Inverse Scattering Theory and Transmission Eigenvalues

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The Helmholtz Equation

We look for solutions of the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad k > 0$$

in the form

$$u(x) = f(k|x|)Y_n^m(\hat{x})$$

where $x \in \mathbb{R}^3$, $\hat{x} = x/|x|$, Y_n^m is a spherical harmonic

$$Y_n^m(\theta, \varphi) := \left(\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} \right)^{1/2} P_n^m(\cos\theta) e^{im\varphi}$$

$m = -n, \cdots, n$, $n = 0, 1, 2, \cdots$, (θ, φ) are the spherical angles of $\hat{x}$ and P_n^m is an associated Legendre polynomial.

Note that $\{Y_n^m\}$ is a complete orthonormal system in $L^2(S^2)$ where $Y_0^0 = \dfrac{1}{\sqrt{4\pi}}$ and

$$S^2 := \{x : |x| = 1\}.$$

The Helmholtz Equation

The function f is a solution of the spherical Bessel equation

$$t^2 f''(t) + 2tf'(t) + \left[t^2 - n(n+1)\right] f(t) = 0$$

with two linearly independent solutions

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)}$$

$$y_n(t) := \frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n+1)(-2n+3)\cdots(-2n+2p-1)}$$

called, respectively, the spherical Bessel function and the spherical Neumann function. The functions

$$h_n^{(1)}(t) := j_n(t) + iy_n(t) \qquad h_n^{(2)}(t) := j_n(t) - iy_n(t)$$

are called spherical Hankel functions of order n.

The Helmholtz Equation

From the series representation of j_n and y_n we have that for $f_n = j_n$ or $f_n = y_n$ we have that

$$f_{n+1}(t) = -t^n \frac{d}{dt}\left\{t^{-n} f_n(t)\right\}$$

for $n = 0, 1, 2, \cdots$ and

$$h_0^{(1,2)}(t) = \frac{e^{\pm it}}{\mp it}.$$

From this we see that the spherical Hankel functions have the asymptotic behavior

$$h_n^{(1,2)}(t) = \frac{1}{t} e^{\pm i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}$$

as $t \to \infty$. In particular, $h_n^{(1)}(kr)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0,$$

i.e. if $u(x) = h_n^{(1)}(kr) Y_n^m(\hat{x})$ then $u(x)e^{-i\omega t}$ (where ω is the frequency and t is time) is an outgoing wave.

The Helmholtz Equation

Solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition uniformly in $\hat{x}$ are called radiating.
The Wronskian of $h_n^{(1)}(t)$ and $h_n^{(2)}(t)$ is given by

$$W\left(h_n^{(1)}(t), h_n^{(2)}(t)\right) := h_n^{(1)}(t)h_n^{(2)'}(t) - h_n^{(2)}(t)h_n^{(1)'}(t) = -\frac{2i}{t^2}.$$

Now let D be a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and assume that ∂D is of class C^2 with unit normal ν directed into the exterior of D. Let

$$\Phi(x, y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \qquad x \neq y$$

be the radiating fundamental solution to the Helmholtz equation.

The Helmholtz Equation

Using Green's second identity

$$\int (u\Delta v - v\Delta u)\,dx = \int_D \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds$$

we can deduce Green's formula for functions $u \in C^2(D) \cap C^1(\overline{D})$:

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}\Phi(x,y) - u\frac{\partial}{\nu_y}\Phi(x,y) \right\} ds$$

$$- \int_D \left\{ \Delta u + k^2 u \right\} \Phi(x,y)dy, \quad x \in D.$$

Theorem

Let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution to the Helmholtz equation in D. Then u is real analytic in D.

The Helmholtz Equation

Holmgren's Theorem

Let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution to the Helmholtz equation in D such that

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \ \Gamma$$

for some open subset $\Gamma \subset \partial D$. Then u is identically zero in D.

For x in the exterior of D we have the following theorem:

Theorem

Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ be a radiating solution to

$$\text{the Helmholtz equation} \qquad \Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}.$$

Then we have Green's formula

$$u(x) = \int_{\partial D} \left\{ u \frac{\partial}{\nu_y} \Phi(x,y) - \frac{\partial u}{\partial \nu} \Phi(x,y) \right\} ds, \qquad x \in \mathbb{R}^3 \setminus \overline{D}.$$

The Helmholtz Equation

Corollary

An entire solution to the Helmholtz equation satisfying the radiation condition must vanish identically.

Corollary

Every radiating solution to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave

$$u(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \qquad |x| \to \infty$$

uniformly in all directions $\hat{x} = x/|x|$.

The function u_∞ defined on the unit sphere S^2 is called the far field pattern of u.

The Helmholtz Equation

Rellich's Lemma

Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D})$ be a solution to the Helmholtz equation satisfying

$$\lim_{r \to \infty} \int\limits_{|x|=r} |u(x)|^2 \, dx = 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Corollary

Assume $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus \overline{D})$ is a radiating solution to the Helmholtz equation such that

$$\operatorname{Im} \int\limits_{\partial D} u \frac{\partial \overline{u}}{\partial \nu} \, ds = 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.