Inverse Scattering Theory and Transmission Eigenvalues

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Scattering by an Inhomogeneous Medium

We consider the propagation of sound waves of small amplitude in $\mathbb{R}^3$ viewed as a problem in fluid dynamics. Let $v(x,t)$, $x \in \mathbb{R}^3$, be the velocity potential of a fluid particle in an inviscid fluid and

$$p(x,t) = \text{pressure}, \ \rho(x,t) = \text{density}, \ S(x,t) = \text{specific entropy}.$$

Then, if there are no external forces, we have

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v + \frac{1}{\rho}\nabla p = 0 \qquad \text{(Euler's equation)}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \qquad \text{(equation of continuity)} \tag{1}$$

$$p = f(\rho, S) \qquad \text{(equation of state)}$$

$$\frac{\partial S}{\partial t} + v \cdot \nabla S = 0 \qquad \text{(adiabatic hypothesis)}$$

where f is a function depending on the fluid.

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Assuming $v(x,t)$, $p(x,t)$, $\rho(x,t)$ and $S(x,t)$ are small, we perturb around the static case $v=0$, $p=p_0=$constant, $\rho=\rho_0(x)$, $S=S_0(x)$ with $p_0=f(\rho_0,S_0)$:

$$v(x,t)=\epsilon v_1(x,t)+O(\epsilon^2), \qquad p(x,t)=p_0+\epsilon p_1(x,t)+O(\epsilon^2), \tag{2}$$

$$\rho(x,t)=\rho_0(x)+\epsilon\rho_1(x,t)+O(\epsilon^2), \quad S(x,t)=S_0(x)+\epsilon S_1(x,t)+O(\epsilon^2),$$

where $0<\epsilon<<1$. Substituting (2) into (1) implies that

$$\frac{\partial v_1}{\partial t}+\frac{1}{\rho_0}\nabla p_1=0, \qquad \frac{\partial \rho_1}{\partial t}+\nabla\left(\rho_0 v_1\right)=0$$

$$\frac{\partial p_1}{\partial t}+c^2(x)\left(\frac{\partial \rho_1}{\partial t}+v_1\cdot\nabla\rho_0\right)$$

where the sound speed c is defined by

$$c^2(x)=\frac{\partial}{\partial\rho}f(\rho_0(x),S_0(x)).$$

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We now have that

$$\frac{\partial^2 p_1}{\partial t^2} = c^2(x)\rho_0(x)\nabla\left(\frac{1}{\rho_0(x)}\nabla p_1\right).$$

If $p_1(x,t) = \text{Re}\left\{u(x)e^{-i\omega t}\right\}$ we have that u satisfies

$$\rho_0(x)\nabla\left(\frac{1}{\rho_0(x)}\nabla p_1\right) + \frac{\omega^2}{c^2(x)}u = 0.$$

Making the further assumption that $\nabla\rho_0$ can be ignored, we arrive at

$$\Delta u + \frac{\omega^2}{c^2(x)}u = 0 \qquad (3)$$

We now assume that the slowly varying inhomogeneous medium is of compact support and is imbedded in $\mathbb{R}^3$ where the sound speed is $c(x) = c_0$ =constant.

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We further assume that the wave motion is caused by an incident field u^i satisfying (3) with $c(x) = c_0$. We then arrive at the scattering problem of determining u such that

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{4}$$

$$u = u^i + u^s \tag{5}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{6}$$

where $n(x) = 1$ outside the inhomogeneous medium, $k = \omega/c_0 > 0$ is the wave number, u^i is an entire solution of the Helmholtz equation $\Delta u + k^2 u = 0$ and u^s is the scattered field. The function $n(x)$ is called the refractive index.

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We now assume that $n(x)$ is piecewise continuous such that $n(x) > 0$ and set

$$m := 1 - n.$$

Let

$$D := \left\{ x \in \mathbb{R}^3 : \; m(x) \neq 0 \right\}.$$

We again let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \qquad x \neq y.$$

Theorem

Given two bounded domains D and G, the volume potential

$$(V\varphi)(x) := \int_D \Phi(x, y)\varphi(y)dy, \qquad x \in \mathbb{R}^3$$

defines a bounded operator $V : L^2(D) \to H^2(G)$.

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We now show that the scattering problem $(4) - (6)$ is equivalent to solving the Lippmann-Schwinger integral equation

$$u(x) = u^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy, \qquad x \in \mathbb{R}^3. \tag{7}$$

Theorem

If $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution of $(4) - (6)$ then u is a solution of (7). Conversely, if $u \in C(\mathbb{R}^3)$ is a solution of (7) then $u \in H^2_{loc}(\mathbb{R}^3)$ and u is a solution of $(4) - (6)$.

Theorem

Suppose that $m(x) = 0$ for $|x| \geq a$ and $k^2 < 2/Ma^2$ where $M = \sup_{|x| \leq a} |m(x)|$. Then there exists a unique solution to the Lippmann-Schwinger integral equation

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From (7) we see that

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) u(y)\, dy, \qquad x \in \mathbb{R}^3$$

and hence

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty$$

where the far field pattern u_∞ is given by

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u(y)\, dy, \qquad \hat{x} = \frac{x}{|x|}.$$

Assuming k is sufficiently small and replacing u by the first term in solving (7) by iteration (the weak scattering assumption) gives the Born approximation

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u^i(y)\, dy.$$

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Unique Continuation Principle

Let G be a domain in $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution of

$$\Delta u + k^2 n(x) u = 0$$

in G such that n is piecewise continuous in G and u vanishes in a neighborhood of some $x_0 \in G$. Then u is identically zero in G.

Theorem

For each $k > 0$ there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ to the scattering problem $(4) - (6)$ and u depends continuously with respect to the maximum norm on the incident field u^i.