Inverse Scattering Theory and Transmission Eigenvalues

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We assume that $D$ (or a reconstruction of $D$) is available.

Then we will show that real transmission eigenvalues can be determined from a knowledge of $u_\infty(\hat{x}, d, k)$ for $\hat{x} \in S^2$, $d \in S^2$ and $k \in [\kappa_0, \kappa_1]$.

Let us consider the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad z \in D$$

where $$(Fg)(\hat{x}) = \int_{S^2} u_\infty(\hat{x}, d, k)g(d)ds_d$$
Theorem
Let \( z \in D \) and assume that \( k \) is not a transmission eigenvalue. Then for every \( \epsilon > 0 \) there exists an approximate solution \( g^\epsilon_z \in L^2(S^2) \)

\[
\|(Fg^\epsilon_z)(\hat{x}) - \Phi_\infty(\hat{x}, z)\|_{L^2(S^2)} < \epsilon
\]
such that the corresponding Herglotz function

\[
v_{g^\epsilon_z}(x) := \int_{S^2} e^{ikx \cdot d} g^\epsilon_z(d) \, ds_d
\]
converges in the \( L^2(D) \) norm to \( v_z \in L^2(D) \) as \( \epsilon \to 0 \) where

\[
\Delta w_z + k^2 n(x) w_z = 0 \quad \text{in} \quad D
\]

\[
\Delta v_z + k^2 v_z = 0 \quad \text{in} \quad D
\]

\[
w_z - v_z = \Phi(\cdot, z) \quad \text{on} \quad \partial D
\]

\[
\frac{\partial w_z}{\partial n} - \frac{\partial v_z}{\partial n} = \frac{\partial \Phi(\cdot, z)}{\partial n} \quad \text{on} \quad \partial D
\]
Theorem

Assume that $k$ is a transmission eigenvalue. Let $z \in D$ and $g_z^\epsilon \in L^2(S^2)$ be an approximate solution of

$$\|(Fg_z^\epsilon)(\hat{x}) - \Phi(\hat{x}, z)\|_{L^2(S^2)} < \epsilon.$$ 

Then except for a set of $z \in D$ without accumulation point, $\|v_{g_z^\epsilon}\|_{L^2(D)}$ can not be bounded as $\epsilon \to 0$.

Proof. On the board.

Remark

In both theorems we have assumed that $0 < n^* < 1$ or $n^* > 1$. Note that the result of these theorems also hold as long as the inhomogeneous media is such that the corresponding interior transmission problem satisfies the Fredholm alternative.
Transmission Eigenvalues

If \( \theta_z \in H^2(D) \) is such that \( \theta = \Phi(\cdot, z) \) and \( \partial \theta_z / \partial \nu = \partial \Phi(\cdot, z) / \partial \nu \), in the ITP for \( u_z = w_z - v_z - \theta_z \in H^2_0(D) \) reads

\[
a(u_z, \varphi) = a(\theta_z, \varphi) \quad \text{for all } \varphi \in H^2_0(D)
\]

\[
a(u_z, \varphi) := \int_D \frac{1}{n-1} (\Delta u_z + k^2 n u_z)(\Delta \varphi + k^2 \varphi) \, dx.
\]

But \( k^2 \) is a transmission eigenvalue, hence ITP has solution if and only if

\[
a(\theta_z, u_0) = 0,
\]

where \( u_0 \) is an eigenfunction.

Integrating by parts gives

\[
\int_{\partial D} \psi(x) \frac{\partial \Phi(\cdot, z)}{\partial \nu} - \Phi(\cdot, z) \frac{\partial \psi(x)}{\partial \nu} \, ds = 0
\]

where \( \psi(x) := \frac{1}{n-1} (\Delta u_0 + k^2 n u_0) \).
Remark

It can be shown that both the above theorems hold true for the Tikhonov regularized solution $g_\delta$ of the far field equation which is the minimizer of the Tikhonov functional

$$\| F_\delta g - \Phi_\infty (\cdot, z) \|^2_{L^2(\Omega)} + \epsilon \| g \|^2_{L^2(\Omega)}.$$

Here $F_\delta$ is the far field operator corresponding to noisy measurements $u_\infty^\delta (\hat{x}, d, k)$ with noise level $\delta > 0$ and the regularization parameter $\epsilon := \epsilon(\delta)$ satisfies $\epsilon(\delta) \to 0$ as $\delta \to 0$.


An alternative way to characterize transmission eigenvalues in terms of the far field operator $F$

Computation of Real TE

averaged over 81 sources

Wave number $k$

$L^2$ norm of $g$
The transmission eigenvalue problem for the anisotropic media is to find $v, w \in H^1(D)$ such that

$$
\nabla \cdot A \nabla w + k^2 w = 0 \quad \text{in} \quad D \\
\Delta v + k^2 v = 0 \quad \text{in} \quad D \\
w = v \quad \text{on} \quad \partial D \\
\nu \cdot A \nabla w = \nu \cdot \nabla v \quad \text{on} \quad \partial D.
$$

Here $A$ is a $3 \times 3$ symmetric invertible matrix, with bounded real-valued entries.

Denote

$$A_* := \inf_{x \in D} \inf_{|\xi|=1} (\xi \cdot A(x) \bar{\xi}) > 0, \quad A^* := \sup_{x \in D} \sup_{|\xi|=1} (\xi \cdot A(x) \bar{\xi}) < \infty$$
We introduce the new unknown vector valued functions:

\[ \mathbf{w} = A \nabla \mathbf{w} \in L^2(D) \quad \text{and} \quad \mathbf{v} = \nabla \mathbf{v} \in L^2(D). \]

Setting \( N := A^{-1} \) and \( k^2 := \tau \), the above problem becomes

\[
\nabla (\nabla \cdot \mathbf{w}) + \tau N \mathbf{w} = 0 \quad \text{in} \quad D \\
\nabla (\nabla \cdot \mathbf{v}) + \tau \mathbf{v} = 0 \quad \text{in} \quad D \\
\nu \cdot \mathbf{w} = \nu \cdot \mathbf{v} \quad \text{on} \quad \partial D \\
\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v} \quad \text{on} \quad \partial D
\]

or if we let \( \mathbf{u} = \mathbf{w} - \mathbf{v} \)

\[
(\nabla \nabla \cdot + \tau)(N - I)^{-1}(\nabla \nabla \cdot + \tau N)\mathbf{u} = 0 \quad \text{in} \quad D
\]

for \( \mathbf{u} \in \mathcal{H}_0(D) \) where

\[
\mathcal{H}_0(D) := \left\{ \mathbf{u} \in H_0(\text{div}, D), \text{ such that } \nabla \cdot \mathbf{u} \in H_0^1(D) \right\}
\]

\[
H_0(\text{div}, D) := \left\{ \mathbf{u} \in L^2(D), \nabla \cdot \mathbf{u} \in L^2(D), \nu \cdot \mathbf{u} = 0 \text{ on } \partial D \right\}
\]
Assuming that $A_\ast > 1$ and $0 < A^* < 1$ we can apply the analysis for the case of isotropic media to the eigenvalue problem

$$(A_\tau - \tau B) u = 0, \quad u \in H_0(D)$$

where

$$(A_\tau u, \varphi) := \int_D (N - I)^{-1}(\nabla \nabla \cdot u + \tau u)(\nabla \nabla \cdot \varphi + \tau \varphi)dx + \tau^2 \int_D u \cdot \varphi dx$$

$$(B_\tau u, \varphi) := \int_D (\nabla \cdot u \nabla \cdot \varphi)dx.$$

**Theorem**

Assume that $0 < A^* < 1$. Then, there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$. Furthermore

$$\tau_1, A^*, D_1 \leq \tau_1, A^*, D \leq \tau_1, A(x), D \leq \tau_1, A_\ast, D \leq \tau_1, A_\ast, D_2.$$ 

where $D_2 \subset D \subset D_1$. 
Transmission Eigenvalues

For a given (unknown) anisotropic media $A$, we find an isotropic homogenous media $a_0$ that has the first transmission eigenvalue the same as the (measured) first transmission eigenvalue for the anisotropic media. Monotonicity properties gives that this $a_0$ is between $A_\ast$ and $A^\ast$.

**Numerical Example:** We consider $D := [-1, 1] \times [-1, 1]$.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\tau_1$</th>
<th>Predicted $a_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag(5.5,6.5)</td>
<td>1.9657</td>
<td>5.95</td>
</tr>
<tr>
<td>diad(5,7)</td>
<td>1.9696</td>
<td>5.79</td>
</tr>
<tr>
<td>diag(6,6.5)</td>
<td>1.9591</td>
<td>6.24</td>
</tr>
<tr>
<td>diag(6,7)</td>
<td>1.9547</td>
<td>6.45</td>
</tr>
</tbody>
</table>
Shape Reconstruction

252 directions, $\pi/k$ ($ka$) are 0.224 (12), 0.112 (24) and 0.075 (42)
Examples of Reconstruction

$\lambda = 1$, wavelength $= 0.7$
Examples of Reconstruction

$N = 16$, $k$ is not TE
Examples of Reconstruction

$N = 16$, $k$ is a TE

$xy$ cut of $g$ for $ka = 1.470000$

$g$ isosurf for $ka = 1.470000$

$1/|g|$, $k = 1.47$
Limited Aperture

Reconstruction of a square.
The aperture

Reconstruction of the coated sphere with $\lambda = 0.1$ with limited aperture data; $k = 3$
The aperture

Reconstruction of the coated sphere with \( \lambda = 0.1 \) with limited aperture data; \( k = 3 \)
The aperture

Reconstruction of the same sphere using the intersection of two reconstructions from different single aperture
Multifrequency Data

Can multifrequency or time domain data help?

- **Multifrequency data** (Guzina, Cakoni, Bellis (2010))
  Domain indicator function is defined as
  \[
  \psi(z) = \int_{k_{\text{min}}}^{k_{\text{max}}} \frac{1}{\|g_{\delta}^{z,k}(\cdot)\|_{L^2(\Omega_0)}^2} dk.
  \]

- **Time domain linear sampling method**
  (Haddar, Lechleiter (2013), Colton, Guo, Monk (2013)).
  on-going research area
Limited aperture reconstructions with 1% added random noise.

The stars indicates the location and number of the transmitters/receivers.
The interface earth-air is at $z = 0$, two layered medium, absorbing background and 5% random noise.
Examples of Reconstruction

Reconstruction by using the near field Linear Sampling Method
Examples of Reconstruction

Ipwich data*

Target ips010: Plexiglass Triangle
Penetrable

Plot of $\frac{1}{\|g\|_{L^2(\Omega)}}$
Sampling region: 40 x 40 grid
Delta 0.22, $\lambda = 3$

*Measured data provided by:
Electromagnetics Technology Division, AFRL/SNH
31 Grenier Street
Hanscom AFB, MA 01731-3010