Inverse Scattering Theory and Transmission Eigenvalues

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The Far Field Operator

Now let $u^i(x) = e^{ikx\cdot d}$, $|d| = 1$, and consider the scattering problem

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1}$$

$$u(x) = u^i(x) + u^s(x) \tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{3}$$

with corresponding far field pattern $u_\infty(\hat{x}) = u_\infty(\hat{x}, d)$ defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right)\right\}.$$

Reciprocity Principle

Let $u_\infty(\hat{x}, d)$ be the far field pattern corresponding to (1) – (3). Then

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}).$$

The Far Field Operator

Proof: Let $D \subset \{x : |x| < a\}$ where $D := \{x : m(x) \neq 0\}$. We now use Green's second identity to obtain

$$0 = \int\limits_{|y|=a} \left\{ u^i(y,d) \frac{\partial}{\partial \nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial \nu} u^i(y,d) \right\} ds(y)$$

$$0 = \int\limits_{|y|=a} \left\{ u^s(y,d) \frac{\partial}{\partial \nu} u^s(y,-\hat{x}) - u^s(y,-\hat{x}) \frac{\partial}{\partial \nu} u^s(y,d) \right\} ds(y).$$

The second corollary to Green's formula in Lecture 2 shows that

$$4\pi u_\infty(\hat{x},d) = \int\limits_{|y|=a} \left\{ u^s(y,d) \frac{\partial}{\partial \nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial \nu} u^s(y,d) \right\} ds(y)$$

$$4\pi u_\infty(-d,-\hat{x}) = \int\limits_{|y|=a} \left\{ u^s(y,-\hat{x}) \frac{\partial}{\partial \nu} u^i(y,d) - u^i(y,d) \frac{\partial}{\partial \nu} u^s(y,-\hat{x}) \right\} ds(y).$$

The Far Field Operator

Subtracting the last of the above equations from the sum of the first three gives

$$4\pi\,[u_\infty(\hat{x}, d) - 4\pi u_\infty(-d, -\hat{x})]$$

$$= \int\limits_{|y|=a} \left\{ u(y, d)\frac{\partial}{\partial\nu}u(y, -\hat{x}) - u(y, -\hat{x})\frac{\partial}{\partial\nu}u(y, d) \right\} ds(y)$$

$$= 0$$

by Green's second identity.

The Far Field Operator

We now define the far field operator $F : L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) ds(d).$$

Since $u_\infty(\hat{x}, d)$ is infinitely differentiable with respect to each of its variables, F is clearly compact.

The corresponding scattering operator $S : L^2(S^2) \to L^2(S^2)$ is defined by

$$S := I + \frac{ik}{4\pi} F.$$

The Far Field Operator

Lemma

For $g, h \in L^2(S^2)$ define the Herglotz wave functions v^i and w^i with kernels g and h respectively by

$$v^i(x) := \int_{S^2} e^{ikx\cdot d} g(d) ds(d) \qquad w^i(x) := \int_{S^2} e^{ikx\cdot d} h(d) ds(d).$$

Let v and w be the solutions of the scattering problem (1) – (3) corresponding to the incident fields v^i and w^i respectively. Then

$$0 = 2\pi\,(Fg, h) - 2\pi\,(g, Fh) - ik\,(Fg, Fh)\,.$$

Theorem

The far field operator f is normal, i.e. $F^*F = FF^*$, and the scattering operator S is unitary, i.e. $SS^* = S^*S = I$.

The Far Field Operator

We now introduce the transmission eigenvalue problem: Determine $k > 0$ and $v, w \in L^2(D)$, $v - w \in H_0^2(D)$ such that v and w are not identically zero and

$$
\begin{aligned}
\Delta w + k^2 n w = 0 \qquad & \text{in} \quad D \\
\Delta v + k^2 v = 0 \qquad & \text{in} \quad D \\
w = v \qquad & \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad & \text{on} \quad \partial D
\end{aligned}
$$

Such values of k are called transmission eigenvalues.
Recall that $D := \{x : n(x) \neq 1\}$ and it is assumed that D is bounded with C^2 boundary ∂D such that $\mathbb{R}^3 \setminus \overline{D}$ is connected.

Theorem

Let F be the far field operator corresponding to the scattering problem $(1) - (3)$. Then F is injective if k is not a transmission eigenvalue

The Far Field Operator

Proof: Suppose $Fg = 0$. Then the far field pattern w_∞ of the scattered field w^s corresponding to the incident field

$$w^i(x) = \int_{S^2} e^{ikx\cdot d} g(d) ds(d)$$

vanishes. By Rellich's lemma $w^s = w - w^i$ vanishes outside D. Then $w = w^s + w^i$ satisfies $\Delta w + k^2 n w = 0$ in $\mathbb{R}^3$ and $w - w^i = 0$ on ∂D, $\frac{\partial}{\partial \nu}(w - w^i) = 0$ on ∂D. If k is not a transmission eigenvalue then $w^i = w = 0$ which implies $g = 0$, i.e. F is injective.

Corollary

Let F be the far field operator corresponding to the scattering problem (1) – (3). Then F has dense range if k is not a transmission eigenvalue.

The Far Field Operator

Proof: From $(Fg, h) = (g, F^*h)$ we have $R(F)^\perp = N(F^*)$.
Hence we must show that $F^*h = 0 \Longrightarrow h = 0$.

To this end, using reciprocity, we have that $F^*h = 0$

$$\Longrightarrow \int_{S^2} \overline{u_\infty(d, \hat{x})} h(d) ds(d) = 0$$

$$\Longrightarrow \int_{S^2} u_\infty(-\hat{x}, -d) \overline{h(d)} ds(d) = 0$$

$$\Longrightarrow \int_{S^2} u_\infty(\hat{x}, d) \overline{h(-d)} ds(d) = 0$$

$$\Longrightarrow h = 0 \quad \text{by the previous theorem.}$$