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The Borg-Marchenko Theorem with a Continuous Spectrum

Tuncay Aktosun and Ricardo Weder

Abstract. The Schrödinger equation is considered on the half line with a selfadjoint boundary condition when the potential is real valued, integrable, and has a finite first moment. It is proved that the potential and the two boundary conditions are uniquely determined by a set of spectral data containing the discrete eigenvalues for a boundary condition at the origin, the continuous part of the spectral measure for that boundary condition, and a subset of the discrete eigenvalues for a different boundary condition. This result provides a generalization of the celebrated uniqueness theorem of Borg and Marchenko using two sets of discrete spectra to the case where there is also a continuous spectrum. The proof employed yields a method to recover the potential and the two boundary conditions, and it also constructs data sets used in various inversion methods. A comparison is made with the uniqueness result of Gesztesy and Simon using Krein's spectral shift function as the inversion data.

1. Introduction

Consider the Schrödinger equation on the half line

$$-\psi'' + V(x)\,\psi = k^2\psi, \qquad x\in\mathbf{R}^+, \tag{1.1}$$

where the prime denotes the derivative with respect to x, the potential V is real valued and measurable, and $\int_0^\infty dx\,(1+x)\,|V(x)|$ is finite. Such potentials are said to make up the Faddeev class. Let H_α for a fixed $\alpha\in(0,\pi]$ denote the unique selfadjoint realization [1] of the corresponding Schrödinger operator on $L^2(\mathbf{R}^+)$ with the boundary condition

$$\sin\alpha\cdot\psi'(k,0)+\cos\alpha\cdot\psi(k,0)=0, \tag{1.2}$$

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which can also be written as

$$\begin{cases} \psi'(k,0) + \cot\alpha \cdot \psi(k,0) = 0, & \alpha \in (0,\pi), \\ \psi(k,0) = 0, & \alpha = \pi. \end{cases}$$

Note that $\alpha \mapsto \cot\alpha$ is a monotone decreasing mapping of $(0,\pi)$ onto $\mathbf{R}$, and hence α is uniquely determined by $\cot\alpha$.

It is known [1,2] that H_α has no positive or zero eigenvalues, has no singular-continuous spectrum, has at most a finite number of (simple) negative eigenvalues, and its absolutely-continuous spectrum consists of $k^2 \in [0,+\infty)$.

Borg [3] and Marchenko [4] independently analyzed (1.1) with the boundary condition (1.2) when there is no continuous spectrum. They showed [3-5] that two sets of discrete spectra associated with two distinct boundary conditions at $x = 0$ (with a fixed boundary condition, if any, at $x = +\infty$) uniquely determine the potential and the two boundary conditions.

A continuous spectrum often appears in applications, and it usually arises when the potential vanishes at infinity. In our paper we present an extension of the celebrated Borg-Marchenko result in the presence of a continuous spectrum; namely, we show that the potential and the two boundary conditions are uniquely determined by an appropriate data set containing the discrete eigenvalues and the continuous part of the spectral measure corresponding to one boundary condition and a subset of the discrete eigenvalues corresponding to a different boundary condition.

Another extension of the Borg-Marchenko result with a continuous spectrum is given by Gesztesy and Simon [6], where a uniqueness result is presented when the corresponding Krein's spectral shift function is used as the data in the class of real-valued potentials that are integrable on $[0,R]$ for all $R > 0$. In our generalization of the Borg-Marchenko theorem, we specify the data in terms of a subset of the spectral measure; namely, the amplitude of the Jost function and the eigenvalues. The connection between the data used in [6] and ours is analyzed in Section 5.

The problem under study has applications in the acoustical analysis of the human vocal tract. The related inverse problem can be described [7-9] as determining a scaled curvature of the duct of the vocal tract when a constant-frequency sound is uttered. Such an inverse problem has important applications in speech recognition.

Our paper is organized as follows. In Section 2 we introduce the preliminary material related to the Jost function, the phase shift, and the spectral measure. In Section 3 we present our generalized Borg-Marchenko theorem. In Section 4 we briefly outline the Gel'fand-Levitan, Marchenko, and the Faddeev-Marchenko procedures to recover the potential. In Section 5 we show how the data used in our theorem uniquely constructs Krein's spectral shift function and vice versa. Finally, in Section 6 we present two examples to illustrate the theory presented in our paper.

2. Preliminaries

Recall that the Jost function associated with H_α is defined as [2,10-12]

$$F_\alpha(k) := \begin{cases} -i[f'(k,0) + \cot\alpha \cdot f(k,0)], & \alpha \in (0,\pi), \\ f(k,0), & \alpha = \pi, \end{cases} \tag{2.1}$$

with $f(k,x)$ denoting the Jost solution to (1.1) satisfying the asymptotics

$$f(k,x) = e^{ikx}[1+o(1)], \quad f'(k,x) = ike^{ikx}[1+o(1)], \qquad x \to +\infty. \tag{2.2}$$

From (1.1) and (2.2) it follows that

$$f(-k,0)=f(k,0)^*,\quad f'(-k,0)=f'(k,0)^*,\qquad k\in\mathbf{R}, \tag{2.3}$$

where the asterisk denotes complex conjugation, and hence for $k\in\mathbf{R}$ we get

$$F_\alpha(-k)=\begin{cases}-F_\alpha(k)^*, & \alpha\in(0,\pi),\\ F_\pi(k)^*, & \alpha=\pi.\end{cases} \tag{2.4}$$

We use $\mathbf{C}^+$ for the upper half complex plane, $\overline{\mathbf{C}^+}:=\mathbf{C}^+\cup\mathbf{R}$ for its closure, and $\mathbf{I}^+:=i(0,+\infty)$ for the positive imaginary axis in $\mathbf{C}^+$.

It is known [2,10-12] that $F_\alpha(k)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, the zeros in $\mathbf{C}^+$ of F_α, if any, can only occur on $\mathbf{I}^+$ and such zeros are all simple, $F_\alpha(k)\neq 0$ for $k\in\mathbf{R}\setminus\{0\}$, and that either $F_\alpha(k)$ is nonzero at $k=0$ (generic case) or it has a simple zero there (exceptional case). Because of (2.4), knowledge of $F_\alpha(k)$ for $k\in\mathbf{R}^+$ is equivalent to that for $k\in\mathbf{R}$. Let $k=i\kappa_{\alpha j}$ for $j=1,\dots,N_\alpha$ represent the zeros of F_α on $\mathbf{I}^+$. Thus, the set $\{-\kappa_{\alpha j}^2\}_{j=1}^{N_\alpha}$ corresponds to the discrete eigenvalues of H_α.

The negative of the phase of the Jost function F_α is usually known as the phase shift ϕ_α, i.e.

$$e^{-i\phi_\alpha(k)}:=\frac{F_\alpha(k)}{|F_\alpha(k)|},\qquad k\in\mathbf{R}, \tag{2.5}$$

where it is understood that $\phi_\alpha(+\infty)=0$. For $k\in\mathbf{R}$, the phase shift ϕ_α satisfies

$$\phi_\alpha(-k)=\begin{cases}\pi-\phi_\alpha(k), & \alpha\in(0,\pi),\\ -\phi_\pi(k), & \alpha=\pi.\end{cases} \tag{2.6}$$

The scattering matrix $S_\alpha(k)$ for $k\in\mathbf{R}$ associated with H_α is defined as

$$S_\alpha(k):=e^{2i\phi_\alpha(k)}=\begin{cases}-\dfrac{F_\alpha(-k)}{F_\alpha(k)}, & \alpha\in(0,\pi),\\[2mm] \dfrac{F_\pi(-k)}{F_\pi(k)}, & \alpha=\pi.\end{cases} \tag{2.7}$$

The number of discrete eigenvalues N_α is related to the phase shift ϕ_α by Levinson's theorem [2,11], i.e.

$$\phi_\alpha(0^+)=\begin{cases}\left(N_\alpha+\dfrac{1+d_\alpha}{2}\right)\pi, & \alpha\in(0,\pi),\\[2mm] \left(N_\pi+\dfrac{d_\pi}{2}\right)\pi, & \alpha=\pi,\end{cases} \tag{2.8}$$

where we have defined

$$d_\alpha:=\begin{cases}0, & \text{if } F_\alpha(0)\neq 0,\\ 1, & \text{if } F_\alpha(0)=0.\end{cases}$$

The spectral measure ρ_α corresponding to H_α can be determined via [2,10-12]

$$d\rho_\alpha(\lambda)=\begin{cases}\dfrac{\sqrt{\lambda}}{\pi}\,\dfrac{1}{|F_\alpha(\sqrt{\lambda})|^2}\,d\lambda, & \lambda>0,\\[2mm] \displaystyle\sum_{j=1}^{N_\alpha} g_{\alpha j}^2\,\delta(\lambda+\kappa_{\alpha j}^2)\,d\lambda, & \lambda<0,\end{cases}$$

where $\delta(\cdot)$ is the Dirac delta distribution, $\lambda := k^2$, and the norming constants $g_{\alpha j}$ are given by

$$g_{\alpha j} := \begin{cases} \dfrac{|f(i\kappa_{\alpha j},0)|}{||f(i\kappa_{\alpha j},\cdot)||}, & \alpha \in (0,\pi), \\ \dfrac{|f'(i\kappa_{\pi j},0)|}{||f(i\kappa_{\pi j},\cdot)||}, & \alpha = \pi, \end{cases} \tag{2.9}$$

with $||\cdot||$ denoting the standard norm in $L^2(0,+\infty)$. It is known [2,10-12] that $\{V,\alpha\}$ is uniquely determined by the corresponding spectral measure ρ_α and that the reconstruction can be achieved by solving the Gel'fand-Levitan integral equation.

3. The Borg-Marchenko Theorem with a Continuous Spectrum

Our generalized Borg-Marchenko theorem consists of identifying the appropriate data set leading to the unique determination of the potential V in (1.1) and the two distinct boundary parameters α and β in (1.2) with $0<\beta<\alpha\le\pi$. Here, we briefly state our theorem and summarize the steps to construct $\{V,\beta,\alpha\}$ and refer the reader to [12] for the proof and further details.

Let the set $\{-\kappa_{\beta j}^2\}_{j=1}^{N_\beta}$ correspond to the discrete eigenvalues of H_β. We use F_β to denote the Jost function associated with H_β, and it is obtained by replacing α with β on the right hand side of (2.1).

Our motivation is as follows. Assume that we are given some data set $\mathcal{D}$, which contains $|F_\alpha(k)|$ for $k\in\mathbf{R}$, the whole set $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$, and a subset of $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$ consisting of N_α elements. Alternatively, our data $\mathcal{D}$ may include $|F_\beta(k)|$ for $k\in\mathbf{R}$ and the sets $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$ and $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$. Does $\mathcal{D}$ uniquely determine $\{V,\alpha,\beta\}$? If not, what additional information do we need to include in $\mathcal{D}$ for the unique determination? Can we also present a constructive method to recover $\{V,\alpha,\beta\}$ from $\mathcal{D}$ or from a data set obtained by some augmentation of $\mathcal{D}$?

Since $0<\beta<\alpha\le\pi$, from the interlacing properties of eigenvalues, it is known [2,10-12] that either $N_\beta=N_\alpha$, in which case we have

$$0<\kappa_{\alpha 1}<\kappa_{\beta 1}<\kappa_{\alpha 2}<\kappa_{\beta 2}<\cdots<\kappa_{\alpha N_\alpha}<\kappa_{\beta N_\alpha}, \tag{3.1}$$

or else we have $N_\beta=N_\alpha+1$, in which case we get

$$0<\kappa_{\beta 1}<\kappa_{\alpha 1}<\kappa_{\beta 2}<\kappa_{\alpha 2}<\cdots<\kappa_{\alpha N_\alpha}<\kappa_{\beta N_\beta}. \tag{3.2}$$

There are eight distinct cases to consider depending on whether $\alpha\in(0,\pi)$ or $\alpha=\pi$, whether $N_\beta=N_\alpha$ or $N_\beta=N_\alpha+1$, and whether the data set used contains $|F_\alpha|$ or $|F_\beta|$. So, let us define the data sets $\mathcal{D}_1,\dots,\mathcal{D}_8$ as follows [12]:

$$\mathcal{D}_1 := \{h_{\beta\alpha}, |F_\alpha(k)| \text{ for } k\in\mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.3}$$

$$\mathcal{D}_2 := \{\beta, |F_\pi(k)| \text{ for } k\in\mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.4}$$

$$\mathcal{D}_3 := \{h_{\beta\alpha}, |F_\alpha(k)| \text{ for } k\in\mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, N_\alpha\text{-element subset of } \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.5}$$

$$\mathcal{D}_4 := \{\beta, |F_\pi(k)| \text{ for } k\in\mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, N_\pi\text{-element subset of } \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.6}$$

$$\mathcal{D}_5 := \{h_{\beta\alpha}, |F_\beta(k)| \text{ for } k\in\mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.7}$$

$$\mathcal{D}_6 := \{|F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.8}$$

$$\mathcal{D}_7 := \{\beta, h_{\beta\alpha}, |F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.9}$$

$$\mathcal{D}_8 := \{\beta, |F_\beta(k)| \text{ for } k \in \mathbf{R}, \{\kappa_{\pi j}\}_{j=1}^{N_\pi}, \{\kappa_{\beta j}\}_{j=1}^{N_\beta}\}, \tag{3.10}$$

where we let

$$h_{\beta\alpha} := \cot\beta - \cot\alpha. \tag{3.11}$$

Note that $h_{\beta\alpha} > 0$ when $0 < \beta < \alpha < \pi$. The data sets $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5, \mathcal{D}_7$ correspond to $\alpha \in (0, \pi)$ and the sets $\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6, \mathcal{D}_8$ to $\alpha = \pi$; the sets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ contain $|F_\alpha|$ whereas the sets $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8$ contain $|F_\beta|$; the sets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_5, \mathcal{D}_6$ correspond to $N_\beta = N_\alpha$ whereas the sets $\mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_7, \mathcal{D}_8$ to $N_\beta = N_\alpha + 1$. Our generalized Borg-Marchenko theorem can be stated as follows.

THEOREM 3.1. *Let the realizations H_α and H_β for some $0 < \beta < \alpha \le \pi$ correspond to a potential V in the Faddeev class with the boundary conditions identified by α and β, respectively. Then, each of the data sets $\mathcal{D}_j$ with $j = 1, \dots, 8$ uniquely determines the corresponding $\{V, \alpha, \beta\}$.*

The proof of the above theorem provides a method to recover α and β as well as $F_\alpha(k)$ and $F_\beta(k)$ for $k \in \overline{\mathbf{C}^+}$, thus also allowing us to construct the data sets used as input in various inversion methods to determine V. For the proof and details we refer the reader to [12], and here we only briefly outline the steps involved. Using $\mathcal{D}_j$ for each of $j = 1, 2, 5, 6, 7, 8$ we first construct $\mathrm{Re}[\Lambda_j(k)]$ for $\mathbf{R}$, where

$$\mathrm{Re}[\Lambda_1(k)] = \frac{k\, h_{\beta\alpha}}{|F_\alpha(k)|^2} \prod_{j=1}^{N_\alpha} \frac{k^2 + \kappa_{\alpha j}^2}{k^2 + \kappa_{\beta j}^2}, \quad \mathrm{Re}[\Lambda_2(k)] = -1 + \frac{1}{|F_\pi(k)|^2} \prod_{j=1}^{N_\pi} \frac{k^2 + \kappa_{\pi j}^2}{k^2 + \kappa_{\beta j}^2},$$

$$\mathrm{Re}[\Lambda_3(k)] = \frac{h_{\beta\alpha}\, k^2}{|F_\alpha(k)|^2} \frac{\prod_{j=1}^{N_\alpha}(k^2 + \kappa_{\alpha j}^2)}{\prod_{j=1}^{N_\beta}(k^2 + \kappa_{\beta j}^2)}, \quad \mathrm{Re}[\Lambda_4(k)] = -1 + \frac{k^2}{|F_\pi(k)|^2} \frac{\prod_{j=1}^{N_\pi}(k^2 + \kappa_{\pi j}^2)}{\prod_{j=1}^{N_\beta}(k^2 + \kappa_{\beta j}^2)},$$

$$\mathrm{Re}[\Lambda_5(k)] = -h_{\beta\alpha} + \frac{k^2\, h_{\beta\alpha}}{|F_\beta(k)|^2} \prod_{j=1}^{N_\beta} \frac{k^2 + \kappa_{\beta j}^2}{k^2 + \kappa_{\alpha j}^2},$$

$$\mathrm{Re}[\Lambda_6(k)] = -1 + \frac{k^2}{|F_\beta(k)|^2} \prod_{j=1}^{N_\beta} \frac{k^2 + \kappa_{\beta j}^2}{k^2 + \kappa_{\pi j}^2},$$

$$\mathrm{Re}[\Lambda_7(k)] = h_{\beta\alpha} - \frac{h_{\beta\alpha}}{|F_\beta(k)|^2} \frac{\prod_{j=1}^{N_\beta}(k^2 + \kappa_{\beta j}^2)}{\prod_{j=1}^{N_\beta - 1}(k^2 + \kappa_{\alpha j}^2)},$$

$$\text{Re}[\Lambda_8(k)] = -1 + \frac{1}{|F_\beta(k)|^2} \frac{\prod_{j=1}^{N_\beta}(k^2+\kappa_{\beta j}^2)}{\prod_{j=1}^{N_\beta-1}(k^2+\kappa_{\alpha j}^2)}.$$

Then, using the Schwarz integral formula

$$\Lambda_j(k) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{dt}{t-k-i0^+}\,\text{Re}[\Lambda_j(t)], \qquad k \in \overline{\mathbf{C}^+},$$

we uniquely construct Λ_j, where 0^+ in the integrand indicates that the values of $\Lambda_j(k)$ for $k \in \mathbf{R}$ must be obtained via a limit from $\mathbf{C}^+$. We get

$$\Lambda_1(k) = -i + i\,\frac{F_\beta(k)}{F_\alpha(k)} \prod_{j=1}^{N_\alpha} \frac{k^2+\kappa_{\alpha j}^2}{k^2+\kappa_{\beta j}^2},$$

$$\Lambda_2(k) = -1 - \frac{1}{k}\frac{F_\beta(0)}{F_\pi(0)} \prod_{j=1}^{N_\pi} \frac{\kappa_{\pi j}^2}{\kappa_{\beta j}^2} + \frac{1}{k}\frac{F_\beta(k)}{F_\pi(k)} \prod_{j=1}^{N_\pi} \frac{k^2+\kappa_{\pi j}^2}{k^2+\kappa_{\beta j}^2},$$

$$\Lambda_3(k) = ik\,\frac{F_\beta(k)}{F_\alpha(k)} \frac{\prod_{j=1}^{N_\alpha}(k^2+\kappa_{\alpha j}^2)}{\prod_{j=1}^{N_\beta}(k^2+\kappa_{\beta j}^2)}, \quad \Lambda_4(k) = -1 + k\,\frac{F_\beta(k)}{F_\pi(k)} \frac{\prod_{j=1}^{N_\pi}(k^2+\kappa_{\pi j}^2)}{\prod_{j=1}^{N_\beta}(k^2+\kappa_{\beta j}^2)},$$

$$\Lambda_5(k) = ik - h_{\beta\alpha} - ik\,\frac{F_\alpha(k)}{F_\beta(k)} \prod_{j=1}^{N_\beta} \frac{k^2+\kappa_{\beta j}^2}{k^2+\kappa_{\alpha j}^2}, \quad \Lambda_6(k) = -1 + k\,\frac{F_\pi(k)}{F_\beta(k)} \prod_{j=1}^{N_\beta} \frac{k^2+\kappa_{\beta j}^2}{k^2+\kappa_{\pi j}^2},$$

$$\Lambda_7(k) = -ik + h_{\beta\alpha} - \frac{i}{k}\,\frac{F_\alpha(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta}\kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta-1}\kappa_{\alpha j}^2} + \frac{i}{k}\,\frac{F_\alpha(k)}{F_\beta(k)} \frac{\prod_{j=1}^{N_\beta}(k^2+\kappa_{\beta j}^2)}{\prod_{j=1}^{N_\beta-1}(k^2+\kappa_{\alpha j}^2)},$$

$$\Lambda_8(k) = -1 - \frac{1}{k}\,\frac{F_\pi(0)}{F_\beta(0)} \frac{\prod_{j=1}^{N_\beta}\kappa_{\beta j}^2}{\prod_{j=1}^{N_\beta-1}\kappa_{\alpha j}^2} + \frac{1}{k}\,\frac{F_\pi(k)}{F_\beta(k)} \frac{\prod_{j=1}^{N_\beta}(k^2+\kappa_{\beta j}^2)}{\prod_{j=1}^{N_\beta-1}(k^2+\kappa_{\alpha j}^2)}.$$

Each $\Lambda_j(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$.

Next, with the help of $\mathcal{D}_j$ and Λ_j, we uniquely construct α, β, F_α, and F_β by using the procedure given in Section 4 of [12]. Having these four quantities, we can construct V by using one of the available inversion methods [2,10-12], three of which are outlined in Section 4. Note that $f(k,0)$ and $f'(k,0)$ can then also be constructed via

$$f(k,0)=\begin{cases}\dfrac{i}{h_{\beta\alpha}}\left[F_\beta(k)-F_\alpha(k)\right], & \alpha,\beta\in(0,\pi),\\ F_\pi(k),\end{cases}$$

$$f'(k,0)=\begin{cases}\dfrac{i}{h_{\beta\alpha}}\left[\cot\beta\cdot F_\alpha(k)-\cot\alpha\cdot F_\beta(k)\right], & \alpha,\beta\in(0,\pi),\\ i\,F_\beta(k)-\cot\beta\cdot F_\pi(k), & \beta\in(0,\pi).\end{cases}$$

The constructions from $\mathcal{D}_j$ with $j=3,4$ involve an extra step because in each of those two cases exactly one of the discrete eigenvalues needed to construct $\mathrm{Re}[\Lambda_j(k)]$ for $k\in\mathbf{R}$ is not contained in $\mathcal{D}_j$. As a result, we first construct a one-parameter family of $\Lambda_j(k)$ for $k\in\overline{\mathbf{C}^+}$ and then determine the eigenvalue missing in $\mathcal{D}_j$ by taking a nontangential limit as $k\to\infty$ on $\mathbf{C}^+$. Once the missing discrete eigenvalue is at hand, the corresponding $\Lambda_j(k)$ for $k\in\overline{\mathbf{C}^+}$ is uniquely determined as well. We can then proceed as in the case with $j=1,2,5,6,7,8$ in order to determine α and β, $F_\alpha(k)$ and $F_\beta(k)$ for $k\in\overline{\mathbf{C}^+}$, the potential V, and any other relevant quantities.

4. Reconstruction of the Potential

In Section 3 we have outlined the construction of the quantities α, β, and $F_\alpha(k)$ and $F_\beta(k)$ for $k\in\overline{\mathbf{C}^+}$ by using each of the data sets $\mathcal{D}_j$ with $j=1,\dots,8$ given in (3.3)-(3.10), respectively. In this section we outline three methods to briefly illustrate the use of such quantities to recover the potential V.

In the Gel'fand-Levitan method [2,10-12], one forms the input data $\mathcal{G}_\alpha$ given by

$$\mathcal{G}_\alpha:=\{|F_\alpha(k)| \text{ for } k\in\mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{g_{\alpha j}\}_{j=1}^{N_\alpha}\},$$

where the $g_{\alpha j}$ are the norming constants appearing in (2.9) and they can also be obtained from (cf. (3.25) of [12])

$$g_{\alpha j}=\begin{cases}\sqrt{\dfrac{2i\kappa_{\alpha j}\,F_\beta(i\kappa_{\alpha j})}{h_{\beta\alpha}\,\dot F_\alpha(i\kappa_{\alpha j})}}, & 0<\beta<\alpha<\pi,\\ \sqrt{\dfrac{2\,\kappa_{\pi j}\,F_\beta(i\kappa_{\pi j})}{\dot F_\pi(i\kappa_{\pi j})}}, & 0<\beta<\alpha=\pi,\end{cases}$$

with an overdot indicating the k-derivative. The corresponding potential V is uniquely recovered via

$$V(x)=2\,\frac{d}{dx}A_\alpha(x,x^-),$$

where $A_\alpha(x,y)$ is obtained by solving the Gel'fand-Levitan integral equation

$$A_\alpha(x,y)+G_\alpha(x,y)+\int_0^x dz\,G_\alpha(y,z)\,A_\alpha(x,z)=0,\qquad 0\le y<x,$$

with the kernel $G_\alpha(x,y)$ for $\alpha\in(0,\pi)$ given by

$$G_\alpha(x,y):=\frac{1}{\pi}\int_{-\infty}^{\infty}dk\,\left[\frac{k^2}{|F_\alpha(k)|^2}-1\right](\cos kx)\,(\cos ky)+\sum_{j=1}^{N_\alpha}g_{\alpha j}^2\,(\cosh\kappa_{\alpha j}x)\,(\cosh\kappa_{\alpha j}y),$$

and the kernel $G_\pi(x,y)$ given by

$$G_\pi(x,y) := \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, \left[\frac{1}{|F_\pi(k)|^2} - 1\right] (\sin kx)\,(\sin ky) + \sum_{j=1}^{N_\pi} \frac{g_{\pi j}^2}{\kappa_{\pi j}^2}\,(\sinh \kappa_{\pi j} x)\,(\sinh \kappa_{\pi j} y)\,.$$

In the Marchenko method [2,11,12] one forms the input data $\mathcal{M}_\alpha$ given by

$$\mathcal{M}_\alpha := \{S_\alpha(k) \text{ for } k \in \mathbf{R}, \{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}, \{m_{\alpha j}\}_{j=1}^{N_\alpha}\},$$

where S_α is the scattering matrix defined in (2.7) and the norming constants $m_{\alpha j}$ can be obtained by using

$$m_{\alpha j} := \frac{1}{||f(i\kappa_{\alpha j},\cdot)||}, \qquad j = 1, \dots, N_\alpha,$$

or equivalently via (cf. (3.26) of [12])

$$m_{\alpha j} = \begin{cases} \sqrt{\dfrac{-2i\kappa_{\alpha j}\, h_{\beta\alpha}}{F_\beta(i\kappa_{\alpha j})\, \dot F_\alpha(i\kappa_{\alpha j})}}, & 0 < \beta < \alpha < \pi, \\ \sqrt{\dfrac{-2\,\kappa_{\pi j}}{F_\beta(i\kappa_{\pi j})\, \dot F_\pi(i\kappa_{\pi j})}}, & 0 < \beta < \alpha = \pi. \end{cases}$$

The potential V is uniquely recovered as

$$V(x) = -2\,\frac{d}{dx} K(x, x^+),$$

where $K(x,y)$ is obtained by solving the Marchenko integral equation

$$K(x,y) + M_\alpha(x+y) + \int_x^\infty dz\, M_\alpha(y+z)\, K(x,z) = 0, \qquad 0 < x < y,$$

with the kernel

$$M_\alpha(y) := \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} dk\, [S_\alpha(k) - 1]\, e^{iky} + \sum_{j=1}^{N_\alpha} m_{\alpha j}^2\, e^{-\kappa_{\alpha j} y}, & \alpha \in (0,\pi), \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} dk\, [1 - S_\pi(k)]\, e^{iky} + \sum_{j=1}^{N_\pi} m_{\pi j}^2\, e^{-\kappa_{\pi j} y}, & \alpha = \pi. \end{cases}$$

In the right Faddeev-Marchenko method [2,11,12], by viewing the potential V on the full line and setting $V \equiv 0$ for $x < 0$, one forms the input data $\mathcal{F}_{\rm r}$ given by

$$\mathcal{F}_{\rm r} := \{L(k) \text{ for } k \in \mathbf{R}, \{\tau_j\}_{j=1}^{N}, \{c_{{\rm r}j}\}_{j=1}^{N}\},$$

where L is the left reflection coefficient given by

$$L(k) = \frac{ik\, f(k,0) - f'(k,0)}{ik\, f(k,0) + f'(k,0)}, \qquad k \in \overline{\mathbf{C}^+},$$

or equivalently

$$L(k)=\begin{cases}\dfrac{(k-i\cot\beta)\,F_\alpha(k)-(k-i\cot\alpha)\,F_\beta(k)}{(k+i\cot\beta)\,F_\alpha(k)-(k+i\cot\alpha)\,F_\beta(k)}, & \alpha,\beta\in(0,\pi),\\[2mm] \dfrac{(k-i\cot\beta)\,F_\pi(k)-F_\beta(k)}{(k+i\cot\beta)\,F_\pi(k)+F_\beta(k)}, & \beta\in(0,\pi).\end{cases}$$

The N positive constants τ_j correspond to the (simple) poles of $L(k)$ on $\mathbf{I}^+$, and the $c_{\mathrm{r}j}$ are the norming constants that can be obtained via

$$c_{\mathrm{r}j}=\sqrt{-i\,\mathrm{Res}(L,i\tau_j)},\qquad j=1,\dots,N.$$

The potential V can be uniquely reconstructed as

$$V(x)=2\,\frac{dB_{\mathrm{r}}(x,0^+)}{dx},$$

where $B_{\mathrm{r}}(x,y)$ is the solution to the right Faddeev-Marchenko integral equation

$$B_{\mathrm{r}}(x,y)+\Omega_{\mathrm{r}}(-2x+y)+\int_0^\infty dy\,\Omega_{\mathrm{r}}(-2x+y+z)\,B_{\mathrm{r}}(x,z)=0,\qquad y>0,$$

with the input data

$$\Omega_{\mathrm{r}}(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,L(k)\,e^{iky}+\sum_{j=1}^N c_{\mathrm{r}j}^2\,e^{-\tau_j y}.$$

5. Krein's Spectral Shift Function

In this section, we indicate the construction of Krein's spectral shift function $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{R}^+\cup\mathbf{I}^+$ with $0<\beta<\alpha\le\pi$, and we also show how to extract α, β, F_α, F_β, and V from $\xi_{\beta\alpha}$.

Krein's spectral shift function $\xi_{\beta\alpha}(k)$ associated with H_α and H_β can be defined in various ways [5,13-15]. We find it convenient to introduce $\xi_{\beta\alpha}$ by relating it to the phase of $F_\alpha(k)/F_\beta(k)$ for $k\in\mathbf{R}\cup\mathbf{I}^+$. Let us note that $\xi_{\beta\alpha}$ in [6] is considered when $0\le\beta<\alpha<\pi$, but studying it for $0<\beta<\alpha\le\pi$ does not present any inconvenience because we can choose $\xi_{0\theta}(k)=\xi_{\theta\pi}(k)$ for $k\in\mathbf{R}^+\cup\mathbf{I}^+$ with $\theta\in(0,\pi)$, as done in our paper.

Let

$$e^{\pi i\xi_{\beta\alpha}(k)}:=\frac{Z_{\beta\alpha}(k)}{|Z_{\beta\alpha}(k)|},\qquad k\in\mathbf{R}^+\cup\mathbf{I}^+,\tag{5.1}$$

with the normalization

$$\xi_{\beta\alpha}(+\infty)=\begin{cases}0, & \alpha\in(0,\pi),\\ 1/2, & \alpha=\pi,\end{cases}\tag{5.2}$$

and the range of $\xi_{\beta\alpha}(k)$ on $\mathbf{I}^+$ restricted to the interval $[0,1]$, where we have defined

$$Z_{\beta\alpha}(k):=\begin{cases}\dfrac{F_\alpha(k)}{F_\beta(k)}, & 0<\beta<\alpha<\pi,\\[2mm] \dfrac{i\,F_\beta(k)}{F_\pi(k)}, & 0<\beta<\alpha=\pi.\end{cases}\tag{5.3}$$

As seen from (2.5) and (5.1)-(5.3), we can express $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{R}^+$ in terms of the phase shifts ϕ_α and ϕ_β as

$$\xi_{\beta\alpha}(k)=\begin{cases}\dfrac{1}{\pi}\left[\phi_\beta(k)-\phi_\alpha(k)\right], & 0<\beta<\alpha<\pi,\\ \dfrac{1}{2}+\dfrac{1}{\pi}\left[\phi_\pi(k)-\phi_\beta(k)\right], & 0<\beta<\alpha=\pi.\end{cases}\tag{5.4}$$

With the help of (2.6) and (5.4) we see that $\xi_{\beta\alpha}(k)$ can be extended from $k\in\mathbf{R}^+$ to $k\in\mathbf{R}$ oddly, i.e.

$$\xi_{\beta\alpha}(-k)=-\xi_{\beta\alpha}(k),\qquad 0<\beta<\alpha\le\pi,\quad k\in\mathbf{R}.\tag{5.5}$$

Using (3.18) of [12], for $k\in\mathbf{R}$ we get

$$\operatorname{Im}\left[Z_{\beta\alpha}(k)\right]=\begin{cases}\dfrac{k\,h_{\beta\alpha}}{|F_\beta(k)|^2}, & 0<\beta<\alpha<\pi,\\ \dfrac{k}{|F_\pi(k)|^2}, & 0<\beta<\alpha=\pi.\end{cases}\tag{5.6}$$

Since $F_\alpha(k)$ is nonzero for $k\in\mathbf{R}^+$, (5.6) implies that $\operatorname{Im}[Z_{\beta\alpha}(k)]>0$ for $k\in\mathbf{R}^+$. Thus, from (5.1) we can conclude that $\xi_{\beta\alpha}(k)\in(0,1)$ when $k\in\mathbf{R}^+$. Furthermore, using (2.8) and (5.4) we get

$$\xi_{\beta\alpha}(0^+)=\begin{cases}N_\beta-N_\alpha+\dfrac{d_\beta-d_\alpha}{2}, & 0<\beta<\alpha<\pi,\\ N_\pi-N_\beta+\dfrac{d_\pi-d_\beta}{2}, & 0<\beta<\alpha=\pi.\end{cases}$$

Having introduced $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{R}\cup\mathbf{I}^+$, let us now analyze the problem of recovering α, β, F_α, F_β, and V from $\xi_{\beta\alpha}$. First, given $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{R}^+\cup\mathbf{I}^+$, from (5.2) we determine whether $\alpha\in(0,\pi)$ or $\alpha=\pi$.

Next, we can recover N_α, N_β, $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$, and $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$ by using the values of $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{I}^+$. In fact, with the help of (2.1), (2.3), and (5.1)-(5.3), we see that $Z_{\beta\alpha}(k)$ is real valued for $k\in\mathbf{I}^+$ and that $\xi_{\beta\alpha}(k)$ on $\mathbf{I}^+$ is equal to either 0 or 1, with jump discontinuities at $k=i\kappa_{\alpha j}$ and $k=i\kappa_{\beta j}$. In other words, in consonant with the interlacing properties given in (3.1) and (3.2), as a result of the simplicity of zeros of F_α and F_β on $\mathbf{I}^+$, we have, when $N_\alpha=N_\beta$ and $0<\beta<\alpha<\pi$

$$\xi_{\beta\alpha}(i\omega)=\begin{cases}0, & \omega\in(0,\kappa_{\alpha1})\cup(\kappa_{\beta N_\beta},+\infty)\cup_{j=2}^{N_\alpha}(\kappa_{\beta(j-1)},\kappa_{\alpha j}),\\ 1, & \omega\in\cup_{j=1}^{N_\alpha}(\kappa_{\alpha j},\kappa_{\beta j}),\end{cases}$$

and we have, when $N_\alpha=N_\beta-1$ and $0<\beta<\alpha<\pi$

$$\xi_{\beta\alpha}(i\omega)=\begin{cases}0, & \omega\in(\kappa_{\beta N_\beta},+\infty)\cup_{j=1}^{N_\alpha}(\kappa_{\beta j},\kappa_{\alpha j}),\\ 1, & \omega\in(0,\kappa_{\beta1})\cup_{j=1}^{N_\alpha}(\kappa_{\alpha j},\kappa_{\beta(j+1)}).\end{cases}$$

On the other hand, we have, when $0<\beta<\alpha=\pi$ and $N_\beta=N_\pi$

$$\xi_{\beta\pi}(i\omega)=\begin{cases}1, & \omega\in(0,\kappa_{\pi1})\cup(\kappa_{\beta N_\beta},+\infty)\cup_{j=2}^{N_\pi}(\kappa_{\beta(j-1)},\kappa_{\pi j}),\\ 0, & \omega\in\cup_{j=1}^{N_\pi}(\kappa_{\pi j},\kappa_{\beta j}),\end{cases}$$

and we have, when $0<\beta<\alpha=\pi$ and $N_\beta=N_\pi+1$

$$\xi_{\beta\pi}(i\omega)=\begin{cases}1, & \omega\in(\kappa_{\beta N_\beta},+\infty)\cup_{j=1}^{N_\pi}(\kappa_{\beta j},\kappa_{\pi j}),\\ 0, & \omega\in(0,\kappa_{\beta 1})\cup_{j=1}^{N_\pi}(\kappa_{\pi j},\kappa_{\beta(j+1)}).\end{cases}\tag{5.7}$$

Thus, we are able to recover N_α, N_β, $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$, and $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$ by analyzing the location of the jumps of $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{I}^+$.

In order to continue with the recovery, let us define the 'reduced' quantities identified with the superscript [0] as follows:

$$F_\alpha^{[0]}(k):=F_\alpha(k)\prod_{j=1}^{N_\alpha}\frac{k+i\kappa_{\alpha j}}{k-i\kappa_{\alpha j}},\qquad \alpha\in(0,\pi].$$

Note that $|F_\alpha^{[0]}(k)|=|F_\alpha(k)|$ for $k\in\mathbf{R}$, and hence as seen from (5.3) it is natural to let

$$Z_{\beta\alpha}^{[0]}(k):=\begin{cases}Z_{\beta\alpha}(k)\displaystyle\prod_{j=1}^{N_\alpha}\frac{k+i\kappa_{\alpha j}}{k-i\kappa_{\alpha j}}\prod_{p=1}^{N_\beta}\frac{k-i\kappa_{\beta p}}{k+i\kappa_{\beta p}}, & 0<\beta<\alpha<\pi,\\ Z_{\beta\pi}(k)\displaystyle\prod_{j=1}^{N_\pi}\frac{k-i\kappa_{\pi j}}{k+i\kappa_{\pi j}}\prod_{p=1}^{N_\beta}\frac{k+i\kappa_{\beta p}}{k-i\kappa_{\beta p}}, & 0<\beta<\alpha=\pi,\end{cases}\tag{5.8}$$

so that $Z_{\beta\alpha}^{[0]}$ has no zeros or poles in $\overline{\mathbf{C}^+}\setminus\{0\}$.

Let $\mathcal{W}^{[0]}$ denote the class of functions $Y(k)$ that are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$ satisfying $Y(-k)=Y(k)^*$ for $k\in\mathbf{R}$ and $1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$, and that Y is either continuous at $k=0$ or has a finite-order zero or pole at $k=0$. Let $\mathcal{W}$ denote the extended class of functions that differ from $Y(k)$ in $\mathcal{W}^{[0]}$ by a finite number of multiplicative factors of the form $\dfrac{k+ia_j}{k+ib_j}$ with real constants a_j and b_j. For such functions $Y(k)$ in $\mathcal{W}$, we let $\log(Y(k))$ denote the branch of the logarithm normalized as $\operatorname{Im}[\log(Y(\infty))]=0$.

THEOREM 5.1. *The quantities $Z_{\beta\alpha}^{[0]}(k)$ for $0<\beta<\alpha<\pi$ and $Z_{\beta\pi}^{[0]}(k)/[ik]$ for $0<\beta<\pi$ each belong to $\mathcal{W}^{[0]}$, and for $k\in\overline{\mathbf{C}^+}$ we have*

$$Z_{\beta\alpha}^{[0]}(k)=\exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty}dt\,\frac{\operatorname{Im}\left[\log\left(Z_{\beta\alpha}^{[0]}(t)\right)\right]}{t-k-i0^+}\right),\qquad 0<\beta<\alpha<\pi,\tag{5.9}$$

$$Z_{\beta\pi}^{[0]}(k)=ik\cdot\exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty}dt\,\frac{\operatorname{Im}\left[\log\left(Z_{\beta\pi}^{[0]}(t)/[it]\right)\right]}{t-k-i0^+}\right),\qquad 0<\beta<\pi.\tag{5.10}$$

PROOF. For any $\alpha\in(0,\pi]$ it is known [2,10-12] that $F_\alpha(k)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, it is nonzero in $\overline{\mathbf{C}^+}$ except for the simple zeros at $k=i\kappa_{\alpha j}$

with $j = 1, \dots, N_\alpha$ and perhaps a simple zero at $k = 0$. By using (3.12) and (3.13) of [12], as $k \to \infty$ in $\overline{\mathbf{C}^+}$ we obtain

$$Z_{\beta\alpha}(k) = \begin{cases} 1 + \dfrac{i\,h_{\beta\alpha}}{k} - \dfrac{h_{\beta\alpha}\,\cot\beta}{k^2} + o(1/k^2), & 0 < \beta < \alpha < \pi, \\ ik + \cot\beta + o(1), \qquad 0 < \beta < \alpha = \pi, \end{cases} \tag{5.11}$$

and hence from (5.8) and (5.11), as $k \to \infty$ in $\overline{\mathbf{C}^+}$, we get

$$Z^{[0]}_{\beta\alpha}(k) = 1 - \frac{i}{k}\left[h_{\beta\alpha} + 2\sum_{j=1}^{N_\pi} \kappa_{\pi j} - 2\sum_{j=1}^{N_\beta} \kappa_{\beta j}\right] + O(1/k^2), \qquad 0 < \beta < \alpha < \pi,$$

$$\frac{Z^{[0]}_{\beta\pi}(k)}{ik} = 1 + \frac{1}{ik}\left[\cot\beta + 2\sum_{j=1}^{N_\pi} \kappa_{\pi j} - 2\sum_{j=1}^{N_\beta} \kappa_{\beta j}\right] + o(1/k), \qquad 0 < \beta < \pi.$$

Using also (2.4), (5.3), and (5.8), it follows that $Z^{[0]}_{\beta\alpha}(k)$ for $0 < \beta < \alpha < \pi$ and $Z^{[0]}_{\beta\pi}(k)/[ik]$ for $0 < \beta < \pi$ each belong to $\mathcal{W}^{[0]}$. Moreover, the logarithms of both these quantities are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$, have at most an integrable singularity at $k = 0$, and are $O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Thus, the Schwarz integral formula can be used to obtain

$$\log\left(Z^{[0]}_{\beta\alpha}(k)\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, \frac{\mathrm{Im}\left[\log\left(Z^{[0]}_{\beta\alpha}(t)\right)\right]}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+},$$

$$\log\left(\frac{Z^{[0]}_{\beta\pi}(k)}{ik}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, \frac{\mathrm{Im}\left[\log\left(Z^{[0]}_{\beta\pi}(t)/[it]\right)\right]}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+},$$

which give us (5.9) and (5.10), respectively. □

From (5.8) we get $|Z^{[0]}_{\beta\alpha}(k)| = |Z_{\beta\alpha}(k)|$ for $k \in \mathbf{R}$, and hence a comparison with (5.1) and (5.2) leads us to let

$$e^{\pi i \xi^{[0]}_{\beta\alpha}(k)} := \frac{Z^{[0]}_{\beta\alpha}(k)}{|Z^{[0]}_{\beta\alpha}(k)|}, \qquad k \in \mathbf{R}^+ \cup \mathbf{I}^+, \tag{5.12}$$

with the normalization

$$\xi^{[0]}_{\beta\alpha}(+\infty) = \begin{cases} 0, & \alpha \in (0,\pi), \\ 1/2, & \alpha = \pi, \end{cases}$$

so that for $0 < \beta < \alpha < \pi$ we have

$$\xi^{[0]}_{\beta\alpha}(k) = \xi_{\beta\alpha}(k) + \frac{1}{\pi i}\log\left(\prod_{j=1}^{N_\alpha} \frac{k + i\kappa_{\alpha j}}{k - i\kappa_{\alpha j}} \prod_{p=1}^{N_\beta} \frac{k - i\kappa_{\beta p}}{k + i\kappa_{\beta p}}\right), \tag{5.13}$$

and for $0 < \beta < \pi$ we get

$$\xi^{[0]}_{\beta\pi}(k) = \xi_{\beta\pi}(k) + \frac{1}{\pi i}\log\left(\prod_{j=1}^{N_\pi} \frac{k - i\kappa_{\pi j}}{k + i\kappa_{\pi j}} \prod_{p=1}^{N_\beta} \frac{k + i\kappa_{\beta p}}{k - i\kappa_{\beta p}}\right). \tag{5.14}$$

Using (5.5), (5.13), and (5.14) we can extend $\xi^{[0]}_{\beta\alpha}(k)$ oddly from $k \in \mathbf{R}^+$ to $k \in \mathbf{R}$, i.e.

$$\xi^{[0]}_{\beta\alpha}(-k) = -\xi^{[0]}_{\beta\alpha}(k), \qquad 0<\beta<\alpha\le\pi, \quad k\in\mathbf{R}. \tag{5.15}$$

From (5.12) and (5.15), for $k\in\mathbf{R}$ we get

$$\xi^{[0]}_{\beta\alpha}(k) = \begin{cases} \dfrac{1}{\pi}\,\mathrm{Im}\left[\log\left(Z^{[0]}_{\beta\alpha}(k)\right)\right], & 0<\beta<\alpha<\pi, \\ \dfrac{1}{\pi}\,\mathrm{Im}\left[\log\left(\dfrac{Z^{[0]}_{\beta\pi}(k)}{ik}\right)\right] + \dfrac{1}{2}\,\mathrm{sign}(k), & 0<\beta<\alpha=\pi, \end{cases} \tag{5.16}$$

and similarly, for $k\in\mathbf{R}$ we have

$$\xi_{\beta\alpha}(k) = \begin{cases} \dfrac{1}{\pi}\,\mathrm{Im}\left[\log\left(Z_{\beta\alpha}(k)\right)\right], & 0<\beta<\alpha<\pi, \\ \dfrac{1}{\pi}\,\mathrm{Im}\left[\log\left(\dfrac{Z_{\beta\pi}(k)}{ik}\right)\right] + \dfrac{1}{2}\,\mathrm{sign}(k), & 0<\beta<\alpha=\pi. \end{cases} \tag{5.17}$$

Since $Z^{[0]}_{\beta\alpha}(k)>0$ for $k\in\mathbf{I}^+$, we have

$$\xi^{[0]}_{\beta\alpha}(k) = 0, \qquad 0<\beta<\alpha\le\pi, \quad k\in\mathbf{I}^+, \tag{5.18}$$

and hence, from (5.13) and (5.14), for $k\in\mathbf{I}^+$ we get

$$\xi_{\beta\alpha}(k) = \begin{cases} \dfrac{1}{\pi i}\log\left(\displaystyle\prod_{j=1}^{N_\alpha}\frac{k-i\kappa_{\alpha j}}{k+i\kappa_{\alpha j}}\prod_{p=1}^{N_\beta}\frac{k+i\kappa_{\beta p}}{k-i\kappa_{\beta p}}\right), & 0<\beta<\alpha<\pi, \\ \dfrac{1}{\pi i}\log\left(\displaystyle\prod_{j=1}^{N_\pi}\frac{k+i\kappa_{\pi j}}{k-i\kappa_{\pi j}}\prod_{p=1}^{N_\beta}\frac{k-i\kappa_{\beta p}}{k+i\kappa_{\beta p}}\right), & 0<\beta<\alpha=\pi. \end{cases} \tag{5.19}$$

Using (5.16) in (5.9) and (5.10), we obtain the following.

Corollary 5.3. *For $0<\beta<\alpha\le\pi$, the quantity $Z^{[0]}_{\beta\alpha}(k)$ for $k\in\overline{\mathbf{C}^+}$ can be obtained from $\xi^{[0]}_{\beta\alpha}(k)$ given for $k\in\mathbf{R}^+$ via (5.15) and*

$$Z^{[0]}_{\beta\alpha}(k) = \begin{cases} \exp\left(\displaystyle\int_{-\infty}^{\infty} dt\,\frac{\xi^{[0]}_{\beta\alpha}(t)}{t-k-i0^+}\right), & 0<\beta<\alpha<\pi, \\ ik\cdot\exp\left(\displaystyle\int_{-\infty}^{\infty} dt\,\frac{\xi^{[0]}_{\beta\pi}(t)-(1/2)\,\mathrm{sign}(t)}{t-k-i0^+}\right), & 0<\beta<\alpha=\pi. \end{cases}$$

Now let us continue with the recovery of α and β from $\xi_{\beta\alpha}$. We have already constructed N_α, N_β, $\{\kappa_{\alpha j}\}_{j=1}^{N_\alpha}$, and $\{\kappa_{\beta j}\}_{j=1}^{N_\beta}$. Next, with the help of (5.13), (5.14), and (5.15) we obtain $\xi^{[0]}_{\beta\alpha}(k)$ for $k\in\mathbf{R}$. Then, using Corollary 5.3 we get $Z^{[0]}_{\beta\alpha}(k)$ and via (5.8) we obtain $Z_{\beta\alpha}(k)$ for $k\in\overline{\mathbf{C}^+}$. Having $Z_{\beta\alpha}(k)$ in hand, we can recover α and β by using (3.11) and (5.11).

Next, we continue with the construction of F_α and F_β. When $\alpha\neq\pi$, we proceed as follows. Having $h_{\beta\alpha}$ and $Z_{\beta\alpha}(k)$ for $k\in\mathbf{R}$, via (5.6) we construct $|F_\beta(k)|^2$ as

$$|F_\beta(k)|^2=\frac{k\,h_{\beta\alpha}}{\text{Im}\,[Z_{\beta\alpha}(k)]},\qquad k\in\mathbf{R}.$$

Knowing $|F_\beta(k)|$ for $k\in\mathbf{R}$, we can then construct F_β via (cf. (3.6) of [12])

$$F_\beta(k)=k\left(\prod_{j=1}^{N_\beta}\frac{k-i\kappa_{\beta j}}{k+i\kappa_{\beta j}}\right)\exp\left(\frac{-1}{\pi i}\int_{-\infty}^{\infty}dt\,\frac{\log|t/F_\beta(t)|}{t-k-i0^+}\right),\qquad k\in\overline{\mathbf{C}^+}. \tag{5.20}$$

On the other hand, if $\alpha=\pi$ we proceed as follows. From (5.6) we get $|F_\pi(k)|^2$ as

$$|F_\pi(k)|^2=\frac{k}{\text{Im}\,[Z_{\beta\pi}(k)]},\qquad k\in\mathbf{R}.$$

Having $|F_\pi(k)|$ for $k\in\mathbf{R}$ at hand, we can then construct F_π via (cf. (3.7) of [12])

$$F_\pi(k)=\left(\prod_{j=1}^{N_\pi}\frac{k-i\kappa_{\pi j}}{k+i\kappa_{\pi j}}\right)\exp\left(\frac{1}{\pi i}\int_{-\infty}^{\infty}dt\,\frac{\log|F_\pi(t)|}{t-k-i0^+}\right),\qquad k\in\overline{\mathbf{C}^+}. \tag{5.21}$$

Finally, if $\alpha\neq\pi$ then via the first line of (5.3) we can recover $F_\alpha(k)$ from the already constructed quantities $F_\beta(k)$ and $Z_{\beta\alpha}(k)$ for $k\in\overline{\mathbf{C}^+}$. Similarly, if $\alpha=\pi$ then via the second line of (5.3) we can recover $F_\beta(k)$ from the already constructed quantities $F_\pi(k)$ and $Z_{\beta\pi}(k)$ for $k\in\overline{\mathbf{C}^+}$. A comparison with (3.3)-(3.10) reveals that we have now constructed the data sets $\mathcal{D}_1,\mathcal{D}_3,\mathcal{D}_5,\mathcal{D}_7$ when $\alpha\in(0,\pi)$ and $\mathcal{D}_2,\mathcal{D}_4,\mathcal{D}_6,\mathcal{D}_8$ when $\alpha=\pi$. Finally, we can recover V via one of the methods described in Section 4.

Let us also note that, by using any one of the eight data sets $\mathcal{D}_j$ given in (3.3)-(3.10), we can construct the quantities $F_\alpha(k)$ and $F_\beta(k)$ for $k\in\overline{\mathbf{C}^+}$ as outlined in Section 3, then obtain $Z_{\beta\alpha}$ for $k\in\overline{\mathbf{C}^+}$ given in (5.3), and also recover $\xi_{\beta\alpha}(k)$ for $k\in\mathbf{R}^+\cup\mathbf{I}^+$ via (5.17) and (5.19).

6. Examples

In this section we present two examples to illustrate the recovery of the potential and the boundary conditions from the data set $\mathcal{D}_3$ given in (3.5) and also from Krein's spectral shift function $\xi_{\beta\alpha}$.

EXAMPLE 6.1. In our first example, assume that we are given $\mathcal{D}_3$ with $h_{\beta\alpha}=5$, $N_\alpha=1$, $N_\beta=2$, $\kappa_{\alpha1}=2$, $\kappa_{\beta2}=4$, and $|F_\alpha(k)|^2=k^2+4$ for $k\in\mathbf{R}$, and we are interested in constructing all the relevant quantities such as $\cot\alpha$, $\cot\beta$, F_α, F_β, and V. By using the method outlined in Section 3, with the details given in the proof of Theorem 2.3 of [12], we get

$$\kappa_{\beta1}=1,\quad \cot\alpha=-8/5,\quad \cot\beta=17/5,$$

$$F_\alpha(k)=k-2i,\quad F_\beta(k)=\frac{(k-i)(k-4i)}{k+2i}.$$

As indicated in Sections 3-5, we can then obtain other relevant quantities such as

$$f(k,0)=\frac{5k-8i}{5(k+2i)},\quad f'(k,0)=\frac{25ik^2+40k+36i}{25(k+2i)},$$

the quantities used in the Gel'fand-Levitan and Marchenko methods, e.g.

$$g_{\alpha 1}=\sqrt{\frac{2}{5}},\quad m_{\alpha 1}=\sqrt{40},\quad S_\alpha(k)=\frac{k+2i}{k-2i},$$

the quantities used in the Faddeev-Marchenko method, e.g.

$$N=1,\quad \tau_1=\frac{(\sqrt{34}+4)i}{5},\quad c_{\mathrm{r}1}=\frac{3}{\sqrt{5\sqrt{34}}},\quad L(k)=\frac{-18}{25k^2-40ik+18},$$

the quantities relevant to Krein's spectral shift function, e.g.

$$Z^{[0]}_{\beta\alpha}(k)=\frac{(k+2i)^2}{(k+i)(k+4i)},\quad \xi^{[0]}_{\beta\alpha}(k)=\frac{1}{\pi}\,\mathrm{Im}\left[\log\left(\frac{(k+2i)^2}{(k+i)(k+4i)}\right)\right],$$

$$Z_{\beta\alpha}(k)=\frac{(k+2i)(k-2i)}{(k-i)(k-4i)},\quad \xi_{\beta\alpha}(k)=\frac{1}{\pi}\,\mathrm{Im}\left[\log\left(\frac{(k+2i)(k-2i)}{(k-i)(k-4i)}\right)\right],$$

and finally the potential and the Jost solution

$$V(x)=-\frac{288e^{4x}}{(9+e^{4x})^2},\quad f(k,x)=e^{ikx}\left[1-\frac{36i}{(k+2i)\,(9+e^{4x})}\right].$$

Example 6.2. As our second example, let us assume that we have as our data Krein's spectral shift function given by

$$\xi_{\beta\alpha}(k)=\begin{cases}\dfrac{1}{2}+\dfrac{1}{\pi}\,\mathrm{Im}\left[\log\left(\dfrac{(k-i)(k+i)(k-3i)}{k(k-2i)(k+2i)}\right)\right], & k\in\mathbf{R}^+,\\ 0, \quad k\in i(0,1)\cup i(2,3),\\ 1, \quad k\in i(1,2)\cup i(3,+\infty),\end{cases}\tag{6.1}$$

and we would like to construct all the relevant quantities such as $\cot\alpha$, $\cot\beta$, F_α, F_β, and V. By letting $k\to+\infty$ in (6.1) we get $\xi_{\beta\alpha}(+\infty)=1/2$, and hence we see from (5.2) that $\alpha=\pi$. Next, we analyze our $\xi_{\beta\alpha}(k)$ on $\mathbf{I}^+$ and see the jump discontinuities at $k=i,2i,3i$. A comparison with (5.7) indicates that $N_\alpha=1$, $N_\beta=2$, $\kappa_{\alpha1}=2$, $\kappa_{\beta1}=1$, and $\kappa_{\beta2}=3$. Next, using (5.14) and (5.18) we obtain

$$\xi^{[0]}_{\beta\alpha}(k)=\begin{cases}\dfrac{1}{2}+\dfrac{1}{\pi}\,\mathrm{Im}\left[\log\left(\dfrac{(k+i)^2(k+3i)}{k(k+2i)^2}\right)\right], & k\in\mathbf{R},\\ 0, \quad k\in\mathbf{I}^+,\end{cases}$$

and then, via Corollary 5.3, we get

$$Z^{[0]}_{\beta\alpha}(k)=\frac{i(k+i)^2(k+3i)}{(k+2i)^2}.\tag{6.2}$$

Next, using (6.2) in (5.8) we obtain

$$Z_{\beta\alpha}(k)=\frac{i(k+i)(k-i)(k-3i)}{(k+2i)(k-2i)}.\tag{6.3}$$

Using (6.3) in (5.11) we get $\cot\beta=3$, and then with the help of (5.20), (5.21), and then (5.3) we obtain

$$F_\pi(k)=\frac{k-2i}{k+i},\quad F_\beta(k)=\frac{(k-i)(k-3i)}{k+2i}.$$

Proceeding as in the first example, we can then construct all the relevant quantities. For example, we have

$$S_\pi(k) = \frac{(k+i)(k+2i)}{(k-i)(k-2i)}, \quad m_{\pi 1} = 4\sqrt{3}, \quad g_{\pi 1} = \sqrt{3},$$

$$L(k) = \frac{-9i}{2k^3+5k+9i}, \quad N = 1, \quad \tau_1 = 2.14444\overline{1}\, i, \quad c_{\text{r}1} = 0.63118\overline{2},$$

where the overline indicates a roundoff at the digit. Finally, we obtain the potential and the Jost solution

$$V(x) = \frac{24e^{-2x} - 480e^{-4x} + 720e^{-6x} - 480e^{-8x} + 600e^{-10x}}{(1-3e^{-2x}+15e^{-4x}-5e^{-6x})^2},$$

$$f(k,x) = e^{ikx}\left[1 + \frac{\dfrac{6i(e^{-2x}-5e^{-6x})}{k+i} + \dfrac{60i(-e^{-4x}+e^{-6x})}{k+2i}}{1-3e^{-2x}+15e^{-4x}-5e^{-6x}}\right].$$

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Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019, USA

E-mail address: aktosun@uta.edu

Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20-726, IIMAS-UNAM, México DF 01000, México

E-mail address: weder@servidor.unam.mx