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Solitons and Inverse Scattering Transform

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Abstract. A review of the inverse scattering transform is presented in solving initial-value problems for nonlinear evolution equations such as the Korteweg-de Vries equation. The derivation of such equations is illustrated by using the Lax method and the AKNS method. The inverse scattering problem is outlined for the one-dimensional Schrödinger equation, and the time evolution of the corresponding scattering data is given. Soliton solutions to the Korteweg-de Vries equation are explicitly written.

1. Introduction

The Korteweg-de Vries equation (KdV) is a nonlinear partial differential equation (PDE) modeling propagation of water waves in long, narrow, and shallow canals. It was formulated [1] in 1895 by Diederik Johannes Korteweg and Gustav de Vries based on the thesis work of de Vries. Korteweg was a well-known Dutch mathematician. He was born in 1848, received his doctorate in 1878 from University of Amsterdam, became a professor there in 1881, retired in 1918, and died in 1941. De Vries wrote his doctoral thesis under Korteweg, defended it in 1894, served as a high school teacher until 1931, and died in 1934. An extensive obituary of Korteweg was published in the 1945/1946 Annals of the Royal Dutch Academy of Arts and Science, but there was no mention of the KdV there. No one could have predicted that the KdV would be a famous equation some 70 years later.

The KdV is usually written in the form

$$u_t - 6uu_x + u_{xxx} = 0, \qquad x \in \mathbf{R}, \quad t > 0, \tag{1.1}$$

where the subscripts denote partial derivatives. The coefficients in (1.1) can be changed at will by scaling x, t, and u. The u_{xxx} term causes dispersion, in general resulting in spatial broadening of an initial wave $u(x,0)$ as time progresses.

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Besides its solutions showing nonlinear and dispersive behavior, the KdV also possesses some special solutions, known as solitary wave solutions or solitons, which would not be expected from a nonlinear or dispersive PDE. A single-soliton solution retains its shape in time and moves to the right, and a multiple-soliton solution asymptotically consists of a train of single solitons, all moving to the right but at different speeds.

The Scottish engineer John Scott Russell (1808-1882) was the first [2] to observe a solitary water wave, which took place in 1834 in the Union Canal between Edinburgh and Glasgow, some 60 years before the formulation of the KdV. Unfortunately, the importance of his discovery was not understood by his contemporaries.

In early 1950s Fermi, Pasta, and Ulam were involved in computational studies of a one-dimensional (1-D) dynamical system of 64 particles with nonlinear interactions. Their aim was to determine the rate of approach to the equipartition of energy among various degrees of freedom. Contrary to expectations, they did not observe any tendency towards the equipartition, and they were puzzled by noticing the almost ongoing recurrence to the initial state. The preprint [3] based on their findings was never published as a journal article, but it appeared in Fermi's Collected Papers [4] after his death and is also available on the internet [5].

The Fermi-Pasta-Ulam puzzle was solved [6] in 1965 by Zabusky and Kruskal, who explained it in terms of solitary-wave solutions to the KdV. Analyzing the KdV numerically, Zabusky and Kruskal observed solitary-wave solutions, named such waves solitons, and witnessed that pulses in a multi-soliton solution interact with each other nonlinearly but come out of their interaction without any change in size or shape. Such unusual nonlinear interactions created a lot of excitement, but no one knew how to solve the KdV, except numerically.

In 1967, Gardner, Greene, Kruskal, and Miura presented [7] a method, known as the inverse scattering transform, to solve the initial-value problem for the KdV. They showed that $u(x,t)$ can be obtained from $u(x,0)$ with the help of the solution to the inverse scattering problem for the 1-D Schrödinger equation. They also explained that soliton solutions to the KdV correspond to a zero reflection coefficient in the scattering data.

In 1972 Zakharov and Shabat showed [8] that the method of inverse scattering transform is applicable also on the nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2u|u|^2 = 0, \qquad x \in \mathbf{R}, \quad t > 0. \tag{1.2}$$

They proved that the initial-value problem for (1.2) can be solved with the help of the solution to the inverse scattering problem for the first-order linear system

$$\begin{cases} \xi' = -i\lambda\xi + V(x)\,\eta, \\ \eta' = i\lambda\eta - \overline{V(x)}\,\xi, \end{cases} \qquad x \in \mathbf{R}, \quad t > 0, \tag{1.3}$$

where the prime is used for the derivative with respect to the spatial coordinate x, V is the potential, λ is the spectral parameter, and an overline denotes complex conjugation. The system (1.3) is known as the Zakharov-Shabat system.

Soon afterwards, again in 1972 Wadati showed in a one-page publication [9] that the modified Korteweg-de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \qquad x \in \mathbf{R}, \quad t > 0, \tag{1.4}$$

can be solved with the help of the inverse scattering problem for the linear system

$$\begin{cases} \xi' = -i\lambda\xi + V(x)\,\eta, \\ \eta' = i\lambda\eta + V(x)\,\xi, \end{cases} \qquad x \in \mathbf{R}, \quad t > 0. \tag{1.5}$$

Next, in 1973 Ablowitz, Kaup, Newell, and Segur showed [10] that the sine-Gordon equation

$$u_{xt} = \sin u, \qquad x \in \mathbf{R}, \quad t > 0, \tag{1.6}$$

can be solved in the same way by exploiting the inverse scattering problem associated with the linear system

$$\begin{cases} \xi' = -i\lambda\xi - \dfrac{1}{2}\,V'(x)\,\eta, \\ \eta' = i\lambda\eta + \dfrac{1}{2}\,V'(x)\,\xi, \end{cases} \qquad x \in \mathbf{R}, \quad t > 0. \tag{1.7}$$

Since then, many other nonlinear PDEs have been discovered to be solvable by the method of inverse scattering transform.

Our review paper is organized as follows. In Section 2 we outline the Lax method and illustrate how the KdV is related to 1-D Schrödinger equation (6.1). Section 3 gives an outline of the AKNS method and illustrates the derivation of the KdV via that method. In Section 4 we briefly review scattering solutions, bound state solutions, and related quantities for the Schrödinger equation (4.1) when its potential belongs to the so-called Faddeev class. In Section 5 we present an outline of the inverse scattering problem for (4.1) and of its solution via a formulation as a Riemann-Hilbert problem and also via the Faddeev-Marchenko theory. In Section 6 we consider the time evolution of the scattering solutions and of scattering data for (6.1) as the potential $V(x)$ evolves from $u(x, 0)$ to $u(x, t)$, the latter of which satisfies the KdV. Finally, in Section 7 we mention some methods to solve the initial-value problem for the KdV and also present two explicit formulas for the N-soliton solution to the KdV.

There are many excellent references for the inverse scattering transform, solitons, and related topics. We refer the reader to [11-14] and to other such references for further study of these topics.

2. The Lax Method

In 1968 Peter Lax presented [15] a method to show that the KdV can be derived as a compatibility condition related to the time evolution of solutions to (6.1).

The basic idea behind the Lax method is the following. Given a linear differential operator $\mathcal{L}$ appearing in the spectral problem $\mathcal{L}\psi = \lambda\psi$, find an operator $\mathcal{A}$ (the operators $\mathcal{A}$ and $\mathcal{L}$ are said to form a Lax pair) such that:

(i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.
(ii) The quantity $\psi_t - \mathcal{A}\psi$ remains a solution to $\mathcal{L}\psi = \lambda\psi$.
(iii) The quantity $\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$ is a multiplication operator, i.e. it is not a differential operator.

From condition (ii) we get

$$\mathcal{L}\,(\psi_t - \mathcal{A}\psi) = \lambda\,(\psi_t - \mathcal{A}\psi)\,, \tag{2.1}$$

and with the help of $\mathcal{L}\psi=\lambda\psi$ and $\lambda_t=0$, from (2.1) we obtain

$$\begin{aligned}\mathcal{L}\psi_t-\mathcal{L}\mathcal{A}\psi&=\lambda\psi_t-\mathcal{A}(\lambda\psi)=\partial_t(\lambda\psi)-\mathcal{A}\mathcal{L}\psi\\&=\partial_t(\mathcal{L}\psi)-\mathcal{A}\mathcal{L}\psi=\mathcal{L}_t\psi+\mathcal{L}\psi_t-\mathcal{A}\mathcal{L}\psi.\end{aligned}\tag{2.2}$$

After canceling the $\mathcal{L}\psi_t$ terms in (2.2), we get

$$(\mathcal{L}_t+\mathcal{L}\mathcal{A}-\mathcal{A}\mathcal{L})\,\psi=0,$$

which, because of (iii), implies

$$\mathcal{L}_t+\mathcal{L}\mathcal{A}-\mathcal{A}\mathcal{L}=0.\tag{2.3}$$

Note that (2.3) is an evolution equation containing a first-order time derivative, and in general it is a nonlinear PDE.

Let us illustrate the Lax method to derive the KdV from the 1-D Schrödinger equation. For this purpose, we write (6.1) as $\mathcal{L}\psi=\lambda\psi$ with $\lambda=k^2$, where

$$\mathcal{L}=-\partial_x^2+u(x,t),\tag{2.4}$$

and try to determine $\mathcal{A}$ by assuming that it has the form

$$\mathcal{A}=\alpha_3\partial_x^3+\alpha_2\partial_x^2+\alpha_1\partial_x+\alpha_0,\tag{2.5}$$

where the coefficients α_j with $j=1,2,3,4$ may depend on x and t, but not on the spectral parameter λ. Note that $\mathcal{L}_t=u_t$. Using (2.4) and (2.5) in (2.3), we obtain

$$(\ )\partial_x^5+(\ )\partial_x^4+(\ )\partial_x^3+(\ )\partial_x^2+(\ )\partial_x+(\ )=0,\tag{2.6}$$

where, because of (iii), each coefficient must vanish. The coefficient of ∂_x^5 vanishes identically. Setting the coefficients of ∂_x^j to zero for $j=4,3,2,1$, we obtain

$$\alpha_3=c_1,\quad \alpha_2=c_2,\quad \alpha_1=c_3-\frac{3}{2}c_1u,\quad \alpha_0=c_4-\frac{3}{4}c_1u_x-c_2u,$$

with c_1, c_2, c_3, and c_4 denoting arbitrary constants. Choosing $c_1=-4$ and $c_3=0$ in the last coefficient in (2.6) and setting that coefficient to zero, we get the KdV equation (1.1). Moreover, by letting $c_2=c_4=0$, we obtain the operator $\mathcal{A}$ as

$$\mathcal{A}=-4\partial_x^3+6u\partial_x+3u_x.\tag{2.7}$$

As for the Zakharov-Shabat system (1.3), we can write its time-evolved version as $\mathcal{L}\psi=\lambda\psi$, where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L}=i\begin{bmatrix}1&0\\0&-1\end{bmatrix}\partial_x-i\begin{bmatrix}0&u(x,t)\\\overline{u(x,t)}&0\end{bmatrix},$$

and the operator $\mathcal{A}$ is obtained as

$$\mathcal{A}=2i\begin{bmatrix}1&0\\0&-1\end{bmatrix}\partial_x^2-2i\begin{bmatrix}0&u\\\overline{u}&0\end{bmatrix}\partial_x-i\begin{bmatrix}-|u|^2&u_x\\\overline{u}_x&|u|^2\end{bmatrix}.$$

It can be verified that the compatibility condition (2.3) is equivalent to the nonlinear Schrödinger equation (1.2).

As for the first-order system (1.5), we can write its time-evolved version as $\mathcal{L}\psi = \lambda\psi$, where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x + i \begin{bmatrix} 0 & -u(x,t) \\ u(x,t) & 0 \end{bmatrix},$$

and the operator $\mathcal{A}$ is obtained as

$$\mathcal{A} = -4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \partial_x^3 + 6 \begin{bmatrix} u^2 & u_x \\ u_x & u^2 \end{bmatrix} \partial_x + \begin{bmatrix} 6uu_x & 3u_{xx} \\ 3u_{xx} & 6uu_x \end{bmatrix}.$$

Then, the compatibility condition (2.3) gives us the modified KdV (1.4).

3. The AKNS Method

In the previous section we have seen how a nonlinear PDE solvable by the inverse scattering method arises as a compatibility condition in the Lax method. There are other methods to derive such nonlinear PDEs. One such method (AKNS method) is due to Ablowitz, Kaup, Newell, and Segur, who first used it to derive the sine-Gordon equation [10].

The basic idea behind the AKNS method is the following. Given a linear operator $\mathcal{X}$ associated with the first-order system $v_x = \mathcal{X}v$, we are interested in finding an operator $\mathcal{T}$ (the operators $\mathcal{X}$ and $\mathcal{T}$ are said to form an AKNS pair) such that:

(i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.
(ii) The quantity $v_t - \mathcal{T}v$ is also a solution to $v_x = \mathcal{X}v$; i.e., $(v_t - \mathcal{T}v)_x = \mathcal{X}(v_t - \mathcal{T}v)$.
(iii) The quantity $\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X}$ is a (matrix) multiplication operator, i.e. it is not a differential operator.

From condition (ii) we get

$$\begin{aligned} v_{tx} - \mathcal{T}_x v - \mathcal{T}v_x &= \mathcal{X}v_t - \mathcal{X}\mathcal{T}v = (\mathcal{X}v)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= (v_x)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v = v_{xt} - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v. \end{aligned} \tag{3.1}$$

Using $v_{tx} = v_{xt}$ and replacing $\mathcal{T}v_x$ by $\mathcal{T}\mathcal{X}v$ in (3.1), we obtain

$$(\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X})v = 0,$$

which in turn, because of (iii), implies

$$\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} = 0. \tag{3.2}$$

We can view (3.2) as an integrable PDE solvable with the help of the inverse scattering for the linear system $v_x = \mathcal{X}v$. Like (2.3), (3.2) is in general a nonlinear evolution equation containing a first-order time derivative. Note that $\mathcal{X}$ contains the spectral parameter λ, and hence $\mathcal{T}$ also depends on λ as well.

Let us derive the KdV from the 1-D Schrödinger equation in order to demonstrate the AKNS method. For this purpose we write (6.1), by replacing the spectral parameter k^2 with λ, as a first-order linear system $v_x = \mathcal{X}v$, where

$$v = \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} 0 & u(x,t) - \lambda \\ 1 & 0 \end{bmatrix}.$$

Let us look for $\mathcal{T}$ in the form

$$\mathcal{T} = \begin{bmatrix} \alpha & \beta \\ \rho & \sigma \end{bmatrix},$$

where the entries α, β, ρ, and σ may depend on x, t, and λ. The compatibility condition (3.2) leads to

$$\begin{bmatrix} -\alpha_x - \beta + \rho(u-\lambda) & u_t - \beta_x + \sigma(u-\lambda) - \alpha(u-\lambda) \\ -\rho_x + \alpha - \sigma & -\sigma_x + \beta - \rho(u-\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.3}$$

The $(1,1)$, $(2,1)$, and $(2,2)$ entries in (3.3) imply

$$\beta = -\alpha_x + (u-\lambda)\rho, \quad \sigma = \alpha - \rho_x, \quad \sigma_x = -\alpha_x. \tag{3.4}$$

Then from the $(1,2)$ entry in (3.3) we obtain

$$u_t + \frac{1}{2}\rho_{xxx} - u_x\rho - 2\rho_x(u-\lambda) = 0. \tag{3.5}$$

Assuming an affine dependence of ρ on the spectral parameter and letting $\rho = \lambda\zeta + \mu$ in (3.5), we get

$$2\zeta_x\lambda^2 + \left(\frac{1}{2}\zeta_{xxx} - 2\zeta_x u + 2\mu_x - u_x\zeta\right)\lambda + \left(u_t + \frac{1}{2}\mu_{xxx} - 2\mu_x u - u_x\mu\right) = 0.$$

Equating the coefficients of each power of λ to zero, we have

$$\zeta = c_1, \quad \mu = \frac{1}{2}c_1 u + c_2, \quad u_t - \frac{3}{2}c_1 u u_x - c_2 u_x + \frac{1}{4}c_1 u_{xxx} = 0, \tag{3.6}$$

with c_1 and c_2 denoting arbitrary constants. Choosing $c_1 = 4$ and $c_2 = 0$, it can be verified that the last equation in (3.6) reduces to the KdV given in (1.1). Moreover, with the help of (3.4) we get

$$\alpha = u_x + c_3, \quad \beta = -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}, \quad \xi = 4\lambda + 2u, \quad \sigma = c_3 - u_x,$$

where c_3 is an arbitrary constant. Choosing $c_3 = 0$, we find

$$\mathcal{T} = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}.$$

Let us now turn our attention to the Zakharov-Shabat system (1.3). We can write its time-evolved form as $v_x = \mathcal{X}v$, where

$$\mathcal{X} = \begin{bmatrix} -i\lambda & u(x,t) \\ -\overline{u(x,t)} & i\lambda \end{bmatrix}.$$

The matrix operator $\mathcal{T}$ can be obtained as

$$\mathcal{T} = \begin{bmatrix} -2i\lambda^2 + i|u|^2 & 2\lambda u + iu_x \\ -2\lambda\overline{u} + i\overline{u}_x & 2i\lambda^2 - i|u|^2 \end{bmatrix}.$$

It can be verified that the compatibility condition (3.2) is identical to the nonlinear Schrödinger equation (1.2).

As for the linear system (1.5), we can write its time-evolved version as $v_x = \mathcal{X} v$, where

$$\mathcal{X} = \begin{bmatrix} -i\lambda & u(x,t) \\ u(x,t) & i\lambda \end{bmatrix}.$$

We obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \begin{bmatrix} -4i\lambda^3 - 2i\lambda u^2 & 4\lambda^2 u + 2i\lambda u_x - u_{xx} + 2u^3 \\ 4\lambda^2 u - 2i\lambda u_x - u_{xx} + 2u^3 & 4i\lambda^3 + 2i\lambda u^2 \end{bmatrix}.$$

It can be checked that the compatibility condition (3.2) is equivalent to the modified KdV (1.4).

As for the AKNS system (1.7), we can write its time-evolved form as $v_x = \mathcal{X} v$, where

$$\mathcal{X} = \begin{bmatrix} -i\lambda & -\frac{1}{2}\, u_x(x,t) \\ \frac{1}{2}\, u_x(x,t) & i\lambda \end{bmatrix},$$

and we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \frac{i}{4\lambda} \begin{bmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{bmatrix}.$$

Then, the compatibility condition (3.2) gives us the sine-Gordon equation (1.6).

4. Schrödinger Equation and the Scattering Data

Consider the Schrödinger equation

$$\psi''(k,x) + k^2\, \psi(k,x) = V(x)\, \psi(k,x), \qquad x \in \mathbf{R}, \tag{4.1}$$

where the potential belongs to the Faddeev class; i.e., V is real valued, measurable, and $\int_{-\infty}^{\infty} dx\, (1+|x|)\, |V(x)|$ is finite.

There are two types of solutions to (4.1); namely, scattering solutions and bound-state solutions. The scattering solutions are those that consist of linear combinations of e^{ikx} and e^{-ikx} as $x \to \pm\infty$, and they occur for $k \in \mathbf{R} \setminus \{0\}$. Two linearly independent scattering solutions f_{l} and f_{r}, known as the Jost solutions from the left and from the right, respectively, satisfy the respective asymptotic conditions

$$f_{\mathrm{l}}(k,x) = e^{ikx}[1+o(1)], \quad f_{\mathrm{l}}'(k,x) = e^{ikx}[ik+o(1)], \qquad x \to +\infty, \tag{4.2}$$

$$f_{\mathrm{r}}(k,x) = e^{-ikx}[1+o(1)], \quad f_{\mathrm{r}}'(k,x) = -e^{-ikx}[ik+o(1)], \qquad x \to -\infty. \tag{4.3}$$

Their spatial asymptotics

$$f_{\mathrm{l}}(k,x) = \frac{e^{ikx}}{T(k)} + \frac{L(k)\, e^{-ikx}}{T(k)} + o(1), \qquad x \to -\infty, \tag{4.4}$$

$$f_{\mathrm{r}}(k,x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k)\, e^{ikx}}{T(k)} + o(1), \qquad x \to +\infty, \tag{4.5}$$

give us the scattering coefficients, namely, the transmission coefficient T and the reflection coefficients L and R, from the left and right, respectively.

Let $\mathbf{C}^+$ denote the upper-half complex plane and put $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$. A bound-state solution to (4.1) is a solution that belongs to $L^2(\mathbf{R})$ in the x variable, and such solutions can occur only at certain k-values on the imaginary axis in $\mathbf{C}^+$. When V is in the Faddeev class, the number of bound states is finite and the multiplicity of each bound state is one. Let us use N to denote the number of bound states, and suppose that the bound states occur at $k = i\kappa_j$ with the ordering $0 < \kappa_1 < \cdots < \kappa_N$. Each bound state corresponds to a pole of T in $\mathbf{C}^+$.

The scattering matrix associated with (4.1) is defined as

$$\mathbf{S}(k) := \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \qquad k \in \mathbf{R}, \tag{4.6}$$

and it can be constructed from the bound-state energies and one of the reflection coefficients. If we start with the right reflection coefficient $R(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, then we get

$$T(k) = \left(\prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \, \frac{\log(1 - |R(s)|^2)}{s - k - i0^+} \right), \qquad k \in \overline{\mathbf{C}^+}, \tag{4.7}$$

where the quantity $i0^+$ indicates that the value for $k \in \mathbf{R}$ must be obtained as a limit from $\mathbf{C}^+$. Then, the left reflection coefficient $L(k)$ can be constructed via

$$L(k) = -\frac{\overline{R(k)}\, T(k)}{\overline{T(k)}}, \qquad k \in \mathbf{R}. \tag{4.8}$$

Similarly, if we start with $L(k)$ for $k \in \mathbf{R}$ and $\{\kappa_j\}_{j=1}^N$, with the help of (4.7) but with $|L(s)|$ instead of $|R(s)|$ there, we can construct T and then obtain $R(k)$ by using (4.8).

The determination of the scattering matrix from the potential is known as the direct scattering problem. For a detailed study of the direct scattering problem for the 1-D Schrödinger equation, we refer the reader to [16-22].

5. Inverse Scattering Problem

The inverse scattering problem for (4.1) consists of the determination of the potential V from an appropriate set of scattering data. If there are no bound states, then either one of the reflection coefficients R and L uniquely determines the corresponding potential in the Faddeev class. However, when there are bound states, for the unique determination of V, in addition to a reflection coefficient and the bound-state energies, one also needs to specify a bound-state norming constant, or equivalently a dependency constant, for each bound state.

The left and right bound-state norming constants $c_{\mathrm{l}j}$ and $c_{\mathrm{r}j}$, respectively, are defined as

$$c_{\mathrm{l}j} := \left[\int_{-\infty}^{\infty} dx \, f_{\mathrm{l}}(i\kappa_j, x)^2 \right]^{-1/2}, \quad c_{\mathrm{r}j} := \left[\int_{-\infty}^{\infty} dx \, f_{\mathrm{r}}(i\kappa_j, x)^2 \right]^{-1/2},$$

and they are related to each other via the residues of T as

$$\mathrm{Res}\,(T, i\kappa_j) = i\, c_{\mathrm{l}j}^2 \, \gamma_j = i\, \frac{c_{\mathrm{r}j}^2}{\gamma_j}, \qquad j = 1, \ldots, N, \tag{5.1}$$

where the γ_j are the dependency constants defined as

$$\gamma_j := \frac{f_{\rm l}(i\kappa_j,x)}{f_{\rm r}(i\kappa_j,x)}, \qquad j=1,\dots,N. \tag{5.2}$$

The sign of γ_j is the same as that of $(-1)^{N-j}$, and hence $c_{{\rm r}j}=(-1)^{N-j}\gamma_j c_{{\rm l}j}$.

A characterization for a specific class of potentials consists of specifying some necessary and sufficient conditions on the scattering data which guarantee that there exists a corresponding unique potential in that class. To recover the potential V uniquely, we can use various sets of scattering data such as the left scattering data $\big\{R,\{\kappa_j\},\{c_{{\rm l}j}\}\big\}$, the right scattering data $\big\{L,\{\kappa_j\},\{c_{{\rm r}j}\}\big\}$, or $\big\{\mathbf{S},\{\gamma_j\}\big\}$, where $\mathbf{S}$ is the scattering matrix given in (4.6). Various characterizations in the Faddeev class can be found in [17,21-23].

As an example, let us mention that a potential in the Faddeev class can uniquely be identified in terms of the left scattering data consisting of the reflection coefficient $R(k)$ for $k\in\mathbf{R}$, the bound-state energies $-\kappa_j^2$ with $0<\kappa_1<\cdots<\kappa_N$, and the bound-state norming constants $c_{{\rm l}j}$ satisfying the following characterization conditions:

(i) R is continuous on $\mathbf{R}$, and $R(-k)=\overline{R(k)}$ for $k\in\mathbf{R}$.
(ii) $|R(k)|\le 1-Ck^2/(1+k^2)$ on $\mathbf{R}$ for some constant $C>0$.
(iii) $R(0)\in[-1,1)$.
(iv) $R(k)=o(1/k)$ as $k\to\pm\infty$.
(v) The function $k/T(k)$, where $T(k)$ is given by (4.7), is continuous in $\overline{\mathbf{C}^+}$.
(vi) The functions $\hat R$ and $\hat L$ are absolutely continuous. Moreover, $\hat R'\in L_1^1(a,+\infty)$ and $\hat L'\in L_1^1(-\infty,a)$ for every $a\in\mathbf{R}$. Here, $\hat R$ and $\hat L$ are defined as

$$\hat R(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,R(k)\,e^{iky},\quad \hat L(y):=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\frac{R(-k)\,T(k)}{T(-k)}\,e^{iky}.$$

Let us mention a few methods to solve the inverse scattering problem for (4.1). The functions $f_{\rm l}(-k,x)$ and $f_{\rm r}(-k,x)$ are also solutions to (4.1) and they can be expressed as linear combinations of the Jost solutions $f_{\rm l}(k,x)$ and $f_{\rm r}(k,x)$ as

$$f_{\rm l}(-k,x)=T(k)\,f_{\rm r}(k,x)-R(k)\,f_{\rm l}(k,x), \qquad k\in\mathbf{R}, \tag{5.3}$$

$$f_{\rm r}(-k,x)=T(k)\,f_{\rm l}(k,x)-L(k)\,f_{\rm r}(k,x), \qquad k\in\mathbf{R}. \tag{5.4}$$

We can view each of (5.3) and (5.4) as a Riemann-Hilbert problem, where, knowing the scattering coefficients for $k\in\mathbf{R}$, the aim is to construct $f_{\rm l}(\cdot,x)$ and $f_{\rm r}(\cdot,x)$ for each fixed $x\in\mathbf{R}$ by exploiting their properties in k. Once $f_{\rm l}(k,x)$ or $f_{\rm r}(k,x)$ is constructed, the potential can be obtained with the help of (4.1) by using one of

$$V(x)=\frac{f_{\rm l}''(k,x)}{f_{\rm l}(k,x)}+k^2,\quad V(x)=\frac{f_{\rm r}''(k,x)}{f_{\rm r}(k,x)}+k^2,$$

where the right hand sides can be evaluated at any particular value of $k\in\overline{\mathbf{C}^+}$, including zero and infinity.

Alternatively, the potential can be constructed as

$$V(x)=-2\,\frac{dK_{\rm l}(x,x^+)}{dx},$$

where $K_{\rm l}(x,y)$ is the solution to the left Faddeev-Marchenko equation

$$K_{\rm l}(x,y)+M_{\rm l}(x+y)+\int_x^\infty dz\, M_{\rm l}(y+z)\,K_{\rm l}(x,z)=0,\qquad x<y<+\infty,$$

with

$$M_{\rm l}(y):=\frac{1}{2\pi}\int_{-\infty}^\infty dk\, R(k)\,e^{iky}+\sum_{j=1}^N c_{{\rm l}j}^2\, e^{-\kappa_j y},$$

which is constructed from the left scattering data $\big\{R,\{\kappa_j\},\{c_{{\rm l}j}\}\big\}$.

Similarly, the potential can be recovered as

$$V(x)=2\,\frac{dK_{\rm r}(x,x^-)}{dx},$$

where $K_{\rm r}(x,y)$ is the solution to the right Faddeev-Marchenko equation

$$K_{\rm r}(x,y)+M_{\rm r}(x+y)+\int_{-\infty}^x dz\, M_{\rm r}(y+z)\,K_{\rm r}(x,z)=0,\qquad -\infty<y<x,$$

with

$$M_{\rm r}(y):=\frac{1}{2\pi}\int_{-\infty}^\infty dk\, L(k)\,e^{-iky}+\sum_{j=1}^N c_{{\rm r}j}^2\, e^{\kappa_j y},$$

which is constructed from the right scattering data $\big\{L,\{\kappa_j\},\{c_{{\rm r}j}\}\big\}$.

For a more detailed study or a review of various methods to recover V from appropriate sets of scattering data, the reader is referred to [16-22].

6. Time Evolution of the Scattering Data

Let us consider the time-evolved Schrödinger equation, where $V(x)$ in (4.1) is replaced by $u(x,t)$, which is the solution to (1.1) with the initial value equal to $V(x)$. We have

$$\psi''(k,x;t)+k^2\,\psi(k,x;t)=u(x,t)\,\psi(k,x;t),\qquad x\in\mathbf{R},\quad t>0. \tag{6.1}$$

We are interested in determining the time evolution of $\psi(k,x;t)$ appearing in (6.1) from its initial state $\psi(k,x)$ appearing in (4.1). The scattering coefficients $T(k;t)$, $R(k;t)$, and $L(k;t)$ and the bound-state norming and dependency constants $c_{{\rm l}j}(t)$, $c_{{\rm r}j}(t)$, and $\gamma_j(t)$ associated with (6.1) are envisioned as evolving from the corresponding quantities $T(k)$, $R(k)$, $L(k)$, $c_{{\rm l}j}$, $c_{{\rm r}j}$, and γ_j of (4.1), respectively, in such a way that

$$u(x,0)=V(x),\quad \psi(k,x;0)=\psi(k,x),$$
$$T(k;0)=T(k),\quad R(k;0)=R(k),\quad L(k;0)=L(k),$$
$$c_{{\rm l}j}(0)=c_{{\rm l}j},\quad c_{{\rm r}j}(0)=c_{{\rm r}j},\quad \gamma_j(0)=\gamma_j.$$

Notice that, as indicated in (i) of Sections 2 and 3, the spectral parameter k and the bound-state energies $-\kappa_j^2$ do not change in time.

As the initial potential $u(x,0)$ evolves to $u(x,t)$, condition (ii) of the Lax method or of the AKNS method allows us to determine the time evolution of any solution to the Schrödinger equation (6.1). For example, let us find the time evolution of $f_{\rm l}(k,x;t)$, the Jost solution from the left. From condition (ii) in Section 2, we see that the quantity $\partial_t f_{\rm l} - \mathcal{A} f_{\rm l}$ remains a solution to (6.1) and hence we can write it as a linear combination of the two linearly independent Jost solutions as

$$\begin{aligned}\partial_t f_{\rm l}(k,x;t) - \big(-4\partial_x^3 + 6u\partial_x + 3u_x\big) f_{\rm l}(k,x;t)& \\ = p(k,t)\, f_{\rm l}(k,x;t) + q(k,t)\, f_{\rm r}(k,x;t),&\end{aligned} \tag{6.2}$$

where the coefficients $p(k,t)$ and $q(k,t)$ are yet to be determined and $\mathcal{A}$ is the operator in (2.7). The asymptotic conditions for the time-evolved Jost solutions are analogous to (4.2)-(4.5); i.e., we have

$$f_{\rm l}(k,x;t) = e^{ikx}[1+o(1)], \quad f_{\rm l}'(k,x;t) = e^{ikx}[ik+o(1)], \qquad x \to +\infty, \tag{6.3}$$

$$f_{\rm r}(k,x;t) = e^{-ikx}[1+o(1)], \quad f_{\rm r}'(k,x;t) = -e^{-ikx}[ik+o(1)], \qquad x \to -\infty, \tag{6.4}$$

$$f_{\rm l}(k,x;t) = \frac{e^{ikx}}{T(k;t)} + \frac{L(k;t)\, e^{-ikx}}{T(k;t)} + o(1), \qquad x \to -\infty, \tag{6.5}$$

$$f_{\rm r}(k,x;t) = \frac{e^{-ikx}}{T(k;t)} + \frac{R(k;t)\, e^{ikx}}{T(k;t)} + o(1), \qquad x \to +\infty. \tag{6.6}$$

For each fixed t, using (6.3), (6.6), and $u(x,t) = o(1)$ and $u_x(x,t) = o(1)$ as $x \to +\infty$, from (6.2) as $x \to +\infty$ we get

$$\partial_t e^{ikx} + 4\partial_x^3 e^{ikx} = p(k,t)\, e^{ikx} + q(k,t) \left[\frac{1}{T(k;t)}\, e^{-ikx} + \frac{R(k;t)}{T(k;t)}\, e^{ikx} \right]. \tag{6.7}$$

Comparing the coefficients of e^{ikx} and e^{-ikx} on both sides of (6.7), we obtain

$$q(k,t) = 0, \quad p(k,t) = -4ik^3.$$

Thus, $f_{\rm l}(k,x;t)$ evolves in time by obeying the linear third-order PDE

$$\partial_t f_{\rm l}(k,x;t) - \mathcal{A} f_{\rm l}(k,x;t) = -4ik^3 f_{\rm l}(k,x;t), \qquad x \in \mathbf{R}, \quad t > 0, \tag{6.8}$$

with $\mathcal{A}$ as in (2.7). Proceeding in a similar manner, we find that $f_{\rm r}(k,x;t)$ evolves in time according to

$$\partial_t f_{\rm r}(k,x;t) - \mathcal{A} f_{\rm r}(k,x;t) = 4ik^3 f_{\rm r}(k,x;t), \qquad x \in \mathbf{R}, \quad t > 0. \tag{6.9}$$

Notice that the time evolution of each Jost solution is fairly complicated. We will see, however, that the time evolution of the scattering data is very simple. Letting $x \to -\infty$ in (6.8), using (6.4), (6.5), and $u(x,t) = o(1)$ and $u_x(x,t) = o(1)$ as $x \to -\infty$, and comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$\partial_t T(k;t) = 0, \quad \partial_t L(k;t) = -8ik^3 L(k;t),$$

which give us

$$T(k;t) = T(k;0) = T(k), \quad L(k;t) = L(k;0)\, e^{-8ik^3 t} = L(k)\, e^{-8ik^3 t}. \tag{6.10}$$

In a similar way, from (6.9) as $x \to +\infty$, we get

$$R(k;t) = R(k;0)\, e^{8ik^3 t} = R(k)\, e^{8ik^3 t}.$$

Thus, the transmission coefficient remains unchanged and only the phase of each reflection coefficient changes as time progresses.

Let us also evaluate the time evolution of the dependency constants $\gamma_j(t)$, which are defined analogous to (5.2) as

$$\gamma_j(t) := \frac{f_{\rm l}(i\kappa_j, x; t)}{f_{\rm r}(i\kappa_j, x; t)}, \qquad j = 1, \dots, N.$$

Evaluating (6.8) at $k = i\kappa_j$ and replacing $f_{\rm l}(i\kappa_j, x; t)$ with $\gamma_j(t)\, f_{\rm r}(i\kappa_j, x; t)$, we get

$$\begin{aligned} f_{\rm r}(i\kappa_j, x; t)\, \partial_t \gamma_j(t) + \gamma_j(t)\, \partial_t f_{\rm r}(i\kappa_j, x; t) - \gamma_j(t) \mathcal{A} f_{\rm r}(i\kappa_j, x; t) \\ = -4\kappa_j^3 \gamma_j(t) f_{\rm r}(i\kappa_j, x; t). \end{aligned} \tag{6.11}$$

On the other hand, evaluating (6.9) at $k = i\kappa_j$, we obtain

$$\gamma_j(t)\, \partial_t f_{\rm r}(i\kappa_j, x; t) - \gamma_j(t)\, \mathcal{A} f_{\rm r}(i\kappa_j, x; t) = 4\kappa_j^3 \gamma_j(t)\, f_{\rm r}(i\kappa_j, x; t). \tag{6.12}$$

Comparing (6.11) and (6.12) we see that $\partial_t \gamma_j(t) = -8\kappa_j^3 \gamma_j(t)$, or equivalently

$$\gamma_j(t) = \gamma_j(0)\, e^{-8\kappa_j^3 t} = \gamma_j\, e^{-8\kappa_j^3 t}.$$

In the light of the first identity in (6.10), from the time-evolved version of (5.1), we get

$$c_{{\rm l}j}(t) = c_{{\rm l}j} \sqrt{\frac{\gamma_j}{\gamma_j(t)}}, \quad c_{{\rm r}j}(t) = c_{{\rm r}j} \sqrt{\frac{\gamma_j(t)}{\gamma_j}},$$

and hence we determine the time evolutions of the norming constants as

$$c_{{\rm l}j}(t) = c_{{\rm l}j}(0)\, e^{4\kappa_j^3 t} = c_{{\rm l}j} e^{4\kappa_j^3 t}, \quad c_{{\rm r}j}(t) = c_{{\rm r}j}(0)\, e^{-4\kappa_j^3 t} = c_{{\rm r}j} e^{-4\kappa_j^3 t}.$$

7. Solution to KdV and Solitons

Let us use $\mathcal{D}(t)$ to denote the scattering data for the Schrödinger equation (6.1) with the time-evolved potential $u(x,t)$:

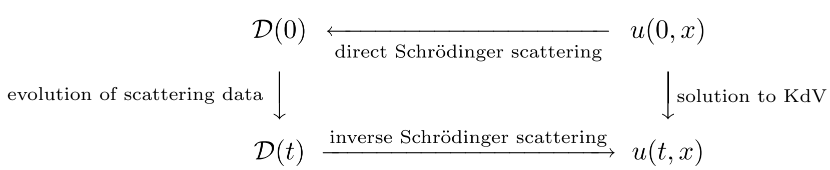
$$\mathcal{D}(t) := \big\{ R(k;t), L(k;t), T(k;t), \{c_{{\rm l}j}(t)\}, \{c_{{\rm r}j}(t)\}, \{\gamma_j(t)\} \big\}.$$

From the previous section we have

$$\begin{cases} R(k;t) = R(k)\, e^{8ik^3 t}, \quad L(k;t) = L(k)\, e^{-8ik^3 t}, \quad T(k;t) = T(k), \\ c_{{\rm l}j}(t) = c_{{\rm l}j}\, e^{4\kappa_j^3 t}, \quad c_{{\rm r}j}(t) = c_{{\rm r}j}\, e^{-4\kappa_j^3 t}, \quad \gamma_j(t) = \gamma_j\, e^{-8\kappa_j^3 t}. \end{cases} \tag{7.1}$$

Note that $\mathcal{D}(0)$ corresponds to the initial scattering data associated with the potential $V(x)$, where $u(x,0) = V(x)$.

The initial-value problem for the KdV consists of finding $u(x,t)$ when $u(x,0)$ is known. Its solution is obtained in three steps as indicated in the following diagram.



(i) From $u(x,0)$, obtain the corresponding scattering data $\mathcal{D}(0)$. The direct scattering problem $V(x) \mapsto \mathcal{D}(0)$ is equivalent to solving (4.1) and obtaining a Jost solution, from which $\mathcal{D}(0)$ can be constructed.

(ii) Let the scattering data evolve in time as $\mathcal{D}(0) \mapsto \mathcal{D}(t)$ according to (7.1).

(iii) With $\mathcal{D}(t)$ as input, solve the inverse scattering problem $\mathcal{D}(t) \mapsto u(x,t)$ for (6.1). Note that $\mathcal{D}(t)$ satisfies the characterization given in Section 5 whenever $\mathcal{D}(0)$ does; hence, if V belongs to the Faddeev class, then for each fixed $t > 0$, the quantity $u(\cdot, t)$ exists, is unique, and also belongs to the Faddeev class.

Any method applicable to solve the inverse scattering problem $\mathcal{D}(0) \mapsto V(x)$ can also be used to solve the inverse scattering problem $\mathcal{D}(t) \mapsto u(x,t)$. For example, as in (5.3) and (5.4) one can solve one of the Riemann-Hilbert problems

$$f_{\rm l}(-k,x;t) = T(k)\, f_{\rm r}(k,x;t) - R(k)\, e^{8ik^3t} f_{\rm l}(k,x;t), \qquad k \in \mathbf{R}, \tag{7.2}$$

$$f_{\rm r}(-k,x;t) = T(k)\, f_{\rm l}(k,x;t) - L(k)\, e^{-8ik^3t} f_{\rm r}(k,x;t), \qquad k \in \mathbf{R}, \tag{7.3}$$

and recover $u(x,t)$ by using one of

$$u(x,t) = \frac{\partial_x^2 f_{\rm l}(k,x;t)}{f_{\rm l}(k,x;t)} + k^2, \quad u(x,t) = \frac{\partial_x^2 f_{\rm r}(k,x;t)}{f_{\rm r}(k,x;t)} + k^2,$$

where the evaluation can be made at any $k \in \overline{\mathbf{C}^+}$, including zero and infinity.

Alternatively, one can solve the time-evolved left Faddeev-Marchenko equation

$$K_{\rm l}(x,y;t)+M_{\rm l}(x+y;t)+\int_x^\infty dz\, M_{\rm l}(y+z;t)\, K_{\rm l}(x,z;t) = 0, \qquad x < y < +\infty, \tag{7.4}$$

with

$$M_{\rm l}(y;t) := \frac{1}{2\pi} \int_{-\infty}^\infty dk\, R(k)\, e^{8ik^3t+iky} + \sum_{j=1}^N c_{{\rm l}j}^2\, e^{8\kappa_j^3 t-\kappa_j y}, \tag{7.5}$$

and recover $u(x,t)$ by using

$$u(x,t) = -2\, \frac{\partial K_{\rm l}(x,x^+;t)}{\partial x}. \tag{7.6}$$

Equivalently, one can solve the time-evolved right Faddeev-Marchenko equation

$$K_{\rm r}(x,y;t) + M_{\rm r}(x+y;t) + \int_{-\infty}^x dz\, M_{\rm r}(y+z;t)\, K_{\rm r}(x,z;t) = 0, \qquad -\infty < y < x,$$

with

$$M_{\rm r}(y;t) := \frac{1}{2\pi} \int_{-\infty}^\infty dk\, L(k)\, e^{-8ik^3t-iky} + \sum_{j=1}^N c_{{\rm r}j}^2\, e^{-8\kappa_j^3 t+\kappa_j y},$$

and obtain $u(x,t)$ by using

$$u(x,t) = 2\, \frac{\partial K_{\rm r}(x,x^-;t)}{\partial x}.$$

Let us now turn our attention to solitons. The soliton solutions to the KdV are obtained when the reflection coefficients are zero. In that case, the left scattering data consists of the sets of bound state energies $\{-\kappa_j^2\}_{j=1}^N$ and norming constants $\{c_{{\rm l}j}\}_{j=1}^N$. Corresponding to this data we obtain the N-soliton solution to the KdV that can be found, e.g., with the help of (7.4)-(7.6). We have

$$u(x,t) = -2\, \frac{\partial}{\partial x} \left[\frac{\det \Lambda}{\det \Gamma} \right], \tag{7.7}$$

where $\det\Lambda$ and $\det\Gamma$ denote the determinants of the respective matrices

$$\Lambda := \begin{bmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_N \\ 1 & 1+\dfrac{\varepsilon_1}{\kappa_1+\kappa_1} & \dfrac{\varepsilon_2}{\kappa_1+\kappa_2} & \cdots & \dfrac{\varepsilon_N}{\kappa_1+\kappa_N} \\ 1 & \dfrac{\varepsilon_1}{\kappa_2+\kappa_1} & 1+\dfrac{\varepsilon_2}{\kappa_2+\kappa_2} & \cdots & \dfrac{\varepsilon_N}{\kappa_2+\kappa_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dfrac{\varepsilon_1}{\kappa_N+\kappa_1} & \dfrac{\varepsilon_2}{\kappa_N+\kappa_2} & \cdots & 1+\dfrac{\varepsilon_N}{\kappa_N+\kappa_N} \end{bmatrix},$$

$$\Gamma := \begin{bmatrix} 1+\dfrac{\varepsilon_1}{\kappa_1+\kappa_1} & \dfrac{\varepsilon_2}{\kappa_1+\kappa_2} & \cdots & \dfrac{\varepsilon_N}{\kappa_1+\kappa_N} \\ \dfrac{\varepsilon_1}{\kappa_2+\kappa_1} & 1+\dfrac{\varepsilon_2}{\kappa_2+\kappa_2} & \cdots & \dfrac{\varepsilon_N}{\kappa_2+\kappa_N} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\varepsilon_1}{\kappa_N+\kappa_1} & \dfrac{\varepsilon_2}{\kappa_N+\kappa_2} & \cdots & 1+\dfrac{\varepsilon_N}{\kappa_N+\kappa_N} \end{bmatrix},$$

with

$$\varepsilon_j := c_{lj}^2 e^{-2\kappa_j x + 8\kappa_j^3 t}, \quad j = 1, \dots, N.$$

On the other hand, if our scattering data consists of the sets of bound state energies $\{-\kappa_j^2\}_{j=1}^N$ and dependency constants $\{\gamma_j\}_{j=1}^N$, then we can solve either one of the Riemann-Hilbert problems (7.2) and (7.3) and obtain

$$u(x,t) = -2\,\frac{\partial}{\partial x}\left[\frac{\det P}{\det Q}\right], \tag{7.8}$$

where

$$P := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \kappa_1^N(1-\omega_1) & \kappa_1^{N-1}(\omega_1+1) & \kappa_1^{N-2}(\omega_1-1) & \cdots & \omega_1-(-1)^N \\ \kappa_2^N(1-\omega_2) & \kappa_2^{N-1}(\omega_2+1) & \kappa_2^{N-2}(\omega_2-1) & \cdots & \omega_2-(-1)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_N^N(1-\omega_N) & \kappa_N^{N-1}(\omega_N+1) & \kappa_N^{N-2}(\omega_N-1) & \cdots & \omega_N-(-1)^N \end{bmatrix},$$

$$Q := \begin{bmatrix} \kappa_1^{N-1}(\omega_1+1) & \kappa_1^{N-2}(\omega_1-1) & \cdots & \omega_1-(-1)^N \\ \kappa_2^{N-1}(\omega_2+1) & \kappa_2^{N-2}(\omega_2-1) & \cdots & \omega_2-(-1)^N \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_N^{N-1}(\omega_N+1) & \kappa_N^{N-2}(\omega_N-1) & \cdots & \omega_N-(-1)^N \end{bmatrix},$$

with

$$\omega_j := \gamma_j\, e^{2\kappa_j x - 8\kappa_j^3 t}, \qquad j = 1, \dots, N,$$

For the details of the derivations of (7.7) and (7.8), see, e.g. [24]. Some Mathematica notebooks based on (7.7) and (7.8) for the evaluation of N-soliton solutions to the KdV and their animations are available at the author's web page [25].

We can use (7.7) or (7.8) to analyze properties of solitons of the KdV. For example, when $N = 1$, from (7.7) or (7.8) we get the single soliton solution to (1.1) as

$$u(x,t) = -2\kappa_1^2 \operatorname{sech}^2\left(\kappa_1 x - 4\kappa_1^3 t + \sqrt{\ln \gamma_1}\right). \tag{7.9}$$

In this case, from (5.1) we have $\gamma_1 = 2\kappa_1/c_{l1}^2$ and hence we can also write (7.9) in terms of κ_1 and c_{l1}. It is seen that the amplitude of this wave is $2\kappa_1^2$, it moves in the positive x direction with speed $4\kappa_1^2$, and the dependency constant γ_1 plays a role in the initial location of the soliton. The width of the soliton is inversely proportional to κ_1; this can be seen by using one of the facts that for one-soliton solution we have

$$\int_{-\infty}^{\infty} dx\, u(x,t) = -4\kappa_1, \quad \int_{-\infty}^{\infty} dx\, u(x,t)^2 = \frac{16\kappa_1^3}{3}.$$

If $u(x,t)$ is the N-soliton solution given in (7.7) and (7.8), we have [26]

$$\int_{-\infty}^{\infty} dx\, u(x,t) = -4\sum_{j=1}^{N} \kappa_j, \quad \int_{-\infty}^{\infty} dx\, u(x,t)^2 = \frac{16}{3}\sum_{j=1}^{N} \kappa_j^3,$$

which, as noted by P. C. Sabatier, also follow from the fact that as $t \to +\infty$ the N-soliton solution to the KdV consists of a train of N separate solitons each traveling at speed its own speed $4\kappa_j^2$.

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