Inverse Scattering Theory and Transmission Eigenvalues

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Anisotropic Media - The Direct Problem

We now are concerned with the scattering problem for anisotropic media, i.e to find v and u^s such that

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{1}$$

$$\nabla \cdot A\nabla v + k^2 n v = 0 \qquad \text{in } D \tag{2}$$

$$v - u^s = u^i \qquad \text{on } \partial D \tag{3}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \qquad \text{on } \partial D \tag{4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{5}$$

where A is a 3×3 matrix with C^1 entries, A is real valued, symmetric and satisfies $\overline{\xi} \cdot A\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}$, $n(x)$ is continuous such that $n(x) > 0$ for all $x \in \overline{D}$, $\partial v/\partial \nu_A := \nu \cdot A\nabla u$ and $u^i(x) = e^{ikx\cdot d}$.

We want to established the existence of a unique solution to $(1) - (5)$.

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Definition

Let X be a Hilbert space. A mapping $a(\cdot,\cdot) : X \times X \to \mathbb{C}$ is called a sesquilinear form if

$$a(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v)$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$, $u_1, u_2 \in X$ and

$$a(u, \mu_1 v_1 + \mu_2 v_2) = \overline{\mu}_1 a(u, v_1) + \overline{\mu}_2 a(u, v_2)$$

for all $\mu_1, \mu_2 \in \mathbb{C}$, $v_1, v_2 \in X$.

Definition

A mapping $F : X \to \mathbb{C}$ is called a conjugate linear functional if

$$F(\mu_1 v_1 + \mu_2 v_2) = \overline{\mu}_1 F(v_1) + \overline{\mu}_2 F(v_2)$$

for all $\mu_1, \mu_2 \in \mathbb{C}$, $v_1, v_2 \in X$.

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Lax-Milgram Lemma

Assume that $a : X \times X \to \mathbb{C}$ is a sesquilinear form for which there exist constants $\alpha, \beta > 0$ such that

1. $|a(u, v)| \leq \alpha \|u\| \|v\| \qquad$ for all $u, v \in X$
2. $|a(u, u)| \geq \beta \|u\|^2 \qquad$ for all $u \in X$. $\qquad$ (6)

Then for every bounded conjugate linear functional $F : X \to \mathbb{C}$ there exists a unique element $u \in X$ such that

$$a(u, v) = F(v)$$

for all $v \in X$. Furthermore

$$\|u\|_X \leq \frac{1}{\beta} \|F\|_{X'}$$

where $\|F\|_{X'} = \sup_{v \in X, v \neq 0} \frac{|F(v)|}{\|v\|_X}$.

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Remark

A sesquilinear form satisfying (6) is said to be strictly coercive.

Definition

The Dirichlet-to-Neuman map T is defined by

$$T : w = \frac{\partial w}{\partial \nu} \qquad \text{on } S_R$$

where w is a radiating solution to the Helmholtz equation $\Delta w + k^2 w = 0$, S_R is the boundary of some ball $B_R := \{x : |x| < R\}$ and ν is the unit outward normal to S_R.

From the definition T maps

$$w = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m \qquad \text{onto} \qquad Tw = \sum_{n=0}^{\infty} \gamma_n \sum_{m=-n}^{n} a_n^m Y_n^m$$

$$\text{where} \qquad \gamma_n := \frac{k h^{(1)'}(kR)}{h_n^{(1)}(kR)}, \qquad n = 0, 1, 2, \cdots$$

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Noting that spherical Hankel functions and their derivatives do not have real zeros since otherwise the Wronskian of $h_n^{(1)}$ and $h_n^{(2)}$ would vanish, we see that T is bijective. Furthermore, from Lecture 1 we can deduce that

$$c_1(n+1) \leq |\gamma_n| \leq c_2(n+1)$$

for all $n \geq 0$ and $0 < c_1 < c_2$.
This implies that $T : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ is bounded. If we define $T_0 : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ by

$$T_0 w = -\frac{1}{R}\sum_{n=0}^{\infty}(n+1)\sum_{m=-n}^{n} a_n^m Y_n^m$$

we clearly have that

$$-\int_{S_R} T_0 w \overline{w}\, ds = \frac{1}{R}\sum_{n=0}^{\infty}(n+1)\sum_{m=-n}^{n} |a_n^m|^2$$

with the integral to be understood as the duality pairing between $H^{1/2}(S_R)$ and $H^{-1/2}(S_R)$.

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This implies that

$$-\int_{S_R} T_0 w\overline{w}\, ds \geq c\|w\|^2_{H^{1/2}(S_R)}$$

for some constant $c > 0$, i.e $-T_0$ is strictly coercive. From the series expansion for $h_n^{(1)}$ we have that

$$\gamma_n = -\frac{n+1}{R}\left\{1 + O\left(\frac{1}{n}\right)\right\}, \qquad n \to \infty$$

which implies that

$$T - T_0 : H^{1/2}(S_R) \to H^{-1/2}(S_R)$$

is compact since it is bounded from $H^{1/2}(S_R)$ into $H^{1/2}(S_R)$ and the imbedding from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$ is compact.

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Setting $f = u^i$, $h = \partial u^i/\partial\nu$ and $u = u^s$ we can now replace the scattering problem $(1) - (5)$ by an equivalent problem for a bounded domain: Find $u \in H^1(B_R \setminus \overline{D})$ and $v \in H^1(D)$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in } B_R \setminus \overline{D} \tag{7}$$

$$\nabla \cdot A\nabla v + k^2 n v = 0 \qquad \text{in } D \tag{8}$$

$$v - u = f \qquad \text{on } \partial D \tag{9}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \qquad \text{on } \partial D \tag{10}$$

$$\frac{\partial u}{\partial \nu} = Tu \qquad \text{on } S_R. \tag{11}$$

We now let $u_f \in H^1(B_R \setminus \overline{D})$ be the unique solution of

$$\Delta u_f + k^2 u_f = 0 \qquad \text{in } B_R \setminus \overline{D}$$

$$u_f = f \;\text{ on } \partial D \qquad \text{and} \qquad u_f = 0 \;\text{ on } S_R$$

and assume without loss of generality that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in $B_R \setminus \overline{D}$.

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Multiplying (7) and (8) by a test function $\varphi \in H^1(B_R)$ and using Green's first identity we obtain the following equivalent variational formulation of the scattering problem (1)-(5): Find $w \in H^1(B_R)$ such that

$$a_1(w,\varphi) + a_2(w,\varphi) = F(\varphi) \qquad \text{for all } \ \varphi \in H^1(B_R)$$

where

$$\begin{aligned} a_1(\psi,\varphi) : &= \int\limits_D \left(\nabla\overline{\varphi}\cdot A\nabla\psi + \overline{\varphi}\psi\right)\, dx + \int\limits_{B_R\setminus D} \left(\nabla\overline{\varphi}\cdot \nabla\psi + \overline{\varphi}\psi\right)\, dx \\ &- \int\limits_{S_R} \overline{\varphi}T_0\psi\, ds \end{aligned}$$

$$a_2(\psi,\varphi) := -\int\limits_D (nk^2+1)\overline{\varphi}\psi\, dx - \int\limits_{B_R\setminus D} (k^2+1)\overline{\varphi}\psi\, dx - \int\limits_{S_R} \overline{\varphi}(T-T_0)\psi\, ds.$$

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$$F(\varphi) := \int_{\partial D} \overline{\varphi} h \, ds + \int_{B_R \setminus D} \left(\nabla \overline{\varphi} \cdot \nabla u_f - k^2 \overline{\varphi} u_f \right) \, dx.$$

Then $v := w|_D$ and $u := w|_{B_R \setminus \overline{D}} - u_f$ is a solution to $(7) - (11)$.

Theorem

There exists a unique solution to the scattering problem $(1) - (5)$.

As in the isotropic case we have that

$$u^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|} \right) \right\}$$

and can define the far field operator $F : L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) \, ds(d).$$

Theorem

The far field operator corresponding to the scattering problem $(1) - (5)$ is normal.