik, x)$ , while for  $x \leq 0$  we have

$$\phi_+(k, x) = A(k) f(-ik, -x) + B_+(k) f(-k, -x) + B_+(k)^* f(k, -x) - \frac{k^2 - 2}{k^5} f(ik, -x),$$

where we have defined

$$f(k, x) := e^{ikx} \left[ 1 + \frac{i}{k(x+1)} \right].$$

In this example there exists exactly one bound state at  $k = \kappa$  with  $\kappa := 0.77\overline{8}(1+i)$  corresponding to the simple pole of  $T(k)$  on the ray  $\arg k = \pi/4$ . Note that we have used an overline to denote the round-off on the digit. An eigenfunction for that simple bound state is a constant multiple of  $\psi_+(\kappa, x)$ . At  $k = 0.47\overline{6} + 1.18\overline{3}i$  in the interior of the sector  $\Omega$ , both  $A$  and  $T$  have simple zeros without  $A/T$  vanishing there; thus, that  $k$ -value does not correspond to a bound state and it corresponds to a nonspectral singularity.

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