Brief notes on Fourier Transform
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P. Kuchment

Contents

1 Some linear algebra

Space $H = \mathbb{C}^n$ of vectors $f = (f^1, ..., f^n)$ with complex components.
Hermitian metric (dot product) in this space $(f, g) = \sum f^i \overline{g^i}$ (notice the complex conjugation!!)
Norm (length) on this space $||f||^2 = (f, f) = \sum |f^i|^2$.

Definition 1.1. *Let e_k, $k = 1, ..., n$ be any ortho-normal basis in H, then for any vector f its **Fourier coefficients** are*

$$f_k = (f, e_k). \tag{1}$$

*Mapping f into the set of its Fourier coefficients $\{f_k\}$ is **Fourier analysis***

Theorem 1.2. *(**Fourier synthesis**) For any vector f*

$$f = \sum f_k e_k. \tag{2}$$

Exercise 1.3.

- *Prove Theorem 1.2*
- *Prove that if the basis vectors e_k are not ortho-normal, but merely orthogonal, then keeping (1) one has to modify (2) as follows:*

$$f = \sum \frac{f_k}{||e_k||^2} e_k.$$

- *Prove:*

$$\|f\|^2 = \sum \frac{|f_k|^2}{\|e_k\|^2}$$

2 Fourier series

Although we consider here only functions of one variable, the considerations can be easily extended to Fourier series for periodic functions in several variables (e.g., [15]).

2.1 Fourier series expansions

Space $H = L_2[-\pi/h, \pi/h]$ of square integrable functions $f(x)$ on $[-\pi/h, \pi/h]$.
Inner product and Hermitian metric

$$(f, g) = \int_{-\pi/h}^{\pi/h} f(x)\overline{g(x)}dx, ||f||^2 = \int_{-\pi/h}^{\pi/h} |f(x)|^2 dx.$$

Consider the sequence of functions $e_k = \frac{1}{\sqrt{2\pi}} \exp(ikhx)$, $k = 0, \pm 1, \pm 2, ...$

Exercise 2.1. *Prove:*

$$(e_k, e_j) = \frac{1}{h}\delta_{ij},$$

where $\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{otherwise} \end{cases}$ *is the Kronecker's delta.*

Theorem 2.2. *(e.g., [9, Section 34]) Functions*

$$\sqrt{\frac{h}{2\pi}}e^{ikhx},\ k = 0, \pm 1, \pm 2, ...$$

form an ortho-normal basis of $L_2[-\pi/h, \pi/h]$.

Definition 2.3. *For function* $f(x) \in L_2[-\pi/h, \pi/h]$, ***Fourier coefficients*** *are*

$$f_k := (f, \frac{1}{\sqrt{2\pi}}e^{ikhx}) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\pi/h}^{\pi/h} f(x)e^{-ikhx}dx. \tag{3}$$

Fourier analysis*:* $f(x) \in L_2[-\pi/h, \pi/h] \mapsto \{f_k\}$.
Fourier synthesis $\{f_k\} \mapsto f(x)$ *(**Fourier series expansion**): for* $f(x) \in L_2[-\pi/h, \pi/h]$,

$$f(x) = \frac{h}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} f_k e^{ikhx} \tag{4}$$

(series converges in L_2*).*

Fourier series represents $f(x)$ on $[-\pi/h, \pi/h]$ (alternatively, a $2\pi/h$-periodic function on $\mathbb{R}$) as a sum of harmonic oscillations (sinusoidal waves).

Exercise 2.4. *Prove the **Parseval/Plancherel theorem***

$$\int\limits_{-\pi/h}^{\pi/h} |f(x)|^2\,dx = h\sum |f_k|^2.$$

Fourier series expansions are common when using polar coordinates: if $f(x)$ is a function on $\mathbb{R}^2$, then in polar coordinates (r, ϕ) it is a 2π-periodic function of the polar angle ϕ. Thus, Fourier series w.r.t. ϕ is $\sum_{k\in\mathbb{Z}} f_k(r)e^{ik\phi}$, which under appropriate conditions converges to f.

2.2 Properties of Fourier series expansions

Remark 2.5. *Functions* e^{ikhx} *with* $k \in \mathbb{Z}$ *are* $\frac{2\pi}{h}$*-periodic. Hence, it is natural to consider the sum in (4) as a* $\frac{2\pi}{h}$*-periodic function as well.*

Considering $f(x)$ as $\frac{2\pi}{h}$-periodic, we can discuss its values on the whole axis.

Exercise 2.6. *Prove that the Fourier coefficients of a $\frac{2\pi}{h}$-periodic function $f(x)$ can be computed as*

$$f_k = \frac{1}{\sqrt{2\pi}} \int\limits_a^{a+2\pi/h} f(x) e^{-ikhx} dx$$

*for **any** value of a.*

Theorem 2.7. *(e.g. [9, Sect. 34]) If $f \in L_2[-\pi/h, \pi/h]$ and $f_N(x)$ is the N-th partial sum of (4), then*

$$\int\limits_{-\pi/h}^{\pi/h} |f(x) - f_N(x)|^2 \, dx \underset{N\to\infty}{\to} 0.$$

Theorem 2.8. *(e.g., [9, Sect. 15-17], [18, Ch1, Sect. 10])*

1. *If $f(x) \in C[-\pi/h, \pi/h]$, has bounded first derivative and is periodic (i.e., $f(-\pi/h) = f(\pi/h)$), then*

$$\lim_{N\to\infty} f_N(x) = f(x)$$

for all $x \in [-\pi/h, \pi/h]$.

2. *If $f(x)$ is continuous on $[-\frac{\pi}{h}, \frac{\pi}{h}]$, except of a finite number of finite jump discontinuities, then*

$$\lim_{N\to\infty} f_N(x) = \begin{cases} \frac{f(x+0)+f(x-0)}{2} & \text{when } x \in (-\pi/h, \pi/h) \\ \frac{f(-\pi/h+0)+f(\pi/h-0)}{2} & \text{when } x = \pm\pi/h \end{cases}.$$

2.3 Smoothness of $f(x)$ ⇔ decay of Fourier coefficients f_k.

In the theorem below, the word "periodic" applied to a function f on $[-\frac{\pi}{h}, \frac{\pi}{h}]$ means $f(-\frac{\pi}{h}) = f(\frac{\pi}{h})$.

Theorem 2.9.

1. *If $f(x)$ is continuous (in fact, $f \in L_1$ suffices), then*

$$|f_k| \le const.$$

2. *If $f(x)$ is periodic and has a continuous first derivative, then*

$$|f_k| \le \text{const}(1 + |k|)^{-1}$$

3. *If $f(x)$ has n continuous derivatives and the first $n-1$ of them are periodic, then*

$$|f_k| \le \text{const}(1 + |k|)^{-n}.$$

4. *If*
$$|f_k| \le \mathrm{const}(1+|k|)^{-\alpha} \text{ for some } \alpha > 1,$$
then $f(x)$ is continuous and periodic.

5. *If*
$$|f_k| \le \mathrm{const}(1+|k|)^{-\alpha} \text{ for some } \alpha > 2,$$
then $f(x)$ is continuous, periodic, and once continuously differentiable.

6. *If*
$$|f_k| \le C_N(1+|k|)^{-N} \text{ for any } N > 0,$$
then $f(x)$ is infinitely differentiable, and all its derivatives are $2\pi/h$-periodic.

7. *If*
$$|f_k| \le Ce^{-A|k|} \text{ for some } A > 0,$$
then f is analytic.

Exercise 2.10. *Prove this theorem.*

2.4 Relations with shifts and derivatives

Functions on $[-\frac{\pi}{h}, \frac{\pi}{h}]$ will be extended $\frac{2\pi}{h}$-periodically to $\mathbb{R}$. Then we can **shift** (translate) them:
$$(T_t f)(x) := f(x+t).$$

Exercise 2.11.

- *Prove that for functions $e_k = e^{ikhx}$, one has $(T_t e_k)(x) = \lambda_{k,t} e_k(x)$ for some constant $\lambda_{k,t}$. Find $\lambda_{k,t}$.*

- *Prove that if a **continuous** function $e(x)$ on $\mathbb{R}$ satisfies $T_t e = \lambda_t e_k$ for all $t \in \mathbb{R}$ and some numbers λ_t, then $e(x) = Ce^{\mu x}$ for some $\mu \in \mathbb{C}$ and a constant C.*
Prove that if such $e(x)$ is $2\pi/h$ periodic, then $e(x) = Ce_k(x)$ for some $k \in \mathbb{Z}$.
These statements are not necessarily true if e is discontinuous.

- *Let B be a linear operator acting on functions of x, such that B commutes with shifts (i.e., $BT_t = T_t B$ for all $t \in \mathbb{R}$). If Be_k is continuous, then $Be_k = \beta_k e_k$ for some constant β_k, which is called the **Fourier multiplier** corresponding to B. In other words, B is diagonal in the basis $\{e_k\}$.*

- *Prove that $\frac{d^l}{dx^l}$ commutes with T_t for any t and find the corresponding Fourier multipliers.*

- *Check when the operator of multiplication by a given function $g(x)$ commutes with the shifts.*

So, if there is a linear transformation B commuting with all shifts T_t, then it has exponents as eigenvectors. I.e., $Be_k = \beta_k e_k$ for some numbers β_k depending on B. In particular, action of B on any function f is easy to write down in terms of the Fourier expansion: if $f(x) = \sum_k f_k e_k$, then $Bf = \sum_k \beta_k f_k e_k$. The common examples of such operations are differentiation and convolution (considered in the next section).

2.5 Convolution of periodic functions

Definition 2.12. *Convolution $f*g$ of two $\frac{2\pi}{h}$-periodic functions (say, belonging to $L_1[-\pi/h, \pi/h]$) is defined as*

$$f * g(x) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} f(y)g(x-y)dy.$$

Exercise 2.13.

1. *Prove that convolution is commutative.*
2. *Prove that in the definition of convolution one can integrate over arbitrary segment of the length of the period.*
3. *Prove that (under appropriate smoothness conditions)*

$$\frac{d}{dx}(f*g) = \left(\frac{df}{dx} * g\right) = \left(f * \frac{dg}{dx}\right)$$

4. *Prove that convolution commutes with shifts:*

$$T_t(f*g) = (T_t f * g)$$

5. *Prove that $(f*g)_k = \sqrt{2\pi} f_k g_k$*

2.6 Convolution on $\mathbb{R}^n$

Definition 2.14. *Convolution of two (sufficiently fast decaying, so the integral converges) functions on $\mathbb{R}^n$ is*

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Exercise 2.15. *Prove the following* ***properties of convolution****:*

1. ***Linearity****: $f(x) * (ag_1(x) + bg_2(x)) = a(f*g_1) + b(f*g_2)$*
2. ***Commutativity****: $f*g = g*f$.*
3. ***Commuting with shifts****: If $(T_a f)(x) := f(x+a)$, then $f * (T_a g) = T_a(f*g)$.*

4. ***Commuting with differentiation**: If $f \in L^1_{loc}(\mathbb{R}^n)$ and g is smooth and compactly supported, then $f * g$ is smooth and $\frac{d^l}{dx^l}(f * g) = (f * \frac{d^l g}{dx^l})$.*

5. *Convolution operation has **no unity** element, i.e., there is no function $i(x)$ such that $i * f = f$ for all functions f (make reasonable assumptions on the functions f in order for this to make sense). (If you know distributions (Section 4), there **is** a distribution, namely Dirac's δ-function (see Section 3.4.2), with this property.))*

One can make precise[1] the following statement (converse to statement 3 of the Exercise):

"Theorem": Any *linear operator A mapping functions of $x \in \mathbb{R}^n$ into functions of $x \in \mathbb{R}^n$ and commuting with shifts, i.e., $AT_a = T_a A$ for all $a \in \mathbb{R}^n$, is a convolution, i.e. $Af = f * g$ for some g.*

3 Fourier transform

Take $f(x)$ on $\mathbb{R}$, restrict it to $[-\pi/h, \pi/h]$ for some $h > 0$, expand into Fourier series (3), and take (formally) the limit when $h \to 0$ to get

Definition 3.1. *Fourier transform of $f(x)$ on $\mathbb{R}$ is defined as*

$$(\mathcal{F}f)(\xi) = \widetilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx. \tag{5}$$

In $\mathbb{R}^n$,

$$\widetilde{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} f(x) dx,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

This is well defined when $f(x)$ decays sufficiently fast, e.g. $f \in L_1(\mathbb{R}^n)$. If $f \in L_2(\mathbb{R}^n)$, then the definition should be carefully adjusted (e.g., [15]).

Theorem 3.2. *(Plancherel's Theorem/Parseval's identity). The identity holds:*

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{\mathbb{R}^n} \left|\widetilde{f}(\xi)\right|^2 d\xi. \tag{6}$$

I.e., operator $\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is isometric.

Exercise 3.3.

- *Prove that the adjoint operator to $\mathcal{F}$ is given by*

$$(\mathcal{F}^* g)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(\xi) e^{i\xi\cdot x} d\xi. \tag{7}$$

[1] considering objects more general than functions

- *Prove that the adjoint $\mathcal{F}^*$ to $\mathcal{F}$ is its inverse.* **Hint:** *Use isometric property of $\mathcal{F}$.*

We get the **Fourier inversion formula**:

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \widetilde{f}(\xi) e^{i\xi\cdot x} d\xi. \tag{8}$$

Exercise 3.4. *Show that this inversion formula for $n = 1$ can be obtained formally as a limit when $h \to 0$ from (4).*

Remark 3.5. *The Fourier inversion is almost the same as the direct Fourier transform, one only needs to flip the sign of the independent variable:*

$$\left(\mathcal{F}^{-1} g\right)(x) = (\mathcal{F} g)(-x).$$

Fourier transform $f \mapsto \widetilde{f}$ provides **Fourier analysis** of a function, i.e., finds the amplitudes with which different oscillating exponents enter the function. Fourier inversion $\widetilde{f} \mapsto f$ provides **Fourier synthesis**, synthesizing the function back from these amplitudes.

3.1 Properties of FT

Exercise 3.6. *Prove the following properties of the Fourier transform in $\mathbb{R}^n$:*

1. **Dilation invariance:** *Let $f_r(x) = f(rx)$. Then $\widetilde{f}_r(\xi) = r^{-n}\widetilde{f}(r^{-1}\xi)$*
2. **Shift invariance:** $\widetilde{T_y f}(\xi) = e^{i\xi\cdot y}\widetilde{f}(\xi)$.
3. **Rotational invariance:** *If A is a rotation in $\mathbb{R}^n$, then $\widetilde{f(Ax)}(\xi) = \widetilde{f(x)}(A\xi)$.*
4. $\widetilde{\frac{D^\alpha f}{dx^l}}(\xi) = (i\xi)^\alpha \widetilde{f}(\xi)$, *where $(\xi)^\alpha = (\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2}\dots(\xi_n)^{\alpha_n}$.*
5. *Find a formula for $\widetilde{x^l f}(\xi)$ in terms of $\widetilde{f}(\xi)$ (**Hint:** use the previous question and the remark above).*
6. $\widetilde{f * g} = (2\pi)^{n/2}\widetilde{f}\widetilde{g}$
7. $\widetilde{fg} = \frac{1}{(2\pi)^{n/2}}\widetilde{f} * \widetilde{g}$.
8. ***Parseval identity:*** $\int f\,\widetilde{g}dx = \int \widetilde{f}\,g dx$

3.2 Some common functions

Heaviside function

$$H(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases}$$

Box function

$$\Pi(x) = \begin{cases} 1 & \text{when } |x| \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Gaussian function

$$G(x) = e^{-\frac{x^2}{2}}$$

Cardinal sine function, or $\sin cx$

$$\sin cx = \begin{cases} \frac{\sin x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

Normal density with the mean μ and standard deviation σ

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Exercise 3.7. *Express the box function in terms of Heaviside function.*

Exercise 3.8. *The B-splines $\Pi_n(x)$ are defined as $\Pi_n(x) = \underbrace{\Pi(x) * ... * \Pi(x)}_{n \text{ times}}$.*

1. *Find and graph $\Pi_1(x)$, $\Pi_2(x)$, and $\Pi_3(x)$*
2. *Determine the support of $\Pi_n(x)$*

Exercise 3.9. *Find the Fourier transform of the box function.*

Fourier transform of the Gaussian. The Gaussian $G(x)$ satisfies the differential equation

$$\frac{dG}{dx} + xG = 0$$

(check this).

Exercise 3.10.

- *Prove that the Fourier transform of the Gaussian satisfies the same equation:*

$$\frac{d\widetilde{G}}{d\xi} + \xi\widetilde{G} = 0. \tag{9}$$

- *Prove that the Fourier transform of the Gaussian is the Gaussian:*

$$\widetilde{G}(\xi) = e^{-\frac{\xi^2}{2}}$$

Hint: *Use (9) and Plancherel's theorem.*

3.3 Paley-Wiener theorem

Several versions of the **Paley-Wiener theorem**, which describe Fourier transforms of various classes of functions, are combined here ($\Im\xi$ denotes the imaginary part of a complex vector ξ):

Theorem 3.11.

1. *Fourier transform $\widetilde{f}(\xi)$ of a function $f \in C_0^\infty(\mathbb{R}^n)$ supported in the ball $\{x \mid |x| \leq A\}$ is an entire function in $\mathbb{C}^n$ and satisfies for any $N > 0$ the estimate:*
$$|\widetilde{f}(\xi)| \leq C_N(1+|\xi|)^{-N} e^{A|\Im\xi|}. \tag{10}$$
The converse statement also holds: any entire function with such estimates is the Fourier transform of a smooth function supported in that ball.

2. *Fourier transform $\widetilde{f}(\xi)$ of a function $f \in L_2(\mathbb{R}^n)$ supported in the ball $\{x \mid |x| \leq A\}$ is an entire function in $\mathbb{C}^n$, which is square integrable along $\mathbb{R}^n$ and satisfies in $\mathbb{C}^n$ the estimate*
$$|\widetilde{f}(\xi)| \leq C e^{A|\xi|}. \tag{11}$$
The converse statement also holds.

3. *A function f belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ if and only if its Fourier transform $\widetilde{f}(\xi)$ belongs to the Schwartz class.*

3.4 Smoothness and decay of Fourier transform. Smoothing

As for the Fourier series, smoothness of a function is tied to the decay of its Fourier transform (although exact theorems must be stated rather carefully). An example:

Exercise 3.12. *Prove:*

1. *If $f \in L_1(\mathbb{R})$, then $\widetilde{f}$ is bounded (in fact, even continuous and tending to zero at infinity).*
2. *If $f, f' \in L_1(\mathbb{R})$, then $\left|\widetilde{f}(\xi)\right| \leq C(1+|\xi|)^{-1}$.*
3. *If $f, f', f'', ..., f^{(n)} \in L_1(\mathbb{R})$, then $\left|\widetilde{f}(\xi)\right| \leq C(1+|\xi|)^{-n}$.*
4. *If $\left|\widetilde{f}(\xi)\right| \leq C(1+|\xi|)^{-\alpha}, \alpha > 1$, then f is bounded and continuous.*
5. *If $\left|\widetilde{f}(\xi)\right| \leq C(1+|\xi|)^{-\alpha}, \alpha > 2$, then f has a bounded and continuous derivative.*
6. *If for a function f on $\mathbb{R}^n$, $|\widetilde{f}(\xi)| \leq C_N(1+|\xi|)^{-N}$ for all $N > 0$ (i.e., $\widetilde{f}$ decays faster than any power), then $f \in C^\infty(\mathbb{R}^n)$.*

3.4.1 Smoothing

If one can guarantee fast decay of the Fourier transform of a function, then the function is smooth. This is the basis of standard smoothing procedures. Namely, suppose function $f(x)$ is not smooth (e.g., $f \in L_2(\mathbb{R})$ only). Assume that we have another function $w(x)$ whose Fourier transform $\widehat{w}$ is smooth and very fast decaying (for instance, even supported on a finite interval). Then taking convolution $f * w$, we get a smooth function. The reason is that $\widetilde{f * w}$ coincides (up to a constant factor) with $\widetilde{f}\widetilde{w}$, which decays due to the decay of $\widetilde{w}$. In other words, multiplication of $\widetilde{f}$ by $\widetilde{w}$ "filters out" high frequencies ξ, making the original function smoother. This is why $\widetilde{w}$ is often called a **filter**, or a **window function** (the window that allows certain frequencies through), while w is called a **mollifier**. There are quite a few window functions used in practice. The simplest one is the box function $\Pi(\xi)$ (the rectangular window). It has the disadvantage that it is not continuous, hence after the convolution the function will not decay fast, and one has to deal with long "tails." One also uses Gaussian filters, where the window function is the Gaussian $G_a(\xi) = \exp(-a\xi^2)$. There are many more commonly used filters.

Can one make the smoothed (**mollified**) function $f * w$ close to the original one? We cannot make it equal to f, since there is no identity element for the convolution. So, the question is whether one can find an approximate identity under the convolution, i.e. a sequence of functions w_n such that $w_n * f \to f$ for a reasonable class of functions f and a reasonable notion of convergence. This can be done. The simplest way of constructing approximate identities is the following:

Theorem 3.13. *Let $w(x)$ be smooth, supported on $[-1, 1]$, and such that $\int_{\mathbb{R}} w(x) = 1$. Define $w_n(x) = nw(nx)$. Then for any continuous function $f(x)$ on $\mathbb{R}$ the convolutions $f_n = w_n * f$ converge when $n \to \infty$ to f, where convergence is uniform on any finite interval.*

Analogous statements hold for different classes of functions, for instance for L_1-functions (then convergence also need to be understood in L_1-sense).

3.4.2 δ-function

If the functions $w_n(x)$ had some limit $\delta(x)$, then it would have the following impossible properties:

$$\delta(x) = 0 \text{ for } x \neq 0, \int_{\mathbb{R}} \delta(x)dx = 1.$$

This, however, did not stop famous physicist Dirac and electrical engineer Heaviside from using it. So, how can we handle such an object? Well, instead of looking at it as a function we look at it as a functional, i.e. an operation that assigns to each function $f(x)$ a number that we will symbolically write as

$$\int_{\mathbb{R}} \delta(x) f(x) dx.$$

So, the Dirac's δ-function $\delta(x)$ does not have values, but it has "integrals" after being multiplied by any continuous function $f(x)$. What should be the value of such an "integral?" The last theorem above implies that for any continuous function $f(x)$,

$$\lim_{n \to \infty} \int_{\mathbb{R}} w_n(x) f(x) dx = f(0).$$

So, we can define the Dirac's δ-function as follows:

Definition 3.14. *The Dirac's δ-function is the functional on the space of all continuous functions that on each continuous function $f(x)$ produces the value $f(0)$. Symbolically, we write this as*

$$< \delta, f >= f(0)$$

or **formally**

$$\int_{\mathbb{R}} \delta(x) f(x) dx = f(0).$$

A mathematician can also recognize the delta function as the unit "atomic" measure concentrated at the point 0.

4 Distributions

Definition 4.1. *A* **distribution** f *on* $\mathbb{R}^n$ *is a continuous linear functional*

$$\phi \mapsto (f, \phi) \in \mathbb{C}$$

on the space $\mathcal{D}(\mathbb{R}^n)$.

For instance, Dirac's δ-function, which acts as $(\delta, \phi) = \phi(0)$ is a distribution.

Any function $f \in L^1_{loc}(\mathbb{R}^n)$ is a distribution defined as $(f, \phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$. This is so called **regular distribution**.

Any distribution f **can be differentiated** as follows:

$$(D^\alpha f, \phi) := (-1)^{|\alpha|}(f, D^\alpha \phi)$$

for any $\phi \in \mathcal{D}(\mathbb{R}^n)$.

This is the so called **distributional derivative**.

Exercise 4.2.

- *Explain the reason for this definition of the derivative.* **Hint:** *try smooth functions f first.*
- *Compute $D^\alpha \delta$.*

5 Sobolev spaces

The relations between smoothness of a function and decay of its Fourier transform is seen best in the **Sobolev spaces** of functions.

Definition 5.1. *(**Sobolev spaces of a positive integer order**) Let $k \geq 0$ be an integer. A function f belongs to the **Sobolev space** $H^k(\mathbb{R}^n)$ if $D^\alpha f \in L^2(\mathbb{R}^n)$ for any $|\alpha| \leq k$ and*

$$\|f\|_{H^k}^2 := \int_{\mathbb{R}^n} \sum_{|\alpha|\leq k} |D^\alpha f(x)|^2 dx.$$

Here the derivatives $D^\alpha f$ are understood in the distributional sense.

The following statement can be proven using the properties of Fourier transform:

Theorem 5.2. *$f \in H^k(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^n} |\widetilde{f}(\xi)|^2 < \xi >^{2k} d\xi < \infty$$

(here $< \xi >= (1 + |\xi|^2)^{1/2}$). The norm $\|f\|_{H^k}$ is equivalent to

$$\left(\int_{\mathbb{R}^n} |\widetilde{f}(\xi)|^2 < \xi >^{2k} d\xi \right)^{1/2}.$$

Now one can define **Sobolev spaces of arbitrary (not necessarily integer and positive) order**:

Definition 5.3. *A function f belongs to the **Sobolev space** $H^s(\mathbb{R}^n), s \in \mathbb{R}$ if*

$$\int_{\mathbb{R}^n} |\widetilde{f}(\xi)|^2 < \xi >^{2s} d\xi < \infty.$$

The norm is defined as

$$\|f\|_{H^s}^2 := \int_{\mathbb{R}^n} |\widetilde{f}(\xi)|^2 < \xi >^{2s} d\xi.$$

6 Singular support of a function. Wavefront sets

We will indicate here some basic notions of microlocal analysis encountered in tomography. For more detailed accounts, refer to the literature Section 9.

6.1 Singular support

Definition 6.1. *A function f is smooth near point $x_0 \in \mathbb{R}^n$ if there is a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x_0) \neq 0$ and $\phi f \in C^\infty(\mathbb{R}^n)$.*

This notion corresponds well to our intuitive understanding of a function being smooth near x_0: it can be smoothly cut-off by a **cut-off function** ϕ to produce a function which is smooth everywhere. The cut-off factor ϕ eliminates all far away parts of f, which might not be smooth. The condition $\phi(x_0) \neq 0$ guarantees that we do not kill a possible singularity at x_0.

Definition 6.2. *The singular support* sing supp$f$ *of function f is the smallest closed set K such that f is smooth near any point of its complement $\mathbb{R}^n \setminus K$.*

So, the singular support set collects all points where f has some kind of singularity.

6.2 Microlocal detection of singularities. Wavefront set

Fourier transform allows one to check smoothness at a point x_0 by going to the Fourier domain:

Exercise 6.3. *Prove that a function f is smooth near x_0 if and only if there exists a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x_0) \neq 0$ and $\widetilde{(\phi f)}(\xi)$ decays at infinity faster than any power of $|\xi|$. In other words, for any $N > 0$, there is a constant C_N such that*

$$|\widetilde{(\phi f)}(\xi)| \leq C_N(1 + |\xi|)^{-N}. \tag{12}$$

So, at the points where f is singular, no cut-off leads to fast decay of the Fourier transform. However, the decay (12) might be possible to achieve at least in **some** of the directions of $\xi \in \mathbb{R}^n$. This leads to the definition:

Definition 6.4. *Let $x_0 \in \mathbb{R}^n$ and $\xi_0 \neq 0, \xi_0 \in \mathbb{R}^n$. The function f is said to be* **microlocally smooth near** (x_0, ξ_0)*, if there exists a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x_0) \neq 0$ and $\epsilon > 0$ such that for any $N > 0$, there is a constant C_N such that the inequality holds*

$$|\widetilde{(\phi f)}(\xi)| \leq C_N(1 + |\xi|)^{-N}. \tag{13}$$

for all ξ in the **conical neighborhood**

$$\left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon$$

of the vector ξ_0.

Definition 6.5. *The* ***Wavefront set*** *$WF(f)$ of function $f(x)$ consists of all points $(x_0, \xi_0) \in \mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus \{0\})$ such that f is* **not** *microlocally smooth near (x_0, ξ_0).*

Exercise 6.6. *Prove that* $WF(f)$ *is a closed conical (w.r.t.* ξ*) subset of* $\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus \{0\})$.

Theorem 6.7. *The projection onto the* x*-space of the wavefront set of* f *is the singular support of* f*. I.e., let* $\pi : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \mapsto \mathbb{R}^n_x$ *be the natural projection. Then* $\pi(WF(f)) = \operatorname{sing\,supp} f$.

Examples:

- $\operatorname{sing\,supp}(\delta) = \{0\}$
- $WF(\delta) = \{0\} \times (\mathbb{R}^n \setminus 0)$
- If $S \subset \mathbb{R}^n$ is a smooth hypersurface and $f(x)$ is infinitely differentiable on both sides of S, but is not smooth through S, then $WF(f)$ contains only pairs (x, ξ) such that $x \in S$ and ξ is normal to S at the point x.

7 Appendix 1. A crude idea of harmonic analysis

We will try to provide here a crude cartoon of the idea of what is called **harmonic (or Fourier) analysis**. The reader might enjoy reading the wonderful historical survey "Harmonic analysis as exploitation of symmetry" by G. Mackey [10].

Let A be an $n \times n$ matrix and e be its eigenvector corresponding to the eigenvalue λ:

$$Ae = \lambda e.$$

Exercise 7.1. *Prove:*

- *Assume that* λ *is a simple eigenvalue, i.e. it has unique (up to a scalar multiple) eigenvector* e*. Let matrix* B *commute with* A*. Then* e *must be an eigenvector of* B *as well.*
- *If* A *has a basis of eigenvectors and all the eigenvalues are simple, then any matrix* B *that commutes with* A *is diagonal in this basis.*

Remark 7.2. *Notice that when* λ *is not simple (i.e., has multiplicity), the conclusion is incorrect: while* Be *still is an eigenvector that corresponds to* λ*, it does not have to be proportional to* e*. In other words,* B *can "move around" eigenvectors corresponding to the same eigenvalue* λ *of* A*.*

Example 7.3. *Let* $A = diag(2, 2, 3)$*, i.e.*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

then any matrix B of the form

$$A = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

(where stars denote arbitrary numbers) commutes with A, but is not diagonal. The reason is the multiplicity of $\lambda = 2$.

This problem of multiplicity (which is rather common) can be alleviated by the following observation, which gives a more general principle:

Exercise 7.4. *Prove: Let $A_1, ..., A_m$ be $n \times n$ matrices and $e_1, ..., e_n$ be a basis such that all its vectors are eigenvectors of all the matrices A_j (i.e., $A_j e_i = \lambda_{ij} e_i$). Assume that for each two of these vectors there is a matrix among A_j whose eigenvalues corresponding to these two vectors are distinct. Let B be a matrix that commutes with all matrices A_j. Then B is diagonal in the basis $\{e_i\}$.*

Example 7.5.

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \; A_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \; A_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that these matrices satisfy the conditions of the above theorem. Prove that any B commuting with all of them is diagonal.

Exercise 7.6. *In the example above matrices A_j commute with each other. Does it always have to be the case under the conditions of the Theorem above?*

The idea of harmonic analysis: **If the operator B that you are interested in commutes with a family of matrices A_j satisfying the conditions of Exercise 7.4, then choosing the basis of joint eigenvectors of A_j simplifies (i.e. diagonalizes) B.**

Particular instances of this general principle are:

- When studying shift-invariant linear problems (e.g., constant coefficient PDEs, or Radon transforms), Fourier transform might be useful.
- When studying rotation-invariant linear problems (e.g., Laplace equation, or Radon transform) in $\mathbb{R}^2$, Fourier series with respect to the polar angle might be useful. (Higher dimensional rotation invariant problems require more sophisticated expansions into the so called spherical harmonics [15]. This is related to the non-commutativity of the higher dimensional groups of rotations.)

8 Appendix 2. Notations

For points $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we denote $|x| = \sqrt{x_1^2 + ...x_n^2}$ and by $< x >= (1 + |x|^2)^{1/2}$. For $x, y \in \mathbb{R}^n, x \cdot y = x_1 y_1 + \ldots x_n y_n$.

A vector $\alpha = (\alpha_1, ...\alpha_n) \in (\mathbb{Z}^+)^n$ – a **multi-index**. We use the following notations:

$$\begin{gathered} |\alpha| = \sum \alpha_j \\ x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_1^{\alpha_1} \quad \text{for } x \in \mathbb{R}^n \end{gathered} \tag{14}$$

and

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \partial_j = \frac{\partial}{\partial x_j}, D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}, \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}}.$$

Definition 8.1. *$\mathcal{D}(\mathbb{R}^n)$ (or $\mathcal{D}(X)$ for some domain X) - the space of all C^∞ functions with compact support in $\mathbb{R}^n$ (in X). Convergence of a sequence $\phi_j \in \mathcal{D}$ means that all supports are contained in a fixed finite ball, and functions uniformly converge with all their derivatives.*

The space $\mathcal{D}$ is also denoted by C_c^∞, C_0^∞, or C_{comp}^∞.

Definition 8.2. *$\mathcal{S}(\mathbb{R}^n)$ - the* **Schwartz space** *of C^∞-functions that decay (with all the derivatives) at infinity faster any power of the argument.*

In other words, $\phi \in \mathcal{S}$ iff it is infinitely differentiable and all the semi-norms

$$d_{M,N}(\phi) = \max_{x \in \mathbb{R}^n, |\alpha| \le M} |D^\alpha \phi(x)| < x >^N$$

are finite.

We remind the notations $L^p(X)$ and $L^p_{loc}(X)$ for a domain $X \subset \mathbb{R}^n$. A measurable function f on X belongs to $L^p(X)$ for some $p \in [1, \infty)$, if its pth power is absolutely integrable over X, i.e.

$$\int_X |f(x)|^p dx < \infty.$$

The $L^p(X)$**-norm** of such a function is defined as

$$\|f\|_{L^p(X)}(\text{ or just } \|f\|_p) = \left(\int_X |f(x)|^p dx \right)^{1/p}.$$

For $p = \infty$, the L^∞ space consists of essentially bounded functions f with the bound

$$\|f\|_{L^\infty(X)}(\text{ or just } \|f\|_\infty) = \operatorname*{ess\,sup}_{x \in X} |f(x)|.$$

A larger space of functions is $L^p_{loc}(\mathbb{R}^n)$ (can be defined analogously on any domain $X \subset \mathbb{R}^n$). It consists of measurable functions such that they belong to

L^p **locally**. I.e., f belongs to $L^p(B_R)$ on any ball of finite radius R in $\mathbb{R}^n$ (in case of a general domain X, f belongs to $L^p(K)$ on any compact $K \Subset X$), while not necessarily $f \in L^p(\mathbb{R}^n)$.

Compactly supported functions are those vanishing outside of a sufficiently large ball (depending on the function).

9 Some remarks on the literature

Disclaimer: There are many other good sources, besides the ones mentioned below. 

Fourier series. [18] for $1D$ and [15] for any dimension.

Fourier transform. A nice introduction to Fourier transform, distributions, and microlocal analysis is [16]. Several books of E. Stein provide Fourier analysis theory on different levels (e.g., [14] and the classics [15]). Körner's book [9] is a wonderful collection of essays, proofs, historical accounts, etc. concerning the Fourier analysis. Hörmander's volume [6] is a comprehensive (albeit very technical) account of Fourier analysis. The classical Natterer's book [11] shows how Fourier analysis works in tomography.

Distributions. [16] provides a nice introduction. The classical book [4] is still of value. An interesting (albeit outdated) account is given in [1]. A contemporary comprehensive account can be found in [6].

Microlocal analysis. We have barely touched the microlocal analysis here. As mentioned above, [16] provides a nice introduction. This can be also said about [12]. Many good more advanced sources are available, e.g. [2, 3, 5, 7, 8, 17, 19].

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